Representation of Geometric Objects

Chapter 4

University of Amsterdam

On Representations

- The most fundamental geometric object is a *point* usually represented by (at least) its *coordinates* w.r.t. some *frame*.
- A representation can contain additional (redundant) information, in which case it is not *minimal*. The number of crucial (scalar) parameters is the number of *degrees of freedom* (DOF).
- Representations "live" on a manifold of dimension #DOF embedded in a higher dimensional space. The shape of the manifold is determined by *constraints* on the representation.
- "Higher" geometric objects are usually thought of as the set (or *locus*) of points fulfilling a certain equation (e.g. in terms of the points' coordinates).
- The set representation is clumsy at not very accessible, mathematically. Therefore we will look at more (or less) clever ones.

Commonalities

- Give the set representation of an object.
- Usually try to represent objects by vectors, because we can use linear algebra on them. Also try to keep them 3D because then the cross-product is available.
- Apply noise to the coordinates of the representation and analyze the covariance matrix. Its rank gives the #DOF of the object.
- For composite objects, perturb the components and analyze the effect on (the covariance matrix of) the object.

Representation

- Introduce homogeneous coordinates. According to Kanatani a vector

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{1}$$

only represents an image point, if $z = 1$.

- This can be achieved by dividing by the actual z.
- Theoretically, while $z \to 0$, $\mathbf{x}$ moves towards infinity in the *direction* of $(x, y)^\top$. $(x, y, 0)^\top$ is called *ideal point*.
- Numerically unstable!

Noise on Point Coordinates

- Kanatani disturbs the 3D vector $\mathbf{x}$, *while* keeping the condition $z = 1$ invariant.

$$\mathbf{x} + \mathbf{\Delta x} = (x + \Delta x, y + \Delta y, 1)^\top, \tag{2}$$

  i.e. noise effectively only occurs on x and y, i.e. orthogonal to $\mathbf{k} = (0, 0, 1)^\top$.

- Consequently the covariance matrix

$$V[\mathbf{x}] = E[\mathbf{\Delta x}\mathbf{\Delta x}^\top] \tag{3}$$

  has rank 2.

- $\mathbf{\Delta x} = (\Delta x, \Delta y, 0)^\top$ is regarded as small random variable of mean $\mathbf{0}$.

Representation

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- Set representation of an image line

$$\{(x, y) | Ax + By + C = 0\} \quad (4)$$

- Vector representation of an image line

$$\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}, ||\mathbf{n}||^2 = 1 \quad (5)$$

- Normalization removes scale ambiguity but leaves sign ambiguity. **n** can be interpreted as the 3D normal vector of the plane joining the viewpoint and the image line.
- Image lines have 2 DOF and therefore "live" on a 2D manifold (the unit sphere) embedded in 3D space.

Noise on Line Coordinates

- Kanatani *first* disturbs $\mathbf{n}$, *then* normalizes, i.e.

$$
\begin{aligned}
1 &= ||\mathbf{n} + \mathbf{\Delta n}||^2 && (6)\\
&= ||\mathbf{n}||^2 + 2(\mathbf{n}, \mathbf{\Delta n}) + ||\mathbf{\Delta n}||^2 && (7)\\
&\approx ||\mathbf{n}||^2 + 2(\mathbf{n}, \mathbf{\Delta n}). && (8)
\end{aligned}
$$

- With the normalization condition $||\mathbf{n}||^2 = 1$ this gives $(\mathbf{n}, \mathbf{\Delta n}) = 0$, i.e. noise occurs only orthogonal to $\mathbf{n}$ to first order approximation.
- Consequently the covariance matrix $V[\mathbf{n}] = E[\mathbf{\Delta n}\mathbf{\Delta n}^\top]$ has rank 2.

Perks of the Representation

- Distance between image line and image point is calculated as

$$D(\mathbf{p}, \mathbf{l}) = \frac{|(\mathbf{n}, \mathbf{x})|}{\sqrt{1 - (\mathbf{k}, \mathbf{x})^2}}, \tag{9}$$

  i.e. the point is on the line if $D(\mathbf{p}, \mathbf{l}) = 0$.

- Intersection point of two lines is

$$\mathbf{x} = \frac{\mathbf{n}_1 \times \mathbf{n}_2}{|\mathbf{n}_1, \mathbf{n}_2, \mathbf{k}|}, \tag{10}$$

  which becomes an ideal point as the denominator approaches zero, but the formula does not fail gracefully.

- A line can be calculated as the *join* of two points as

$$\mathbf{n} = \pm N[\mathbf{x}_1 \times \mathbf{x}_2]. \tag{11}$$

Noise on Component Points

- Perturbing two points $\mathbf{x}_1$ and $\mathbf{x}_2$ by noise as above, the join of the two is perturbed as well. We spell out the formula once; future formulas will follow a derivation along the same lines.
- Kanatani performs an approximation to first order, giving

$$\begin{aligned}
\mathbf{n} + \mathbf{\Delta n} &= \pm N[(\mathbf{x}_1 + \mathbf{\Delta x}_1) \times (\mathbf{x}_2 + \mathbf{\Delta x}_2)] \\
&= \frac{\pm(\mathbf{x}_1 + \mathbf{\Delta x}_1) \times (\mathbf{x}_2 + \mathbf{\Delta x}_2)}{||\mathbf{x}_1 \times \mathbf{x}_2||} \\
&= \frac{\pm(\mathbf{x}_1 \times \mathbf{x}_2 + \mathbf{\Delta x}_1 \times \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{\Delta x}_2 + \mathbf{\Delta x}_1 \times \mathbf{\Delta x}_2)}{||\mathbf{x}_1 \times \mathbf{x}_2||} \\
&\approx \frac{\mathbf{x}_1 \times \mathbf{x}_2}{||\mathbf{x}_1 \times \mathbf{x}_2||} + \frac{\mathbf{\Delta x}_1 \times \mathbf{x}_2 + \mathbf{x}_1 \times \mathbf{\Delta x}_2}{||\mathbf{x}_1 \times \mathbf{x}_2||} \qquad (12)
\end{aligned}$$

Noise on Component Points, cont'd

- The covariance matrix of the image line becomes

$$
\begin{aligned}
V[\mathbf{n}] &= E[\mathbf{\Delta n \Delta n}^\top] && (13)\\
&= \frac{P_{\mathbf{n}}(\mathbf{x}_1 \times V[\mathbf{x}_2] \times \mathbf{x}_1 + \mathbf{x}_2 \times V[\mathbf{x}_1] \times \mathbf{x}_2)P_{\mathbf{n}}}{||\mathbf{x}_1 \times \mathbf{x}_2||^2} && (14)
\end{aligned}
$$

Representation

- Homogeneous coordinates again. All remarks regarding image points hold.

Representation

- Set representation of a space line

$$\{\mathbf{r} | (\mathbf{r} - \mathbf{r}_H) \times \mathbf{m} = 0\} \tag{15}$$
$$\{\mathbf{r} | \mathbf{r} \times \mathbf{m} = \mathbf{r}_H \times \mathbf{m}\} \tag{16}$$

- Concatenation (i.e. direct sum) of $\mathbf{m}$ and $\mathbf{r}_H$ represents the line $\mathbf{l} = \mathbf{m} \oplus \mathbf{r}_H$.
- Above equation holds for any scalar multiple of the direction vector $\mathbf{m}$ and for *any point on the line* $\mathbf{r}_H$. Therefore Kanatani requires

$$||\mathbf{m}|| = 1, \quad (\mathbf{m}, \mathbf{r}_H) = 0 \tag{17}$$

- Therefore the 6D vector $\mathbf{l} = \mathbf{m} \oplus \mathbf{r}_H$ has only 4 DOF.

Alternative Representation, Plücker Coordinates

- If we vary the given point on the line $\mathbf{r}_H$, we will always end up with a scalar multiple of the same vector $\mathbf{n} = \mathbf{r}_H \times \mathbf{m}$, which yields an alternative representation by $\mathbf{l} = \mathbf{p} \oplus \mathbf{n}$.
- For some reason, the direction vector $\mathbf{m}$ is now called $\mathbf{p}$.
- Approaching the matter reversely, not every pair of vectors $\{\mathbf{p}, \mathbf{n}\}$ represents a line. Require

$$||\mathbf{p}||^2 + ||\mathbf{n}||^2 = 1, \quad (\mathbf{p}, \mathbf{n}) = 0. \tag{18}$$

- Consequently the 6D vector $\mathbf{l} = \mathbf{p} \oplus \mathbf{n}$ also has 4 DOF.

Noise on Line Coordinates

- Perturbing the 3D vectors $\mathbf{m}$ and $\mathbf{r}_H$ by - *not necessarily independent* - zero mean random variables $\boldsymbol{\Delta}\mathbf{m}$ and $\boldsymbol{\Delta}\mathbf{r}$, respectively yields

$$V[\mathbf{l}] = V[\mathbf{m} \oplus \mathbf{r}_H] = \begin{pmatrix} V[\mathbf{m}] & V[\mathbf{m}, \mathbf{r}_H] \\ V[\mathbf{r}_H, \mathbf{m}] & V[\mathbf{r}_H] \end{pmatrix}, \qquad (19)$$

with

$$(\mathbf{m}, \boldsymbol{\Delta}\mathbf{m}) = 0, \quad (\boldsymbol{\Delta}\mathbf{m}, \mathbf{r}_H) + (\mathbf{m}, \boldsymbol{\Delta}\mathbf{r}_H) = 0. \qquad (20)$$

- Due to these constraints, the 6×6 covariance matrix has rank 4.

Noise on Plücker Coordinates

- If the line is represented by Plücker coordinates, we have

$$V[\mathbf{l}] = V[\mathbf{p} \oplus \mathbf{n}] = \begin{pmatrix} V[\mathbf{p}] & V[\mathbf{p}, \mathbf{n}] \\ V[\mathbf{n}, \mathbf{p}] & V[\mathbf{n}] \end{pmatrix} \tag{21}$$

with

$$(\mathbf{p}, \mathbf{\Delta p}) + (\mathbf{n}, \mathbf{\Delta n}) = 0, \quad (\mathbf{\Delta p}, \mathbf{n}) + (\mathbf{p}, \mathbf{\Delta n}) = 0. \tag{22}$$

- Due to these constraints, the 6×6 covariance matrix has rank 4.
- There is a hugely complicated formula which relates the two covariance matrices (19) and (21), (Kanatani: 4.46).

Perks of the Representation

- Distance between a (space) point and a (space) line

$$D(P, L) = \frac{||\mathbf{r} \times \mathbf{p} - \mathbf{n}||}{||\mathbf{p}||} = ||P_{\mathbf{m}}\mathbf{r} - \mathbf{r}_H|| \qquad (23)$$

- Intersection between two (space) lines

$$D(L, L') = \begin{cases} \frac{|\mathbf{m}, \mathbf{m}', \mathbf{r}_H - \mathbf{r}'_H|}{||\mathbf{m} \times \mathbf{m}'||} & \text{if } \mathbf{m} \times \mathbf{m}' \neq 0 \\ ||\mathbf{r}_H - \mathbf{r}'_H|| & \text{if } \mathbf{m} \times \mathbf{m}' = 0 \end{cases} \qquad (24)$$

- As $\mathbf{m} \times \mathbf{m}'$ approaches zero, the distance becomes infinitely large and the point of intersection becomes an ideal point. But due to the division the respective formulas do not fail gracefully.

Perks of the Representation, cont'd

- Distance between two (space) lines in Plücker coordinates

$$D(L, L') = \begin{cases} \frac{(\mathbf{p},\mathbf{n}')+(\mathbf{p}',\mathbf{n})}{||\mathbf{p}\times\mathbf{p}'||} & \text{if } \mathbf{p}\times\mathbf{p}' \neq 0 \\ ||\frac{\mathbf{n}}{||\mathbf{p}||} - \frac{\mathbf{n}}{||\mathbf{p}'||} & \text{if } \mathbf{p}\times\mathbf{p}' = 0 \end{cases} \tag{25}$$

- Point of intersection between two (co-planar space) lines

$$\mathbf{r} = \frac{(\mathbf{m}\times\mathbf{r}_H)\times(\mathbf{m}'\times\mathbf{r}'_H)}{|\mathbf{m},\mathbf{m}',\mathbf{r}_H|} = \frac{\mathbf{n}\times\mathbf{n}'}{(\mathbf{p}',\mathbf{n})} \tag{26}$$

- More convenient to work with - because it makes sense to detect whether direction vectors are parallel as a first step - is

$$\mathbf{r} = \mathbf{r}_H + \frac{(\mathbf{m},\mathbf{r}_H)+(\mathbf{m},\mathbf{m}')(\mathbf{m}',\mathbf{r}_H)}{||\mathbf{m}\times\mathbf{m}'||^2}\mathbf{m} \tag{27}$$

$$= \frac{1}{||\mathbf{p}||^2}\left(\frac{||\mathbf{p}||^2|\mathbf{p},\mathbf{p}',\mathbf{n}'| - (\mathbf{p},\mathbf{p}')|\mathbf{p},\mathbf{p}',\mathbf{n}|}{||\mathbf{p}\times\mathbf{p}'||^2}\mathbf{p} + \mathbf{p}\times\mathbf{n}\right) \tag{28}$$

Perks of the Representation, cont'd

- Lines can be calculated as join of two points as

$$\mathbf{m} = N[\mathbf{r}_1 - \mathbf{r}_2], \quad \mathbf{r}_H = \frac{(\mathbf{m}, \mathbf{r}_1)\mathbf{r}_2 - (\mathbf{m}, \mathbf{r}_2)\mathbf{r}_1}{||\mathbf{r}_1 - \mathbf{r}_2||} \tag{29}$$

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{n} \end{pmatrix} = N[\begin{pmatrix} \mathbf{r}_1 - \mathbf{r}_2 \\ \mathbf{r}_2 \times \mathbf{r}_1 \end{pmatrix}] \tag{30}$$

- Under assumption of independent uncertain $\mathbf{r}_1$ and $\mathbf{r}_2$ the line's covariance matrix becomes

$$V[\mathbf{p} \oplus \mathbf{n}] = \frac{1}{||\mathbf{r}_1 - \mathbf{r}_2||^2 + ||\mathbf{r}_2 \times \mathbf{r}_1||^2} P_{\mathbf{p}\oplus\mathbf{n}} \, M \, P_{\mathbf{p}\oplus\mathbf{n}}$$

$$M = \begin{pmatrix} V[\mathbf{r}_1] + V[\mathbf{r}_2] & V[\mathbf{r}_1] \times \mathbf{r}_2 + V[\mathbf{r}_2] \times \mathbf{r}_1 \\ \mathbf{r}_2 \times V[\mathbf{r}_1] + \mathbf{r}_1 \times V[\mathbf{r}_2] & \mathbf{r}_2 \times V[\mathbf{r}_1] \times \mathbf{r}_2 + \mathbf{r}_1 \times V[\mathbf{r}_2] \times \mathbf{r}_1 \end{pmatrix} \tag{31}$$

Representation

- Set representation of a space plane

$$\{(x, y, z) | Ax + By + Cz = d\} \tag{32}$$

- Equivalent formulations are $\{\mathbf{r} | (\mathbf{r}, \mathbf{n}) = d\}$ and $\{\rho | (\rho, \nu) = 0\}$. In the latter case remarks about homogeneous coordinates apply.
- The equations leave a freedom of scale. Instead of fixing the last coordinate, we normalize the 4D vector, imposing

$$||\nu|| = 1 \tag{33}$$

- This leaves the 4D vector ν with 3 DOF.

Noise on Plane Coordinates

- The covariance matrix

$$V[\nu] = \frac{1}{1+d^2} P_\nu \begin{pmatrix} V[\mathbf{n}] & -V[\mathbf{n}, d] \\ -V[\mathbf{n}, d] & V[d] \end{pmatrix} P_\nu \qquad (34)$$

has rank 3.

- Perturbing the 4D vector ν and applying the constraints we get to a first order approximation

$$\boldsymbol{\Delta}\mathbf{n} = \sqrt{1+d^2} P_\nu \begin{pmatrix} \Delta\nu_1 \\ \Delta\nu_2 \\ \Delta\nu_3 \end{pmatrix}, \quad \Delta d = -\sqrt{(1+d^2)^3} \Delta\nu_4 \qquad (35)$$

Perks of the Representation

- Distance between point and plane is calculated as

$$D(P, \Pi) = |(\mathbf{n}, \mathbf{r}) - d| = \frac{|(\nu, \rho)|}{\sqrt{1 - (\kappa, \nu)^2}} \tag{36}$$

- A line and a plane are incident to each other if and only if

$$(\mathbf{n}, \mathbf{m}) = 0, \quad (\mathbf{n}, \mathbf{r}_H) = d \tag{37}$$

or, equivalently

$$(\nu, \mathbf{m} \oplus 0) = 0, \quad (\nu, \mathbf{r}_H \oplus 1) = 0 \tag{38}$$

- Note how Plücker and homogeneous coordinates are basically incompatible and the “gap” between the two has to be patched up.

Perks of the Representation, cont'd

- The line of intersection between two non-parallel planes is calculated as

$$\mathbf{m} = N[\mathbf{n}_1 \times \mathbf{n}_2],$$

$$\mathbf{r}_H = \frac{(d_1 - (\mathbf{n}_1, \mathbf{n}_2)d_2)\mathbf{n}_1 + (d + 2 - (\mathbf{n}_1, \mathbf{n}_2)d_1)\mathbf{n}_2}{||\mathbf{n}_1 \times \mathbf{n}_2||^2} \quad (39)$$

  or as

$$\begin{pmatrix} \mathbf{p} \\ \mathbf{n} \end{pmatrix} = N[\begin{pmatrix} \mathbf{n}_1 \times \mathbf{n}_2 \\ d_2\mathbf{n}_1 - d_1\mathbf{n}_2 \end{pmatrix}]. \quad (40)$$

- Planes can be calculated as the join of
  - 3 space points (Kanatani: 4.76)
  - a space line and a space point (Kanatani: 4.77)
  - two intersecting space lines (Kanatani: 4.78)

Representation

- A conic is the intersection of a cone with a plane. Depending on the type of intersection it can be a point, a line, two lines, a hyperbola, a parabola, a circle, an ellipse
- All can be represented by one equation

$$\{(x, y)|Ax^2 + Bxy + Cy^2 + 2Dx + 2Ey + F = 0\} \tag{41}$$

- This equation is indeterminate only in x and y, so we assume the intersecting plane as fixed (e.g. the image plane) and specify points in it by 2 coordinates.
- The equation is quadratic in x and y and therefore algebraically it has more than the obvious solutions, namely *complex valued* ones. This leads to *imaginary conics*.
- The parameters are only determined up to a common scale, so a conic has 5 DOF.

Representation, cont'd

- The 6 parameters can be stacked into a 3×3 matrix Q. Firstly, we can then use it on 3D object, such as points. Secondly, this form facilitates analysis of the conic.
- Points $\mathbf{x}$ on the conic then fulfill

$$(\mathbf{x}, Q\mathbf{x}) = 0, \quad Q = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} \tag{42}$$

- If $\det Q = 0$, the conic defines two (real or imaginary) lines.
- If $\det Q \neq 0$, the conic defines *real conic* (i.e. an ellipse, a parabola or a hyperbola), if and only if its *signature* is $(2, 1)$ or $(1, 2)$ (two positive and one negative eigenvalue or vice versa). Specifically, if we choose $\det Q \leq 0$
  - if $AC - B^2 > 0$, then
    - if $A + C > 0$, the conic is a real ellipse
    - if $A + C < 0$, the conic is an imaginary ellipse
  - if $AC - B^2 = 0$, the conic is a parabola
  - if $AC - B^2 < 0$, the conic is a hyperbola

Pole-Polar-Relationship

- For a given image point $\mathbf{x}_p$ the line

$$\mathbf{n}_p = \pm N[Q\mathbf{x}_p] \tag{43}$$

is called the *polar* of $\mathbf{x}_p$ w.r.t. Q.

- For an image line $(\mathbf{n}_p, \mathbf{x}) = 0$, the point

$$\mathbf{x}_p = \frac{Q^{-1}\mathbf{n}_p}{(\mathbf{k}, Q^{-1}\mathbf{n}_p)}, \quad \mathbf{k} = (0, 0, 1)^\top \tag{44}$$

is called the *pole* of the line w.r.t. Q.

Canonical Forms

- Because our matrix representation has more parameters than DOF, different matrices represent the same conic. We are allowed to apply certain transformations to Q that preserve certain invariants, such as determinant and signature.
- Using these *congruence transformations* we can bring the matrix Q representing a conic into a standard (or canonical) form. This step is equivalent to the normalization step performed on vector representations.
- This makes it easier to compare conics and "read off" certain information, such as *principal axes*, *radii* and *eccentricity*.
- For details, refer to Kanatani, section 4.4.2.

Space Conics

- Now we leave the image plane and obtain a conic section by intersecting an arbitrary space plane with a cone.
- Kanatani restricts himself to conics that are the intersection between an arbitrary space plane and a cone *with vertex at the viewpoint*.
- We specify the space plane and in it points $\mathbf{r}$ that fulfill (41) by

$$\{\mathbf{r} | (\mathbf{n}, \mathbf{r}) = d \text{ and } (\mathbf{r}, Q\mathbf{r}) = 0\} \tag{45}$$

- We need additional parameters to specify the space plane, and Kanatani does not weave them into the matrix representation, but uses the *set* $\{\mathbf{n}, d, Q\}$ to represent a conic in space.
- The matrix Q is the matrix that represents the conic *in the image plane* which results from the space conic being projected there.

Representation

- A quadric is a 2D surface in 3D space made up by points satisfying a quadratic equation in 3 coordinates.
- It should have 10 parameters, but - since its equation is only determined up to a common scale - only 9 DOF.
- Instead of squeezing those into a single matrix, we split off a *symmetric* 3×3 matrix S (6 DOF) and a 3D vector (3 DOF), giving the set representation of a quadric

$$\{\mathbf{r} | (\mathbf{r} - \mathbf{r}_C, S(\mathbf{r} - \mathbf{r}_C)) = 1\} \tag{46}$$

Representation, cont'd

- While $\mathbf{r}_C$ gives the quadrics center (i.e. its location in 3D space), the matrix S determines its shape. Specifically
  - if S is positive definite, the quadric is an ellipsoid.
  - if S is negative definite, the quadric is an empty set.
  - if S has signature $(2, 1)$, the quadric is a hyperboloid of one sheet.
  - if S has signature $(1, 2)$, the quadric is a hyperboloid of two sheets.
  - if S is singular, the quadric is degenerate.
  - the eigenvalues of S are the quadric's *principal axes*.
  - the reciprocal of the square roots of the positive eigenvalues are its *radii*.

Pole-Polar-Relationship

- Just like conics, quadrics have poles (space points) and polars (space planes) defined by

$$\mathbf{n}_p = N[S(\mathbf{r}_p - \mathbf{r}_C)], \quad d_p = \frac{1}{||S(\mathbf{r}_p - \mathbf{r}_C)||} + (\mathbf{r}_C, \mathbf{n}_p) \quad (47)$$

$$\mathbf{r}_p = \frac{S^{-1}\mathbf{n}}{d_p - (\mathbf{r}_C, \mathbf{n}_p)} \quad (48)$$

- The *conjugate direction* is the direction $\mathbf{n}^\dagger$ from the quadric's center $\mathbf{r}_C$, in which the point on the quadric lies having a surface normal that matches a given direction $\mathbf{n}$.

Perks of the Representation

- As the covariance matrix of a (3D) random variable $\mathbf{r}$ is symmetric positive semi-definite, it can be taken to represent a quadric('s shape). If we locate the quadric at the mean of $\mathbf{r}$, $\hat{\mathbf{r}}$, we can visualize the covariance matrix as the ellipsoid

$$(\mathbf{r} - \hat{\mathbf{r}}, V[\hat{\mathbf{r}}]^{-1}(\mathbf{r} - \hat{\mathbf{r}})) = 1 \tag{49}$$

- If we lower the dimension, the quadric degenerates into a conic, and finally into a line segment, all three of which are called the random variable's *standard confidence region*.

Perks of the Representation, cont'd

- If we raise the dimension,

$$(\mathbf{r} - \hat{\mathbf{r}}, V[\hat{\mathbf{r}}]^{-}(\mathbf{u} - \hat{\mathbf{u}})) = 1 \tag{50}$$

  is a quadric that is generally singular, but becomes non-singular, if it is restricted to $T_{\hat{\mathbf{r}}}(\mathcal{U})$, the tangent space at $\hat{\mathbf{r}}$ to the manifold $\mathcal{U}$ in which $\hat{\mathbf{r}}$ lies.
- In higher dimensions we cannot easily visualize the quadric and revert to vector pairs along its principal axes to indicate the standard confidence region.

Rigid Body and Perspective Transformations

- Kanatani gives formulas for what happens to the different representations of different objects under a change of coordinates, i.e. a transformation of the object.
- Transformation considered are *rigid body motions* (i.e. rotations and translations) as well as *projective transformations* (which can change ideal objects into regular ones and vice versa).
- The formulas given are dependent on the object that is transformed.
- For details, refer to Kanatani section 4.6.