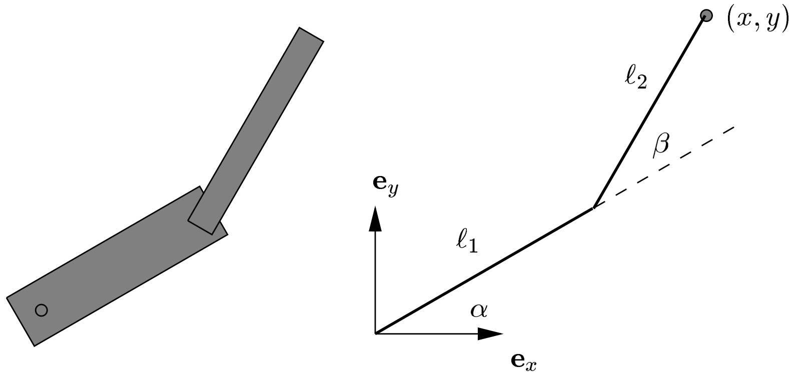
Robot coordinate transformations and their Jacobians

This is a two link robot, with a shoulder angle α and an elbow angle β. The upper arm has length ℓ_1, the lower arm length ℓ_2. The 'hand' of the robot is at a point (x, y) in a Cartesian coordinate system.



We control the robot by changing α and β, but for a task we are interested in where the hand is.

(1) Mathematically, we are interested in the mapping f from (α, β)-coordinates to (x, y)-coordinates, and in its 'inverse' g from (x, y)-coordinates to (α, β)-coordinates. Can you give physical reasons why I put quotes around 'inverse', instead of just calling $g = f^{-1}$?

Hint: is the inverse *unique*, i.e. is there always a unique set of angles that will put the hand at (x, y)?

(2) What is the mapping g? Write $x = g_x(\alpha, \beta)$ and $y = g_y(\alpha, \beta)$ out in formulas. You should view ℓ_1 and ℓ_2 as *parameters* of these formulas, and α and β as *variables* of the functions.

Answer: Straightforward goniometry yields:

$$\begin{cases} x = \ell_1 \cos(\alpha) + \ell_2 \cos(\alpha + \beta) \\ y = \ell_1 \sin(\alpha) + \ell_2 \sin(\alpha + \beta) \end{cases} \tag{1}$$

(3) Is this a linear mapping? For instance, if α and β become twice as big, do x and y become twice as big as well? Can you write it as a matrix? (**Answer**: no!)

(4) Differentiate the formulas eq.(1). You have to use *partial differentiation*, since x depends on both variables α and β. Remember the partial differentiation formula:

$$dx = \frac{\partial g_x}{\partial \alpha} d\alpha + \frac{\partial g_x}{\partial \beta} d\beta \quad \text{and} \quad dy = \frac{\partial g_y}{\partial \alpha} d\alpha + \frac{\partial g_y}{\partial \beta} d\beta \tag{2}$$

and apply them.

Answer:

$$\begin{cases} dx = -\ell_1 \sin(\alpha) d\alpha - \ell_2 \sin(\alpha + \beta) d\alpha - \ell_2 \sin(\alpha + \beta) d\beta \\ dy = \ell_1 \cos(\alpha) d\alpha + \ell_2 \cos(\alpha + \beta) d\alpha + \ell_2 \cos(\alpha + \beta) d\beta \end{cases} \tag{3}$$

(5) Is the mapping from $(d\alpha, d\beta)$ to (dx, dy) linear? If so, write it as a matrix equation.

Answer: yes it is, even though it depends non-linearly on α and β. The matrix form of eq.(3) is:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} -\ell_1 \sin(\alpha) - \ell_2 \sin(\alpha + \beta) & -\ell_2 \sin(\alpha + \beta) \\ \ell_1 \cos(\alpha) + \ell_2 \cos(\alpha + \beta) & \ell_2 \cos(\alpha + \beta) \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \tag{4}$$

(6) The matrix you have introduced has partial derivatives as its elements. It is called the *Jacobian* $\mathbf{J}$, and it is very useful for computing the effect of small changes.

(7) We can use eq.(4) as follows: if there are small changes in α and β of $(\delta\alpha, \delta\beta)$, then we get a small changes δx and δy, given by the formula $(\delta x, \delta y)^T = \mathbf{J}(\delta\alpha, \delta\beta)^T$. (This is because we can develop $g = (g_x, g_y)$ in a Taylor series around (α, β), and then eq.(4) is the first term, which gives the most important displacements for small changes).

(8) Let us do an example.

(a) Take $\ell_1 = \ell_2 = 1$ (in meters). Let us assume the robot is in the state $(\alpha, \beta) = (\pi/4, \pi/4)$. Draw this! What is the Jacobian in this state?
Answer:

$$\begin{pmatrix} (-\frac{1}{\sqrt{2}} - 1) & -1 \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \tag{5}$$

(b) Let the robot wave a little, from the elbow, so take $(\delta\alpha, \delta\beta) = (0, \pm\epsilon)$, with $\epsilon = 0.02$ (0.02 radians). What is $(\delta x, \delta y)$ approximately? So in which direction is the main motion (to first order)?

Answer: Use the Jacobian and obtain: $(\delta x, \delta y) = (\mp\epsilon, 0)$. So the motion is in the x-direction only (which you should understand from your drawing), and just opposite to the change ϵ of the angle β: when β increases a little, x decreases a little. The amount of the displacement is equal to the change in radians, because ℓ_2 is equal to 1 meter.

(9) When the robot is fully folded up or stretched out (so $\beta = 0$ or $\beta = \pi$), there is no total freedom om motion anymore: the hand will have to move tangentially to the robot arm, it cannot easily get closer to the shoulder joint. It is almost as if it has lost a degree of freedom of motion. How do you expect the Jacobian to change?

(10) Purely mathematically, the degeneracy of the Jacobian matrix can be determined from a certain property of the matrix. Which one?

(11) So, compute when the determinant of the Jacobian equals zero. This should tell you at which angles (α, β) the robot becomes rather restrained in its movement.

Answer: You need to use some goniometric formulas to simplify the Jacobian determinant. This is good practice, so do it. The result is: $\det(\mathbf{J}) = \ell_1\ell_2 \sin\beta$. This is 0 (so $\mathbf{J}$ is degenerate) when $\beta = 0$ or $\beta = \pi$, just as expected.

(12) Compute the Jacobian in the degenerate cases, and using this to compute (dx, dy).
Answer

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} -(\ell_1 \pm \ell_2)\sin(\alpha) & \mp\ell_2\sin(\alpha) \\ (\ell_1 \pm \ell_2)\cos(\alpha) & \pm\ell_2\cos(\alpha) \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \tag{6}$$

(13) Back to the example to get the intuition:

(a) Give eq.(6) in the example, where $\ell_1 = \ell_2 = 1$. Since everything that happens at an angle $\alpha = \phi$ is the same as what happens at $\alpha = 0$, but just turned over ϕ, we can take $\alpha = 0$ to study the essence. Treat both of the cases we found above: first take $(\alpha, \beta) = (0, 0)$, then $(\alpha, \beta) = (0, \pi)$.
Answer:

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d\alpha \\ d\beta \end{pmatrix} \tag{7}$$

(b) You see from eq.(7) that $dx = 0$, always, in both situations. Draw both, and make sure you agree with this outcome.

(c) Now look at your drawings again, and derive eq.(7) immediately, without computation. Make sure you get the signs right!