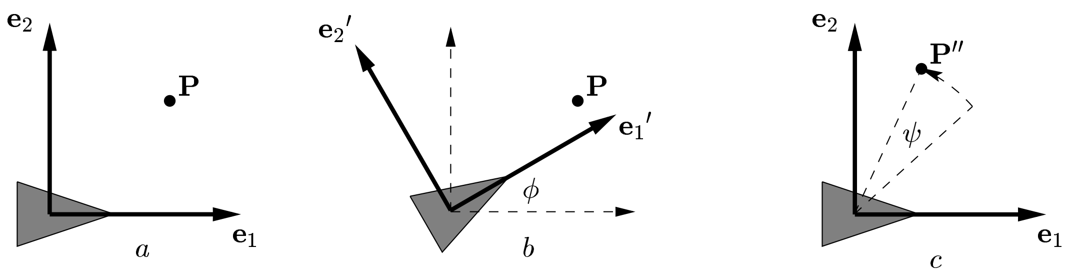
2D rotations as coordinate transformations

1 A problem with rotations

We have a triangular robot, placed somewhere in the world. Let us choose a coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$, fixed in the world, and chosen such that the robot's 'eye' is at the origin. The robot also has a coordinate system to indicate where points in the world are; for the moment, that system $(\mathbf{e}_1', \mathbf{e}_2')$ coincides with the world coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$ (see *a* in the figure).



The robot sees a point $\mathbf{P}$ from the corner of its eye, and has determined that it is at location (p_1, p_2) relative to its coordinate system. We thus know that it also at (p_1, p_2) in the world. The robot should follow this point. As it moves, it will see it at a different place, because of its own motion and the point motion. It is important to distinguish among those. In this problem, we study how the coordinates of the point *as the robot sees it* change when the robot rotates.

(1) Turning robot
Assume $\mathbf{P}$ is static, and that the robot turns over an angle ϕ. As a consequence, its internal coordinate system $(\mathbf{e}_1', \mathbf{e}_2')$ has turned relative to the world coordinate system.

Because of this turn, the robot now sees the object at a different location, denoted by (p_1', p_2') in its own coordinates. These numbers are each a function of the old numbers (p_1, p_2), and ϕ. What are they? (Try this now; hints below)

If you obtain a linear system of equations (how do you check that?) you should be able to express those as a matrix multiplication. Do so! The rotation matrix is orthonormal - check that!

(2) Turning point
Now the point $\mathbf{P}$ turns around the robot over an angle ψ to become point $\mathbf{P}''$, see figure *c*. This changes its coordinates for the robot to new numbers p_1'' and p_2'', obviously both functions of p_1, p_2, and ψ. What are they?

(3) Everything turns
What if both turn? How should a robot turn to 'follow' a turning point?

2 Hints

(1) Turning robot
First we need to cast the problem into the proper mathematical terms. We are looking for some function giving p_1' in terms of p_1, p_2, ϕ, so $p_1' = f(p_1, p_2, \phi)$.

(a) Express the position of the point $\mathbf{P}$ using $(\mathbf{e}_1, \mathbf{e}_2)$ and the numbers p_1 and p_2.

(b) Express the position of the point $\mathbf{P}$ using $(\mathbf{e}_1', \mathbf{e}_2')$ and the numbers p_1' and p_2'.

(c) These two must be the same, because $\mathbf{P}$ does not change. That gives an equation relating p_1' and p_2' to p_1 and p_2. Write that equation down.

Answer: $p_1'\mathbf{e}_1' + p_2'\mathbf{e}_2' = p_1\mathbf{e}_1' + p_2\mathbf{e}_2'$.

(d) Take the dot product with $\mathbf{e}_1'$ of all terms on the left and right of the equation. We said that $(\mathbf{e}_1', \mathbf{e}_2')$ is an *orthonormal* coordinate system. What does that mean for the values of $\mathbf{e}_1' \cdot \mathbf{e}_1'$ and $\mathbf{e}_1' \cdot \mathbf{e}_2'$? Use that, and you obtain an equation for p_1' in terms of p_1, p_2, $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_1'$. So, p_2' and $\mathbf{e}_2'$ have been eliminated: they do not affect the value of p_1'.

Answer: $p_1' = p_1(\mathbf{e}_1 \cdot \mathbf{e}_1') + p_2(\mathbf{e}_2 \cdot \mathbf{e}_1')$.

(e) In a similar way, derive an equation for p_2'.

Answer: $p_2' = p_2(\mathbf{e}_1 \cdot \mathbf{e}_2') + p_2(\mathbf{e}_2 \cdot \mathbf{e}_2')$.

(f) There are still terms like $\mathbf{e}_1 \cdot \mathbf{e}_1'$. Use the definition of the dot product as $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \, \cos\phi(\mathbf{a}, \mathbf{b})$ to express all these terms in $\cos\phi$ or $\sin\phi$. Be careful about the signs! This should give you expressions for p_1' and p_2' in terms of p_1, p_2 and ϕ.

Answer: $\mathbf{e}_1 \cdot \mathbf{e}_1' = \mathbf{e}_2 \cdot \mathbf{e}_2' = \cos\phi$, $\mathbf{e}_1 \cdot \mathbf{e}_2' = -\mathbf{e}_2 \cdot \mathbf{e}_1' = \cos(\frac{\pi}{2} - \phi) = \sin\phi$

(g) The problem is now solved: you know what the new numbers p_1' and p_2' are that the robot needs to find the point $\mathbf{P}$ after it has turned. Do an example: the point $\mathbf{P}$ at $(1, 2)$ in the old coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$; and a turn of $\pi/2$. What are p_1' and p_2' according to your formula? And what according to a drawing?

(h) When you take the expressions for p_1' and p_2' together, you should have a linear set of equations. If we denote $\mathbf{P}$ by the vector $(p_1, p_2)^T$, and the new coordinates (p_1', p_2') by $\mathbf{P}'$, then you can write $\mathbf{P}' = \mathbf{A}\mathbf{P}$, with $\mathbf{A}$ a matrix. Give that matrix.

Answer:

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{1}$$

(i) $\mathbf{A}$ is a *rotation matrix* (it represents the rotation). Mathematically, it is an *orthonormal matrix*. Check that. Can you prove it from the way in which we found it (the dot products)?

(j) What is the inverse of $\mathbf{A}$? How is the inverse useful to robotics?

Answer: It is the action of: turning the opposite way.

(k) From linear algebra, you know that the inverse of an orthonormal matrix is equal to its transpose: $\mathbf{A}^{-1} = \mathbf{A}^T$. Prove that from your dot product expressions. How would you use this fact in robotics to simplify your computations?

Answer: Remember how inverse or transpose are computed, and which would lead to the simpler, less time-consuming program.

(l) The numbers in rotation matrices are easily remembered as follows: the first column is the new $\mathbf{e}_1'$, expressed in the old coordinate system $(\mathbf{e}_1, \mathbf{e}_2)$, and the second column is the new $\mathbf{e}_2'$ in the old coordinates. Can you see that this is indeed correct, from the way you derived it? Now write down the matrix for the rotation over $-60°$ (so $-\frac{\pi}{3}$ radians) without using your earlier derivations.

(2) Turning point

If you observe that this problem is just the same as the previous, from the robot's point of view at least, if we assume that the robot turned over an angle $-\psi$, then you will have no problem finding the solution.

(3) Turning both

Same remark!