The convolution structure for Jacobi function expansions

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Abstract

The product $\varphi_\lambda^{(\alpha,\beta)}(t_1)\varphi_\lambda^{(\alpha,\beta)}(t_2)$ of two Jacobi functions is expressed as an integral in terms of $\varphi_\lambda^{(\alpha,\beta)}(t_3)$ with explicit non-negative kernel, when $\alpha \geqq \beta \geqq -\frac{1}{2}$. The resulting convolution structure for Jacobi function expansions is studied. For special values of α and β the results are known from the theory of symmetric spaces.

1. Introduction

This paper deals with harmonic analysis for Jacobi function expansions, which was initiated in [4]. The functions which we call Jacobi functions and which we denote by $\varphi_\lambda^{(\alpha,\beta)}(t)$ can be expressed as hypergeometric functions and are the non-compact analogues of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$. Similar to results of Gasper for Jacobi series ([5], [6]) we will present here results concerning the convolution structure for Jacobi function expansions.

The Fourier-Jacobi transform is reduced to the classical Fourier-cosine transform in the case $\alpha = \beta = -\frac{1}{2}$. The transform is known as the (generalized) Mehler-transform when $\alpha = \beta$. For certain discrete values of α and β Jacobi functions have an interpretation as spherical functions on non-compact symmetric spaces of rank one. In this group theoretic context all the results presented here are well known, i.e. the product formula, the positivity and commutativity of the convolution product, the phenomenon of holomorphic functions in a strip as the duals of L^p-functions, and the structure of the convolution algebra of L^1-functions. We will give the results for all α and β such that $\alpha \geqq \beta \geqq -\frac{1}{2}$, only using analytic methods, although the group theoretic interpretation was an important guide in our research.

In our opinion Jacobi functions deserve as much interest as Jacobi polynomials.

They probably constitute the most complicated continuous orthogonal system of functions in one variable for which all significant aspects of the classical Fourier transform can be generalized in a nice way.

Section 2 of this paper contains some preliminaries. Section 3 deals with properties of the Fourier-Jacobi transform for L^p-functions ($1 \leqq p \leqq 2$); in particular, the injectivity of this transform is proved. The product formula for Jacobi functions is an important tool for obtaining a convolution structure. In section 4 we derive this product formula, which is analogous to Gelfand's product formula for spherical functions on homogeneous spaces. The formula is proved from the integral representation for Jacobi functions by using a new series expansion for the product of two Jacobi functions. This expansion, which generalizes a formula for Jacobi polynomials due to Bateman, may have some interest of its own. In the second part of section 4 the product formula is rewritten in kernel form and an explicit expression is obtained for the (non-negative) kernel. The methods used in this section belong to the field of classical analysis.

A functional analytic approach is used again in the last two sections. Section 5 contains a number of properties of the convolution product. In particular, estimates are given for $\|f * g\|_r$, where $f \in L^p$ and $g \in L^q$. Generalizing a result of Kunze and Stein [10] we can improve the well-known classical estimates in certain cases. Finally, section 6 deals with the structure of the convolution algebra of L^1-functions. It turns out that all the non-zero continuous characters on this Banach-algebra can be expressed by means of Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(t)$, where λ lies in a certain strip in the complex plane.

In subsequent papers the first author will give a group theoretic interpretation for the convolution structure when α and β are half integers. The second author will give an elementary proof of the inversion formula for the Fourier-Jacobi transform.

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2. Preliminaries

For $\alpha \geqq \beta \geqq -\frac{1}{2}$, for $\lambda \in \mathbf{C}$ and for $t \in [0, \infty)$ let the Jacobi function $\varphi_\lambda^{(\alpha,\beta)}(t)$ (or $\varphi_\lambda(t)$) be defined by

$$\varphi_\lambda(t) = \varphi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1(\tfrac{1}{2}(\varrho + i\lambda), \tfrac{1}{2}(\varrho - i\lambda); \alpha + 1; -(\operatorname{sh} t)^2), \tag{2.1}$$

where $\varrho = \alpha + \beta + 1$, and ${}_2F_1$ denotes the hypergeometric function (see [3], ch. 2)[1]). It follows by transformation of the hypergeometric differential equation that $\varphi_\lambda^{(\alpha,\beta)}(t)$ satisfies

[1]) In [4] the parameters $p = 2\alpha - 2\beta$ and $q = 2\beta + 1$ were used instead of α and β.

$$(\omega_{\alpha,\beta} + \varrho^2 + \lambda^2)\varphi_\lambda^{(\alpha,\beta)}(t) = 0, \tag{2.2}$$

where $\omega_{\alpha,\beta}$ is the differential operator defined by

$$\omega_{\alpha,\beta} f = \Delta(t)^{-1} \frac{d}{dt}\left(\Delta(t)\frac{df}{dt}\right) \tag{2.3}$$

and

$$\Delta(t) = 2^{2\varrho}(\operatorname{sh} t)^{2\alpha+1}(\operatorname{ch} t)^{2\beta+1}. \tag{2.4}$$

Let

$$d\mu(t) = \frac{1}{\sqrt{2\pi}}\Delta(t)dt. \tag{2.5}$$

It was proved in [4] that the mapping $f \to f^\wedge$ defined by

$$f^\wedge(\lambda) = \int_0^\infty f(t)\varphi_\lambda^{(\alpha,\beta)}(t)d\mu(t) \tag{2.6}$$

is a bijection of the space C_0^∞, consisting of the even C^∞-functions $f(t)$ of compact support, onto the space of even, entire, rapidly decreasing functions $f^\wedge(\lambda)$ of exponential type. It was shown that the measure in the inverse mapping

$$f(t) = \int_0^\infty f^\wedge(\lambda)\varphi_\lambda^{(\alpha,\beta)}(t)d\nu(\lambda) \tag{2.7}$$

is given by

$$d\nu(\lambda) = \frac{1}{\sqrt{2\pi}}|c(\lambda)|^{-2}d\lambda, \tag{2.8}$$

where

$$c(\lambda) = \frac{2^{\varrho-i\lambda}\Gamma(i\lambda)\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}(\varrho+i\lambda))\Gamma(\frac{1}{2}(\varrho+i\lambda)-\beta)}. \tag{2.9}$$

The mappings (2.6) and (2.7) extend to an isomorphism between the L^2-spaces with respect to $d\mu(t)$ and $d\nu(\lambda)$ ([4], prop. 3).

For $\alpha = \beta = -\frac{1}{2}$ we have

$$\varphi_\lambda^{(-\frac{1}{2},-\frac{1}{2})}(t) = \cos \lambda t. \tag{2.10}$$

This classical case will not be considered here. If either $\alpha > \beta \geqq -\frac{1}{2}$ or $\alpha \geqq \beta > -\frac{1}{2}$ then $\varrho > 0$. The cases $\beta = -\frac{1}{2}$ and $\alpha = \beta$ are connected by the quadratic transformation

$$\varphi_{\frac{1}{2}\lambda}^{(\alpha,-\frac{1}{2})}(2t) = \varphi_\lambda^{(\alpha,\alpha)}(t). \tag{2.11}$$

The functions $\varphi_\lambda^{(\alpha,\alpha)}(t)$ can be expressed in terms of Gegenbauer functions $C_\alpha^\nu(x)$ or associated Legendre functions $P_\nu^\mu(x)$ by

$$\varphi_\lambda^{(\alpha,\alpha)}(t) = C^{\alpha+\frac{1}{2}}_{\frac{1}{2}(i\lambda-\varrho)}(\mathrm{ch}\, 2t)/C^{\alpha+\frac{1}{2}}_{\frac{1}{2}(i\lambda-\varrho)}(1) = 2^\alpha \Gamma(\alpha+1)(\mathrm{sh}\, 2t)^{-\alpha} P^{-\alpha}_{\frac{1}{2}(i\lambda-1)}(\mathrm{ch}\, 2t)$$

(see [3], ch. 3).

In the case of general α and β we will also use the notation

$$\varphi_\lambda^{(\alpha,\beta)}(t) = R^{\alpha,\beta}_{\frac{1}{2}(i\lambda-\varrho)}(\mathrm{ch}\, 2t), \tag{2.12}$$

where

$$R_\mu^{(\alpha,\beta)}(z) = {}_2F_1\left(-\mu, \mu+\alpha+\beta+1; \alpha+1; \frac{1-z}{2}\right). \tag{2.13}$$

Note that for $n = 0, 1, 2, \ldots$ the function

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1)$$

is a Jacobi polynomial.

For $\alpha > \beta > -\frac{1}{2}$ there is the integral representation (Koornwinder [8], formula (4))

$$\varphi_\lambda^{(\alpha,\beta)}(t) = \int_{r=0}^{1}\int_{\psi=0}^{\pi} [(\mathrm{ch}\, t)^2 + (\mathrm{sh}\, t)^2 r^2 + \mathrm{sh}\, 2t \cdot r \cdot \cos\psi]^{\frac{1}{2}(i\lambda-\varrho)} dm(r,\psi), \tag{2.14}$$

where

$$dm(r,\psi) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})}(1-r^2)^{\alpha-\beta-1} r^{2\beta+1}(\sin\psi)^{2\beta} dr d\psi. \tag{2.15}$$

For the proof of this formula see Askey [1] and Flensted-Jensen [4]. If $\alpha = \beta$ or $\beta = -\frac{1}{2}$ then (2.14) degenerates to a single integral.

Let $\lambda = \xi + i\eta \in \mathbf{C}$. In [4] the following estimates are proved for $t \in [0, \infty)$

$$|\varphi_\lambda(t)| \leqq \varphi_{i\eta}(t) \qquad \text{for all } \lambda \in \mathbf{C}, \tag{2.16}$$

$$|\varphi_\lambda(t)| \leqq 1 \qquad \text{for all } |\eta| \leqq \varrho, \tag{2.17}$$

$$|\varphi_\lambda(t)| \leqq K(1+t)e^{(|\eta|-\varrho)t} \qquad \text{for all } \lambda \in \mathbf{C}. \tag{2.18}$$

In (2.18) K denotes a positive constant independent of λ.

LEMMA 2.1. *φ_λ is bounded if and only if $|\eta| \leq \varrho$.*

Proof. The condition is necessary by (2.17). Assume $\eta > \varrho$ and that φ_λ is bounded. From the discussion in [4], section 2.1. of the asymptotic behaviour of solutions to the differential equation (2.2), it is seen that φ_λ and Φ_λ are linearly dependent. But this contradicts the fact that the Wronski determinant $W(\varphi_\lambda, \Phi_\lambda) = -2i\lambda c(\lambda)$ is different from zero, ([4], proof of lemma 8). Q.e.d.

3. The Fourier-Jacobi transform of L^p-spaces

Let $1 \leqq p < 2$ and take q such that $1/p + 1/q = 1$. Let D_p be the strip in the complex λ-plane defined by

$$D_p = \left\{ \lambda = \xi + i\eta \in \mathbf{C} \,\middle|\, |\eta| < \left(\frac{2}{p} - 1\right)\varrho \right\}.$$

Note that by (2.18) $\varphi_\lambda \in L^q(d\mu)$ for all $\lambda \in D_p$. More precisely, $\|\varphi_\lambda\|_q$ is uniformly bounded in any closed strip contained in D_p and, by (2.17), $\|\varphi_\lambda\|_\infty = 1$ for all λ in the closure $\bar{D}_1$ of D_1.

LEMMA 3.1. *Let $1 \leqq p < 2$, $1/p + 1/q = 1$ and $f \in L^p(d\mu)$, then $f^\wedge(\lambda)$ is well defined and holomorphic in D_p, and for all $\lambda \in D_p$*

$$|f^\wedge(\lambda)| \leqq \|f\|_p \|\varphi_\lambda\|_q. \tag{3.1}$$

In particular if $f \in L^1(d\mu), f^\wedge(\lambda)$ is continuous also on $\bar{D}_1$, and for all $\lambda \in \bar{D}_1$

$$|f^\wedge(\lambda)| \leqq \|f\|_1. \tag{3.2}$$

Proof. Formulas (3.1) and (3.2) are proved by using Hölder's inequality. Observe that $\varphi_\lambda(t)$ is holomorphic in λ. Hence, by applying Fubini's theorem and Cauchy's formula we conclude that

$$\frac{1}{2\pi i} \oint \frac{f^\wedge(\lambda)}{\lambda - \lambda_0}\, d\lambda = f^\wedge(\lambda_0)$$

for all $\lambda_0 \in D_p$, where the contour is taken around λ_0 inside D_p. Thus it follows that $f^\wedge(\lambda)$ is holomorphic in D_p. The rest is now clear. Q.e.d.

Let $M = M([0, \infty))$ be the set of bounded mesures on $[0, \infty)$. For $\gamma \in M$ define $\gamma^\wedge$ by

$$\gamma^\wedge(\lambda) = \int_0^\infty \varphi_\lambda(t)\, d\gamma(t).$$

THEOREM 3.2. *The Fourier-Jacobi transform is injective on $L^p(d\mu)$ for $1 \leqq p \leqq 2$, and likewise on M.*

Proof. For $p = 2$ the result follows from the L^2-isomorphism mentioned in section 2. So assume $1 \leqq p < 2$. Take q such that $1/p + 1/q = 1$. For $f \in L^p(d\mu)$ and $g \in C_0^\infty$ we have the inequalities

$$|(f, g)| = \left| \int_0^\infty f(t)\overline{g(t)}d\mu(t) \right| \leqq \|f\|_p \|g\|_q$$

and

$$|(f^\wedge, g^\wedge)| = \left| \int_0^\infty f^\wedge(\lambda)\overline{g^\wedge(\lambda)}d\nu(\lambda) \right| \leqq \|f^\wedge\|_\infty \|g^\wedge\|_1 \leqq \text{const.} \cdot \|f\|_p \|g^\wedge\|_1.$$

Therefore the mappings $f \to (f, g)$ and $f \to (f^\wedge, g^\wedge)$ are continuous functionals on L^p. Now $(f, g) = (f^\wedge, g^\wedge)$ for all $f \in L^p \cap L^2$ and by continuity for all $f \in L^p$. Assume that $f \in L^p$ and that $f^\wedge = 0$, then for all $g \in C_0^\infty$ we have $(f, g) = (f^\wedge, g^\wedge) = 0$ and therefore $f = 0$.

Let $\gamma \in M$ then by the methods of lemma 3.1, $\gamma^\wedge(\lambda)$ is bounded in $\overline{D_1}$ and holomorphic in D_1. Assume that $\gamma^\wedge = 0$. By Fubini's theorem we have for $f \in C_0^\infty$.

$$\int_0^\infty f(t)d\gamma(t) = \int_0^\infty f^\wedge(\lambda)\gamma^\wedge(\lambda)d\nu(\lambda) = 0,$$

therefore $\gamma = 0$. Q.e.d.

4. The product formula for Jacobi functions

The first part of this section contains a new proof of the following theorem.

THEOREM 4.1. *Jacobi functions* $R_\mu^{(\alpha,\beta)}(x)$ *satisfy the product formula*

$$R_\mu^{(\alpha,\beta)}(x)R_\mu^{(\alpha,\beta)}(y) = \int_0^1\int_0^\pi R_\mu^{(\alpha,\beta)}(\tfrac{1}{2}(x+1)(y+1) + \tfrac{1}{2}(x-1)(y-1)r^2 + \sqrt{(x^2-1)(y^2-1)}\, r\cos\psi - 1)\, dm(r, \psi), \tag{4.1}$$

where $x \geqq 1$, $y \geqq 1$, $\mu \in \mathbf{C}$, $\alpha > \beta > -\frac{1}{2}$. *The notation from* (2.13) *and* (2.15) *is used.*

In the second part of this section we will rewrite formula (4.1) in the so-called kernel form

$$\varphi_\lambda(t_1)\varphi_\lambda(t_2) = \int_0^\infty \varphi_\lambda(t_3)K(t_1, t_2, t_3)d\mu(t_3), \tag{4.2}$$

where the notation from (2.1) and (2.5) is used. The kernel K will be obtained in an explicit way.

It was pointed out in [8], where formula (4.1) first occured, that (4.1) can be proved by analytic continuation with respect to μ if the formula is known for $\mu = 0, 1, 2, \ldots$ For these values of μ the product formula was obtained in [8] as a corollary to the addition formula for Jacobi polynomials. In a forthcoming paper [9] the product formula for Jacobi polynomials is directly proved from the Laplace type integral representation by using an identity for Jacobi polynomials due to Bateman. Here a similar proof of (4.1) will be given for general complex μ.

From (2.14) and (2.12) we obtain

$$R_\mu^{(\alpha,\beta)}(x) = \int_0^1 \int_0^\pi [\tfrac{1}{2}(x+1) + \tfrac{1}{2}(x-1)r^2 + \sqrt{x^2-1}\, r\cos\psi]^\mu dm(r,\psi), \tag{4.3}$$

where $x \geqq 1$, $\mu \in \mathbf{C}$, $\alpha > \beta > -\frac{1}{2}$. It follows from (4.3) that

$$(x+y)^\mu R_\mu^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) = \int_0^1 \int_0^\pi [\tfrac{1}{2}(x+1)(y+1) + \tfrac{1}{2}(x-1)(y-1)r^2 + \sqrt{(x^2-1)(y^2-1)}\cdot r \cdot \cos\psi]^\mu dm(r,\psi), \tag{4.4}$$

where $x \geqq 1$, $y \geqq 1$.

In our proof formula (4.1) will be derived from (4.4) by applying a generalization of Bateman's formula

$$R_n^{(\alpha,\beta)}(x) R_n^{(\alpha,\beta)}(y) = \sum_{k=0}^n a_k (x+y)^k R_k^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right).$$

THEOREM 4.2. *Let* $x \geqq 1$, $y \geqq 1$, $x \neq y$, $2\mu + \varrho$ *non-integer*, $\alpha \geqq \beta \geqq -\frac{1}{2}$. *Then the expansion*

$$\begin{aligned} R_\mu^{(\alpha,\beta)}(x) R_\mu^{(\alpha,\beta)}(y) = \sum_{n=0}^\infty A_n \left(\frac{x+y}{2}\right)^{\mu-n} R_{\mu-n}^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) + \\ + \sum_{n=0}^\infty B_n \left(\frac{x+y}{2}\right)^{-\mu-\varrho-n} R_{-\mu-\varrho-n}^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right) \end{aligned} \tag{4.5}$$

is valid, where the coefficients A_n *and* B_n *are determined from the case* $y = 1$, *i.e.*

$$R_\mu^{(\alpha,\beta)}(x) = \sum_{n=0}^\infty A_n \left(\frac{x+1}{2}\right)^{\mu-n} + \sum_{n=0}^\infty B_n \left(\frac{x+1}{2}\right)^{-\mu-\varrho-n} \tag{4.6}$$

The expansion (4.6) is obtained from [3], § 2.10 (3), which is, in fact, the same as the formula

$$\varphi_\lambda(t) = c(\lambda)\Phi_\lambda(t) + c(-\lambda)\Phi_{-\lambda}(t)$$

in [4]. The coefficients in (4.6) are

$$A_n = C \cdot \frac{\Gamma(-\mu+n)\Gamma(-\mu-\beta+n)}{\Gamma(-2\mu-\varrho+1+n)\Gamma(1+n)} \tag{4.7}$$

and

$$B_n = -C \cdot \frac{\Gamma(\mu+\varrho+n)\Gamma(\mu+\alpha+1+n)}{\Gamma(2\mu+\varrho+1+n)\Gamma(1+n)}, \tag{4.8}$$

where

$$C = \frac{\pi\Gamma(\alpha+1)}{\sin(\pi(2\mu+\varrho))\Gamma(\mu+\alpha+1)\Gamma(\mu+\varrho)\Gamma(-\mu)\Gamma(-\mu-\beta)}. \tag{4.9}$$

The expansion (4.6) is valid and convergent for non-ineteger values of $2\mu+\varrho$ and for all $x \in \mathbf{C}$ such that $|x+1| > 2$ and $|\arg(x+1)| < \pi$.

Proof of theorem 4.1. Suppose first that $x \neq y$ and $2\mu+\varrho$ non-integer. Putting $x = \operatorname{ch} t_1$ and $y = \operatorname{ch} t_2$ we have

$$1 < \operatorname{ch}(t_1 - t_2) = xy - \sqrt{(x^2-1)(y^2-1)} \leqq$$
$$\leqq \tfrac{1}{2}(x+1)(y+1) + \tfrac{1}{2}(x-1)(y-1)r^2 + \sqrt{(x^2-1)(y^2-1)} \cdot r \cdot \cos\psi - 1 \leqq$$
$$\leqq xy + \sqrt{(x^2-1)(y^2-1)} = \operatorname{ch}(t_1+t_2)$$

for $0 \leqq r \leqq 1$ and $0 \leqq \psi \leqq \pi$. Hence the integrand in the right hand side of (4.1) can be expanded by using (4.6), and summation and integration may be interchanged because the series converges uniformly in r and ψ. Next, formula (4.4) can be applied to each term and the resulting series is the right hand side of (4.5). When $x = y$ or $2\mu+\varrho$ is integer, formula (4.1) is proved by continuity. Q.e.d.

Remark. For $\alpha = \beta$ or $\beta = -\frac{1}{2}$ formula (4.1) degenerates to a single integral.

We still have to prove theorem 4.2. This will be done in several steps. First observe that the function $R_\mu^{(\alpha,\beta)}(z)$, defined by (2.13), is holomorphic in the complex z-plane with cut $(-\infty, -1]$ and satisfies the differential equation

$$(D_z + \mu(\mu+\varrho))R_\mu^{(\alpha,\beta)}(z) = 0 \tag{4.10}$$

with

$$D_z = (1-z^2)\frac{d^2}{dz^2} + (\beta - \alpha - (\alpha+\beta+2)z)\frac{d}{dz}. \tag{4.11}$$

The function $F(z, w) = R_\mu^{(\alpha,\beta)}(z)R_\mu^{(\alpha,\beta)}(w)$ is clearly a solution of the partial differential equation

$$(D_z - D_w)F(z, w) = 0. \tag{4.12}$$

LEMMA 4.3. *The function*

$$F(z, w) = (z + w)^\mu R_\mu^{(\alpha, \beta)}\left(\frac{1 + zw}{z + w}\right)$$

is a solution of (4.12).

Proof. The equation (4.12) transforms under the substitution

$$u = \frac{1 + zw}{z + w}, \quad v = z + w$$

into

$$\left(D_u + v^2 \frac{\partial^2}{\partial v^2} + (\alpha + \beta + 2)v \frac{\partial}{\partial v}\right) F(z, w) = 0$$

and the function $v^\mu R_\mu^{(\alpha, \beta)}(u)$ is a solution of this equation. Q.e.d.

LEMMA 4.4. *Let* $F(z, w)$ *be a solution of* (4.12), *analytic in a neighbourhood of* $(z_0, 1)$. *Then* $F(z, w)$ *is uniquely determined by its values* $F(z, 1)$ *for* z *in a neighbourhood of* z_0.

Proof. Expanding the function $F(z, w)$ as $F(z, w) = \sum_{k=0}^\infty F_k(z)(w - 1)^k$ we obtain that

$$(D_z - D_w)F(z, w) =$$
$$= \sum_{k=0}^\infty (w - 1)^k[(D_z + k(k + \varrho))F_k(z) + 2(k + 1)(k + \alpha + 1)F_{k+1}(z)] = 0.$$

Hence all functions $F_k(z)$ can be expressed in terms of $F_0(z) = F(z, 1)$ by means of differential recurrence relations (for $\alpha \neq -1, -2, -3, \ldots$). Q.e.d.

LEMMA 4.5. *Let* $\alpha > \beta > -\frac{1}{2}$. *For every* $x \geqq 1$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ *there exists a* $\delta > 0$ *such that the following holds:*

If $z \in \mathbf{C}$, $|z - x| < \delta$, $\mu = \xi + i\eta$, $\xi \geqq 0$ *then*

$$|R_\mu^{(\alpha, \beta)}(z)| \leqq (x + \sqrt{x^2 - 1})^\xi e^{\varepsilon_1\xi + \varepsilon_2|\eta|}. \tag{4.13}$$

(This lemma is a kind of extension of the estimate (2.18) to complex values of t.)

Proof. Formula (4.3) can be written as

$$R_\mu^{(\alpha, \beta)}(z) = \int_0^1 \int_0^\pi (\Lambda(z; r, \psi))^\mu dm(r, \psi) \tag{4.14}$$

with $z \geqq 1$ and

$$A(z; r, \psi) = \tfrac{1}{2}(z + 1) + \tfrac{1}{2}(z - 1)r^2 + \sqrt{z^2 - 1}\, r \cos \psi.$$

For $z \in \mathbf{C}$, $0 \leqq r \leqq 1$, $0 \leqq \psi \leqq \pi$ the function $A(z; r, \psi)$ is continuous and

$$x - \sqrt{x^2 - 1} \leqq A(x; r, \psi) \leqq x + \sqrt{x^2 - 1} \quad \text{for} \quad x \geqq 1.$$

Hence, for every $x \geqq 1$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ there exists a $\delta > 0$ such that the following holds:

If $z \in \mathbf{C}$ and $|z - x| < \delta$ then

$$e^{-\varepsilon_1}(x - \sqrt{x^2 - 1}) < |A(z; r, \psi)| < e^{\varepsilon_1}(x + \sqrt{x^2 - 1}) \tag{4.15}$$

and

$$|\arg A(z; r, \psi)| < \varepsilon_2.$$

The function $A(z; r, \psi)$ is two-valued for z around 1, but this branching singularity is removed by the integration in (4.14) with respect to ψ. Choosing $x, \varepsilon_1, \varepsilon_2, \delta$ the same as above we conclude that both the left hand side and the right hand side of (4.14) are analytic in z for $z \in \mathbf{C}$ and $|z - x| < \delta$. By analytic continuation formula (4.14) holds for these values of z. The estimate (4.13) is finally obtained from (4.14) and (4.15). Q.e.d.

LEMMA 4.6. *Let $F(x, y)$ denote the right hand side of* (4.5). *If $x \geqq 1$, $y \geqq 1$, $x \neq y$, then the series represented by $F(z, w)$ (z, w complex) converges absolutely and uniformly in a certain neighbourhood of (x, y).*

Proof. It follows from (4.7) and (4.8) that $A_n = O(n^{\alpha-1})$ and $B_n = O(n^{\alpha-1})$ for $n \to \infty$ (see [3], § 1.18 (4)).

Choose $\varepsilon > 0$. There exists a $\delta > 0$ such that for $z \in \mathbf{C}$, $w \in \mathbf{C}$, $|z - x| < \delta$, $|w - y| < \delta$, $n \geqq \operatorname{Re}(\mu + \varrho)$ we have

$$\left| R_{\mu-n}^{(\alpha,\beta)}\left(\frac{1 + zw}{z + w}\right)\right| = \left| R_{n-\mu-\varrho}^{(\alpha,\beta)}\left(\frac{1 + zw}{z + w}\right)\right| \leqq K e^{\varepsilon n}\left(\frac{1 + xy}{x + y} + \sqrt{\left(\frac{1 + xy}{x + y}\right)^2 - 1}\right)^n$$

(see lemma 4.5) and

$$\left|\left(\frac{z + w}{2}\right)^{\mu - n}\right| \leqq K e^{\varepsilon n}\left(\frac{2}{x + y}\right)^n.$$

Here K is a certain positive constant. Combining the three estimates we obtain that for $n \to \infty$:

$$A_n\left(\frac{z + w}{2}\right)^{\mu - n} R_{\mu-n}^{(\alpha,\beta)}\left(\frac{1 + zw}{z + w}\right) = O\left(n^{\alpha-1}e^{2\varepsilon n}\left(\frac{2(1 + xy) + \sqrt{(x^2 - 1)(y^2 - 1)}}{(x + y)^2}\right)^n\right),$$

uniformly in z and w for $|z - x| < \delta$, $|w - y| < \delta$. Observe that for $x \neq y$

$$\frac{2(1+xy)+2\sqrt{(x^2-1)(y^2-1)}}{(x+y)^2}=\frac{2(1+xy)+2\sqrt{(x^2-1)(y^2-1)}}{2(1+xy)+(x^2-1)+(y^2-1)}<1.$$

Choosing ε small enough we find:

$$A_n\left(\frac{z+w}{2}\right)^{\mu-n}R_{\mu-n}^{(\alpha,\beta)}\left(\frac{1+zw}{z+w}\right)=O(n^{\alpha-1}e^{-\varepsilon n})$$

for $n\to\infty$, $|z-x|<\delta(\varepsilon)$, $|w-y|<\delta(\varepsilon)$.

A similar estimate holds for

$$B_n\left(\frac{z+w}{2}\right)^{-\mu-\varrho-n}R_{-\mu-\varrho-n}^{(\alpha,\beta)}\left(\frac{1+zw}{z+w}\right).$$ Q.e.d.

Proof of theorem 4.2. Let $F(x, y)$ denote the right hand side of (4.5) and $G(x, y)$ the left hand side. By lemma 4.6 the function $F(x, y)$ is analytic for $x\geqq 1$, $y\geqq 1$, $x\neq y$. Since $F(z, w)$ is a locally uniform convergent sum of analytic solutions of (4.12) (cf. lemma 4.3), F satisfies the equation $(D_z-D_w)F(z,w)=0$ itself. The function $G(z, w)$ is clearly an analytic solution of the same differential equation and $F(x, 1)=G(x, 1)$ (cf. formula (4.6)). Hence, by lemma (4.4) and by using analytic continuation with respect to y we conclude that $F(x, y)=G(x, y)$ for $x>y\geqq 1$, and also for $y>x\geqq 1$ because of the symmetry. Q.e.d.

Remark. Let $Q_\mu^{(\alpha,\beta)}(x)$ be a second solution of (4.10) with expansion

$$Q_\mu^{(\alpha,\beta)}(x)=\sum_{n=0}^{\infty}A_n\left(\frac{1+x}{2}\right)^{\mu-n}.$$

It follows from the proof of theorems 4.1 and 4.2 that for $x>y\geqq 1$

$$Q_\mu^{(\alpha,\beta)}(x)R_\mu^{(\alpha,\beta)}(y)=\sum_{n=0}^{\infty}A_n\left(\frac{x+y}{2}\right)^{\mu-n}R_{\mu-n}^{(\alpha,\beta)}\left(\frac{1+xy}{x+y}\right)=$$

$$=\int_0^1\int_0^\pi Q_\mu^{(\alpha,\beta)}(\tfrac{1}{2}(x+1)(y+1)+\tfrac{1}{2}(x-1)(y-1)r^2+$$
$$+\sqrt{(x^2-1)(y^2-1)}\,r\cos\psi-1)\,dm(r,\psi).$$

We next come to the second part of this section and will derive an explicit expression for the kernel $K(t_1, t_2, t_3)$ in formula (4.2). Let $f(\operatorname{ch} 2t)$ be a function which is absolutely integrable in every finite t-interval. Let the substitution $(r,\psi)\to(t_3,\chi)$ be defined by

$$\operatorname{ch} t_3\, e^{i\chi}=\operatorname{ch} t_1 \operatorname{ch} t_2+\operatorname{sh} t_1 \operatorname{sh} t_2\, re^{i\psi}.$$

Then by making this substitution of the integration variables, it follows easily (cf. [9], section 5) that

$$\int_0^1 \int_0^\pi f(2[(\operatorname{ch} t_1)^2(\operatorname{ch} t_2)^2 + (\operatorname{sh} t_1)^2(\operatorname{sh} t_2)^2 r^2 + \tag{4.16}$$

$$+ 2 \operatorname{sh} t_1 \operatorname{ch} t_1 \operatorname{sh} t_2 \operatorname{ch} t_2 r \cos \psi] - 1) dm(r, \psi) = \int_0^\infty f(\operatorname{ch} 2t_3) K(t_1, t_2, t_3) d\mu(t_3),$$

where for $t_3 \notin (|t_1 - t_2|, \ t_1 + t_2)$

$$K(t_1, t_2, t_3) = 0$$

and for $|t_1 - t_2| < t_3 < t_1 + t_2$

$$K(t_1, t_2, t_3) = \frac{2^{(3/2)-2\varrho}\Gamma(\alpha+1)}{\Gamma(\alpha-\beta)\Gamma(\beta+\frac{1}{2})} (\operatorname{sh} t_1 \operatorname{sh} t_2 \operatorname{sh} t_3)^{-2\alpha} \cdot \tag{4.17}$$

$$\int_0^\pi (1 - (\operatorname{ch} t_1)^2 - (\operatorname{ch} t_2)^2 - (\operatorname{ch} t_3)^2 + 2 \operatorname{ch} t_1 \operatorname{ch} t_2 \operatorname{ch}_3 t \cos \chi)_+^{\alpha-\beta-1} \cdot (\sin \chi)^{2\beta} d\chi.$$

Here $(x)_+ = x$ for $x > 0$ and $(x)_+ = 0$ for $x \leqq 0$. Taking

$$f(\operatorname{ch} 2t) = \varphi_\lambda(t) = R^{(\alpha,\beta)}_{\frac{1}{2}(i\lambda-\varrho)}(\operatorname{ch} 2t)$$

we obtain formula (4.2) by substituting (4.16) for the right hand side of (4.1) with $\mu = \frac{1}{2}(i\lambda - \varrho)$. Thus we have an explicit expression for the kernel $K(t_1, t_2, t_3)$ in (4.2).

The kernel K can be expressed as a hypergeometric function (see Gasper [6] and Koornwinder [9]). Writing

$$B = \frac{(\operatorname{ch} t_1)^2 + (\operatorname{ch} t_2)^2 + (\operatorname{ch} t_3)^2 - 1}{2 \operatorname{ch} t_1 \operatorname{ch} t_2 \operatorname{ch} t_3} \tag{4.18}$$

we have for $|t_1 - t_2| < t_3 < t_1 + t_2$ (i.e. $|B| < 1$):

$$K(t_1, t_2, t_3) = \frac{2^{\frac{1}{2}-2\varrho}\Gamma(\alpha+1)(\operatorname{ch} t_1 \operatorname{ch} t_2 \operatorname{ch} t_3)^{\alpha-\beta-1}}{\Gamma(\alpha+\frac{1}{2})(\operatorname{sh} t_1 \operatorname{sh} t_2 \operatorname{sh} t_3)^{2\alpha}} \cdot \cdot (1-B^2)^{\alpha-\frac{1}{2}} {}_2F_1\left(\alpha+\beta, \ \alpha-\beta; \ \alpha+\tfrac{1}{2}; \ \frac{1-B}{2}\right). \tag{4.19}$$

The function $K(t_1, t_2, t_3)$ is non-negative, and it is symmetric in the three variables. It is a C^∞-function, singular or identically zero according to whether the sum of two of the variables is greater, equal to or less than the third variable. Since $\varphi_{i\varrho}(t) = 1$ we conclude from (4.2) that

$$\int_0^\infty K(t_1, t_2, t_3)d\mu(t_3) = 1. \tag{4.20}$$

$K(t_1, t_2, t_3)$ is not well-defined as a function if one of the variables is zero; but in this case $K(0, t_2, t_3)$ can be considered as a distribution and it follows from (4.16) that

$$\int_0^\infty f(t_3)K(0, t_2, t_3)d\mu(t_3) = f(t_2). \tag{4.21}$$

In section 5 we shall use the kernel K to define a convolution structure associated with the Fourier-Jacobi transform. We shall need the following lemma.

LEMMA 4.7. *The function*

$$H(t_1, t_2, s_1, s_2) = \int_0^\infty K(t_1, t_2, \tau)K(s_1, s_2, \tau)d\mu(\tau)$$

is well-defined if no two of the numbers $|t_1 - t_2|$, $t_1 + t_2$, $|s_1 - s_2|$, $s_1 + s_2$ *are equal. Moreover* H *is symmetric in the four variables.*

Proof. It follows from (4.20) that $K(t_1, t_2, \tau)K(s_1, s_2, \tau)$ is integrable with respect to $d\mu(\tau)$ if $K(t_1, t_2, \tau)$ and $K(s_1, s_2, \tau)$ do not have a singularity in common. Hence the function $H(t_1, t_2, s_1, s_2)$ is well-defined. Using Fubini's theorem, and formula (4.2) we find that

$$H(\cdot, t_2, s_1, s_2)^\wedge(\lambda) = \varphi_\lambda(t_2)\varphi_\lambda(s_1)\varphi_\lambda(s_2),$$

and similar results hold for the Fourier-Jacobi transform in the other variables.

By applying theorem 3.2 it is proved that the function $H(t_1, t_2, s_1, s_2)$ is symmetric. Q.e.d.

5. The convolution product

DEFINITION 5.1. *Let* f *be a suitable function on* $[0, \infty)$ *and let* $x \in [0, \infty)$. *The* generalized translation operation T_x *is defined by*

$$(T_xf)(y) = \int_0^\infty f(z)K(x, y, z)d\mu(z). \tag{5.1}$$

Obviously $T_xf(y) = T_yf(x)$.

LEMMA 5.2. *For* $1 \leqq p \leqq \infty$, $f \in L^p(d\mu)$ *and* $x \geqq 0$

$$\|T_x f\|_p \leqq \|f\|_p. \tag{5.2}$$

Proof. For $p = \infty$ (5.2) follows from (4.16). For $1 \leqq p < \infty$ we use Hölder's inequality for the bounded measure $K(x, y, z)d\mu(z)$ and we obtain

$$\|T_x f\|_p^p \leqq \int_0^\infty \int_0^\infty |f(z)|^p K(x, y, z) d\mu(z) d\mu(y) = \|f\|_p^p. \qquad \text{Q.e.d.}$$

DEFINITION 5.3. *For suitable functions* f *and* g *the* convolution product $f * g$ *is defined by*

$$f * g(x) = \int_0^\infty f(y) T_x g(y) d\mu(y) = \int_0^\infty \int_0^\infty f(y) g(z) K(x, y, z) d\mu(z) d\mu(y). \tag{5.3}$$

The following properties of the convolution product are easily proved from results in section 4 on the functions K and H:

$$\begin{aligned} &\text{(i)} \ f * g = g * f, \\ &\text{(ii)} \ (f * g) * h = f * (g * h), \\ &\text{(iii)} \ \text{If } f \geqq 0 \text{ and } g \geqq 0 \text{ then } f * g \geqq 0, \\ &\text{(iv)} \ (f * g)^\wedge(\lambda) = f^\wedge(\lambda) g^\wedge(\lambda) \end{aligned} \tag{5.4}$$

whenever these Fourier-Jacobi transforms are well-defined.

THEOREM 5.4. *Let* p, q, r *be such that* $1 \leqq p, q, r \leqq \infty$ *and* $1/p + 1/q - 1 = 1/r$. *For* $f \in L^p(d\mu)$ *and* $g \in L^q(d\mu)$ $f * g$ *is a well-defined element in* $L^r(d\mu)$ *and*

$$\|f * g\|_r \leqq \|f\|_p \|g\|_q. \tag{5.5}$$

Proof. (The idea of the proof is from [11], p. 278). For $r = \infty$ the result follows from (5.2) and Hölder's inequality.

Assume $r < \infty$, which implies $p, q < \infty$. First take $f, g \in C_0$ (continuous of compact support), and let $s = p(1 - 1/q)$ and $1/q + 1/q' = 1$. Then $0 \leqq s < 1$ and $sq' = p$. We assume $s > 0$ or equivalently $q > 1$. (In the case $s = 0$ the proof is almost the same except for some obvious modifications.) Using (5.2) and Hölder's inequality we find

$$\begin{aligned} |f * g(x)|^q &\leqq \int_0^\infty |T_x f(y)|^{(1-s)q} |g(y)|^q d\mu(y) \cdot \left[\int_0^\infty |T_x f(y)|^{sq'} d\mu(y)\right]^{q/q'} \leqq \\ &\leqq \|f\|_p^{sq} \cdot \int_0^\infty h_y(x) d\mu(y), \end{aligned} \tag{5.6}$$

where $h_y(x) = |T_x f(y)|^{(1-s)q}|g(y)|^q$.

Clearly the mapping $y \to h_y$ is continuous and of compact support from $[0, \infty)$ into $L^\alpha(\mu)$ for $1 \leqq \alpha < \infty$. Using vector integration we get

$$\left\| \int_0^\infty h_y d\mu(y) \right\|_\alpha \leqq \int_0^\infty \|h_y\|_\alpha d\mu(y).$$

After taking L^α-norms in both sides of (5.6) if follows by some calculations that

$$\|f * g\|_{\alpha q}^q \leqq \|g\|_q^q \, \|f\|_{(1-s)q\alpha}^{(1-s)q} \, \|f\|_p^{sq}.$$

This inequality is reduced to (5.5) by the substitution $\alpha = r/q$ (observe that $(1 - s)r = p$). Now the theorem is proved by using the fact that C_0 is dense in $L^\gamma(d\mu)$ for $1 \leqq \gamma < \infty$. Q.e.d.

We can get the following improved inequalities for the convolution. In the special case $\alpha = \beta = 0$ theorem 5.5 was proved by Kunze and Stein [10].

THEOREM 5.5. *Let* $1 \leqq p < 2$ *and* $1/p + 1/q = 1$. *There exists a constant* $A_p > 0$ *such that*

(i) *If* $f \in L^2(d\mu)$ *and* $g \in L^p(d\mu)$ *then* $f * g \in L^2(d\mu)$ *and*

$$\|f * g\|_2 \leqq A_p \|f\|_2 \|g\|_p,$$

(ii) *If* $f, g \in L^2(d\mu)$ *then* $f * g \in L^q(d\mu)$ *and*

$$\|f * g\|_q \leqq A_p \|f\|_2 \|g\|_2.$$

Proof. (i) Let $f, g \in C_0^\infty$ then by lemma 3.1 and formula (2.16)

$$\|f * g\|_2^2 = \|f^\wedge \cdot g^\wedge\|_2^2 \leqq \|g^\wedge\|_\infty^2 \|f^\wedge\|_2^2 \leqq \|g\|_p^2 \|\varphi_0\|_q^2 \|f\|_2^2.$$

Since C_0^∞ is dense in L^2 and L^p the result follows with $A_p = \|\varphi_0\|_q$.

(ii) Let $k \in L^p(d\mu)$ and $f, g \in C_0$ then from (i)

$$\left| \int f * g(x) k(x) d\mu(x) \right| \leqq \int |g(x)| \, |k| * |f|(x) d\mu(x) \leqq \|g\|_2 \, \| \, |k| * |f| \, \|_2 \leqq \|g\|_2 A_p \|k\|_p \|f\|_2.$$

Taking supremum over $\{k \in L^p(d\mu) \mid \|k\|_p \leqq 1\}$ we get

$$\|f * g\|_q \leqq A_p \|f\|_2 \|g\|_2$$

and the result follows. Q.e.d.

COROLLARY 5.6. *Let* $1 \leqq p_1 < 2$, $1 \leqq p_2 \leqq 2$ *such that* $1/p_1 + 1/p_2 \leqq 3/2$. *Let* r *be determined by* $1/p_1 + 1/p_2 - 1 = 1/r$. *Suppose that* $f \in L^{p_1}(d\mu)$ *and* $g \in L^{p_2}(d\mu)$. *Then* $f * g \in L^s(d\mu)$ *for all* $s \in [2, r]$.

Proof. Clearly $L^2 \cap L^r \subset L^s$ for $s \in [2, r]$. By theorem 5.4 $f * g \in L^r$. It remains to prove that $f * g \in L^2$. Write $g = g_1 + g_2$ where $g_1 = g \cdot 1_{\{|g| \geqq 1\}}$ and $g_2 = g \cdot 1_{\{|g| < 1\}}$. Then $g_1 \in L^1 \cap L^{p_2}$ and $g_2 \in L^{p_2} \cap L^\infty \subset L^2$. By theorem 5.4 $f * g_1 \in L^{p_1} \cap L^r \subset L^2$, by theorem 5.5 (i) $f * g_2 \in L^2$. Thus

$$f * g = f * g_1 + f * g_2 \in L^2. \qquad \text{Q.e.d.}$$

6. The Banach algebra $(L^1(d\mu), *)$

It is clear from theorem 5.4 that $(L^1(d\mu), *)$ is a commutative Banach algebra. From (5.4) and (3.2) it is seen that the functional χ_λ defined by

$$\chi_\lambda(f) = f^\wedge(\lambda)$$

is a continuous character on $L^1(d\mu)$ for all $\lambda \in \bar{D}_1 = \{\xi + i\eta \in \mathbf{C} \mid |\eta| \leqq \varrho\}$.

Theorem 3.2 implies that $L_1(d\mu)$ is semisimple. Obviously complex conjugation is an isometric involution. It was shown in [4], that $L^1(d\mu)$ has an approximate identity.

Lemma 6.1. *Let* $f \in C^\infty$ *and* $\omega = \omega_{\alpha,\beta}$ *then*

$$(\omega_x - \omega_y) T_x f(y) = 0.$$

Proof. Since the kernel $K(x, y, z)$, if considered as a function of z, has compact support it is easily seen that it is sufficient to prove the lemma for $f \in C_0^\infty$. Using (4.2) we can write for such f

$$T_x f(y) = \int_0^\infty f^\wedge(\lambda) \varphi_\lambda(x) \varphi_\lambda(y) d\nu(\lambda),$$

therefore

$$\omega_x T_x f(y) = \omega_y T_x f(y) = \int_0^\infty -(\lambda^2 + \varrho^2) f^\wedge(\lambda) \varphi_\lambda(x) \varphi_\lambda(y) d\nu(\lambda). \qquad \text{Q.e.d.}$$

Lemma 6.2. *Every non-zero continuous character on* $(L^1(d\mu), *)$ *has the form*

$$\chi(f) = f^\wedge(\lambda)$$

for some $\lambda \in \bar{D}_1$.

Proof. (The idea of the proof is taken from [7], p. 400.) Assume $\chi \neq 0$ is a continuous character on $L^1(d\mu)$. Since the dual space of L^1 is L^∞, there exists a function $g \in L^\infty(d\mu)$ such that $\chi(f) = \int_0^\infty f(x) g(x) d\mu(x)$. In view of the identity

$\chi(f_1 * f_2) = \chi(f_1)\chi(f_2)$ it follows by a straight-forward calculation that for almost all x, y

$$g(x)g(y) = \int_0^\infty g(z)K(x, y, z)d\mu(z) = T_x g(y). \tag{6.1}$$

Choose $\psi \in C_0^\infty$ such that $\int_0^\infty g(x)\psi(x)d\mu(x) = c \neq 0$. Then for almost all y we get

$$g(y) = \frac{1}{c}\int_0^\infty g(z)\int_0^\infty \psi(x)K(x, y, z)d\mu(x)d\mu(z) = \frac{1}{c}(g * \psi)(y).$$

It follows easily from the definition of the convolution that $g * \psi$ is a C^∞-function. Thus we can assume that g is a C^∞-function.

From (6.1) it is seen that $g(0) = 1$. Lemma 6.1 applied to (6.1) gives

$$(\omega g)(x)g(y) = g(x)(\omega g)(y).$$

By taking $y = 0$ it is clear that g is an eigenfunction of ω with eigenvalue $\omega g(0)$. Therefore $g(x) = \varphi_\lambda(x)$ for some $\lambda \in \mathbf{C}$. Since g is bounded it follows from lemma 2.1 that $\lambda \in \bar{D}_1$. Q.e.d.

THEOREM 6.3. *$(L^1(d\mu), *)$ is a semisimple, commutative Banach algebra with involution and approximate identity.*

The maximal ideal space is $\bar{D}_1$ with λ and $-\lambda$ identified. The set of self-adjoint maximal ideals are given by $\{\lambda \in \mathbf{C} \mid \lambda^2 + \varrho^2 \geqq 0\}$, with λ and $-\lambda$ identified.

Proof. The only thing left to be proved is that for $\lambda \in \bar{D}_1$ χ_λ is self-adjoint if and only if $\lambda^2 + \varrho^2 \geqq 0$. But this follows easily from the fact that φ_λ is real if and only if λ^2 is real. Q.e.d.

COROLLARY 6.4. *The function $\varphi_\lambda(x)$ for $\lambda \in \mathbf{C}$ is characterized by the integral equation* (6.1).

This follows from the proof of lemma 6.2.

Remark. It was pointed out by H. Chébli in [2], that Weinberger's maximum property for differential equations [12] can be applied in order to prove the positivity of the generalized translation operation (5.1).

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