New proof of the positivity of generalized translation for Jacobi series *)

by

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ABSTRACT

Weinberger's maximum theorem for hyperbolic differential equations is applied to obtain a new proof of Gasper's result concerning the positivity of generalized translation for Jacobi series.

KEY WORDS & PHRASES: *Positivity of generalized translation for Jacobi series, Weinberger's maximum theorem for hyperbolic differential equations.*

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1. INTRODUCTION

Let $\alpha,\beta > -1$. Jacobi polynomials $R_n^{(\alpha,\beta)}(x)$ are orthogonal polynomials of degree n on the interval $(-1,1)$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$ and with the normalization $R_n^{(\alpha,\beta)}(1) = 1$.

Let us consider functions u defined on $[0,\pi/2] \times [0,\pi/2]$ of the form

$$u(s,t) = \sum_{k=0}^{n} c_k\, R_k^{(\alpha,\beta)}(\cos 2s)\, R_k^{(\alpha,\beta)}(\cos 2t), \tag{1.1}$$

where $n = 0,1,2,\ldots$ and $c_0,c_1,\ldots,c_n$ are arbitrary real constants. The function $u(.,t)$ is called the generalized translate of the function $u(.,0)$. The purpose of the present paper is to give a new proof of the following theorem of GASPER [5], [6].

<u>THEOREM 1.1</u>. *Let* $\alpha \geq \beta \geq -\frac{1}{2}$. *If* $u(s,t)$ *has the form* (1.1) *and if* $u(s,0) \geq 0$ *for each* $s \in [0,\pi/2]$ *then* $u(s,t) \geq 0$ *for each* $s,t \in [0,\pi/2]$.

We mention three possible approaches to prove Theorem 1.1.

A. BOCHNER [2] pointed out that in the case $\alpha = \beta \geq -\frac{1}{2}$ the positivity result follows from the product formula for Gegenbauer polynomials. The product formula for Jacobi polynomials (cf. KOORNWINDER [7]) has a similar corollary in the case $\alpha \geq \beta \geq -\frac{1}{2}$.

B. GASPER [5], [6] explicitly calculated the kernel of generalized translation and he expressed the kernel in terms of hypergeometric functions. This enabled him to prove that generalized translation for Jacobi series is positive if and only if $\alpha \geq \beta \geq -\frac{1}{2}$ or $\alpha \geq |\beta|$, $\beta > -1$.

C. Functions of the form (1.1) are solutions of a hyperbolic differential equation. A maximum property of such solutions implies that if $\alpha \geq \beta > -1$, $\alpha + \beta + 1 \geq 0$ and if $u(s,0) \geq 0$ for each $s \in [0,\pi/2]$ then $u(s,t) \geq 0$ for $0 \leq t \leq s \leq \pi/2 - t$. This result is due to WEINBERGER [8] in the case $\alpha = \beta \geq -\frac{1}{2}$ and to ASKEY [1, p.81] in the general case. If $\alpha = \beta \geq -\frac{1}{2}$ then the positivity of generalized translation follows from the case $0 \leq t \leq s \leq \pi/2 - t$ by the identities $u(s,t) = u(t,s)$ and $u(s,t) = u(\pi/2-s,\pi/2-t)$. However, if $\alpha > \beta$ then this method fails since the second identity no longer holds.

In the present note we shall prove Theorem 1.1 by using the approach described in C. The proof is a refinement of Askey's argument used in [1, p.81].

It is of interest to compare these results with the corresponding case of Jacobi functions $\phi_\lambda^{(\alpha,\beta)}(t)$. FLENSTED-JENSEN & KOORNWINDER [4] proved that the generalized translation for Jacobi function expansions is positive in the case $\alpha \geq \beta \geq -\frac{1}{2}$ by using approach A. CHÉBLI [3] independently obtained the positivity result for $\alpha \geq \beta \geq -\alpha - 1$ by using approach C. In the case of Jacobi functions one needs only to consider the region $\{(s,t) \mid 0 \leq t \leq s\}$. Hence, Chébli could obtain his result without using the more intricate argument we shall need.

2. THE POSITIVITY OF GENERALIZED TRANSLATION FOR $\alpha \geq \beta \geq -\frac{1}{2}$

Let us write

$$w(s) = w_{\alpha,\beta}(s) = (\sin s)^{2\alpha+1}(\cos s)^{2\beta+1} \qquad (0 < s < \pi/2)$$

and

$$a(s,t) = a_{\alpha,\beta}(s,t) = w_{\alpha,\beta}(s)w_{\alpha,\beta}(t).$$

Jacobi polynomials satisfy the differential equation

$$(w_{\alpha,\beta}(s))^{-1} \frac{d}{ds}\left[w_{\alpha,\beta}(s) \frac{d}{ds} R_n^{(\alpha,\beta)}(\cos 2s)\right] =$$

$$= -4n(n+\alpha+\beta+1)\, R_n^{(\alpha,\beta)}(\cos 2s).$$

Hence, any function u of the form (1.1) is a solution of the hyperbolic differential equation

$$(2.1) \qquad (a\, u_s)_s - (a\, u_t)_t = 0.$$

Using Bateman's integral for Jacobi polynomials ASKEY [1, p.82] proved:

LEMMA 2.1. *Let* $\alpha \geq \beta > -1$ *and let* $u(s,t)$ *have the form* (1.1). *If* $u(s,0) \geq 0$ *for each* $s \in [0,\pi/2]$ *then* $u(\pi/2,t) \geq 0$ *for each* $t \in [0,\pi/2]$.

Using approach C we shall prove:

LEMMA 2.2. *Let* $\alpha \geq \beta \geq -\frac{1}{2}$ *and let* $u(s,t)$ *have the form* (1.1). *If* $u(s,0) > 0$ *for each* $s \in [0,\pi/2]$ *and if* $u(\pi/2,t) > 0$ *for each* $t \in [0,\pi/2]$ *then* $u(s,t) > 0$ *for each* (s,t) *such that* $0 \leq t \leq s \leq \pi/2$.

Lemma 2.1 and Lemma 2.2 together imply theorem 1.1.

PROOF OF LEMMA 2.2. Let $O = (0,0)$, $C = (\pi/2,0)$, $D = (\pi/2,\pi/2)$, $E = (0,\pi/2)$. Choose a point P in the closed triangular region OCD and let A and B be points on AC or CD such that the slopes of AP and BP are 1 and -1, respectively.

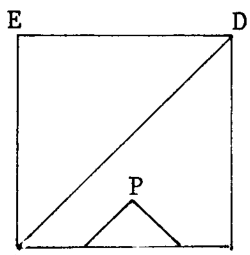

Figure 1

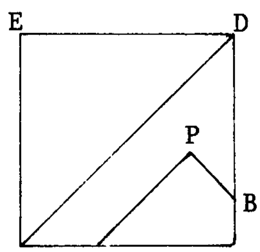

Figure 2

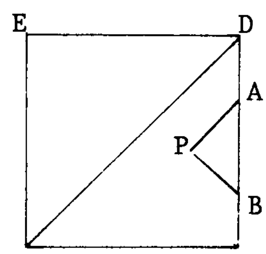

Figure 3

For any possible contour ABP (cf. figure 1,2,3) we have by (2.1) and by Gauss's theorem:

$$0 = \iint_{ABP} [(a\, u_s)_s - (a\, u_t)_t]\, ds\, dt$$

$$= \pm \oint_{ABPA} (a\, u_s\, dt + a\, u_t\, ds) = \mp \left(\int_{AP} + \int_{BP} \right) a\, du.$$

If u is positive on OC and CD then $a(A)u(A) + a(B)u(B) \geq 0$. Hence, integration by parts gives

$$(2.2) \qquad 2a(P)u(P) \geq \int_{AP} u(a_s + a_t)\, dt + \int_{BP} u(-a_s + a_t)\, dt.$$

It follows by a simple calculation that

$$\pm a_s(s,t) + a_t(s,t) = a(s,t)(\text{cotg } t \pm \text{cotg } s) \cdot$$

$$\cdot ((2\alpha+1) \mp (2\beta+1)\text{tgs tgt}).$$

Hence, $-a_s + a_t > 0$ if $0 < t < s < \pi/2$. Let Γ denote the curve $\{(s,t) \mid 2\alpha + 1 - (2\beta+1)\text{tgs tgt} = 0\}$, cf. figure 2.

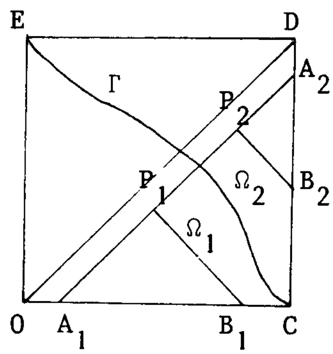

Figure 2

Any line inside OCDE with slope 1 intersects Γ in one and only one point. The curve Γ separates the region OCDE in two connected regions Ω_1 and Ω_2 on which $a_s + a_t$ is positive, respectively negative. Suppose now that u is positive on OC and CD but that $u(s,t) \le 0$ for some (s,t), $0 < t \le s < \pi/2$. Then, by continuity, there is a line A_1A_2 (A_1 on OC and A_2 on CD) with slope 1 and there are points P_1 and P_2 on A_1A_2 (cf. figure 2) such that $u(P_1) = 0 = u(P_2)$ and $u(s,t) > 0$ on the open region A_1CA_2 and on the open line segments A_1P_1 and A_2P_2. Let B_i $(i=1,2)$ be on OC or CD such that P_iB_i has slope -1. In at least one of the two cases $i = 1,2$ the open line segment A_iP_i is contained in the region Ω_i. For this choice of A_i, B_i, P_i formula (2.2) gives a contradiction. □

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