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REPRESENTATIONS OF LOCALLY COMPACT GROUPS WITH APPLICATIONS

PART II

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INDUCED REPRESENTATIONS OF FINITE GROUPS

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1. INTRODUCTION

The concept of induction is a most powerful tool in representation theory. For finite groups, this method of obtaining representations of a group by means of representations of its subgroups was designed by Frobenius in 1898. In the period 1939-1950, Wigner, Bargmann and others used induction in an implicit manner, in papers which dealt with the representations of special noncompact groups, such as the Lorentz group (cf. references of Ch.X). It was G.W. Mackey in the years around 1950, who constructed a unified theory of induced representations for general locally compact groups. He also developed an extension of the important concept of imprimitivity to locally compact groups. Imprimitivity is closely related to representation theory, in particular to the theory of induced representations. Finally, Mackey showed how to apply induction and imprimitivity to obtain irreducible unitary representations of locally compact semidirect products from certain proper subgroups ("little groups"). For an important class of semidirect products*) these results are fairly complete (see Ch.X). This method, known as the little group method, had been used earlier by Wigner in connection with the Poincaré group.

In this chapter we plan to discuss rather extensively the basic features of induction on finite groups. In the first place we aim to provide a motivation for the theory of Mackey, which is to be discussed in the chapters IX, X and XI. Secondly, this chapter could serve as a simple introduction for people who take interest in advanced representation theory of finite groups. For further reading in this direction we refer to the excellent exposé of SERRE [3], where among other things the important theorems of Artin and Brauer are discussed, which ensue from the induction process.

We will start with reviewing briefly some basic facts from the general representation theory of finite groups. Next the inducing construction will be presented, first for characters only (§3), and then for representations (§4). Finally we will prove a useful theorem, which provides us with a way of deciding whether an arbitrary representation is induced from a subgroup. The extension of this theorem to locally compact groups will be given in Chapter X.

We emphasize that G will always denote a finite group, unless otherwise stated. Furthermore, all vector spaces are assumed to be complex and

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*) For instance, the Poincaré group and the Euclidean groups belong to this class.

finite-dimensional.

The reader may keep in mind that many of the results apply to compact groups as well. This can be seen by replacing expressions of the form

$$\frac{1}{|G|}\sum_{x\in G}\cdots \quad \text{by} \quad \int_G \cdots\, dx,$$

where $|G|$ denotes the cardinality of a finite group G and dx the normalized Haar measure on a compact group G.

2. GENERAL REPRESENTATION THEORY FOR FINITE GROUPS

Let V be a finite-dimensional complex vector space. By $Gl(V)$ we will denote the group of invertible linear operators on V. A homomorphism τ from a finite group G into $Gl(V)$ is called a *representation* of G on V.

Suppose that there exists a linear subspace V' of V which is stable under the action of τ, i.e. $\tau(x)\, V' = V'$ for all x in G. Then, denoting the operators $\tau(x)$ restricted to V' by $\tau'(x)$, we obtain a new representation of G; $\tau': G \to Gl(V')$. We call τ' a *subrepresentation of* τ. If τ admits no non-trivial stable subspaces, then τ is said to be *irreducible*. Let τ' be a subrepresentation of τ on V'. By π_0 we denote the "average" of a mapping π from V into itself with $\pi(V) = V'$ and $\pi^2 = \pi$, that is,

$$\pi_0 := \frac{1}{|G|}\sum_{x\in G}\tau(x)\pi\tau(x)^{-1}.$$

Clearly $\pi_0(V) = V'$, and one verifies easily that the complement V'' in V of V' corresponding to π_0 (i.e. $V'' = \text{kernel}(\pi_0)$) is stable under τ. The subrepresentation τ'' corresponding to V'' is called *complementary* to τ', and τ is called the *direct sum* of τ' and τ''. This is denoted by $\tau = \tau' \oplus \tau''$. Conversely, if we are given two representations τ and σ of G on spaces V and W respectively, we can form in an obvious way a new representation $\tau \oplus \sigma$ on the direct sum $V \oplus W$.

By iterating the construction of complementary subrepresentations given above, we see that any representation τ of G can be written as a direct sum of irreducible subrepresentations. This result is known as the theorem of Maschke. Unfortunately, such a decomposition is not always unique, as a simple counterexample may show. We will say more about this below.

Let $\mathcal{Rep}(G)$ denote the set of all representations of G. We define an equivalence relation in $\mathcal{Rep}(G)$ by calling $\tau,\sigma \in \mathcal{Rep}(G)$ *equivalent* (notation $\tau \simeq \sigma$) if there exists an invertible linear mapping $T: V(\tau) \to V(\sigma)$ such that

$$T\tau(x) = \sigma(x)T, \quad \forall x \in G. \tag{2.1}$$

(By $V(\tau)$ we denote the representation space of a representation τ.) It is clear that an equivalence class containing an irreducible representation can contain only irreducible representations. The set of equivalence classes of irreducible representation is called the *dual* of G and denoted by $\hat{G}$.

Let $\tau \in \mathcal{Rep}(G)$. The complex-valued function χ on G defined by

$$\chi(x) = \text{trace}(\tau(x)), \quad x \in G,$$

is called the *character* of τ. One verifies easily the following properties of characters.

<u>LEMMA 2.1</u>. *Let $\tau,\sigma \in \mathcal{Rep}(G)$ and let χ and ϕ denote their respective characters. Then*

(i) $\chi(e) = $ *dimension* $(V(\tau))$;

(ii) $\chi(x^{-1}) = \overline{\chi(x)}$, $\forall x \in G$;

(iii) $\chi(yxy^{-1}) = \chi(x)$, $\forall x,y \in G$;

(iv) *the character of* $\tau \oplus \sigma$ *equals* $\chi + \phi$;

(v) $\tau \simeq \sigma \Rightarrow \chi = \phi$.

We continue with discussing several important consequences of this simple lemma, especially of (iii).

Two elements x and y of G are said to be *conjugate* if $x = zyz^{-1}$ for some z in G. This defines an equivalence relation in G, so we can partition G into equivalence classes, which are called *conjugacy classes*. We shall see below that the number of conjugacy classes, the so-called *class number* of G, is an important feature of the group G. From Lemma 2.1 (iii) it follows that characters are constant on conjugacy classes. In general, we call a complex-valued function on G which satisfies this condition a *class function* (or *central function*). The set of all class functions on G, denoted by $\mathcal{Cl}(G)$, is a linear subspace of the space $\ell^2(G)$ of all complex-valued functions on G. The latter space can be equipped with an inner product, defined by

$$(\phi,\psi) := \frac{1}{|G|} \sum_{x\in G} \phi(x)\overline{\psi(x)}, \quad \phi,\psi \in \ell^2(G).$$

With an irreducible character we mean the character of an irreducible representation. The set of all irreducible characters of G will be denoted by $\mathcal{Irr}(G)$. The following lemma exposes the distinguished role played by $\mathcal{Irr}(G)$ in the space $\mathcal{Cl}(G)$.

LEMMA 2.2. *The elements of* $\mathcal{Irr}(G)$ *form an orthonormal basis for* $\mathcal{Cl}(G)$.

COROLLARY 2.3. *A class function* ϕ *is a character if and only if for each* χ *in* $\mathcal{Irr}(G)$ *the number* (ϕ,χ) *is a nonnegative integer.*

PROOF. Clear from the theorem of Maschke, mentioned above, Lemma 2.1 (iv) and Lemma 2.2. □

We continue this preliminary subsection with a discussion of the proof of Lemma 2.2, and some of its corollaries. First we need the celebrated lemma of Schur. We will take the elements of $\hat{G}$ to be proper representations, for convenience. By virtue of Lemma 2.1 (v) we can unambiguously speak about the character of $\tau \in \hat{G}$.

LEMMA 2.4 (Schur). *Let* $\tau,\sigma \in \hat{G}$, *and suppose we are given a nonzero linear mapping* $T\colon V(\tau) \to V(\sigma)$, *which satisfies*

$$T\tau(x) = \sigma(x)T, \quad \forall x \in \sigma.$$

Then $\tau = \sigma$ *and* T *is a scalar multiple of the identity on the representation space*)*.

PROOF. The obvious observation that the kernel and the range of T are invariant subspaces for τ and σ, respectively, shows that T is either zero or invertible. In the second case we have $\tau = \sigma$. Moreover, if T is invertible and if λ is any eigenvalue of T, then iteration of the preceding argument yields $T-\lambda I = 0$, where I denotes the identity on $V(\tau) = V(\sigma)$. □

Next we choose a basis in $V(\tau)$ and in $V(\sigma)$ for $\tau,\sigma \in \hat{G}$. Then τ and σ can be written in matrix form: $\tau(x) = (\tau_{ij}(x))$ and $\sigma(x) = (\sigma_{ij}(x))$. The Schur lemma implies the following orthogonality relations between matrix elements of τ and σ.

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*) This lemma has an infinite dimensional counterpart; see chapter VII.

COROLLARY 2.5.

(i) *For* $\tau \neq \sigma$ *one has*

$$\frac{1}{|G|} \sum_{x \in G} \tau_{ij}(x)\sigma_{kl}(x^{-1}) = 0, \quad \forall i,j,k,l.$$

(ii) $$\frac{1}{|G|} \sum_{x \in G} \tau_{ij}(x)\tau_{kl}(x^{-1}) = \frac{1}{\dim(V(\tau))} \delta_{i\ell}\delta_{jk}.$$

PROOF. Let $T = (T_{ij})$ be a linear mapping from $V(\tau)$ into $V(\sigma)$. Then

$$T^0 := \frac{1}{|G|} \sum_{x \in G} \sigma(x^{-1})T\tau(x)$$

is also a linear mapping from $V(\tau)$ into $V(\sigma)$. Moreover, one checks easily that T^0 satisfies relation (2.1). Since

$$\text{trace}(T^0) = \frac{1}{|G|} \sum_{x \in G} \text{trace}(\sigma(x^{-1})T\tau(x)) = \text{trace}(T),$$

the eigenvalues of T^0 are all equal to $(\dim V(\tau))^{-1} \cdot \text{trace}(T)$. Finally, choosing for T the matrix with $T_{rs} = \delta_{rj}\delta_{sk}$, the identities stated in the corollary are readily verified. □

Let χ and ϕ be irreducible characters of G. After choosing the indices in the orthogonality relations stated above conveniently, we find $(\chi,\chi) = 1$ and $(\chi,\phi) = 0$ for $\chi \neq \phi$. In order to finish the proof of Lemma 2.2, we have to check completeness of the system $\mathcal{Irr}(G)$ in $\mathcal{Cl}(G)$.

Let $\alpha \in \mathcal{Cl}(G)$, and let τ be an irreducible representation of G with character χ. The operator $\tau(\alpha)$ on $V(\tau)$ defined by

$$\tau(\alpha) = \sum_{x \in G} \alpha(x)\tau(x)$$

satisfies (2.1), and is therefore a scalar multiple of the identity on $V(\tau)$ (possibly zero). We have

$$\text{trace}(\tau(\alpha)) = \sum_{x \in G} \alpha(x)\chi(x) = |G| \cdot (\alpha,\bar{\chi}),$$

where $\bar{\chi}(x) := \overline{\chi(x)}$. Hence,

$$\tau(\alpha) = \frac{|G|}{\dim(V(\tau))} (\alpha,\bar{\chi}) \cdot I.$$

Next, suppose $(\alpha,\bar{\chi}) = 0$ for all $\chi \in \mathcal{Irr}(G)$. Then $\tau(\alpha) = 0$ for all $\tau \in \hat{G}$. If we define $\sigma(\alpha)$ for an arbitrary representation of G, we have again $\sigma(\alpha) = 0$, by direct sum decomposition. In order to finish our argument, we need the following example.

EXAMPLE 2.6. Let λ be the representation of G on the space $\ell^2(G)$, defined by

$$(\lambda(x)f)(y) = f(x^{-1}y), \quad f \in \ell^2(G).$$

A basis for $\ell^2(G)$ is formed by the functions $\{\varepsilon_x\}_{x\in G}$, defined by

$$\varepsilon_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\lambda(x)\varepsilon_y = \varepsilon_{xy}$. The representation λ is called the *left regular representation* of G. The *right regular representation* ρ of G is defined on $\ell^2(G)$ by

$$(\rho(x)f)(y) = f(yx), \quad f \in \ell^2(G).$$

For σ in the paragraph preceding this example we take λ. Then

$$0 = \lambda(\alpha)\varepsilon_e = \sum_{x\in G} \alpha(x)\lambda(x)\varepsilon_e = \sum_{x\in G} \alpha(x)\varepsilon_x.$$

Hence, $\alpha(x) = 0$ for all x in G. Thus, we proved that any function in $\mathcal{Cl}(G)$ which is orthogonal to the system $\{\bar{\chi}; \chi \in \mathcal{Irr}(G)\}$ must be zero. Clearly this implies the same for the system $\mathcal{Irr}(G)$, so we are through with Lemma 2.2. This lemma has important consequences. First, note that it follows from the orthogonality relations for the irreducible characters that non-equivalent irreducible representations have different characters. This fact yields

LEMMA 2.7. *The number of non-equivalent irreducible representations of* G *equals the class number of* G.

PROOF. The cardinality of $\hat{G}$ is equal to that of $\mathcal{Irr}(G)$, by the observation made above. The number of elements in $\mathcal{Irr}(G)$ is, in its turn, equal to the dimension of $\mathcal{Cl}(G)$, which obviously is the class number of G. □

Next, let τ be any representation of G, and let

(2.2) $$\tau = \sigma_1 \oplus \dots \oplus \sigma_n$$

be a decomposition of τ into irreducible representations. Write $\chi, \chi_1, \ldots, \chi_n$ for the characters of $\tau, \sigma_1, \ldots, \sigma_n$, respectively. The following lemma establishes the degree of uniqueness of decomposition (2.2).

LEMMA 2.8. *The number of* σ_j *equivalent to a certain* σ_i $(1 \le i,j \le n)$ *is equal to the number* (χ, χ_i). *In particular, it does not depend on the chosen decomposition.*

PROOF. We have $(\chi, \chi_i) = \sum_{j=1} (\chi_j, \chi_i)$, and the result follows from the orthonormality relations for irreducible characters. □

The character χ of the regular representation ρ is readily found to be given by $\chi(e) = |G|$ and $\chi(x) = 0$ if $x \neq e$. Let ψ be an irreducible character of G. Then

$$(\chi, \psi) = \frac{1}{|G|} \sum_{x \in G} \chi(x)\overline{\psi(x)} = \psi(e).$$

Hence, each τ in $\hat{G}$ occurs in the direct sum decomposition of ρ, with multiplicity equal to $\dim(V(\tau))$. (We call the number of subrepresentations equivalent to a given irreducible representation τ which occur in a representation σ, the *multiplicity* of τ in σ.) This observation implies the following lemma.

LEMMA 2.9. $\sum_{\tau \in \hat{G}} (\dim(V(\tau)))^2 = |G|$.

PROOF. $\mathrm{Dim}(\ell^2(G)) = |G|$. □

Last but not least we notice that the converse of Lemma 2.1 (v) follows from Lemma 2.8. Thus we have

LEMMA 2.10. *Two representations of* G *are equivalent if and only if they have the same character.*

EXAMPLE 2.11. Let $S_3 = \{(1),(12),(13),(23),(123),132)\}$ be the permutation group of an ordered set of three elements. This group is isomorphic to the dihedral group D_3, which consists of those rotations and reflections of the real plane that preserve a regular triangle. If we set $s = (12)$ and $r = (123)$, we get $s^2 = (1) = e$, $r^3 = e$, $sr = r^2 s$ and $rs = sr^2$. The conjugacy classes are readily seen to be $K_1 = \{e\}$, $K_2 = \{s, sr, rs\}$ and $K_3 = \{r, r^2\}$. Hence, there are three irreducible characters. Furthermore, we must have

$$\sum_{\tau\in\hat{S}_3} (\dim \tau)^2 = \sum_{\chi\in Irr(S_3)} (\chi(e))^2 = |S_3| = 6.$$

Therefore, two of the irreducible characters are one-dimensional and one is two-dimensional. Let χ_1 be the trivial character ($\chi_1 = 1$) and let χ_2 be the one-dimensional character that can be defined on all permutation groups: $\chi_2(x) = 1$ if x is even and $\chi_2(x) = -1$ if x is odd. (We call a permutation even (odd) if it contains an even (odd) number of inversions.) For S_3 we get $\chi_2(K_2) = -1$ and $\chi_2(K_3) = 1$, denoting by $\chi(K)$ the constant value of χ on a conjugacy class K. The third character can now be reconstructed from the orthogonality relations, knowing that $\chi_3(e) = 2$:

$$(\chi_1,\chi_3) = \frac{1}{6}(2 + 3\chi_3(K_2) + 2\chi_3(K_3)) = 0,$$

$$(\chi_2,\chi_3) = \frac{1}{6}(2 - 3\chi_3(K_2) + 2\chi_3(K_3)) = 0.$$

Hence, $\chi_3(K_2) = 0$ and $\chi_3(K_3) = -1$. It is convenient to store our knowledge in a so-called *character table*, that is, a matrix, with at the ij-th place the value of the i-th character on the j-th conjugacy class:

Table 1

	K_1	K_2	K_3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

The representation τ_3 corresponding to χ_3 can be realized in $\mathbb{C}^2$ by the aforementioned isomorphism of S_3 on D_3. Choosing the regular triangle conveniently in $\mathbb{R}^2 \subset \mathbb{C}^2$, we obtain as the generators of D_3 two matrices X and Y which are given by

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{pmatrix},$$

respectively a reflection and a rotation through an angle $\frac{2}{3}\pi$. Clearly s corresponds with X and r with Y. Hence, $\tau_3(s) = X$, $\tau_3(s^2) = \tau_3(e) = I$ (the identity matrix), $\tau_3(r) = Y$, $\tau_3(r^2) = Y^2$, $\tau_3(sr) = XY$, $\tau_3(rs) = YX$.

The group S_3 contains an invariant subgroup of index two, namely

$A_3 := \{e,r,r^2\}$, the so-called *alternating group,* which contains all even permutations. This subgroup is cyclic, and its character table is easily verified to be:

Table 2

	$\{e\}$	$\{r\}$	$\{r^2\}$
ψ_1	1	1	1
ψ_2	1	ω	ω^2
ψ_3	1	ω^2	ω

Here $\omega = e^{\frac{2i\pi}{3}}$. Note that it is in general not true that a subgroup inherits the conjugacy class structure from the original group.

3. INDUCTION OF CHARACTERS

Restricting representations of G to a subgroup H yields representations of H, with the same representation space. In general this restriction can not be reversed, that is, it is not always possible to extend representations of H to representations of G with the same representation space. For instance, the representations of A_3 corresponding to its nontrivial irreducible characters (Example 2.11) cannot be extended to one-dimensional representations of S_3. However, there is a canonical construction which assigns a representation of G to every representation of H, and which is in some sort dual to the process of restriction. It proceeds by extending the representation space of a given representation of H to a larger space in which a representation of G can be defined (§4). For the sake of clarity we will show by means of characters that such a construction is possible, before discussing it in detail. The sense of duality in this context is to be explained at the end of this subsection.

Thus, let τ be a representation of H and let χ be its character. We will show how χ can be extended to a character of G. The most natural way, perhaps, would be to produce a function $\dot{\chi}$: $G \to \mathbb{C}$ by the following definition:

$$\dot{\chi}(x) := \begin{cases} \chi(x) & \text{if } x \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately, this yields in general not even a class function: take for example any irreducible character of $A_3 \subset S_3$. Another possible step is to *centralize* $\dot{\chi}$:

$$(3.1) \qquad \hat{\chi}(x) := \frac{1}{|G|} \sum_{y \in G} \dot{\chi}(y^{-1}xy).$$

Here we have a class function on G, but is it a character? To check this we compute its Fourier coefficients in the space $\mathcal{C}\ell(G)$. Let ϕ be in $\mathcal{Irr}(G)$. By $(-,-)_G$ and $(-,-)_H$ we denote the inner products in $\mathcal{C}\ell(G)$ and $\mathcal{C}\ell(H)$, respectively.

$$\begin{aligned}
(\hat{\chi},\phi)_G &= \frac{1}{|G|} \sum_{x \in G} \hat{\chi}(x)\overline{\phi(x)} = \\
&= \frac{1}{|G|} \sum_{x \in G} \left(\frac{1}{|G|} \sum_{y \in G} \dot{\chi}(y^{-1}xy)\right)\overline{\phi(x)} = \\
&= \frac{1}{|G|^2} \sum_{x,y \in G} \dot{\chi}(y^{-1}xy)\overline{\phi(y^{-1}xy)} = \\
&= \frac{1}{|G|} \sum_{x \in G} \dot{\chi}(x)\overline{\phi(x)} = \\
&= \frac{1}{|G|} \sum_{x \in H} \chi(x)\overline{\phi(x)} = \\
&= \frac{|H|}{|G|} (\chi,\phi|_H)_H.
\end{aligned}$$

Here $\phi|_H$ denotes the character of H obtained by restricting ϕ. From Corollary 2.3 we see that taking $(|G|/|H|)\cdot\hat{\chi}$ instead of $\hat{\chi}$ yields a character of G. Denoting this character by χ^G, it follows from (3.1) that

$$(3.2) \qquad \chi^G(x) = \frac{1}{|H|} \sum_{y \in G} \dot{\chi}(y^{-1}xy), \qquad x \in G.$$

DEFINITION 3.1. The character χ^G defined by (3.2) is said to be *induced* on G by χ. The corresponding representation is denoted by τ^G. It is also called induced on G (by τ).

PROPOSITION 3.2 (Frobenius reciprocity theorem). *If χ and ϕ are characters of* H *and* G *respectively,* H *being a subgroup of* G, *then*

$$(3.3) \qquad (\chi^G,\phi)_G = (\chi,\phi|_H)_H.$$

The proof of this proposition follows directly from the above computation, in which we did not use the irreducibility of ϕ. It provides us with information about the decomposition of χ^G when χ is irreducible. For suppose that

$$\chi^G = \sum_{\psi\in \mathcal{I}rr(G)} m_{\chi,\psi}\,\psi \quad \text{and} \quad \phi|_H = \sum_{\eta\in \mathcal{I}rr(H)} n_{\phi,\eta}\,\eta.$$

Then one has for all ϕ in $\mathcal{I}rr(G)$ and all χ in $\mathcal{I}rr(H)$:

$$m_{\chi,\phi} = (\chi^G,\phi)_G = (\chi,\phi|_H)_H = n_{\phi,\chi}.$$

Hence, we find the following corollary to Proposition 3.2:

COROLLARY 3.3. *If* τ *and* σ *are irreducible representations of* H *and* G, *respectively, then the multiplicity of* σ *in* τ^G *equals the multiplicity of* τ *in* $\sigma|_H$.

Using formula (3.2) the reader will find no difficulty in verifying the following results:

PROPOSITION 3.4. *Let* χ *and* ϕ *be characters of the subgroup* $H \subset G$. *Then*

$$(\chi+\phi)^G = \chi^G + \phi^G$$

and, if ψ *is a character of* G,

$$\chi^G\psi = (\chi\cdot\psi|_H)^G.$$

COROLLARY 3.5. *For representations* τ *and* σ *of* H *and a representation* ν *of* G, *one has*

$$(\tau \oplus \sigma)^G \simeq \tau^G \oplus \sigma^G$$

and

$$\tau^G \otimes \nu \simeq (\tau \otimes \nu|_H)^G.$$

COROLLARY 3.6. *If the induced representation* τ^G *is irreducible, then* τ *is irreducible.*

Unfortunately, the converse of this statement is in general false (cf. Example 3.8).

PROPOSITION 3.7 (Induction in stages). *If* H_1 *and* H_2 *are subgroups of* G *such that* $H_1 \subset H_2$, *and if* τ *is a representation of* H_1, *then*

$$(\tau^{H_2})^G \simeq \tau^G .$$

REMARK

(i) If n is the dimension of a representation τ of H, then the dimension of τ^G is n.d, where d is the index of H in G, that is, the number of different left H-cosets. This follows from (3.2).

(ii) We can define a linear mapping

$$\mathcal{R}es_H : \mathcal{C}\ell(G) \to \mathcal{C}\ell(H),$$

which sends a class function on G to its restriction to H. Formula (3.2) may be considered as a definition of ϕ^G for all ϕ in $\mathcal{C}\ell(G)$, and the resulting mapping $\phi \to \phi^G$:

$$\mathcal{I}nd^H : \mathcal{C}\ell(H) \to \mathcal{C}\ell(G)$$

is then linear, and, moreover, it is the adjoint of $\mathcal{R}es_H$ by (3.3). In this sense, restriction and induction are dual actions.

EXAMPLE 3.8. If we take $H \subset G$ to be the trivial subgroup $\{e\}$, and if we induce the trivial one-dimensional representation of $\{e\}$ (denote it by 1_e), then we obtain

$$1_e^G(x) = \begin{cases} |G| & \text{if } x = e, \\ 0 & \text{otherwise.} \end{cases}$$

This is just the character of the regular representation of G. Application of Proposition 3.7 shows that induction of the regular representation of any subgroup results in the regular representation of G.

EXAMPLE 3.9. Consider the subgroup A_3 of S_3 discussed in Example 2.11. Inducing the caracter ψ_2 of A_3 on S_3 yields

$$\psi_2^{S_3}(K_1) = 2, \quad \psi_2^{S_3}(K_2) = 0 \quad \text{and} \quad \psi_2^{S_3}(K_3) = -1,$$

since $1 + \omega + \omega^2 = 0$, $\omega = e^{\frac{2i\pi}{3}}$. Thus we obtain the only irreducible character of S_3 of dimension greater than one. In general we call a group *monomial* whenever all its irreducible representations are induced by one-dimensional representations.

EXAMPLE 3.10. Suppose that there are two subgroups N and H of G, such that

(i) N is invariant,

(ii) $G = N \cdot H$, and

(iii) $N \cap H = \{e\}$.

Then G is called a *semidirect product* (of N and H). Note that (ii) and (iii) imply that every element of G can be written uniquely as the product of an element of N and an element of H. If the additional condition

(iv) N is commutative

is satisfied, then G enjoys the property of having all of its irreducible representations induced from subgroups of the form $N \cdot H'$, where H' is a subgroup of H (a *little group*). This is also true for infinite locally compact semidirect products satisfying (iv), be it under a certain restriction of a measure theoretical kind. We will come to this in Chapter XI. Note that $S_3 = A_3 \cdot \{e,s\}$ is an example of a semidirect product.

4. THE INDUCING CONSTRUCTION

We will now explicitly construct the representation τ^G, induced by a given representation τ of a subgroup H of G. First we define a representation $\hat{\tau}$ of G in terms of τ and then we prove that its character equals χ^G, where χ is the character of τ. Except for a lot of technical complications of a mainly measure theoretical kind, the following procedure is the same as that for locally compact groups.

Let $V = V(\tau)$ be the representation space of τ. Define F_τ as the linear space of all functions $f: G \to V$ that satisfy

$$(4.1) \qquad f(xy) = \tau(y^{-1})f(x), \qquad \forall x \in G,\ \forall y \in H.$$

In F_τ we define an action $\hat{\tau}(y)$ for y in G, by

$$(4.2) \qquad (\hat{\tau}(y)f)(x) := f(y^{-1}x), \qquad f \in F_\tau.$$

Obviously, for all y in G and all f in F_τ the new function $\hat{\tau}(y)f$ belongs to F_τ as well. Moreover, $\hat{\tau}(e)$ is the identity and, for all x, y and z in G:

$$(\hat{\tau}(y)\hat{\tau}(z)f)(x) = (\hat{\tau}(z)f)(y^{-1}x) = f(z^{-1}y^{-1}x) =$$

$$= f((yz)^{-1}x) = (\hat{\tau}(yz)f)(x).$$

In particular, it follows that $(\hat{\tau}(y))^{-1} = \hat{\tau}(y^{-1})$, so $\hat{\tau}(y)$ is invertible for all y in G. Hence $\hat{\tau}$ is a homomorphism of G into the group of all invertible linear mappings of F_τ into itself. Consequently, $\hat{\tau}$ is a representation of G.

Fix a set of representatives of left H-cosets xH, say $\{x_i\}_{i=1}^d$, with $d = |G/H|$; the index of H in G. Thus, $G = x_1H \cup \ldots \cup x_dH$, and $x_iH \cap x_jH = \emptyset$ if $i \neq j$. Clearly the functions in F_τ are determined by their values on the x_i. Hence, the mapping $f \to (f(x_1),\ldots,f(x_d))$ defines a vector space isomorphism from F_τ onto $V^d = V \oplus \ldots \oplus V$. In order to compute the character of $\hat{\tau}$, it is convenient to lift the action of $\hat{\tau}$ on F_τ to an action on V^d, also denoted by $\hat{\tau}$, by means of this isomorphism. The action of $\hat{\tau}(y)$ on V^d can be represented by a d×d-array $(\hat{\tau}_{ij}(y))$ of operators on V. That is, for all y in G we have

$$(4.3) \qquad (\hat{\tau}(y)f)(x_i) = \sum_{j=1}^{d} \hat{\tau}_{ij}(y)f(x_j).$$

Let now x_ℓ be the representative of the coset containing $y^{-1}x_i$. Then $x_\ell^{-1}y^{-1}x_i \in H$, or, saying it in a different way, $x_j^{-1}y^{-1}x_i \in H$ if and only if $j = \ell$. Hence, using (4.1) and (4.2) we obtain

$$(\hat{\tau}(y)f)(x_i) = f(y^{-1}x_i) = f(x_\ell x_\ell^{-1}y^{-1}x_i) =$$

$$= \tau(x_i^{-1}yx_\ell)f(x_\ell) = \sum_{j=1}^{d} \dot{\tau}(x_i^{-1}yx_j)f(x_j)$$

where

$$\dot{\tau}(x) = \begin{cases} \tau(x) & \text{if } x \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Combining this with (4.3) yields $\hat{\tau}_{ij}(y) = \dot{\tau}(x_i^{-1}yx_j)$. Now we are in a position to compute the trace of $\hat{\tau}(y)$:

$$\text{trace } (\hat{\tau}(y)) = \sum_{i=1}^{d} \text{trace } (\hat{\tau}_{ii}(y)) =$$

$$= \sum_{i=1}^{d} \text{trace } \dot{\tau}(x_i^{-1}yx_i)) =$$

$$= \sum_{i=1} \dot{\chi}(x_i^{-1}yx_i).$$

Since $\chi(z^{-1}yz) = \chi(y)$ for all $y \in G$ and all $z \in H$, we may rewrite this expression as

$$\sum_{i=1}^{d} \frac{1}{|H|} \sum_{z\in H} \dot{\chi}(z^{-1}x_i^{-1}yx_iz).$$

If i runs from 1 to d and z runs through H, x_iz runs precisely once through G, so we have

$$\text{trace } (\hat{\tau}(y)) = \frac{1}{|H|} \sum_{x\in G} \dot{\chi}(x^{-1}yx) = \chi^G(y).$$

Since the trace of the lifted operator $\hat{\tau}(y)$ equals the trace of $\hat{\tau}(y)$ in F_τ, we have proved that $\hat{\tau} \simeq \tau^G$.

REMARK. Formula (4.2) defines an action similar to the left regular representation, be it in a different space. If we take $H = \{e\}$ and for τ the trivial representation of H, we get $F_\tau = \ell^2(G)$. Hence the above construction is in fact a generalization of the regular representation. Generalizing in the same way the right regular representation we obtain an alternative approach.

(4.1)' $\quad f(yx) = \tau(y)f(x), \quad \forall y \in H, \forall x \in G,$

and

(4.2)' $\quad (\hat{\tau}(y)f(x) = f(xy), \quad x,y \in G.$

However, it is easily verified that $\hat{\tau}$ and $\hat{\tau}'$ are equivalent. If we take H to be an arbitrary subgroup of G, we can also induce the trivial representation. In that case we have that $F_\tau = \ell^2(G/H)$, the space of all complex-valued functions on G which are constant on left cosets of H. The induced representation acts in this space just as the left regular representation. It is often called the permutation representation of G corresponding to H.

EXAMPLE 4.1. Let τ be the representation of $A_3 \subset S_3$ corresponding to the character ψ_2 (Example 2.11). Note that $\tau(x) = \psi_2(x)\cdot 1_{\mathbb{C}}$ for all $x \in A_3$, since ψ_2 is a one-dimensional character. We will construct τ^{S_3} explicitly.

Choosing e and s as representatives of the left A_3-cosets in S_3, we can identify F_τ with $\mathbb{C}^2$, by sending $f \in F_\tau$ to $(f(e),f(s)) \in \mathbb{C}^2$. Using (4.1) and (4.2), the action of τ^{S_3} on $\mathbb{C}^2$ can be computed:

$$\begin{cases} (\tau^{S_3}(e)f)(e) = f(e), \\ (\tau^{S_3}(e)f)(s) = f(s), \end{cases}$$

$$\begin{cases} (\tau^{S_3}(s)f)(e) = f(s), \\ (\tau^{S_3}(s)f)(s) = f(e), \end{cases}$$

and

$$\begin{cases} (\tau^{S_3}(r)f)(e) = f(r^2) = \tau(r)f(e) = \omega f(e), \\ (\tau^{S_3}(r)f)(s) = f(r^2 s) = f(sr) = \tau(r^2)f(s) = \omega^2 f(s). \end{cases}$$

In the same way one finds

$$\tau^{S_3}(r^2):\ (f(e),f(s)) \longrightarrow (\omega^2 f(e),\omega f(s)),$$

$$\tau^{S_3}(sr):\ (f(e),f(s)) \longrightarrow (\omega^2 f(s),\omega f(e))$$

and

$$\tau^{S_3}(rs):\ (f(e),f(s)) \longrightarrow (\omega f(s),\omega^2 f(e)).$$

Hence, with respect to the basis (1,0), (0,1) of $\mathbb{C}^2$, we can realize τ^{S_3} as follows:

$$\tau^{S_3}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau^{S_3}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^{S_3}(r) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix},$$

$$\tau^{S_3}(r^2) = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}, \quad \tau^{S_3}(sr) = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad \tau^{S_3}(rs) = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}.$$

This unitary representation is clearly equivalent to the one we presented in Example 2.11, where it was called τ_3.

5. FINITE SYSTEMS OF IMPRIMITIVITY

We start this section with some preliminary remarks on so-called G-spaces. Suppose that we are given a (not necessarily finite) group G, and a set Γ on which G acts in the following way. Each $x \in G$ defines a bijection $\gamma \to x(\gamma)$ of Γ such that (i) $e(\gamma) = \gamma$ for all $\gamma \in \Gamma$ and (ii) $x(y(\gamma)) = (xy)(\gamma)$ for all $x,y \in G$. Then Γ is said to be a G-*space*. Furthermore, Γ is said to be a *trivial* G-space if each mapping $\gamma \to x(\gamma)$ is the identity on Γ. It is called a *transitive* G-space if for any pair $\gamma,\gamma' \in \Gamma$ there exists an $x \in G$ with $x(\gamma) = \gamma'$. An example of this situation is provided by taking $\Gamma = G/H$, where H is a subgroup of G. For, let the G-action be defined by $y\colon xH \to y(xH) := (yx)H$. Obviously, G/H is a transitive G-space. On the other hand, any transitive G-space can be written as G/H for some subgroup H. Indeed, fix $\gamma_0 \in \Gamma$ and let H be the stabilizer in G of γ_0, that is,

$$H := \{x \in G;\ x(\gamma_0) = \gamma_0\}.$$

Then $f: xH \to x(\gamma_0)$ is a well-defined bijection from G/H onto Γ, such that for all x and y in G $f(y(xH)) = y(f(xH))$.

From now on we assume again that G is a finite group. Let $V = V(\tau)$ be the representation space of a representation τ of G. Suppose that there exist a G-space Γ, and a family of linear subspaces of V, indexed by Γ, say $\{V_\gamma\}_{\gamma\in\Gamma}$, with

(i) $$V = \sum_{\gamma\in\Gamma}^{\oplus} V_\gamma \qquad \text{(as a vector space direct sum)},$$

and

(ii) $$\tau(x)V_\gamma = V_{x(\gamma)} \qquad (\forall x \in G,\ \forall\gamma \in \Gamma)$$

(i.e., the spaces V_γ are permuted by the action of τ in V). Then we will call this family $\{V_\gamma\}_{\gamma\in\Gamma}$ a *system of imprimitivity* (s.o.i.) for τ. In that case, we say that γ admits a s.o.i. Moreover, we will call the system *trivial* or *transitive* according to Γ being a trivial or transitive G-space. It will turn out that we can obtain a lot of information about τ by means of the systems of imprimitivity admitted by τ.

For instance, it is clear that if τ admits no s.o.i. except the obvious one in which Γ has only one element, then τ is irreducible. Indeed, any direct sum decomposition of V in τ-invariant subspaces forms a (trivial) s.o.i. Such representations are often called *primitive*. It is in general not true that irreducibility implies primitivity.

EXAMPLE 5.1. Consider the left regular representation λ in $\ell^2(G)$. Define subspaces of $\ell^2(G)$ by

$$\ell^2_x(G) := \{f \in \ell^2(G);\ f(y) = 0 \ \text{ if } \ y \neq x\}, \qquad x \in G.$$

Clearly we have

$$\ell^2(G) = \sum_{x\in G}^{\oplus} \ell^2_x(G).$$

Moreover,

$$\lambda(y)\ell^2_x(G) = \ell^2_{yx}(G) \qquad (\forall y, x \in G).$$

Hence we have a s.o.i. for λ with $\Gamma = G$, and the action of G on itself is defined by left multiplication with a fixed element. Obviously, this system is transitive.

The next theorem is the so-called imprimitivity theorem, stated here for finite groups.

THEOREM 5.2. *Let* τ *be a representation of* G. *The following statements are equivalent*:

(i) τ *admits a transitive system of imprimitivity.*

(ii) *There exist a subgroup* $H \subset G$ *and a representation* σ *of* H *such that* τ *is equivalent to* σ^G.

PROOF. (ii) $\Rightarrow$ (i). Suppose that $\tau = \sigma^G$. Let $\Gamma = G/H$, and denote the elements of Γ by $\bar{x} := xH$. Consider for each $\bar{x} \in \Gamma$ the subspaces $F_{\bar{x}}$ of F_σ defined by

$$F_{\bar{x}} := \{f \in F \ ; \ f(y) = 0 \quad \text{if} \quad y \notin \bar{x}\}.$$

As mentioned above, Γ is a transitive G-space, under the action $y\bar{x} := \overline{yx}$. Furthermore, it is clear that $y^{-1}z \notin \bar{x}$ iff $z \notin \overline{yx}$, for all x, y and z in G. Hence,

$$\tau(y)F_{\bar{x}} = \sigma^G(y)F_{\bar{x}} = F_{\overline{yx}} .$$

Finally, we have $F_\sigma = \sum^{\oplus}_{\bar{x}\in\Gamma} F_{\bar{x}}$.

(i) $\Rightarrow$ (ii). Let τ be a representation of G, admitting a transitive s.o.i., say $\{V_\gamma\}_{\gamma\in\Gamma}$. Then Γ can be identified with G/H, where H is a subgroup of G, stabilizing some fixed point $\gamma_0 \in \Gamma$. Accordingly, we may write $\Gamma = \{x_1 = e, x_2, \ldots, x_d\}$, if $\{x_i\}_{i=1}^d$ is a fixed set of left H-coset representatives. The identity $y(\gamma) = \gamma'$ reduces to $y(x_i) = x_j$, where $\gamma = x_i\gamma_0$ and $\gamma' = x_j\gamma_0$. Thus, $y(x_i) = x_j$ if and only if $x_j^{-1}yx_i \in H$.

Since every $\tau(x)$ is an isomorphism of $V(\tau)$ we can conclude from the transitivity of the system that all spaces V_{x_i} have the same dimension, say n. Hence, $\tau(y)$ may be written as a d×d-array of n-dimensional linear mappings

$$\tau_{ij}(y) := \tau(y)|V_{x_i} : V_{x_i} \to V_{x_j}.$$

Obviously, $\tau_{ij}(y)$ is the zero mapping if $y(x_i) \neq x_j$, or, equivalently, if $x_j^{-1}yx_i \notin H$. Therefore, in order to compute the trace of $\tau(y)$, we only have to take into account $\tau_{ii}(y)$ for those values of i for which $x_i^{-1}yx_i \in H$.

Furthermore, clearly $\tau_{ii}(y)$ and $\tau_{11}(x_i^{-1}yx_i)$ have the same trace.

Let a representation σ of H be defined by

$$\sigma(y) := \tau(y)\,|_{V_{x_1}}, \quad y \in H$$

(so $V(\sigma) = V_{x_1}$). Using the preceding paragraph we can make the following computation:

$$\text{trace } (\tau(y)) = \sum_{i=1}^{d} \text{trace } (\tau_{ii}(y)) =$$

$$= \sum_{x_i^{-1}yx_i \in H} \text{trace } (\tau_{11}(x_i^{-1}yx_i)) =$$

$$= \sum_{i=1}^{d} \text{trace } (\dot{\sigma}(x_i^{-1}yx_i)) =$$

$$= \frac{1}{|H|} \sum_{z \in G} \text{trace } (\dot{\sigma}(z^{-1}yz)) =$$

$$= \frac{1}{|H|} \sum_{z \in G} \dot{\chi}(z^{-1}yz),$$

where χ is the character of σ. Hence $\sigma^G \simeq \tau$. □

COROLLARY 5.3. *All irreducible representations of* G *are induced by primitive representations.*

PROOF. A s.o.i. admitted by an irreducible representation is necessarily transitive. Therefore, the result follows via complete induction from the imprimitivity theorem 5.2, the stages theorem 2.18 and Corollary 3.6. □

REMARK. The imprimitivity theorem gives rise to an alternative definition of induced representations, which is, however, less constructive than the one we used. In order to deepen the insight into the inducing process, we will make a few remarks on this different approach.

Let τ be a representation of G in a space $V = V(\tau)$. Suppose that we are given a subgroup $H \subset G$ and a linear subspace $W \subset V$, such that

(i) $\quad \tau(x)W = W, \quad \forall x \in H,$

and

(ii) $$V = \sum_{i=1}^{d} {}^{\oplus}\, \tau(x_i) W, \quad \text{where } G/H = \{x_i H\}_{i=1}^{d}.$$

Then we shall say that τ is *induced* by $\sigma := (\tau_{|H})_{|W}$ (cf. SERRE [3, Chapter 7]). The lack of constructiveness is easily repaired.

Indeed, let σ be a representation of $H \subset G$, in a space $W = W(\sigma)$. Consider the tensor product $\ell^2(G) \otimes W$ of the space of all complex-valued functions on G, and W. For f in $\ell^2(G)$ we define two new functions on $\ell^2(G)$, ${}_y f$ and f_y, by

$${}_y f(x) := f(y^{-1}x) \quad \text{and} \quad f_y(x) := f(xy).$$

In $\ell^2(G) \otimes W$ we define the equivalence relation $\sim$ as follows:

$$f \otimes v \sim g \otimes w \text{ if for some } y \in H: \begin{cases} g = f_y, \\ w = \sigma(y) v. \end{cases}$$

The space of equivalence classes is denoted usually by $\ell^2(G) \otimes_H W(\sigma)$. Writing $f \otimes v$ for the equivalence class containing this element, a representation τ of G can be defined in this space by

$$\tau(y)(f \otimes v) := {}_y f \otimes v, \quad f \otimes v \in \ell^2(G) \otimes_H W(\sigma).$$

It is readily verified that τ is equivalent to σ^G.

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VII

GENERAL REPRESENTATION THEORY

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CONTENTS

This chapter deals with the general theory of unitary representations of locally compact groups. Bij "general" we mean that no use is made of any special structure of the group, for instance that the group is a semi-simple Lie group. The chapter falls apart in two different parts. Section 1 presents a quick survey of the theory. It is a written version of the lecture given at the colloquium. The other sections contain a rigorous and rather self-contained account of the theory, be it with two important restrictions. First, we only discuss that part of the theory which can be formulated in terms of direct sums rather than direct integrals. Second, we mainly restrict ourselves to the case of type I representations.

In the discussion of direct sum decompositions it is inessential that the representations under consideration are actually representations of groups or unitary representations. We will formulate the theory in such a generality that, for instance, representations of involutive Banach algebras and representations of σ-algebras of sets (i.e. projection-valued measures) are included.

The main reference for this chapter is Chapter I in MACKEY's Chicago Lecture notes [6]. We also made important use of ARVESON [1].

Conventions. Throughout it will be assumed that all Hilbert spaces under consideration are separable. The inner product on a Hilbert Space H, denoted by (v,w) or $(v,w)_H$ ($v,w \in H$), is supposed to be linear in the first argument and conjugate-linear in the second argument. The corresponding vector norm on H is given by $\|v\|$ or $\|v\|_H$ ($v \in H$). The algebra of all bounded linear operators on a Hilbert space H will be denoted by $L(H)$. If $A \in L(H)$ then its adjoint is written as A^*. Self-adjoint or hermitian operators on H are always supposed to be bounded. A projection operator P on H (that is, an operator $P \in L(H)$ such that $P^2 = P$) is always supposed to be self-adjoint. This implies that the null space and range of P are orthogonal to each other.

1. A QUICK SURVEY

Remember (cf. Ch.I) that a *unitary representation* π of a locally compact group G on a Hilbert space $H = H(\pi)$ is a mapping $x \to \pi(x): G \to L(H)$ such that

(i) $\pi(x)$ is a unitary operator for all $x \in G$;

(ii) π is a homomorphism, that is, $\pi(xy) = \pi(x)\pi(y)$ for all $x,y \in G$;

(iii) π is (strongly) continuous, that is, the mapping $x \to \pi(x)v: G \to H$ is continuous for each $v \in H$.

The conditions (i), (ii) imply that $\pi(e) = I$ and $\pi(x^{-1}) = \pi(x)^{-1} = (\pi(x))^*$ for all $x \in G$. Condition (iii) can be replaced by the equivalent condition:

(iii)' π is weakly continuous, that is, the function $x \to (\pi(x)v,w): G \to \mathbb{C}$ is continuous for all $v,w \in H$.

Clearly (iii) implies (iii)'. For the proof of the converse note that

$$\|\pi(x)v - \pi(y)v\|^2 =$$
$$= (\pi(x)v,\pi(x)v) + (\pi(y)v,\pi(y)v) - 2\,\mathrm{Re}\,(\pi(x)v,\pi(y)v) =$$
$$= 2\,\mathrm{Re}\,\{(\pi(y)v,\pi(y)v) - (\pi(x)v,\pi(y)v)\}, \quad x,y \in G,\ v \in H,$$

where we have only used the fact that $(\pi(x)v,\pi(x)v) = (v,v) = (\pi(y)v,\pi(y)v)$, because $\pi(x)$ and $\pi(y)$ are unitary operators. Hence, if the function $x \to (\pi(x)v,\pi(y)v)$ is continuous at y then the mapping $x \to \pi(x)v$ is continuous at y.

In the definition of a unitary representation we admit the case that $H(\pi) = \{0\}$ and $\pi(x)0 = 0$ for all $x \in G$. Then π is called the *zero representation* of G (notation $\pi = 0$).

In general representation theory it is an important problem to classify all unitary representations of G up to equivalence in terms of irreducible representations (or, in case G is not a type I group, in terms of primary representations). This problem has been solved, although the complete story would be much too long for this colloquium. A further question, the classification of the irreducible (or primary) representations, can only be answered, if more structural facts are known about G, for instance that G is a semi-simple Lie group.

In this introductory section we start with the definitions of subrepresentation, equivalence and direct sum, which are basic concepts in representation theory. Then we discuss the building blocks of general representations: irreducible, primary and multiplicity free representations. Finally we define type I representations and we state a canonical direct sum decomposition of type I representations in terms of multiplicity free representations.

1.1. The basic concepts

Let π be a unitary representation of G on H. A linear subspace H_1 of H

is called *invariant* if $\pi(x)\, H_1 \subset H_1$ for all $x \in G$. If H_1 is a closed invariant subspace then the unitary representation π_1 of G on H, defined by

$$\pi_1(x) := \pi(x)|_{H_1}, \quad x \in G,$$

is called a *subrepresentation* of π. In this case the orthoplement $H_2 := H_1^{\perp}$ is again a closed invariant subspace of H. Denote the subrepresentation corresponding to H_2 by π_2. Then π_1 and π_2 are called *complementary subrepresentations* of π, H is the direct sum of H_1 and H_2 (notation $H = H_1 \oplus H_2$) and we say that π is the *direct sum* of π_1 and π_2 (notation $\pi = \pi_1 \oplus \pi_2$). The representation π on H is called *irreducible* if $\pi \neq 0$ and if $\{0\}$ and H are the only closed invariant subspaces of H.

Let π_1 and π_2 be unitary representations of G on H_1 and H_2, respectively. A bounded linear operator A: $H_1 \to H_2$ is called an *intertwining operator* for π_1 and π_2 if $A\pi_1(x) = \pi_2(x)A$ for all $x \in G$. The linear space of all such intertwining operators is denoted by $R(\pi_1,\pi_2)$. If $\pi = \pi_1 = \pi_2$ then we write $R(\pi)$ instead of $R(\pi,\pi)$. The representations π_1 and π_2 are called (*unitarily*) *equivalent* (notation $\pi_1 \simeq \pi_2$) if $R(\pi_1,\pi_2)$ contains an operator A which maps H_1 isometrically onto H_2. Then

$$\pi_2(x) = A\pi_1(x)A^{-1}, \quad x \in G.$$

We already defined the direct sum of two unitary representations. Now let π be a unitary representation of G on H and let $H_1, H_2, \ldots$ be countably many, mutually orthogonal, closed, invariant subspaces of H such that their linear span is dense in H. Let π_i be the corresponding subrepresentation on H_i. Then H is the direct sum of the H_i's (notation: $H = H_1 \oplus H_2 \oplus \ldots = \Sigma_i^{\oplus} H_i$) and we say that π is the *direct sum* of the π_i's (notation: $\pi = \pi_1 \oplus \pi_2 \oplus \ldots = \Sigma_i^{\oplus}\pi_i$). Conversely, if countably many unitary representations π_i of G are given and if $H_i := H(\pi_i)$ then let

$$H := \{(v_1,v_2,\ldots)\,|\,v_i \in H_i,\ \sum_i \|v_i\|^2_{H_i} < \infty\}.$$

H becomes a Hilbert space in a natural way and the H_i's can be isometrically imbedded in H. Thus we have constructed the direct sum H of the H_i's and the corresponding direct sum π of the π_i's is given by

$$\pi(x)v := (\pi_1(x)v_1, \pi_2(x)v_2, \ldots),$$

where $v = (v_1,v_2,\ldots) \in H$ and $x \in G$. Finally, if π is a unitary representation then $m\pi$ ($m \in \{1,2,\ldots,\infty\}$) will denote the m-fold direct sum of π.

1.2. Decompositions of finite-dimensional representations

For reasons of motivation we first define multiplicity free and primary representations in the finite-dimensional case. Remember:

LEMMA 1.1 (Schur). *Let* π *and* ρ *be finite-dimensional irreducible unitary representations of* G. *Then:*

(a) *If* $\pi \not\simeq \rho$ *then* $R(\pi,\rho) = \{0\}$.

(b) *If* $\pi \simeq \rho$ *then* $R(\pi,\rho) = \{\lambda T \mid \lambda \in \mathbb{C}\}$, *where* T *is an intertwining isometry for* π *and* ρ. *In particular,* $R(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$.

If π is any finite-dimensional unitary representation of G then π has a direct sum decomposition

$$\pi \simeq m_1\pi_1 \oplus m_2\pi_2 \oplus \ldots \oplus m_p\pi_p , \tag{1.1}$$

where the representations π_i are irreducible and mutually inequivalent and the m_i's are natural numbers. By an application of Schur's lemma it can be shown that this decomposition is unique in the following sense. Let ρ also be a finite-dimensional unitary representation of G with direct sum decomposition

$$\rho \simeq n_1\rho_1 \oplus n_2\rho_2 \oplus \ldots \oplus n_r\rho_r , \tag{1.2}$$

where the ρ_i's are irreducible and mutually inequivalent. Then $\pi \simeq \rho$ if and only if (i) $p = r$ and (ii) there is a permutation f of $\{1,\ldots,p\}$ such that $\pi_{f(i)} \simeq \rho_i$ and $m_{f(i)} = n_i$ for $i = 1,\ldots,p$. Thus (1.1) is unique up to equivalence and the ordering of terms.

Two representations π and ρ of G with decomposition (1.1) respectively (1.2) are called *disjoint* (notation $\pi \,\check{\circ}\, \rho$) if $\pi_i \not\simeq \rho_j$ for all $i = 1,\ldots,p$, $j = 1,\ldots,q$. A finite-dimensional unitary representation σ of G is called *primary* if $\sigma \simeq m\pi$ for some irreducible unitary representation π and natural number m. A finite-dimensional representation τ of G is called *multiplicity free* if $\tau \simeq \pi_1 \oplus \ldots \oplus \pi_p$ for certain irreducible, mutually inequivalent unitary representations $\pi_1,\pi_2,\ldots,\pi_p$.

Let π be a finite-dimensional unitary representation with decomposition (1.1). Instead of immediately decomposing π in terms of irreducible representations, we may consider two intermediate decompositions. Let

(1.3) $\sigma_i := m_i\pi_i, \quad i = 1,\dots,p,$

(1.4) $\tau_j := \sum_{\{i|m_i=j\}}^{\oplus} \pi_i, \quad j = 1,2,3,\dots .$

Then σ_i is primary and τ_j is zero or multiplicity free. Now we have

(1.5) $\pi \simeq \sigma_1 \oplus \sigma_2 \oplus \dots \oplus \sigma_p ,$

(1.6) $\pi \simeq \tau_1 \oplus 2\tau_2 \oplus 3\tau_3 \oplus \dots .$

Hence the decomposition (1.1) can be obtained in either of the following ways:

(i) First apply the decomposition (1.5) on π, where the σ_i's are primary and mutually disjoint (this decomposition is unique up to equivalence and the ordering of terms) and next decompose each of the primary representations σ_i in terms of irreducible representations according to (1.3) (again unique up to equivalence).

(ii) First apply the decomposition (1.6) on π, where the τ_j's are zero or multiplicity free (unique up to equivalence) and next decompose each of the τ_j's in terms of irreducible representations (unique up to equivalence and the ordering of terms).

We give the various decompositions once more in the following table.

π	→	$\sigma_1 \oplus \dots \oplus \sigma_p$ (σ_i's primary, disjoint)	σ_i (primary)
↓		↓	↓
$\tau_1 \oplus 2\tau_2 \oplus \dots$ (τ_j's 0 or multiplicity free and disjoint)	→	$m_1\pi_1 + \dots + m_p\pi_p$ (π_i's irreducible, inequivalent)	$m_i\pi_i$ (π_i irreducible)
τ_j (multiplicity free)	→	$\sum_{\{i\|m_i=j\}}^{\oplus} \pi_i$ (m_i's irreducible, inequivalent)	

Table 1.

Decompositions in horizontal direction are canonical decompositions in terms of primary representations. Decompositions in vertical directions are canonical decompositions in terms of multiplicity free representations. Since representations which are both multiplicity free and primary are irreducible, two successive decompositions in horizontal and vertical directions yield irreducible representations.

1.3. Characterization of multiplicity free and primary representations by means of the algebra of intertwining operators

Let π be any unitary representation of G on H. If A, B $\in$ R(π), λ, $\mu \in \mathbb{C}$ then, obviously, $\lambda A + \mu B$, AB and $A^* \in R(\pi)$. Clearly R(π) contains I. Hence R(π) is a $*$-subalgebra with identity of $L(H)$. It is called the *commuting algebra* of π. The center CR(π) of R(π) is defined as the set $\{A \in R(\pi) \mid AB = BA$ for all $B \in R(\pi)\}$. It is a commutative $*$-subalgebra of R(π) which contains all scalar multiples of I. R(π) is a commutative algebra if and only if CR(π) = R(π).

THEOREM 1.2. *Let π and ρ be nonzero finite-dimensional unitary representations of* G. *Then:*

(a) $\pi \,\rotatebox[origin=c]{180}{$\perp$}\, \rho \iff R(\pi,\rho) = \{0\}$.

(b) π *is irreducible* $\iff R(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$.

(c) π *is primary* $\iff CR(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$.

(d) π *is multiplicity free* $\iff$ R(π) *is commutative.*

PROOF. Suppose that π and ρ are irreducible. Then it follows from Schur's lemma 1.1 that $R(\pi,\rho) = \{0\}$ if $\pi \not\simeq \rho$ and $R(\pi,\rho) = \{\lambda T \mid \lambda \in \mathbb{C}\}$ if $\pi \simeq \rho$, where T is an intertwining isometry from $H(\pi)$ onto $H(\rho)$. Now let π and ρ be finite-dimensional unitary representations with decompositions $\pi \simeq \pi_1 \oplus \dots \oplus \pi_k$ and $\rho \simeq \rho_1 \oplus \dots \oplus \rho_\ell$ in terms of irreducible representations. Let a linear mapping A: $H(\pi) \to H(\rho)$ have block matrix

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1k} \\ \vdots & & \vdots \\ A_{\ell 1} & & A_{\ell k} \end{pmatrix}$$

with respect to these decompositions. Then $A \in R(\pi,\rho)$ if and only if $A_{ij} \in R(\pi_j,\rho_i)$ for all i,j. Hence: $R(\pi,\rho) = \{0\} \iff R(\pi_j,\rho_i) = \{0\} (\forall i,j) \iff \pi_j \not\simeq \rho_i (\forall i,j) \iff \pi \,\rotatebox[origin=c]{180}{$\perp$}\, \rho$. This proves (a).

In order to prove the other three statements let π be a finite-dimensional unitary representation with decomposition (1.1). We will calculate the commuting algebra R(π). First observe that $R(m_k\pi_k)$ consists of all block matrices

$$(1.7)\qquad \begin{pmatrix} \lambda_{11}^{(k)}T_{11}^{(k)} & \cdots & \lambda_{1m_k}^{(k)}T_{1m_k}^{(k)} \\ \vdots & & \vdots \\ \lambda_{m_k1}^{(k)}T_{m_k1}^{(k)} & \cdots & \lambda_{m_km_k}^{(k)}T_{m_km_k}^{(k)} \end{pmatrix}, \quad \lambda_{ij}^{(k)} \in \mathbb{C},$$

where $T_{ij}^{(k)}$ is the intertwining isometry from the j^{th} copy of $\mathcal{H}(\pi_k)$ onto the i^{th} copy of $\mathcal{H}(\pi_k)$. Since $m_k\pi_k \not\approx m_\ell\pi_\ell$ if $k \neq \ell$, it follows from (a) that

$$(1.8)\qquad R(\pi) = \begin{pmatrix} R(m_1\pi_1) & & \Theta \\ & \ddots & \\ \Theta & & R(m_p\pi_p) \end{pmatrix}.$$

Hence, if $R(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$ then $p = 1$ and $m_1 = 1$, i.e., π is irreducible. This result, together with part (b) of Schur's lemma yields (b). Next, it follows from (1.7) and (1.8) that $CR(m_k\pi_k)$ consists of all block matrices

$$\begin{pmatrix} \mu^{(k)}T_{11}^{(k)} & & \Theta \\ & \ddots & \\ \Theta & & \mu^{(k)}T_{m_km_k}^{(k)} \end{pmatrix}, \quad \mu^{(k)} \in \mathbb{C},$$

and

$$CR(\pi) = \begin{pmatrix} CR(m_1\pi_1) & & \Theta \\ & \ddots & \\ \Theta & & CR(m_p\pi_p) \end{pmatrix}.$$

Hence $CR(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$ if and only if $p = 1$, and $CR(\pi) = R(\pi)$ if and only if $m_1 = m_2 = \ldots = m_p = 1$. This proves (c) and (d). □

If π and ρ are arbitrary (possible infinite-dimensional) unitary representations then we use parts (a), (b), (c) and (d) of Theorem 1.2 as definitions of *disjoint, irreducible, primary* and *multiplicity free*, respectively. By an application of the spectral theorem for hermitian operators it can be shown that this definition of irreducibility is consistent with the one we gave earlier in §1.1 (see also Theorem 4.7).

1.4. Canonical decompositions of general unitary representations

What are the analogues of the decompositions in Table 1 if π is not necessarily finite-dimensional? Generally, the decomposition of π in horizontal direction (in terms of primary representations) still holds, but in the form of a *direct integral* rather than a direct sum. However, the decomposition of π in vertical direction (in terms of multiplicity free representations) fails, except if π is a so-called *type I* representation.

Roughly, a direct integral of representations can be defined as follows. Let $(X,\mathcal{S},\mu)$ be a measure space and let $\mathcal{H}_0$ be a separable Hilbert space. Let $\mathcal{H} := L^2(X,\mu;\mathcal{H}_0)$ be the Hilbert space consisting of all mappings $f: X \to \mathcal{H}_0$ which (i) are weakly measurable, and (ii) satisfy

$$\int_X \|f(\alpha)\|^2_{\mathcal{H}_0} \, d\mu(\alpha) < \infty,$$

cf. §V.1.17. Let G be a locally compact group and, for each $\alpha \in X$, let π_α be a unitary representation of G on $\mathcal{H}_0$. Then a unitary representation π of G on $\mathcal{H}$ is called the direct integral of the representations π_α with respect to the measure μ (notation $\pi = \int_X^{\oplus} \pi_\alpha \, d\mu(\alpha)$) if for each $x \in G$ and $f \in \mathcal{H}$ we have

$$(\pi(x)f)(\alpha) = \pi_\alpha(x)(f(\alpha)) \quad \text{a.e. } [\mu].$$

This definition has to be adjusted if the Hilbert spaces $\mathcal{H}(\pi_\alpha)$ do not all have the same dimension. Then the direct integral of the π_α's is defined as the direct sum over $n = 1,2,\ldots,\infty$ of the direct integrals of the π_α's for which $\dim \mathcal{H}(\pi_\alpha) = n$. For further details on direct integrals, see §VIII.7.

Now the following parts of Table 1 can be generalized for an arbitrary unitary representation π of G. First, there is a canonical direct integral decomposition

$$\pi \simeq \int_X \sigma_\alpha \, d\mu(\alpha), \tag{1.9}$$

where the σ_α's are disjoint primary representations (cf. DIXMIER [3, §8.4.2, §18.7.6], MACKEY [6, Ch.2]). Next we can classify the primary representations σ of G as follows (cf. MACKEY [6, Ch.1]):

(i) σ is of *type I* , that is, $\sigma \simeq n_\tau$ for some irreducible unitary representation τ and some $n \in \{1,2,\ldots,\infty\}$. This is the only case occuring for finite-dimensional primary representations.

(ii) σ is of *type III*, that is, σ is not irreducible and any nonzero subrepresentation of σ is equivalent to σ.

(iii) σ is of *type II*, that is, there are uncountably many, mutually inequivalent subrepresentations of σ.

If all primary representations σ_α occuring in the canonical decomposition (1.9) of π are of type I then π is called a representation of *type I*. Type I representations have a canonical decomposition in terms of multiplicity free representations which is analogous to (1.6) (cf. §8.4):

(1.10) $$\pi \simeq \tau_1 \oplus 2\tau_2 \oplus 3\tau_3 \oplus \dots \oplus \infty\, \tau_\infty,$$

where τ_n is zero or multiplicity free and the τ_n's are mutually disjoint. Finally, if π is a multiplicity free representation then the canonical decomposition (1.9) becomes

(1.11) $$\pi \simeq \int_X \pi_\alpha \, d\mu(\alpha),$$

where the π_α's are mutually inequivalent irreducible representations.

If a locally compact group has the property that all its unitary representations are of type I (or, equivalently, that all its primary representations are of type I) then G is called a *type I group*. Compact groups (cf. the Peter-Weyl theorem), abelian groups (cf. Ch.VIII), connected semi-simple Lie groups (cf. HARISH-CHANDRA [5]) and nilpotent Lie groups (cf. DIXMIER [2]) are known to be type I groups. For a unitary representation π of a type I group we have the following analogue of Table 1:

π (type I)	$\longrightarrow$	$\int_X \sigma_\alpha \, d\mu(\alpha)$ (σ_α's primary, type I, disjoint)		σ_α (primary, type I)
$\downarrow$		$\downarrow$		$\downarrow$
$\tau_1 \oplus 2\tau_2 \oplus \dots \oplus \infty\, \tau_\infty$ (τ_j's 0 or multiplicity free, disjoint)	$\longrightarrow$	$\int_X m_\alpha \pi_\alpha \, d\mu(\alpha)$ (π_α's irreducible, inequivalent)		$m_\alpha \pi_\alpha$ (π_α irreducible, $m_\alpha \in \{1,2,\dots,\infty\}$)
τ_j (multiplicity free)	$\longrightarrow$	$\int_{\{\alpha \in X \mid m_\alpha = j\}} \pi_\alpha \, d\mu(\alpha)$ (π_α's irreducible, inequivalent)		

Table 2

The terminology "type I, II, III" was first introduced in the context of *von Neumann algebras*, these are weakly closed $*$-subalgebras with identity of $L(H)$ (H an Hilbert space). With any unitary representation π of G we can associate the von Neumann algebra which is the weak closure in $L(H(\pi))$ of the linear span of $\pi(G)$. To a large extent, the analysis of a given representation π of G is in one-to-one correspondence with the analysis of the corresponding von Neumann algebra. For instance, primary representations correspond to so-called factors, which have first been classified by MURRAY & VON NEUMANN [7].

2. THE BASIC DEFINITIONS REVISITED

If a unitary representation of a locally compact group is analyzed only by means of direct sum decompositions then the group structure and the topological structure of G do not play any role at all. Therefore we may develop the theory for "representations" of much more general objects as well. In this section we will start with such a theory, not using §1.1, although there will be some repetition. At the end of this section we will list a number of examples which fit into this more general description.

Let G be a nonempty set. For the moment we define a *representation* π of G on a Hilbert space $H = H(\pi)$ as an arbitrary mapping $x \to \pi(x): G \to L(H)$. If $\dim H(\pi) = 0$ then π is called the *zero representation* of G (notation $\pi = 0$).

Let π be a representation of G on H. A subspace H_1 of H is called *invariant* under π if $\pi(x)H_1 \subset H_1$ for all $x \in G$. A representation π_1 of G on a closed invariant subspace H_1 of H is called a *subrepresentation* of π (notation*) $\pi_1 \leq \pi$) if $\pi(x)|_{H_1} = \pi_1(x)$ for all $x \in G$. (Note that a closed subspace of a Hilbert space is again a Hilbert space.) Instead of π_1 we will write π_{H_1} or π_P, where P is the projection from H onto H_1. Two subrepresentations π_1 and π_2 of π are called *complementary subrepresentations* of π if $H(\pi_2)$ is the orthoplement in H of $H(\pi_1)$. The representation π is called *irreducible* if $\pi \neq 0$ and $\{0\}$ and H are the only closed invariant subspaces of H.

It is not generally true that for each representation π and each subrepresentation π_1 of π there exists a complementary subrepresentation of π.

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*) See Table 3 in §6 for a list of notations used for relations between representations.

A sufficient condition is the existence of an *involution* on G (i.e. a bijection $x \to x^{\sim}: G \to G$ with the property that $(x^{\sim})^{\sim} = x$ for all $x \in G$) such that

$$(2.1) \qquad \pi(x^{\sim}) = \pi(x)^*, \quad x \in G.$$

Indeed, if H_1 is a closed invariant subspace of $H(\pi)$ and if $v \in H_1^{\perp}$ then

$$(\pi(x)v,w) = (v,\pi(x)^*w) = (v,\pi(x^{\sim})w) = 0$$

for all $w \in H_1$, $x \in G$, which shows that $H_1^{\perp}$ is an invariant subspace. This situation is encountered for instance in the case of a unitary representation of a locally compact group, with $x^{\sim} := x^{-1}$.

Let π_1 and π_2 be representations of G on H_1 and H_2, respectively. A bounded linear operator $A: H_1 \to H_2$ is called an *intertwining operator* for π_1 and π_2 if $A\pi_1(x) = \pi_2(x)A$ for all $x \in G$. The linear space of all such intertwining operators is denoted by $R(\pi_1,\pi_2)$. If $\pi = \pi_1 = \pi_2$, then we write $R(\pi) := R(\pi,\pi)$. The representations π_1 and π_2 are called *(unitarily) equivalent* (notation $\pi_1 \simeq \pi_2$) if $R(\pi_1,\pi_2)$ contains an operator which maps H_1, isometrically onto H_2.

Equivalence thus defined is an equivalence relation for the collection of all representations of G, that is $\pi \simeq \pi$; $\pi_1 \simeq \pi_2 \Rightarrow \pi_2 \simeq \pi_1$; $\pi_1 \simeq \pi_2$ & $\pi_2 \simeq \pi_3 \Rightarrow \pi_1 \simeq \pi_3$. Many theorems in representation theory are (or can be) formulated in terms of equivalence classes of representations rather than individual representations.

In §1.1 we gave two definitions of a direct sum of Hilbert spaces. Consider the second one, i.e., the *direct sum* $H = \Sigma^{\oplus} H_i = H_1 \oplus H_2 \oplus \ldots$ of countably many Hilbert spaces H_i consists of all elements $v = (v_1,v_2,\ldots)$ such that $v_i \in H_i$ $(i = 1,2,\ldots)$ and $\Sigma_i \|v_i\|^2_{H_i} < \infty$. With respect to the inner product $(v,w) := \Sigma_i (v_i,w_i)_{H_i}$ $(v,w \in H)$, H becomes a Hilbert space itself. (The completeness of H is easily shown.)

<u>LEMMA</u> 2.1. *Let* $H = \Sigma_i^{\oplus} H_i$ *and let* $A_i \in L(H_i)$ $(i = 1,2,\ldots)$. *Then the mapping* $A: (v_1,v_2,\ldots) \to (A_1v_1,A_2v_2,\ldots)$ *is in* $L(H)$ *iff* $\sup_i \|A_i\| < \infty$. *In the case that* $A \in L(H)$ *we have*

$$(2.2) \qquad \|A\| = \sup_i \|A_i\| .$$

<u>PROOF.</u> First, let $A: (v_1,v_2,\ldots) \to (A_1v_1,A_2v_2,\ldots)$ be in $L(H)$. Then, for each i and for each $v_i \in H_i$ we have

$$\|A_iv_i\|_{H_i} = \|(0,\ldots,0,A_iv_i,0,\ldots,0)\|_H = \|A(0,\ldots,0,v_i,0,\ldots,0)\|_H \le$$

$$\le \|A\| \ \|(0,\ldots,0,v_i,0,\ldots,0)\|_H = \|A\| \ \|v_i\|_{H_i}.$$

Hence $\|A_i\| \le \|A\|$. Conversely, if $\sup_i\|A_i\| = M < \infty$ then

$$\sum_i \|A_iv_i\|^2_{H_i} \le \sum_i \|A_i\|^2\|v_i\|^2_{H_i} \le M^2\|v\|^2_H.$$

Hence $Av = (A_1v_1,A_2v_2,\ldots)$ is in H for all $v \in H$ and $\|A\| \le M$.

Finally, for the proof of (2.2) let $A \in L(H)$. For each i there is a sequence $v_{i,1}, v_{i,2},\ldots \in H_i$ such that $\|v_{i,n}\| = 1$ and $\|A_iv_{i,n}\| \to \|A_i\|$ as $n \to \infty$. Now let $w_{i,n} \in H$ having $v_{i,n}$ as its i^{th} coordinate and all other coordinates zero. Then

$$\|A\| \ge \sup_{i,n} \|Aw_{i,n}\|_H = \sup_{i,n} \|A_iv_{i,n}\|_{H_i} = \sup_i \|A_i\|. \quad \square$$

If $A \in L(H)$ and $A_i \in L(H_i)$, $i = 1,2,\ldots$, are as above then we call A the *direct sum* of the A_i's (notation $A = \Sigma_i^{\oplus} A_i = A_1 \oplus A_2 \oplus \ldots$). If $\pi,\pi_1,\pi_2,\ldots$ are representations of G on H,H_1,H_2, respectively, then π is called the *direct sum* of the π_i's (notation $\pi = \Sigma_i^{\oplus} \pi_i = \pi_1 \oplus \pi_2 \oplus \ldots$) if $\pi(x) = \Sigma_i^{\oplus} \pi_i(x)$ for all $x \in G$. Note that the direct sum $\Sigma_i^{\oplus} \pi_i$ of the representations $\pi_1,\pi_2,\ldots$ exists if and only if $\sup_i\|\pi_i(x)\| < \infty$ for all $x \in G$. The n-fold direct sum ($n=1,2,\ldots,\infty$) of a representation π of G is denoted by $n\pi$.

If π is a representation of G on H and if H is the closure of the linear span of mutually orthogonal closed invariant subspaces H_i then $\pi \simeq \Sigma_i^{\oplus} \pi_{H_i}$. The intertwining isometry A from $\Sigma_i^{\oplus} H_i$ onto H is given by $A(v_1,v_1,\ldots) := v_1 + v_2 + \ldots$.

Let $H = \Sigma_i^{\oplus} H_i$ and $A = \Sigma_i^{\oplus} A_i \in L(H)$, where $A_i \in L(H_i)$. Then $A^* = \Sigma_i^{\oplus} A_i^*$. Hence, if $x \to x^{\sim}$ is an involution of G and if, for each i, π_i is a representation of G on H_i satisfying (2.1) such that $\pi := \Sigma_i^{\oplus} \pi_i$ exists, then the representation π also satisfies (2.1). Next, suppose that A_i is unitary for all i. Then $\|A_i\| = 1$ and $A_i^*A_i = A_iA_i^* = I$. Thus $A = \Sigma_i^{\oplus} A_i$ exists and $A^*A = AA^* = I$, so A is unitary as well. Hence, if, for each i, π_i is a representation of G on H_i such that $\pi_i(x)$ is unitary for all $x \in G$ then $\pi = \Sigma_i^{\oplus} \pi_i$ exists and $\pi(x)$ is unitary for all $x \in G$.

LEMMA 2.2. *Let G be a locally compact group with unitary representations*[*)] π_i *on Hilbert spaces* H_i $(i = 1,2,\ldots)$. *Then* $\pi = \Sigma_i^{\oplus} \pi_i$ *exists and is a unitary representation of* G.

PROOF. In view of the previous paragraph, continuity of π is the only non-trivial thing to be proved. Let $v = (v_1, v_2, \ldots)$, $w = (w_1, w_2, \ldots) \in H := \Sigma_i^{\oplus} H_i$. It suffices to prove that

$$x \to (\pi(x)v,w)_H = \sum_i (\pi_i(x)v_i,w_i)_{H_i}$$

is a continuous function on G. Let $x_0 \in G$, $\varepsilon > 0$. There is a natural number j such that

$$\sum_{i=j+1}^{\infty} \|v_i\|_{H_i}^2 < \tfrac{1}{3}\varepsilon > \sum_{i=j+1}^{\infty} \|w_i\|_{H_i}^2 \text{ . Hence}$$

$$\Big|\sum_{i=j+1}^{\infty} (\pi_i(x)v_i,w_i)_{H_i}\Big| \le \Big(\sum_{i=j+1}^{\infty} \|v_i\|_{H_i}^2\Big)^{\frac12}\Big(\sum_{i=j+1}^{\infty} \|w_i\|_{H_i}^2\Big)^{\frac12} < \tfrac{1}{3}\varepsilon$$

for all x, where we used Schwarz's inequality and the fact that $\|\pi_i(x)\| = 1$. It follows that

$$|(\pi(x)v,w)_H - (\pi(x_0)v,w)_H| \le$$

$$\Big|\sum_{i=1}^{j} ((\pi_i(x)v_i,w_i)_{H_i} - (\pi_i(x_0)v_i,w_i)_{H_i})\Big| +$$

$$\Big|\sum_{i=j+1}^{\infty} (\pi_i(x)v_i,w_i)_{H_i}\Big| + \Big|\sum_{i=j+1}^{\infty} (\pi_i(x_0)v_i,w_i)_{H_i}\Big| < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon$$

for x in some neighbourhood of x_0 in G, where we used that $x \to \pi_i(x)$ is weakly continuous for all i. □

Now we will specialize to the case of G provided with an involution $x \to \tilde{x}$ and we will consider subclasses *Rep* of representations of G which meet the following requirements:

ASSUMPTION 2.3.

(i) If $\pi \in$ *Rep* then $\pi(\tilde{x}) = \pi(x)^*$ for all $x \in G$.

(ii) If $\pi \in$ *Rep* and $\pi_1 \le \pi$ then $\pi_1 \in$ *Rep*.

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*) See §1 for the definition of a unitary representation of a locally compact group.

(iii) If $\pi_1, \pi_2, \ldots \in Rep$ then $\Sigma_i^{\oplus} \pi_i$ exists and belongs to Rep.

(iv) If $\pi_1 \in Rep$ and $\pi_1 \simeq \pi_2$ then $\pi_2 \in Rep$.

Note that (i) and (ii) imply that if $\pi \in Rep$, $\pi_1 \leq \pi$ then there is a complementary subrepresentation π_2 which also belongs to Rep. Furthermore, condition (iii) implies that, for each $x \in G$, $\sup_{\pi \in Rep} \|\pi(x)\| < \infty$.

EXAMPLE 2.4. Let us consider some examples of sets G with involution and with a class Rep of representations satisfying Assumption 2.3. The first three examples are rather abstract; the other examples are more concrete, with G having additional structure which is preserved by the representations.

(a) Let G be a set with involution and let $r: G \to [0,\infty)$ be a function on G. Define Rep as the class of all representations π of G satisfying $\pi(x^{\sim}) = \pi(x)^*$ and $\|\pi(x)\| \leq r(x)$ for all $x \in G$.

(b) Let G be a set with involution $x \to x^{\sim}$ and fix a representation π_0 of G which satisfies (2.1). Define Rep as the smallest element in the family of all classes of representations of G which satisfy Assumption 2.3 and which contain π_0. (This family is nonempty, since it contains the class defined in (a) with $r(x) := \|\pi_0(x)\|$. Rep can be obtained as the intersection of all classes in the family.) Rep can alternatively be described as the class of all representations of G which are equivalent to a direct sum of subrepresentations of π_0. (Indeed, it is clear that this last class is included in Rep, that it contains π_0 and that it satisfies properties (i), (iii) and (iv) of Assumption 2.3. Verification of property (ii) is slightly more difficult. It follows by an application of Lemma 4.6 and Zorn's lemma. We will not give the details.)

(c) Let G be a self-adjoint subset of $L(H_0)$ for some Hilbert space H_0 (i.e., $A^* \in G$ iff $A \in G$) and define involution on G by $A^{\sim} := A^*$. Let π_0 be the natural representation of G on H_0. Define Rep as in the previous example (b). In particular, the cases that G consists of one hermitian operator or that G is a von Neumann algebra (cf. §3), are significant. It is possible to do the analysis of a hermitian operator or of a von Neumann algebra in representation theoretic terms by studying the properties of the natural representation π_0 of this object as an element of the class Rep just defined.

(d) Let G be a locally compact group (or just a topological group) with involution $x^{\sim} := x^{-1}$. Define Rep as the class of all unitary representations of G. (Use the definition of unitary representation as given in the beginning of §1.)

(e) Let G be an involutive Banach algebra A (cf. Ch.VIII for the definition) and put $x^{\sim} := x^*$, $x \in A$. Define Rep as the class of all $*$-homomorphisms from A into $L(H)$, where H is an arbitrary Hilbert space. (This coincides with the usual definition of a representation of an involutive Banach algebra.) We have $\|\pi(x)\| \leq \|x\|$ for all $x \in A$, $\pi \in Rep$ (cf. DIXMIER [3, §1.3.7]). Hence the class Rep satisfies Assumption 2.3. Of particular importance are the cases that A is a C^*-algebra or the convolution algebra $L^1(G)$ of a locally compact group (cf. Ch.VIII).

(f) Let G be a σ-algebra S of subsets of a set X (cf. V.1.1 for the definition and put $E^{\sim} := E$, $E \in S$. Define Rep as the class of all projection-valued measures $E \to P_E$ on the measurable space (X,S) (cf. Ch.VIII for the definition).

(g) As a combination of examples (d) and (f) consider a set $G \cup B$ which is the union of a locally compact group G and the σ-algebra B of Borel sets on the homogeneous space G/H, where H is some closed subgroup of G. Put $x^{\sim} := x^{-1}$, $x \in G$; $E^{\sim} := E$, $E \in B$. Define Rep as the class of all pairs (π,P) of unitary representations π of G and projection-valued measures $E \to P_E$ on G/H such that $(G/H,\pi,P)$ is a system of imprimitivity for G (cf. Ch.X for the definition).

The following proposition will be useful in Ch.VIII.

<u>PROPOSITION 2.5.</u> *For* i = 1,2 *let* G_i *be a set with involution and let* Rep_i *be a class of representations of* G_i *satisfying Assumption 2.3. Let* Φ: $Rep_1 \to Rep_2$ *be a mapping such that* $H(\Phi(\pi)) = H(\pi)$ *for all* $\pi \in Rep_1$ *and* $R(\Phi(\pi),\Phi(\rho)) = R(\pi(\rho)$ *for all* $\pi,\rho \in Rep_1$. *Then, for* $\pi,\rho,\pi_1,\pi_2,\ldots \in Rep_1$ *the following holds:*

(i) $H(\pi)$ *has the same closed invariant subspaces with respect to* π *and* $\Phi(\pi)$, *respectively.*

(ii) $\rho \leq \pi$ *iff* $\Phi(\rho) \leq \Phi(\pi)$.

(iii) $\pi \simeq \rho$ *iff* $\Phi(\pi) \simeq \Phi(\rho)$. *For both equivalences the same intertwining isometry can be chosen.*

(iv) $\pi = \Sigma_i^{\oplus} \pi_i$ *iff* $\Phi(\pi) = \Sigma_i^{\oplus} \Phi(\pi_i)$.

<u>PROOF.</u>

(i) $H(\pi) = H(\Phi(\pi))$ and $R(\pi)$ and $R(\Phi(\pi))$ contain the same projection operators.

(ii) Suppose that $\rho \leq \pi$ or $\Phi(\rho) \leq \Phi(\pi)$. Then $H(\rho) = H(\Phi(\rho))$ is a closed invariant subspace of $H(\pi) = H(\Phi(\pi))$. Let P be the projection operator mapping $H(\pi)$ onto $H(\rho)$ and consider P as an element of $L(H(\pi),H(\rho))$. Then: $\rho \le \pi \Leftrightarrow P \in R(\pi,\rho) \Leftrightarrow P \in R(\Phi(\pi),\Phi(\rho)) \Leftrightarrow \Phi(\rho) \le$ $\le \Phi(\pi)$.

(iii) This is obvious from $R(\pi,\rho) = R(\Phi(\pi),\Phi(\rho))$ and the fact that $\pi \simeq \rho$ iff $R(\pi,\rho)$ contains an isometry from $H(\pi)$ onto $H(\rho)$.

(iv) $\pi = \Sigma_i^{\oplus} \pi_i \Leftrightarrow \pi_i \le \pi$ for all i and $H(\pi) = \Sigma_i^{\oplus} H(\pi_i) \Leftrightarrow \Phi(\pi_i) \le \Phi(\pi)$ for all i and $H(\Phi(\pi)) = \Sigma_i^{\oplus} H(\Phi(\pi_i)) \Leftrightarrow \Phi(\pi) = \Sigma_i^{\oplus} \Phi(\pi_i)$. □

For instance, in Ch.VIII such a mapping Φ will be considered from the class of all unitary representations of a locally compact group G into the class of all representations (in the sense of Example 2.4(e)) of the corresponding algebra $L^1(G)$.

Let G be a set with involution $x \to x^{\sim}$ and let $\mathcal{Rep}$ be a class of representations of G which satisfies Assumption 2.3. Consider some $\pi \in \mathcal{Rep}$. Then the closed linear subspace H_1 of H which is spanned by all elements $\pi(x)v(x \in G, v \in H)$ is clearly invariant under π. Let $H_0 := H_1^{\perp}$. If $v \in H_0, x \in G$, then $\pi(x)v \in H_0 \cap H_1$. Hence $\pi(x)v = 0$. Conversely, if $v \in H$ and $\pi(x)v = 0$ for all $x \in G$ then

$$0 = (w,\pi(x^{\sim})v) = (\pi(x)w,v)$$

for all $w \in H$ and $x \in G$, so $v \in H_1^{\perp} = H_0$.

We conclude that for any representation π of G on H we can write H as the direct sum of two invariant subspaces H_1 and H_0, where H_1 is the closed linear span of $\{\pi(x)v \mid x \in G, v \in H\}$ and $H_0 := \{v \in H \mid \pi(x)v=0 \text{ for all } x \in G\}$. We call π a *degenerate* representation if $\dim H_0 > 0$ and a *nondegenerate* representation if $\dim H_0 = 0$. Note that unitary representations of groups and representations of σ-algebras by means of projection-valued measures are always nondegenerate. However, representations of involutive algebras may be degenerate. Define $\mathcal{Rep}_1$ as the class of all nondegenerate representations in $\mathcal{Rep}$. Then $\mathcal{Rep}_1$ again satisfies Assumption 2.3.

3. THE VON NEUMANN ALGEBRA ASSOCIATED WITH A REPRESENTATION

In the remainder of this chapter we will assume the following conventions: G *is a fixed but arbitrary set with involution* $x \to x^{\sim}$. $\mathcal{Rep}$ *is a fixed but arbitrary class of nondegenerate representations of* G *which satisfies Assumption 2.3. "A representation of* G*" will always mean an element of* $\mathcal{Rep}$.

Let π be a representation of G on a Hilbert space H. Then $R(\pi)$ consists of all $A \in L(H)$ which commute with all operators in $\pi(G) := \{\pi(x) | x \in G\}$. The set $\pi(G)$ is self-adjoint, that is, $A^* \in \pi(G)$ if $A \in \pi(G)$. For an arbitrary self-adjoint subset V of $L(H)$ the *commutant* V' of V is defined as the set $V' := \{A \in L(H) | AB=BA \text{ for all } B \in V\}$. Clearly. V' is a linear subspace of $L(H)$ which contains the identity operator I and for which $AB \in V'$ and $A^* \in V'$ whenever A, $B \in V'$. Hence V' is a self-adjoint subalgebra with identity of $L(H)$. In particular, this holds for $R(\pi) := \pi(G)'$.

The *bicommutant* V" of a self-adjoint subset of $L(H)$ is the commutant (V')' of V'. Clearly, V" is also a self-adjoint subalgebra with identity of $L(H)$ and $V \subset V"$. The center of V' is defined by $CV' := \{A \in V' | AB=BA \text{ for all } B \in V'\}$. We have $CV' = V' \cap V"$ and CV' is a self-adjoint commutative algebra with identity. In particular, the center of $R(\pi)$ is denoted by $CR(\pi)$.

Remember that the *weak topology* on $L(H)$ is the weakest topology on $L(H)$ such that all functions $A \to (Av,w)$, $v,w \in H$, are continuous on $L(H)$. The commutant V' of a self-adjoint subset V of $L(H)$, which is evidently closed in the operator norm topology on $L(H)$, is also closed in the weak topology on $L(H)$. For the proof note that for $A,B \in L(H)$ we have AB = BA if and only if $(ABv,w) = (Av,B^*w)$ for all $v,w \in H$. Now V', being the intersection of the weakly closed sets $\{A \in L(H) | (ABv,w)=(Av,B^*w)\}$, $B \in V$, $v,w \in H$, is weakly closed itself. Generally, a self-adjoint subalgebra with identity of $L(H)$ which is weakly closed in $L(H)$ is called a *von Neumann algebra*.

<u>PROPOSITION 3.1</u> (cf. ARVESON [1, Theorem 1.2.1]).

(a) *A self-adjoint subalgebra* A *of* $L(H)$ *is a von Neumann algebra if and only if* $A = A"$.

(b) *The bicommutant* V" *of a self-adjoint subset* V *of* $L(H)$ *equals the weak closure in* $L(H)$ *of the subalgebra with identity of* $L(H)$ *generated by* V.

If $\pi \in Rep$ then $\pi(G)"$ is called the von Neumann algebra associated with π.

4. DISJOINT REPRESENTATIONS AND THE ANALOGUE OF SCHUR's LEMMA IN THE INFINITE-DIMENSIONAL CASE

In sections 1.2 and 1.3 we already met two possible ways of defining disjoint, primary and multiplicity free representations: in terms of the canonical decomposition for finite-dimensional representations (§1.2) or in terms of the intertwining operators (§1.3). In this and the following section we introduce definitions of these concepts in terms of equivalence of subrepresentations. These considerations are independent from §1.

DEFINITION 4.1. Two representations σ and τ of G are *disjoint* (notation $\sigma \,\natural\, \tau$) if no nonzero subrepresentation of σ is equivalent to a subrepresentation of τ.

THEOREM 4.2. *Two representations σ and τ of G are disjoint if and only if $R(\sigma,\tau) = \{0\}$.*

The "if" part of this theorem is easily proved as follows. Suppose that σ and τ are not disjoint. Let $\sigma_1 \le \sigma$, $\tau_1 \le \tau$ such that $\sigma_1 \simeq \tau_1 \neq 0$. Then the intertwining isometry for σ_1 and τ_1 can obviously be extended to some nonzero operator in $R(\sigma,\tau)$. This is a contradiction. The "only if" part of Theorem 4.2 is a corollary of

LEMMA 4.3. *Let σ and τ be representations of G on H_1 and H_2, respectively. and let $A \in R(\sigma,\tau)$. Then the null space N of A and the closure $\bar{R}$ of the range R of A are invariant subspaces of H_1 and H_2, respectively, and $\sigma_{N^\perp} \simeq \tau_{\bar{R}}$.*

PROOF. The invariance of N and $\bar{R}$ is evident. Now we look for the polar decomposition $A|_{N^\perp} = UH$. First we construct H. Let $A^*: H_2 \to H_1$ be such that $(Av,w) = (v,A^*w)$ for all $v \in H_1, w \in H_2$. Then $A^* \in R(\tau,\sigma)$ and $A^*A: H_1 \to H_1$ belongs to $R(\sigma)$. Clearly A^*A is hermitian and positive (i.e., $(A^*Av,v) \ge 0$ for all $v \in H_1$). The null space of A^*A is N. (Indeed, for $v \in H_1$ we have: $A^*Av = 0 \iff (A^*Av,w) = 0$ for all $w \in H_1 \iff (Av,Aw) = 0$ for all $w \in H_1 \iff Av = 0 \iff v \in N$.) Since A^*A is hermitian, the closure of its range is $N^\perp$, the orthoplement of its null space. Now let $H := (A^*A)^{1/2}$, that is, H is the unique positive hermitian operator such that $H^2 = A^*A$. This operator may be constructed by the use of the spectral theorem. However, the following more elementary argument is given in REED & SIMON [8, pp.195-196]: The operator H is given by the power series

$$H = \|A^*A\|^{1/2} \sum_{k=0}^{\infty} c_k (I - \|A^*A\|^{-1} A^*A)^k,$$

where $\sqrt{1-z} = \Sigma_{k=0}^{\infty} c_k z^k$ is the power series of $\sqrt{1-z}$ around the origin, which converges absolutely for $|z| \le 1$. Since $A^*A \in R(\sigma)$, also $H \in R(\sigma)$. The null space of H is N. (Indeed, for $v \in H_1$ we have: $Hv = 0 \iff (Hv,Hw) = 0$ for all

$w \in H_1 \Longleftrightarrow (H^2v,w) = 0$ for all $w \in H_1 \Longleftrightarrow A^*Av = 0 \Longleftrightarrow v \in N$.) Hence the range of H has closure $N^\perp$.

Next we construct U. Let U: range (H) $\to$ range (A) = R be defined by $U(Hv) := Av$. Then U is a well-defined linear operator, since H and A have the same null space N. Furthermore, U is isometric, since, for $v,w \in H_1$:

$$(U(Hv),U(Hw)) = (Av,Aw) = (A^*Av,w) = (H^2v,w) = (Hv,Hw).$$

Hence U can be uniquely extended to an isometry from $N^\perp$ onto $\bar{R}$. Finally we prove that $U \in R(\sigma_{N^\perp},\tau_{\bar{R}})$. Let $v \in H_1$, $x \in G$. Then

$$(U\sigma(x))(Hv) = UH\sigma(x)v = A\sigma(x)v = \tau(x)Av = (\tau(x)U)(Hv),$$

where we used that $H \in R(\sigma)$ and $A \in R(\sigma,\tau)$. □

Note that the "only if" part of Theorem 4.2 is a generalization of part (a) of Schur's lemma 1.1.

The following simple observation is quite important. *Let* π *be a representation of* G *on* H. *Let* H_1 *be a closed linear subspace of* H *and let* P *be the projection operator from* H *onto* H_1. *Then* H_1 *is an invariant subspace if and only if* $P \in R(\pi)$.

As corollaries of Theorem 4.2 we obtain:

PROPOSITION 4.4. *Let* π *be a representation of* G *on* H *and let* H_1 *and* H_2 *be invariant subspaces of* H. *If the subrepresentations* π_{H_1} *and* π_{H_2} *are disjoint then* H_1 *is orthogonal to* H_2.

PROOF. Let P_1,P_2 be the projection operators on H_1,H_2, respectively. The operator P_1P_2 maps H_2 into H_1 and satisfies $P_1P_2\pi(x)v = \pi(x)P_1P_2v$, $x \in G$, $v \in H_2$. Hence $P_1P_2 \in R(\pi_{P_2},\pi_{P_1})$. It follows from Theorem 4.2 that $P_1P_2 = 0$. □

PROPOSITION 4.5. *Let* P *be a projection operator in* $R(\pi)$. *Then the complementary subrepresentations* π_P *and* π_{I-P} *of* π *are disjoint if and only if* $P \in CR(\pi)$.

PROOF. Let π be a representation of G on H. Let P be a projection in $R(\pi)$ with range H_1 and with null space H_2. Let $\pi_1 := \pi_{H_1}$, $\pi_2 := \pi_{H_2}$. If $A \in L(H)$ then write A as a block matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then

$$\pi(x) = \begin{pmatrix} \pi_1(x) & 0 \\ 0 & \pi_2(x) \end{pmatrix}, \quad x \in G,$$

and

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $A \in R(\pi)$ if and only if $A_{11} \in R(\pi_1)$, $A_{22} \in R(\pi_2)$, $A_{12} \in R(\pi_2,\pi_1)$, $A_{21} \in R(\pi_1,\pi_2)$. Now we have the following equivalent statements:
$PA = AP$ for all $A \in R(\pi) \iff$

$$\iff \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} \text{ for all } A_{11} \in R(\pi_1),\ A_{12} \in R(\pi_2,\pi_1),\ A_{21} \in R(\pi_1,\pi_2) \iff$$

$\iff R(\pi_1,\pi_2) = \{0\}$ & $R(\pi_2,\pi_1) = \{0\} \iff$ (in view of Theorem 4.2)
$\iff \pi_1$ and π_2 are disjoint. □

If P is a projection in $CR(\pi)$ then P is called a *central projection* for π. The corresponding subrepresentation π_P is called a *central subrepresentation*.

<u>LEMMA 4.6</u>. *Let* $\rho, \pi_1, \pi_2, \ldots \in \mathcal{R}ep$. *If* $\rho \overset{|}{\circ} \pi_i$ *for all* i *then* $\rho \overset{|}{\circ} \Sigma_i^{\oplus} \pi_i$.

<u>PROOF</u>. Assume $\rho \overset{|}{\circ} \pi_i$ for all i and write $\pi := \Sigma_i^{\oplus} \pi_i$. Suppose that $A \in R(\rho,\pi)$. Let P_i be the projection operator from $\mathcal{H}(\pi)$ onto $\mathcal{H}(\pi_i)$. Then $Av = \Sigma_i P_i Av$ for all $v \in \mathcal{H}(\rho)$ and $P_i A \in R(\rho,\pi_i)$. Hence, in view of Theorem 4.2, $P_i A = 0$ for all i. Thus $A = 0$. A second application of Theorem 4.2 shows that $\rho \overset{|}{\circ} \pi$. □

As a generalization of part (b) of Schur's lemma 1.1 we have:

<u>THEOREM 4.7</u>. *A nonzero representation* π *of* G *is irreducible if and only if* $R(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$.

The proof of Theorem 4.7 starts with the observation that π is irreducible if and only if 0 and I are the only projection operators in $R(\pi)$. This yields the "if" part of the theorem. The proof of the "only if" part in the finite-dimensional case is based on the fact that $A \in R(\pi)$ has an eigenvector. In the infinite-dimensional case we have to use the spectral theorem for hermitian operators (cf. RUDIN [9, Ch.12], see also Theorem II.1.6). In fact, we need only the following version of the spectral theorem, which is contained in RUDIN [9, §12.23, §12.24].

PROPOSITION 4.8. *Let* A *be a hermitian operator on a Hilbert space* H. *Then there is a collection* $\{P_\lambda\}$ *of projection operators on* H *such that each* P_λ *is in the bicommutant of* $\{A\}$ *and* A *can be approximated in norm by finite linear combinations of operators* P_λ.

Now suppose that π is irreducible. Then 0 and I are the only projection operators in $R(\pi)$. The next step is to prove that the operators λI, $\lambda \in \mathbb{R}$, are the only hermitian operators in $R(\pi)$. Indeed, if $A \in R(\pi)$ is hermitian and the P_λ's are as in Proposition 4.8 then the P_λ's commute with all operators commuting with A, in particular $P_\lambda \pi(x) = \pi(x) P_\lambda$, $x \in G$, hence $P_\lambda \in R(\pi)$. Thus $P_\lambda = 0$ or I and $A = \lambda I$ for some $\lambda \in \mathbb{R}$, since A can be approximated in norm by linear combinations of 0 and I. Finally, the case of general $A \in R(\pi)$ can be reduced to the hermitian case by writing $A = \frac{1}{2}(A+A^*) + i.\frac{1}{2i}(A-A^*)$, since $\frac{1}{2}(A+A^*)$ and $\frac{1}{2i}(A-A^*)$ both are hermitian operators in $R(\pi)$. This completes the proof of Theorem 4.7.

As a corollary of Theorem 4.7 we have:

THEOREM 4.9. *Let* π *be an irreducible representation of* G *on* H *such that* $\pi(x)\pi(y) = \pi(y)\pi(x)$ *for all* $x, y \in G$. *Then* $\dim H = 1$.

PROOF. For all $x \in G$, $\pi(x) \in R(\pi)$. Hence $\pi(x) = \lambda(x)\, I$ for some $\lambda(x) \in \mathbb{C}$ (cf. Theorem 4.7). Thus each 1-dimensional subspace of H is invariant, i.e., $\dim H = 1$. □

5. PRIMARY AND MULTIPLICITY FREE REPRESENTATIONS

DEFINITION 5.1. A nonzero representation π is *primary* if no two nonzero complementary subrepresentations of π are disjoint.

DEFINITION 5.2. A nonzero representation π is *multiplicity free* if any two nonzero complementary subrepresentations of π are disjoint.

There is a striking contrast between the definitions of primary and multiplicity free representations. This contrast is also clear from the criteria in Theorems 5.3 and 5.4 below.

THEOREM 5.3. *A nonzero representation π is primary if and only if* $CR(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$.

THEOREM 5.4. *A nonzero representation π is multiplicity free if and only if the algebra* $R(\pi)$ *is commutative (that is,* $CR(\pi) = R(\pi)$*).*

Observe that the center $CR(\pi)$ of $R(\pi)$ is as small as possible if π is primary and as large as possible if π is multiplicity free.

The proofs of Theorems 5.3 and 5.4 are based on Proposition 4.5. It follows from this proposition that a nonzero representation π is primary if and only if 0 and I are the only central projections and that it is multiplicity free if and only if all projections in $R(\pi)$ are central. Now Theorems 5.3 and 5.4 can be proved along the same lines as Theorem 4.7.

Proof of Theorem 5.3. If $CR(\pi) = \{\lambda I \mid \lambda \in \mathbb{C}\}$ then 0 and I are the only central projections. Hence π is primary in view of Proposition 4.5. Conversely, let π be primary. Then 0 and I are the only central projections. Let A be a hermitian operator in $CR(\pi)$. It follows from Proposition 4.8 that A can be approximated in norm by finite linear combinations of certain projection operators P_λ which commute with all operators commuting with A. This implies that all P_λ are in $CR(\pi)$. Hence $A = \lambda I$ for some $\lambda \in \mathbb{R}$. Finally, if A is a general operator belonging to $CR(\pi)$ then $A = B + iC$ with $B, C \in CR(\pi)$ and hermitian. □

Proof of Theorem 5.4. If $R(\pi)$ is commutative then all projection operators in $R(\pi)$ are central. Hence, by the use of Proposition 4.5, π is multiplicity free. Conversely, let π be multiplicity free. Then all projection operators in $R(\pi)$ are central. Let $A \in R(\pi)$ and hermitian. Then an application of Proposition 4.8 shows that A can be approximated in norm by linear combinations of certain projection operators P_λ which are in $R(\pi)$. Hence these P_λ's are in $CR(\pi)$ and the same holds for A. Finally, if A is a general element of $R(\pi)$ then $A = B + iC$ with $B, C \in R(\pi)$ and hermitian. Hence $B, C \in CR(\pi)$ and thus $A \in CR(\pi)$. □

6. INCLUSION, COVERING AND QUASI-EQUIVALENCE

Consider finite-dimentional representations π with decomposition (1.1) and ρ with decomposition (1.2). Then π is *equivalent* to ρ, π is *included* in ρ, π is *quasi-equivalent* to ρ or π is *covered* by ρ, respectively, if and only if after possible rearranging of the summands in (1.1) and (1.2) we have:

(a) (equivalence) $p = r$ and $\pi_i \simeq \rho_i$, $m_i = n_i$ $(i = 1,\dots,p)$.

(b) (inclusion) $p \le r$ and $\pi_i \simeq \rho_i$, $m_i \le n_i$ $(i = 1,\dots,p)$.

(c) (quasi-equivalence) $p = r$ and $\pi_i \simeq \rho_i$ $(i = 1,\dots,p)$.

(d) (covering) $p \le r$ and $\pi_i \simeq \rho_i$ $(i = 1,\dots,p)$.

The last three concepts are new. We now define them in the general case.

<u>DEFINITION 6.1</u>. A representation ρ *includes* a representation π (notation $\pi \lesssim \rho$ or $\rho \gtrsim \pi$) if π is equivalent to some subrepresentation of ρ.

<u>THEOREM 6.2</u>. *The relation $\lesssim$ is a partial ordering for the collection of equivalence classes in $\mathcal{Rep}$, that is, for $\pi_1, \pi_2, \pi_3 \in \mathcal{Rep}$ we have:*

(i) *if $\pi_1 \simeq \pi_2$ then $\pi_1 \lesssim \pi_2$;*

(ii) *if $\pi_1 \lesssim \pi_2$ and $\pi_2 \lesssim \pi_3$ then $\pi_1 \lesssim \pi_3$;*

(iii) *if $\pi_1 \lesssim \pi_2$ and $\pi_2 \lesssim \pi_1$ then $\pi_1 \simeq \pi_2$.*

<u>PROOF</u>. Properties (i) and (ii) are evident. Consider (iii). Let $H_1 := H(\pi_1)$, $H_2 := H(\pi_2)$. It follows from the inclusion relations that there are intertwining isometries A from H_1 onto an invariant subspace AH_1 of H_2 and B from H_2 onto an invariant subspace BH_2 of H_1. We try to find invariant subspaces V_0 of H_1 and W_0 of H_2 such that $AV_0 = W_0$ and $B(W_0^\perp) = V_0^\perp$. Then the linear mapping $C: H_1 \to H_2$ defined by $C|_{V_0} := A$ and $C|_{V_0^\perp} := B^{-1}$ is an intertwining isometry from H_1 onto H_2, which will settle property (iii). In order to find such subspaces V_0 and W_0 we use the mapping F which associates with each invariant subspace V of H_1 a new invariant subspace

$$F(V) := (B(AV)^\perp)^\perp. \tag{6.1}$$

Clearly, if $F(V) = V$ for some invariant subspace V of H_1 then we can take $V_0 := V$ and $W_0 := AV_0$. We will prove the existence of such a fixed point V_0 of F.

Let V denote the class of all invariant subspaces of H_1 and let $F(V)\,(V \in V)$ be defined by (6.1). Since A and B are linear isometries, they map closed, hence complete, linear subspaces of $\mathcal{H}_1$ respectively $\mathcal{H}_2$ onto complete, hence closed, linear subspaces of $\mathcal{H}_2$ respectively $\mathcal{H}_1$. Hence, since A and B are also intertwining operators, they map invariant subspaces onto invariant subspaces and so does $\mathcal{F}$. Note that the mapping $\mathcal{F}\colon \mathcal{V} \to \mathcal{V}$ is monotone with respect to inclusion: if $V_1 \subset V_2$ then $\mathcal{F}(V_1) \subset \mathcal{F}(V_2)$. Consider the set $\mathcal{V}_0$ of all $V \in \mathcal{V}$ such that $\mathcal{F}(V) \supset V$. Then $\mathcal{V}_0$ is nonempty, since $\{0\} \in \mathcal{V}_0$. Let V_0 be the least upper bound of $\mathcal{V}_0$ in $\mathcal{V}$, i.e. the intersection of all $V \in \mathcal{V}$ such that $V \supset V'$ for all $V' \in \mathcal{V}_0$. We will show that $\mathcal{F}(V_0) = V_0$. First note that $\mathcal{F}(V_0) \supset \mathcal{F}(V) \supset V$ for all $V \in \mathcal{V}_0$. Hence $\mathcal{F}(V_0)$ is an upper bound of $\mathcal{V}_0$, so $\mathcal{F}(V_0) \supset V_0$. But, by monotony of $\mathcal{F}$, also $\mathcal{F}(\mathcal{F}(V_0)) \supset \mathcal{F}(V_0)$, hence $\mathcal{F}(V_0) \in \mathcal{V}_0$. Thus $\mathcal{F}(V_0) \subset V_0$. It follows that $\mathcal{F}(V_0) = V_0$. □

<u>REMARK 6.3.</u> The above proof that $\mathcal{F}$ has a fixed point in $\mathcal{V}$ can be applied to each partially ordered set $\mathcal{V}$ and monotone mapping $\mathcal{F}\colon \mathcal{V} \to \mathcal{V}$, provided that $\mathcal{V}$ contains a minimal element and that each subset of $\mathcal{V}$ has a least upper bound in $\mathcal{V}$.

<u>REMARK 6.4.</u> See for instance DRAKE [4, Ch.2, Ex.4.10(1)] for an analogous proof of the Schröder-Bernstein theorem in set theory. A more familiar proof of the Schröder-Bernstein theorem can be found in DRAKE [4, Ch.2, Theorem 4.5]. This last-mentioned proof also has its analogue in the case of representations (cf. MACKEY [6, p.14]; the reader should be aware for a slight error in the proof given there).

<u>DEFINITION 6.5.</u> Let π and ρ be representations of G. Then π is *covered* by ρ (notation $\pi \prec \rho$ or $\rho \succ \pi$) if each nonzero subrepresentation of π contains a nonzero subrepresentation which is equivalent to some subrepresentation of ρ.

An equivalent formulation is: π is covered by ρ if no nonzero subrepresentation of π is disjoint from ρ. Obviously, if $\pi \lesssim \rho$ then $\pi \prec \rho$, and if $\pi \prec \rho$ and $\rho \prec \sigma$ then $\pi \prec \sigma$.

<u>DEFINITION 6.6.</u> Two representations π and ρ of G are called *quasi-equivalent* (notation $\pi \sim \rho$) if both $\pi \prec \rho$ and $\rho \prec \pi$.

Note that equivalence implies quasi-equivalence. The relation $\sim$ is an equivalence relation. The equivalence classes with respect to the relation $\sim$ are called *quasi-equivalence classes*. Any quasi-equivalence class is a union of ($\simeq$) equivalence classes.

In Table 3 below we list the symbols and names of the various relations between two representations π and ρ of G.

notation	name of the relation	reference
$\pi \leq \rho$	π is subrepresentation of ρ	§2
$\pi \simeq \rho$	π is equivalent to ρ	§2
$\pi \between \rho$	π is disjoint from ρ	Def. 4.1
$\pi \lesssim \rho$	π is equivalent to a subrepresentation of ρ	Def. 6.1
$\pi \prec \rho$	π is covered by ρ	Def. 6.5
$\pi \sim \rho$	π is quasi-equivalent to ρ	Def. 6.6

Table 3

LEMMA 6.7. *For any representation* π *of* G *and* $n \in \{1,2,\ldots,\infty\}$ *we have* $\pi \sim n\pi$.

PROOF. Since π is a subrepresentation of $n\pi$ we have $\pi \prec n\pi$. Conversely, Lemma 4.6 shows that no nonzero subrepresentation of $n\pi$ is disjoint from π. Hence $n\pi \prec \pi$.

We now derive some rather technical results which will be useful later. In the proof of the next proposition we will first meet an application of Zorn's lemma (cf. for instance ZAANEN [10, Ch.1, §2, Theor.1]). In subsequent proofs this kind of argument will be used repeatedly.

PROPOSITION 6.8. *Let* π *be a representation of* G *on* H *with subrepresentation* ρ. *Then there is a unique central subrepresentation of* π, *denoted by* $\bar{\rho}$, *such that* $\bar{\rho} \sim \rho$. *Furthermore,* $\bar{\rho}$ *is the smallest central subrepresentation of* π *which contains* ρ *and* $\bar{\rho}$ *is the complement of the largest subrepresentation of* π *which is disjoint from* ρ.

PROOF. First we show that if $\bar{\rho}$ exists then it is unique. Indeed, let σ and τ be central subrepresentations of π such that $\sigma \prec \tau$. Since the projections on $H(\sigma)$ and $H(\tau)$ commute, we have $\sigma = \sigma_1 \oplus \sigma_2$ with $\sigma_1 \leq \tau$ and $H(\sigma_2)$ orthogonal to $H(\tau)$. It follows from Proposition 4.5 that $\sigma_2 \between \tau$. Since $\sigma_2 \leq \sigma \prec \tau$, we must have $\sigma_2 = 0$. Hence $\sigma \leq \tau$. Thus two quasi-equivalent central subrepresentations are equal.

Next we define $\bar{\rho}$. Consider all families $\{\tau_i\}$ of mutually orthogonal nonzero subrepresentations τ_i of π such that $\tau_i \between \rho$. Since H is separable, all such families are countable. Let the collection of these families be partially ordered by inclusion . Application of Zorn's lemma shows that there is a maximal family $\{\tau_i\}$. Let $\tau := \Sigma_i^{\oplus} \tau_i$. It follows from Lemma 4.6 that $\tau \overset{|}{\circ} \rho$. Let $\bar{\rho}$ be the subrepresentation of π which is complementary to τ. Then $\bar{\rho} \overset{|}{\circ} \tau$, for otherwise $\bar{\rho}$ has a nonzero subrepresentation which is equivalent to a subrepresentation of τ and hence disjoint from ρ, thus contradicting the maximality of the family $\{\tau_i\}$. It follows from Propisition 4.5 that $\bar{\rho}$ is a central subrepresentation. Next observe that $\bar{\rho} \sim \rho$. Indeed, $\rho \le \bar{\rho}$ in view of Proposition 4.4, hence $\rho \prec \bar{\rho}$. Also $\bar{\rho} \prec \rho$, since no nonzero subrepresentation of $\bar{\rho}$ is disjoint from ρ.

Now we show that τ is the largest subrepresentation of π which is disjoint from ρ. Let $\sigma \le \pi$, $\sigma \overset{|}{\circ} \rho$. Since τ is central, the projections on $H(\tau)$ and on $H(\sigma)$ commute. Hence $\sigma = \sigma_1 \oplus \sigma_2$ with $\sigma_1 \le \bar{\rho}$, $\sigma_2 \le \tau$. Then $\rho \overset{|}{\circ} \sigma_1 \le \bar{\rho}$ implies that $\sigma_1 = 0$. It follows that $\sigma \le \tau$.

Finally we have to prove that $\bar{\rho}$ is the smallest central subrepresentation of π containing ρ. Let σ_1 be a central subrepresentation of π containing ρ. Let σ_2 be the subrepresentation of π which is complementary to σ_1. It follows from Proposition 4.5 that $\sigma_2 \overset{|}{\circ} \sigma_1$, hence $\sigma_2 \overset{|}{\circ} \rho$. This implies that $\sigma_2 \le \tau$, hence $\sigma_1 \ge \bar{\rho}$. □

<u>LEMMA 6.9</u>. *Let* π *and* ρ *be representations of* G. *Then there are complementary subrepresentations* π_1, π_2 *of* π *and* ρ_1, ρ_2 *of* ρ *such that* $\pi_1 \simeq \rho_1$, $\pi_2 \overset{|}{\circ} \rho_2$.

<u>PROOF.</u> By Zorn's lemma there is a maximal family $\{(\sigma_i,\tau_i)\}$ of pairs (σ_i,τ_i) of nonzero subrepresentations σ_i of π and τ_i of ρ such that both the σ_i's and the τ_i's are mutually orthogonal and $\sigma_i \simeq \tau_i$ for each i. Let $\pi_1 := \Sigma_i^{\oplus} \sigma_i$, $\rho_i := \Sigma_i^{\oplus} \tau_i$ and let π_2 respectively ρ_2 be the complementary subrepresentations to π_1 in π, respectively to ρ_1 in ρ. Then $\pi_1 \simeq \rho_1$ and $\pi_2 \overset{|}{\circ} \rho_2$, since, otherwise, the maximality of the family $\{(\sigma_i,\tau_i)\}$ would be contradicted. □

7. PRIMARY REPRESENTATIONS OF TYPE I

First we derive some general results about primary representations.

<u>LEMMA 7.1.</u> *If* π *is primary and* $\rho \sim \pi$ *then* ρ *is primary.*

<u>PROOF</u>. Suppose that ρ is not primary. Then ρ has two nonzero disjoint subrepresentations. Since $\pi \sim \rho$, there are nonzero disjoint subrepresentations π_1 and π_2 of π. In view of Proposition 6.8, $\bar{\pi}_1 \overset{|}{\circ} \pi_2$, hence $H(\bar{\pi}_1) \perp H(\pi_2)$. It follows that $\bar{\pi}_1$ is a central subrepresentations of π, not equal to 0 or π. This contradicts the fact that π is primary. □

LEMMA 7.2. *If π is primary and $0 \neq \rho \leq \pi$ then $\rho \sim \pi$ and ρ is primary.*

PROOF. Since π is primary, its only nonzero central subrepresentation is π. Hence $\bar{\rho} = \pi$ and $\rho \sim \pi$ (cf. Proposition 6.8). Then ρ is primary by Lemma 7.1. □

PROPOSITION 7.3. *If π and ρ are primary then either $\pi \,\delta\, \rho$ or $\pi \sim \rho$. In the latter case $\pi \lesssim \rho$ or $\rho \lesssim \pi$.*

PROOF. Suppose that π and ρ are not disjoint. Let $\pi = \pi_1 \oplus \pi_2$, $\rho = \rho_1 \oplus \rho_2$ as in Lemma 6.9. Then π_1 and ρ_1 are nonzero. Suppose that both π_2 and ρ_2 are nonzero. Since π is primary, π_2 has a nonzero subrepresentation equivalent to a subrepresentation of π_1, hence equivalent to a subrepresentation of ρ_1. Since ρ is primary, we conclude that there are nonzero equivalent subrepresentations of π_2 and ρ_2. This contradicts the disjointness of π_2 and ρ_2. Hence π_2 or $\rho_2 = 0$, i.e., $\pi \lesssim \rho$ or $\rho \lesssim \pi$. Then $\pi \sim \rho$ because of Lemma 7.2. □

In §1.4 we defined primary representations of type I, II or III. Let us discuss primary representations of type I in some more details. Remember that a primary representation ρ is called of type I if $\rho \simeq n\pi$ for some irreducible representation π and some $n \in \{1,2,\ldots,\infty\}$.

THEOREM 7.4. *Let π and π_1 be irreducible representations. Then:*
(a) $\rho \sim \pi$ *if and only if* $\rho \simeq n\pi$ *for some* $n \in \{1,2,\ldots,\infty\}$.
(b) *All representations* $n\pi (n \in \{1,2,\ldots,\infty\})$ *are primary.*
(c) *If* $n\pi \simeq m\pi_1$ *then* $\pi \simeq \pi_1$ *and* $n = m$.

PROOF.
(a) If $\rho \simeq n\pi$ then $\rho \sim \pi$ by Lemma 6.7. If $\rho \sim \pi$ then each nonzero subrepresentation of ρ contains a subrepresentation which is equivalent to π (since π is irreducible). By Zorn's lemma there exists a maximal family $\{\rho_i\}$ of nonzero mutually orthogonal subrepresentations ρ_i of ρ which are all equivalent to π. Then $\rho = \Sigma_i^{\oplus} \rho_i$, since otherwise the maximality of the family $\{\rho_i\}$ would be contradicted. Hence $\rho \simeq n\pi$ for some n.
(b) Apply part (a) and Lemma 7.1.
(c) If $n\pi \simeq m\pi_1$ then $\pi \sim \pi_1$ (cf. Lemma 6.7). Hence $\pi \simeq \pi_1$, since π and π_1 are irredicible. Next we show that $\dim R(n\pi) = n^2$ if n is finite. Let $\rho = \Sigma_{i=1}^{n\oplus} \pi_i$, where $\pi_i \simeq \pi$ for all i. Any $A \in L(H(\rho))$ can be written as a block matrix

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

with respect to the subspaces $H(\pi_i)$ of $H(\rho)$. It follows from Theorem 4.7 that $A \in R(\rho)$ if and only if for all i,j $A_{ij} = \lambda_{ij}T_{ij}$, where T_{ij} is a fixed intertwining isometry from $H(\pi_j)$ onto $H(\pi_i)$ and λ_{ij} is some complex constant. Hence $\dim R(\rho) = n^2$. It follows that $n\pi \not\simeq m\pi$ if $n \neq m$ and n,m are finite. Also $n\pi \not\simeq \infty\pi$ for all finite n for, otherwise, $n\pi \lesssim (n+1)\pi \lesssim \infty\pi \simeq n\pi$ for some finite n, hence $n\pi \simeq (n+1)\pi$ by Theorem 6.2(iii). □

8. GENERAL REPRESENTATIONS OF TYPE I

In this section we introduce general representations of type I and we prove the canonical decomposition of type I representations in terms of multiplicity free representations. See ARVESON [1, §2.1] for a related approach.

8.1. Some properties of multiplicity free representations

LEMMA 8.1. *Let π be multiplicity free and $\rho \lesssim \pi$, $\rho \neq 0$. Then ρ is multiplicity free.*

PROOF. Let $\rho \le \pi$ and $\pi = \rho \oplus \sigma$. Let $\rho = \rho_1 \oplus \rho_2$. Then $\pi = \rho_1 \oplus (\rho_2 \oplus \sigma)$. Since π is multiplicity free, we have $\rho_1 \overset{|}{\circ} (\rho_2 \oplus \sigma)$, hence $\rho_1 \overset{|}{\circ} \rho_2$. Thus ρ is multiplicity free. □

LEMMA 8.2. *Let $\pi = \Sigma_i^{\oplus} \pi_i$ be a direct sum of mutually disjoint multiplicity free representations π_i. Then π is multiplicity free.*

PROOF. First note that for each i $\pi_i \overset{|}{\circ} \Sigma_{j\neq i}^{\oplus} \pi_j$, hence the projection from $H(\pi)$ onto $H(\pi_i)$ is in $CR(\pi)$ (cf. Lemma 4.6 and Prop. 4.5). Let $\pi = \rho \oplus \sigma$. Let ρ_i and σ_i be the subrepresentations on $H(\rho) \cap H(\pi_i)$ and $H(\sigma) \cap H(\pi_i)$, respectively. Then $\rho = \Sigma_i^{\oplus} \rho_i$, $\sigma = \Sigma_i^{\oplus} \sigma_i$, $\pi_i = \rho_i \oplus \sigma_i$. Now $\rho_i \overset{|}{\circ} \sigma_i$ for all i, since π_i is multiplicity free, and $\rho_i \overset{|}{\circ} \sigma_j$ if $i \neq j$ since $\pi_i \overset{|}{\circ} \pi_j$. Again applying Lemma 4.6 we obtain $\rho \overset{|}{\circ} \sigma$. Hence π is multiplicity free. □

PROPOSITION 8.3. *If π is multiplicity free and $\rho \succ \pi$ then $\rho \gtrsim \pi$.*

PROOF. Let $\pi = \pi_1 \oplus \pi_2$, $\rho = \rho_1 \oplus \rho_2$ as in Lemma 6.9, that is, $\pi_1 \simeq \rho_1$, $\pi_2 \overset{|}{\circ} \rho_2$. Since π is multiplicity free, $\pi_2 \overset{|}{\circ} \pi_1$, hence $\pi_2 \overset{|}{\circ} \rho_1$. Thus $\pi_2 \overset{|}{\circ} \rho$ by Lemma 4.6. Since $\pi \prec \rho$, we must have $\pi_2 = 0$. Hence $\pi \lesssim \rho$. □

It follows from Prop. 8.3 that a multiplicity free representation π is minimal with respect to the partial ordering $\precsim$ in its quasi-equivalence class. In particular, combination with Theorem 6.2 shows:

COROLLARY 8.4. *If* π,ρ *are multiplicity free and* $\pi \sim \rho$ *then* $\pi \simeq \rho$.

8.2. Generalities about type I representations

We define type I representations as the elements of the quasi-equivalence classes of multiplicity free representations:

DEFINITION 8.5. A representation is of *type I* if it is quasi-equivalent to some multiplicity free representation.

In view of Theorem 7.4 (a) this definition is compatible with the definition of a primary type I representation.

In Theorem 8.8 below we will give a few other characterizations of type I representations. For this purpose we need two auxiliary results.

LEMMA 8.6. *If* π *is of type I then* π *contains a multiplicity free subrepresentation* ρ *such that* $\rho \sim \pi$.

PROOF. Use Prop. 8.3. □

PROPOSITION 8.7. *Let* ρ *be a nonzero subrepresentation of a type I representation* π. *Then* ρ *is of type I.*

PROOF. Let σ be a multiplicity free subrepresentation of π such that $\sigma \sim \pi$. Let $\pi = \bar{\rho} \oplus \tau$. Since this decomposition is central, we have $\sigma = \sigma_1 \oplus \sigma_2$ with $\sigma_1 \leq \bar{\rho}$, $\sigma_2 \leq \tau$. Now $\bar{\sigma}_1 \leq \bar{\rho}$, $\bar{\sigma}_2 \leq \tau$ (cf. Prop. 6.8) and $\bar{\sigma}_1 \oplus \bar{\sigma}_2 = \bar{\sigma} = \pi = \bar{\rho} + \tau$. Hence $\bar{\sigma}_1 = \bar{\rho}$, i.e. $\sigma_1 \sim \rho$, and σ_1 is multiplicity free (cf. Lemma 8.1).

THEOREM 8.8. *Let* π *be a nonzero representation. The following statements are equivalent:*

(a) π *is of type I.*

(b) π *is a direct sum of multiplicity free representations.*

(c) *Each nonzero central subrepresentation of* π *contains a multiplicity free subrepresentation.*

(d) *Each nonzero subrepresentation of* π *contains a multiplicity free subrepresentation.*

PROOF.

(a) ⇒ (d). Use Lemma 8.6 and Prop. 8.7.

(d) ⇒ (b). By Zorn's lemma there is a maximal family $\{\rho_i\}$ of mutually orthogonal multiplicity free subrepresentations of π. Let $\rho := \Sigma_i^{\oplus} \rho_i$, $\pi = \rho \oplus \sigma$. Then $\sigma = 0$, for otherwise it would contain a multiplicity free subrepresentation, in contradiction with the maximality of the family $\{\pi_i\}$.

(b) ⇒ (c). Let $\pi = \Sigma_i^{\oplus} \pi_i$, all π_i's multiplicity free, and let ρ be a nonzero central subrepresentation of π. Denote the projections on $\mathcal{H}(\pi_i)$ and $\mathcal{H}(\rho)$ by P_i respectively Q. Then Q commutes with all P_i and the projection $QP_i \in R(\pi)$ is nonzero for at least one i. Denote the corresponding subrepresentation by σ_i. Then $\sigma_i \neq 0$, $\sigma_i \leq \pi_i$, $\sigma_i \leq \rho$. Hence, by Lemma 8.1, σ_i is a multiplicity free subrepresentation of ρ.

(c) ⇒ (a). By Zorn's lemma there is a maximal family $\{\rho_i\}$ of mutually disjoint, multiplicity free subrepresentations of π. Let $\rho := \Sigma_i^{\oplus} \rho_i$, $\pi = \rho \oplus \sigma$. Then ρ is multiplicity free (cf. Lemma 8.2) and $\bar{\rho} = \pi$, for otherwise the nonzero central subrepresentation σ would contain a multiplicity free subrepresentation disjoint from all ρ_i's, thus contradicting the maximality of the family $\{\rho_i\}$. □

COROLLARY 8.9. *A direct sum of type I representations is again of type I.*

8.3. Representations of multiplicity n

In the canonical decomposition for type I representations there will occur multiples $n\pi$ of multiplicity free representations π. In order to justify Def, 8.11 below for representations of this form, we need the following proposition.

PROPOSITION 8.10. *Let* π *and* ρ *be multiplicity free and* $n, m \in \{1,\dots,\infty\}$. *If* $n\pi \simeq m\rho$ *then* $\pi \simeq \rho$ *and* $n = m$.

PROOF. Application of Lemma 6.7 and Corollary 8.4 shows that $\pi \simeq \rho$. Now suppose that $n\pi \simeq m\pi$ and $n > m$. (Thus, m is finite.) We will obtain a contradiction by constructing a complex $m\times m$ matrix A such that $A^m \neq 0$ and $A^{m+1} = 0$. Then $A^{m+1} = 0$ would imply that all eigenvalues of A are zero. Hence A can be brought in Jordan normal form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix},$$

of which the m^{th} power is zero. We turn to the construction of A. Write the elements of $H(n\pi)$ as $(v_1,v_2,\dots)$, where all v_i's are in $H(\pi)$. Let

$$T(v_1,v_2,\dots,v_m,v_{m+1},*) := (0,v_1,\dots,v_{m-1},v_m,0,0,\dots),$$

then $T \in R(n\pi)$,

$$T^m(v_1,\dots,v_{m+1},*) = (0,\dots,0,v_1,0,0,\dots),$$

hence $T^m \neq 0$, and $T^{m+1} = 0$. Here we used that $n \geq m+1$. Since $n\pi \simeq m\pi$, $R(n\pi)$ and $R(m\pi)$ are isomorphic as C^*-algebras. Hence there is an operator $S \in R(m\pi)$ such that $S^m \neq 0$, $S^{m+1} = 0$. The bounded linear transformations U of $H(m\pi)$ can be written as m×m block matrices

$$\begin{pmatrix} U_{11} & \cdots & U_{1m} \\ U_{m1} & \cdots & U_{mm} \end{pmatrix}$$

with respect to the m subspaces $H(\pi)$ of $H(m\pi)$. Then $U \in R(m\pi)$ if and only if $U_{ij} \in R(\pi)$ for all i,j. If ϕ is a $*$-homomorphism from $R(\pi)$ to $\mathbb{C}$ then the mapping

$$\Phi\colon U \to \begin{pmatrix} \phi(U_{11}) & \cdots & \phi(U_{1m}) \\ \phi(U_{m1}) & \cdots & \phi(U_{mm}) \end{pmatrix}$$

is a $*$-homomorphism from $R(m\pi)$ to the C^*-algebra of all complex m×m matrices. Now consider S. Since $S^m \neq 0$, $(S^m)_{ij} \neq 0$ for some (i,j). We use that $R(\pi)$ is a commutative C^*-algebra (cf. Theorem 5.4). Hence there exists a $*$-homomorphism ϕ from $R(\pi)$ onto $\mathbb{C}$ such that $\phi((S^m)_{ij}) \neq 0$ (cf. RUDIN [9, Theor. 11.18]). Let $A := \Phi(S)$. Then $A^m = \Phi(S^m) \neq 0$, $A^{m+1} = \Phi(S^{m+1}) = 0$. □

DEFINITION 8.11. A representation π is said to have *multiplicity* n ($n \in \{1,2,\dots,\infty\}$) if $\pi \simeq n\rho$ for some multiplicity free representation ρ.

LEMMA 8.12. *Let* π *be a representation of multiplicity* n ($n \in \{1,2,\dots,\infty\}$). *If* σ *is a nonzero central subrepresentation of* π *then* σ *also has multiplicity* n.

PROOF. π is the direct sum of n mutually equivalent multiplicity free subrepresentations π_i. We have the nontrivial central decomposition $\pi = \sigma \oplus \tau$. Then $\pi_i = \sigma_i \oplus \tau_i$, where $\sigma_i \leq \sigma$, $\tau_i \leq \tau$. Also $\sigma_i \,\rlap{\,|}\circ\, \tau_j$ for all i,j (cf. Prop. 4.5). Combining this with $\sigma_i \oplus \tau_i \simeq \sigma_j \oplus \tau_j$ we obtain $\sigma_i \precsim \sigma_j$ and $\sigma_j \precsim \sigma_i$ (cf. Prop. 4.4). Hence $\sigma_i \simeq \sigma_j$ (cf. Theorem 6.2). The σ_i's are multiplicity free (cf. Lemma 8.1) and $\sigma = \Sigma_i^{\oplus} \sigma_i$. □

8.4. <u>The canonical decomposition of type I representations</u>

The following two lemmas give the final preparation for the canonical decomposition of type I representations.

<u>LEMMA 8.13</u>. *Let* π *be a representation of type I. Then* π *contains a nonzero central subrepresentation of some multiplicity* n $(n \in \{1,2,\ldots,\infty\})$.

<u>PROOF</u>. Let ρ_0 be a multiplicity free subrepresentation of π such that $\rho_0 \sim \pi$ (cf. Lemma 8.6). By Zorn's lemma there is a maximal family $\{\rho_i\}$ of mutually orthogonal subrepresentations ρ_i of π such that $\rho_i \simeq \rho_0$. Let $\rho := \Sigma_i^{\oplus} \rho_i$ and $\pi = \rho \oplus \sigma$. Let τ be such that $\pi = \tau \oplus \bar{\sigma}$. Then $\tau \neq 0$, for otherwise $\sigma \sim \pi \sim \sim \rho_0$, so σ would contain a subrepresentation equivalent to ρ_0 (cf. Prop. 8.3) in contradiction to the maximality of the family $\{\rho_i\}$. Now τ is a nonzero central subrepresentation of ρ and ρ is a representation of certain multiplicity n. Finally apply Lemma 8.12. □

<u>LEMMA 8.14</u>. *Let* π *be a representation of type I. Then* π *is a direct sum of mutually disjoint central subrepresentations* π_i*, where each* π_i *has some multiplicity* n.

<u>PROOF</u>. By Zorn's lemma there exists a maximal family $\{\pi_i\}$ of mutually orthogonal central subrepresentations π_i of π, where each π_i has some multiplicity n. Then $\pi_i \between \pi_j$ $(i \neq j)$ by Prop. 4.5. It follows from Prop. 4.5 and Lemma 4.6 that $\Sigma_i^{\oplus} \pi_i$ is a central subrepresentation of π. Let $\pi = (\Sigma_i^{\oplus} \pi_i) \oplus \rho$. Then $\rho = 0$, for otherwise the type I representation ρ contains some central subrepresentation of certain multiplicity (cf. Lemma 8.13 and Prop. 8.7) in contradiction to the maximality of the family $\{\pi_i\}$. □

<u>THEOREM 8.15</u>. *Let* π *be a representation of type I. Then there are unique central subrepresentations* $\pi_1, \pi_2, \ldots, \pi_\infty$ *of* π*, where, for each* i*,* $\pi_i = 0$ *or of multiplicity* i*, such that*

$$\pi = \pi_1 \oplus \pi_2 \oplus \ldots \oplus \pi_\infty. \tag{8.1}$$

PROOF. Let $\pi = \Sigma_i^{\oplus} \rho_i$ be a central decomposition as in Lemma 8.14. Now, for each $n \in \{1,2,\dots,\infty\}$ let π_n be the direct sum of all ρ_i's having multiplicity n. Then π_n is a central subrepresentation of π which is either zero or of multiplicity n (cf. Lemma 8.2) and (8.1) holds. In order to prove uniqueness let $\pi = \sigma_1 \oplus \sigma_2 \oplus \dots \oplus \sigma_\infty$ be another decomposition of the required form. Then we have the central decomposition $\pi = \Sigma_{i,j \in \{1,\dots,\infty\}} \tau_{ij}$, where $\tau_{ij} \le \pi_i$, $\tau_{ij} \le \sigma_j$. Then $\tau_{ij} = 0$ if $i \neq j$ for otherwise, in view of Lemma 8.12, τ_{ij} has both multiplicity i and j, which contradicts Prop. 8.10. It follows that $\pi_i = \sigma_i$ for all i.

It follows from Theorem 8.15 that a type I representation π has the decomposition

$$(8.2) \qquad \pi = \mu_1 \oplus 2\mu_2 \oplus \dots \oplus \infty\mu_\infty ,$$

where the μ_i's are zero or multiplicity free and mutually disjoint. Conversely, let us start with a sequence $\{\mu_1,\mu_2,\dots,\mu_\infty\}$ of mutually disjoint representations μ_n which are zero or multiplicity free and let π be defined by (8.2). Then π is of type I (cf. Theor. 8.8), each subrepresentation $n\mu_n$ is disjoint from its complementary subrepresentation (cf. Lemma 4.6) and hence each subrepresentation $n\mu_n$ is central (cf. Prop. 4.5). By definition $n\mu_n$ is zero or of multiplicity n. Thus (8.2) is the canonical decomposition of π according to Theorem 8.15. It follows from Prop. 8.10 that π determines the μ_n's up to equivalence. Thus (8.2) establishes a one-to-one correspondence between equivalence classes of type I representations π and sequences of equivalence classes of multiplicity free or zero representations μ_n $(n \in \{1,2,\dots,\infty\})$ such that $\mu_n \,\text{\textdiscount}\, \mu_m$ if $n \neq m$.

9. CYCLIC REPRESENTATIONS

Let π be a nonzero representation of G on $\mathcal{H}$. In order to find closed invariant subspaces of $\mathcal{H}$ we may pick some nonzero $v \in \mathcal{H}$ and construct the closure of the linear span of the set $\{\pi(x_1)\pi(x_2)\dots\pi(x_k)v \mid k \in \{0,1,2,\dots\}, x_1,\dots,x_k \in G\}$. Clearly this is the smallest closed invariant subspace of $\mathcal{H}$ which contains v. The nonzero representation π is called *cyclic* with *cyclic vector* v if the smallest closed invariant subspace of $\mathcal{H}$ which contains v is equal to $\mathcal{H}$.

PROPOSITION 9.1. *Each representation π can be written as a direct sum of cyclic subrepresentations.*

PROOF. By Zorn's lemma there exists a maximal family $\{\pi_i\}$ of mutually orthogonal cyclic subrepresentations of π. Let $\pi = (\Sigma_i^{\oplus} \pi_i) \oplus \rho$. Then $\rho = 0$, since, otherwise, ρ would contain a cyclic subrepresentation in contradiction to the maximality of the family $\{\pi_i\}$. □

PROPOSITION 9.2. *Each multiplicity free representation π is cyclic.*

PROOF. By Prop. 9.1, π is the direct sum of cyclic subrepresentations π_i. Let v_i be a cyclic vector for π_i and renormalize the v_i's such that $\Sigma_i \|v_i\|^2 < \infty$. Let $v_0 := \Sigma_i v_i$. Let P_i be the projection on $H(\pi_i)$. Since π is multiplicity free, P_i is central. Hence, if $x_1,\ldots,x_k \in G$ then $\pi(x_1) \ldots \pi(x_k)v_i = \pi(x_1) \ldots \pi(x_k)P_i v_0 = P_i(\pi(x_1) \ldots \pi(x_k)v_0)$ belongs to the representation space H_0 of the cyclic subrepresentation π_0 with cyclic vector v_0. It follows that $H(\pi_i) \subset H_0$ for all i. Hence $H(\pi) = H_0$. □

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VIII

REPRESENTATIONS OF LOCALLY COMPACT ABELIAN GROUPS

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CONTENTS

In this chapter we present the representation theory of locally compact abelian groups and the related representation theory of commutative C^*-algebras. Primarily we include these topics, because they are nice, important and relatively simple illustrations of the general representation theory of Chapter VII. However, a second purpose of this chapter is to state and to prove some preliminaries for the representation theory of semidirect products with abelian normal subgroup, which will be treated in Chapter XI. In particular, the SNAG theorem will be relevant there. In Section 1, a bird's eye view will be given of the results of this chapter. For reasons of motivation, this will be done in an order which is converse to the arrangement of the detailed presentation in later sections.

Notation. Throughout, the term "lcsc. space" will be used as an abbreviation for "locally compact Hausdorff space satisfying the second axiom of countability" and "lcsc. group" will mean a topological group which is a lcsc. space as a topological space. If X is a lcsc. space then $K(X)$ will denote the set of all complex-valued continuous functions on X with compact support. All Hilbert spaces are assumed to be separable.

1. INTRODUCTION

Consider the additive group $\mathbb{R}$ of real numbers. The regular representation λ of $\mathbb{R}$ is given by

(1.1) $$(\lambda(x)f)(y) := f(y-x), \quad f \in L^2(\mathbb{R}), \quad x,y \in \mathbb{R},$$

where $L^2(\mathbb{R})$ is taken with respect to the Lebesgue measure (= Haar measure) on $\mathbb{R}$. The Fourier transform

(1.2) $$\hat{f}(\alpha) := (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx, \quad \alpha \in \mathbb{R},$$

establishes an isometric mapping from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. (The indefinite integral in (1.2) converges in L^2-sense.) If we define a unitary representation $\hat{\lambda}$ of $\mathbb{R}$ on $L^2(\mathbb{R})$ by

$$\hat{\lambda}(x)\hat{f} := (\lambda(x)f)^{\wedge}, \qquad f \in L^2(\mathbb{R}),\ x \in \mathbb{R},$$

then $\hat{\lambda}$ is equivalent to λ and

$$(1.3) \qquad (\hat{\lambda}(x)\hat{f})(\alpha) = e^{ix\alpha}\hat{f}(\alpha), \qquad \hat{f} \in L^2(\mathbb{R}),\ x,\alpha \in \mathbb{R}.$$

Formula (1.3) still defines a unitary representation of $\mathbb{R}$ if we replace $L^2(\mathbb{R})$ by a Hilbert space $L^2(\mathbb{R},\mu)$, where μ is an arbitrary nonzero Borel measure on $\mathbb{R}$. Let us denote this representation by π_μ:

$$(1.4) \qquad (\pi_\mu(x)f)(\alpha) := e^{ix\alpha}f(\alpha), \qquad f \in L^2(\mathbb{R},\mu),\ x,\alpha \in \mathbb{R}.$$

Note that the functions $x \to e^{i\alpha x}$ $(\alpha \in \mathbb{R})$ are precisely the characters of the group $\mathbb{R}$, which can be identified with the irreducible unitary representations of $\mathbb{R}$ (cf. § I. 4). All irreducible unitary representations of $\mathbb{R}$ and all (countable) direct sums of mutually disjoint irreducible unitary representations of $\mathbb{R}$ can be written in the form (1.4). However, the collection of all representations of the form (1.4) is much more general: up to equivalence it can be shown to consist of all multiplicity free representations of $\mathbb{R}$.

Similar results hold for an arbitrary lcsc. abelian group G. A *character* on G is a continuous homomorphism from G into $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. The set $\hat{G}$ of all characters on G becomes an abelian group with respect to the product $(\alpha\beta)(x) := \alpha(x)\beta(x)$ $(\alpha,\beta \in \hat{G},\ x \in G)$. It is possible to define a certain canonical lcsc. topology on $\hat{G}$ such that $\hat{G}$ becomes a topological group. If $G = \mathbb{R}$ then $\hat{G} \simeq \mathbb{R}$ and the topology on $\hat{G}$ coincides with the natural (Euclidean) topology on $\mathbb{R}$. For $G = \mathbb{T}$ we have $\hat{G} \simeq \mathbb{Z}$ (the additive group of integers with the discrete topology).

Let μ be a Borel measure on $\hat{G}$. (For convenience we suppose that μ is a finite measure, i.e., $\mu(\hat{G}) < \infty$.) Analogous to (1.4), the formula

$$(1.5) \qquad (\pi_\mu(x)f)(\alpha) := \alpha(x)f(\alpha), \qquad f \in L^2(\hat{G},\mu),\ x \in G,\ \alpha \in \hat{G},$$

defines a unitary representation π_μ of G on $L^2(\hat{G},\mu)$. An important result of this chapter is that for any unitary representation π of G the following three statements are equivalent:

(a) π is equivalent to π_μ for some finite Borel measure μ on $\hat{G}$.

(b) π is multiplicity free.

(c) π is cyclic.

Furthermore, two representations π_μ and π_ν of the form (1.5) are equivalent iff the measures μ and ν are equivalent (cf. V.1.8), and π_μ and π_ν are disjoint iff μ and ν are mutually singular, that is, iff $\mu(E) = 0 = \nu(\hat{G}\setminus E)$ for some Borel set E in $\hat{G}$.

The equivalence of (b) and (c) implies that every unitary representation π of G is of type I. The canonical decomposition (VII.8.2) of type I representations implies that for given π there exist mutually disjoint finite Borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ on $\hat{G}$ (uniquely determined by π up to equivalence) such that π is equivalent to

(1.6) $$\pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus 3\pi_{\mu_3} \oplus \ldots \oplus \infty\pi_{\mu_\infty}.$$

Completely analogous results hold for the representations of a commutative separable C^*-algebra (which, for the moment, we again denote by G). Let $\hat{G}$ be the set of all nonzero multiplicative linear functionals on G. Under the so-called Gelfand topology $\hat{G}$ is a lcsc. space. Now for any finite Borel measure μ on $\hat{G}$ formula (1.5) defines a nondegenerate representation π_μ of the C^*-algebra G on $L^2(\hat{G},\mu)$. Furthermore, for any nondegenerate representation π of G the statements (a), (b) and (c) are equivalent.

This is not just an accidental analogy between the representation theories for lcsc. abelian groups and for commutative separable C^*-agebras, but there is, in fact, a causal relationship. We will derive the results in the group case from the corresponding results in the C^*-algebra case by the following steps. Let G be a lcsc. abelian group:

(i) The formula

(1.7) $$\tilde{\pi}(f) = \int_G f(x)\pi(x)dx, \quad f \in L^1(G),$$

establishes a one-to-one correspondence between the unitary representations π of G and the nondegenerate representations $\tilde{\pi}$ of the commutative involutive Banach algebra $L^1(G)$.

(ii) Specialization of (1.7) to the case that $\pi = \alpha \in \hat{G}$ establishes a one-to-one correspondence between $\hat{G}$ and the space of multiplicative linear functionals on $L^1(G)$. For the Gelfand transform $\hat{f}$ of $f \in L^1(G)$

we have $\hat{f}(\alpha) = \tilde{\alpha}(f)$, $\alpha \in \hat{G}$. The Gelfand topology on $\hat{G}$ induced by $L^1(G)$ and the group topology on $\hat{G}$ induced by G coincide.

(iii) The Gelfand transform $f \to \hat{f}$ is a one-to-one mapping from $L^1(G)$ onto a dense subalgebra of the commutative C^*-algebra $C_0(\hat{G})$ (the space of all continuous functions on G which vanish at infinity).

(iv) There is a one-to-one correspondence between the nondegenerate representations $\tilde{\pi}$ of $L^1(G)$ and $\tilde{\tilde{\pi}}$ of $C_0(\hat{G})$ such that

$$(1.8) \qquad \tilde{\tilde{\pi}}(\hat{f}) = \tilde{\pi}(f), \quad f \in L^1(G).$$

(v) If π_μ is defined by (1.5) then

$$(1.9) \qquad (\tilde{\tilde{\pi}}_\mu(f)\xi)(\alpha) = f(\alpha)\xi(\alpha), \quad f \in C_0(\hat{G}),\ \xi \in L^2(\hat{G},\mu),\ \alpha \in \hat{G}.$$

(vi) If π_1 and π_2 are unitary representations of G then the intertwining spaces $R(\pi_1,\pi_2)$, $R(\tilde{\pi}_1,\tilde{\pi}_2)$ and $R(\tilde{\tilde{\pi}}_1,\tilde{\tilde{\pi}}_2)$ coincide.

Let us next discuss the so-called SNAG theorem. It gives a spectral decomposition for any unitary representation π of a lcsc. abelian group G. The SNAG theorem is an important tool in the representation theory of semidirect products of locally compact groups, where the normal subgroup is abelian (cf. Chapter XI). The theorem involves the concept of a projection-valued measure:

DEFINITION 1.1. Let X be a lcsc. space and let $\mathcal{B}$ denote the σ-algebra of Borel sets of X. A *projection-valued measure* P on X acting in a Hilbert space $\mathcal{H}$ is a mapping $E \to P_E$ from $\mathcal{B}$ into the set of projection operators on $\mathcal{H}$ such that

$$(1.10) \quad \begin{cases} P_\emptyset = 0, \quad P_X = I; \\ P_{E_1 \cap E_2} = P_{E_1}P_{E_2} \quad (E_1,E_2 \in \mathcal{B}); \\ P_E = \sum_{i=1}^\infty P_{E_i} \qquad \text{(strong convergence)} \\ \qquad \text{if } E \text{ is the union of mutually disjoint Borel sets } E_i. \end{cases}$$

Let P be as above. For each $v,w \in \mathcal{H}$ we can define a complex Borel measure (cf. V.1.13) $P_{v,w}$ on X by

$$(1.11) \qquad P_{v,w}(E) := (P_E v,w), \quad E \in \mathcal{B}.$$

In particular, $P_{v,v}$ ($v \in H$) is a finite positive Borel measure on X. It can be shown that for each bounded Borel function f on X there is a unique operator $A \in L(H)$ such that

$$(1.12) \qquad (Av,w) = \int_X f \, dP_{v,w}, \quad v,w \in H.$$

We denote this operator A by

$$\int_X f \, dP \quad \text{or} \quad \int_X f(\alpha)\, dP(\alpha) \quad \text{(symbolically)}.$$

Now we are ready to formulate:

<u>THEOREM 1.2</u> (SNAG theorem). *Let* G *be a lcsc. abelian group. The formula*

$$(1.13) \qquad \pi(x) = \int_{\hat{G}} \alpha(x)\, dP(\alpha), \quad x \in G,$$

establishes a one-to-one correspondence between the unitary representations π *of* G *and the projection-valued measures* P *on* $\hat{G}$.

We will derive this theorem from an analogous theorem for commutative C^*-algebras (Theorem 3.6). If the nondegenerate representation $\tilde{\tilde{\pi}}$ of $C_0(\hat{G})$ is related to the unitary representation π of G by (1.7) and (1.8) and if, according to Theorem 3.6, $\tilde{\tilde{\pi}}$ is related to a projection-valued measure P by

$$(1.14) \qquad \tilde{\tilde{\pi}}(f) = \int_{\hat{G}} f(\alpha)\, dP(\alpha), \quad f \in C_0(\hat{G}),$$

then it can be shown that π is related to P by (1.13), cf. Section 6.

Theorem 1.2 is due to Stone, Naimark, Ambrose and Godement, which four names can be abbreviated as SNAG. First, the theorem was proved in the special case G = $\mathbb{R}$ by STONE [14]. In fact, he proved that there is a one-to-one correspondence between unitary representations π of $\mathbb{R}$ on a Hilbert space H and self-adjoint (generally unbounded) operators A on H such that $\pi(x) = \exp(ixA)$, $x \in \mathbb{R}$. (Then iA is the so-called infinitesimal generator of the one-parameter group $\{\pi(x) \mid x \in \mathbb{R}\}$.) Now combination with the spectral theorem II.1.6 for unbounded self-adjoint operators yields (1.13). For general locally compact abelian groups the theorem was proved about 1943, independently by NAIMARK [9], AMBROSE [1] and GODEMENT [5].

The remainder of this chapter has the following contents. In §2 we introduce Banach algebras, in particular involutive Banach algebras and

C^*-algebras, and we discuss the properties of the Gelfand transform for a commutative Banach algebra. In §3 we classify the representations of a commutative C^*-algebra up to equivalence and we settle a relationship like (1.14) for representations of a commutative C^*-algebra. Section 4 deals with general (not necessarily abelian) lcsc. groups G. We discuss the convolution product of two functions on G and we settle the relationship (1.7) between representations of G and of $L^1(G)$. Section 5 contains preliminaries on lcsc. abelian groups G: the dual group $\hat{G}$ and the Fourier transform $f \to \hat{f}$ for $f \in L^1(G)$. In §6 all preceding results can rapidly be combined in order to yield the classification up to equivalence for unitary representations of a lcsc. abelian group and the SNAG theorem. Finally, in §7 we apply results from §3 in order to prove some properties of diagonal and decomposable operators on a direct integral of Hilbert spaces. We conclude §7 with the definition of a direct integral of representations. Further results on this topic would easily fill another chapter, but they fall outside the scope of this colloquium. The interested reader is referred to the literature.

2. PRELIMINARIES ABOUT BANACH ALGEBRAS

A *Banach algebra* A is a complex Banach space (i.e. a complete normed complex linear space) on which a product $(x,y) \to xy$ is defined which is bilinear and associative such that the norm $\|\cdot\|$ on A satisfies

(2.1) $\quad \|xy\| \le \|x\|\,\|y\|, \quad x,y \in A.$

A mapping $x \to x^*: A \to A$ is said to be an *involution* on the Banach algebra A if

(2.2) $\quad (x^*)^* = x; \quad (ax+by)^* = \bar{a}x^* + \bar{b}y^*; \quad (xy)^* = y^*x^*.$

A Banach algebra A is called *involutive* if it is provided with an involution $x \to x^*$ such that

(2.3) $\quad \|x^*\| = \|x\|, \quad x \in A.$

If a Banach algebra A is provided with an involution which satisfies the stronger norm equality

(2.4) $\quad \|xx^*\| = \|x\|^2, \quad x \in A,$

then A is called a C^*-*algebra*. The standard example of a C^*-algebra is $L(H)$, where H is a Hilbert space. As general references about Banach algebras we mention HEWITT & ROSS [6, Appendix C], RUDIN [12, Ch.10,11], LOOMIS [8, Ch.IV,V], DIXMIER [4,§1].

A Banach algebra with *unit* e is a Banach algebra A with a (necessarily unique) element e such that $ex = xe = x$ for all $x \in A$ and $\|e\| = 1$. If A is a Banach algebra without unit then we can extend A to a Banach algebra A_1 with unit as follows. Let A_1 consist of all pairs (x,λ), $x \in A$, $\lambda \in \mathbb{C}$, with the obvious linear operations and with the product

$$(x,\lambda)(y,\mu) := (xy + \lambda y + \mu x, \lambda\mu).$$

Define a norm on A_1 by $\|(x,\lambda)\| := \|x\| + |\lambda|$. Then A_1 is a Banach algebra with unit $(0,1)$ and the subalgebra of all elements $(x,0)$, $x \in A$, is isomorphic with A (cf. [6,(C.3)]). If A is an involutive Banach algebra then the involution on A has a unique extension $(x,\lambda)^* := (x^*,\bar{\lambda})$ to A_1 such that A_1 becomes an involutive Banach algebra.

Let A be a Banach algebra with unit e. We say that an element $x \in A$ has an *inverse* x^{-1} in A if there exists a (necessarily unique) element $x^{-1} \in A$ such that $x^{-1}x = xx^{-1} = e$. The *spectrum* of an element $x \in A$ (notation $\sigma(x)$ or $\sigma_A(x)$) is the set of all complex λ such that $x-\lambda e$ has no inverse in A. If A is a Banach algebra without unit then the spectrum $\sigma(x)$ is defined as the spectrum of $(x,0)$ in the Banach algebra with unit A_1. In this last case $\sigma(x)$ clearly always contains 0.

THEOREM 2.1 (cf. [12, Theorem 10.13]). *If A is a Banach algebra and $x \in A$ then*

(a) *the spectrum $\sigma(x)$ of x is compact and nonempty;*

(b) *the spectral radius $\rho(x) := \sup\{|\lambda| \mid \lambda \in \sigma(x)\}$ satisfies*

$$\rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n} \leq \|x\|. \tag{2.5}$$

(The limit in (2.5) *always exists.)*

Let A be a commutative Banach algebra. A *multiplicative linear functional* on A is a nonzero linear functional α on A such that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x,y \in A$. Every multiplicative linear functional α on A is bounded and $\|\alpha\| \leq 1$ (cf. [6, (C.21)]). Let X be the set of all multiplicative linear functionals on A; X is called the *structure space* (or *maximal* *ideal space*) for A. If A has no unit then each $\alpha \in X$ has a unique extension to a multiplicative linear functional on A_1. The only element of the structure space X_1 of A_1 which is not in X, is the functional α_∞ defined by $\alpha_\infty(x,\lambda) := \lambda$, $(x,\lambda) \in A_1$ (cf. [6, (C.14)]).

For every element x of a commutative Banach algebra A the *Gelfand transform* $\hat{x}$ is the function on X defined by

$$(2.6) \qquad \hat{x}(\alpha) := \alpha(x), \qquad \alpha \in X.$$

The *Gelfand topology* for X is the weakest topology on X under which all functions $\hat{x}$ $(x \in A)$ are continuous. Note that the class of all finite intersections of sets $\{\alpha \in A \mid |\hat{x}(\alpha)-\hat{x}(\alpha_0)| < \varepsilon\}$, $\alpha_0 \in X$, $x \in A$, $\varepsilon > 0$, forms a base of open sets for the Gelfand topology on X.

Let Y be a locally compact space. Let $C_0(Y)$ be the space of continuous complex-valued functions f on Y such that $\lim_{\alpha\to\infty} f(\alpha) = 0$, that is, for each $\varepsilon > 0$ there is a compact subset K of Y such that $|f(\alpha)| < \varepsilon$ outside K. $C_0(Y)$ becomes a commutative C^*-algebra if λf, $f+g$ and fg are defined pointwise, if $f^*(\alpha) := \overline{f(\alpha)}$ and if the norm is given by $\|f\| := \sup_{\alpha\in X}|f(\alpha)|$. If Y is compact then $C_0(Y) = C(Y)$, the space of all continuous functions on Y, and $C_0(Y)$ has a unit in this case.

We collect the fundamental facts about the Gelfand transform in the following theorem.

THEOREM 2.2 (cf. [6, (C.25),(C.26)],[12, Theorem 11.9]). *Let A be a commutative Banach algebra with structure space* A.

(a) X *is a locally compact Hausdorff space under the Gelfand topology.*

(b) *The mapping* $x \to \hat{x}$ *is a homomorphism from the algebra A onto a subalgebra $\hat{A}$ of $C_0(X)$. The algebra $\hat{A}$ separates the points of* X.

(c) *For each* $x \in A$, $\text{range}(\hat{x})\setminus\{0\} = \sigma(x)\setminus\{0\}$ *and*

$$(2.7) \qquad \|\hat{x}\| = \rho(x) \le \|x\|.$$

(d) *If A has a unit* e *then* X *is compact,* $\hat{e}(\alpha) = 1$ *for all* $\alpha \in X$ *and* $\text{range}(\hat{x}) = \sigma(x)$ *for all* $x \in A$.

(e) *If A has no unit and if* $X_1 = X \cup \{\alpha_\infty\}$ *is the structure space of* A_1 *then* X_1 *is the one-point compactification of* X.

In the case of a commutative C^*-algebra the situation becomes very nice:

THEOREM 2.3 (cf. [6,(C.28)]). *If A is a commutative C^*-algebra then the mapping $x \to \hat{x}$ is a norm-preserving $*$-isomorphism from A onto the C^*-algebra $C_0(X)$.*

Let Y be a locally compact Hausdorff space and let X be the structure space of the commutative C^*-algebra $C_0(Y)$. It is a natural question to ask for the relationship between X and Y. Obviously, with each $y \in Y$ we can associate an element $\delta_y \in X$ defined by $\delta_y(f) := f(y)$, $f \in C_0(Y)$.

THEOREM 2.4 (cf. [6,(C.32)]). *If Y is a locally compact Hausdorff space then the mapping $y \to \delta_y$ is a homeomorphism from Y onto the structure space of $C_0(Y)$.*

If A is a commutative involutive Banach algebra then the Gelfand transform is not necessarily a $*$-homomorphism, as in the C^*-algebra case. (See the counterexample in NAIMARK [10,§14.1,Example 3].) However, it will turn out that a convolution algebra $A = L^1(G)$ (G abelian) always has this property. In such cases the following lemma is useful.

LEMMA 2.5. *Let A be a commutative involutive Banach algebra with structure space X and assume that $(x^*)\hat{}\,(\alpha) = \overline{\hat{x}(\alpha)}$ for all $x \in A$, $\alpha \in X$. Then $\hat{A}$ is dense in $C_0(X)$.*

PROOF. The subalgebra $\hat{A}$ of $C_0(X)$ separates the points of X (cf. Theorem 2.2(b)) and it is closed under complex conjugation. Now apply the Stone-Weierstrass theorem (cf. SIMMONS [15,§38,Theorem B]). □

A *representation* π of an involutive Banach algebra A on a Hilbert space H is a $*$-homomorphism from A into $L(H)$.

THEOREM 2.6. *Let π be a representation of an involutive Banach algebra A on a Hilbert space H. Then*

(2.8) $\qquad \|\pi(x)\| \le \|x\|, \quad x \in A.$

If, in addition, A is commutative then

(2.9) $\qquad \|\pi(x)\| \le \|\hat{x}\|, \quad x \in A.$

PROOF. First observe that for a hermitian operator A on H we have $\rho(A) = \|A\|$. Indeed, $\|A^2\| = \|AA^*\| = \|A\|^2$. Hence, by recurrence, $\|A^{2n}\| = \|A\|^{2n}$, $n \in \mathbb{N}$. Then (2.5) implies the result.

Without loss of generality we may assume that the representation π is nondegenerate. Then $\pi(e) = I$ if A has a unit e, and π uniquely extends to a nondegenerate representation of A_1 if A has no unit. Thus we may also assume that A has a unit.

If $x \in A$ and $x-\lambda e$ has an inverse in A then $\pi(x) - \lambda I$ has an inverse in $L(H)$. Hence $\sigma_{L(H)}(\pi(x)) \subset \sigma_A(x)$.

Let $x \in A$. Then: $\|\pi(x)\|^2 = \|\pi(x)\pi(x)^*\| = \|\pi(xx^*)\| = \rho_{L(H)}(\pi(xx^*)) \leq \rho_A(xx^*) \leq \|xx^*\| \leq \|x\|\|x^*\| = \|x\|^2$. This proves (2.8).

Now assume that A is also commutative. Then (2.7) together with the above inequalities implies:

$$\|\pi(x)\|^2 \leq \rho_A(xx^*) = \|(xx^*)\hat{}\| =$$

$$= \sup_{\alpha\in X}|\alpha(xx^*)| \leq \sup_{\alpha\in X}|\alpha(x)| \cdot \sup_{\alpha\in X}|\alpha(x^*)| =$$

$$= \|\hat{x}\|\|(x^*)\hat{}\| = \rho(x)\rho(x^*) = (\rho(x))^2 = \|\hat{x}\|^2,$$

where we used that the spectrum of x is complex conjugate to the spectrum of x^*. □

In Example VII.2.4(e) it was observed that the class of all representations of an involutive Banach algebra satisfies the conditions of Assumption VII.2.3. Thus all definitions and results of Ch.VII apply to the representations of such an algebra.

In this chapter we will restrict ourselves to locally compact spaces and groups which are second countable (lcsc.). The reason for this is that we like to have all Borel measures on the space regular (cf. V.2.3). The following three lemmas deal with these matters.

<u>LEMMA 2.7</u>. *If A is a separable commutative Banach algebra then its structure space X is a separable metric space. Hence X is a lcsc. space in that case.*

<u>PROOF</u>. Let $\{x_n\}$ be a countable dense subset of the unit ball in A. Then the mapping

$$\alpha \to \{2^{-\frac{1}{2}}\hat{x}_1(\alpha), 2^{-1}\hat{x}_2(\alpha), \ldots, 2^{-\frac{1}{2}n}\hat{x}_n(\alpha), \ldots\}$$

is an injection from X into the separable Hilbert space ℓ^2. The metric on ℓ^2 induces a topology on X under which all functions $\hat{x}_n$, and hence, by density, all functions $\hat{x}$ $(x \in A)$, are continuous. Thus the topology induced on X by the metric is finer than the Gelfand topology. Conversely, we will show that each ball around $\alpha_0 \in X$ with respect to the metric contains a neighbourhood of α_0 with respect to the Gelfand topology. Indeed, let $\varepsilon > 0$ and choose the natural number N such that $2^{-N} < \frac{1}{8}\varepsilon^2$. Then, if $|\hat{x}_k(\alpha) - \hat{x}_k(\alpha_0)| < \frac{1}{2}\varepsilon$ for $k = 1,\dots,N$, we have

$$d(\alpha,\alpha_0) := \left(\sum_{n=1}^{\infty} 2^{-n}|\hat{x}_n(\alpha) - \hat{x}_n(\alpha_0)|^2\right)^{\frac{1}{2}} <$$

$$< \left(\sum_{n=1}^{N} 2^{-n}(\tfrac{1}{2}\varepsilon)^2 + \sum_{n=N+1}^{\infty} 2^{-n}\cdot(2\|\hat{x}_n\|)^2\right)^{\frac{1}{2}} <$$

$$< (\tfrac{1}{4}\varepsilon^2 + 2^{-N}\cdot 4)^{\frac{1}{2}} < \varepsilon.$$

We conclude that both topologies coincide. □

<u>LEMMA 2.8</u>. *Let* X *be a lcsc. space. Then the commutative* C^**-algebra* $C_0(X)$ *is separable.*

<u>PROOF</u>. Choose a countable base $\mathcal{U}$ for the topology of X, consisting of relatively compact, open sets. Let $\mathcal{U}^*$ denote the collection of all pairs (V,U) such that V and U are elements of $\mathcal{U}$ and $\bar{V} \subset U$. Then $\mathcal{U}^*$ is countable. Clearly, if $\alpha,\beta \in X$ and $\alpha \neq \beta$ then we can find $(V,U) \in \mathcal{U}^*$ such that $\alpha \in V$, $\beta \notin U$. For each $(V,U) \in \mathcal{U}^*$ choose a function $f_{V,U} \in K(X)$ such that $0 \le f_{V,U}(\alpha) \le 1$ for all $\alpha \in X$, $f_{V,U}(\alpha) = 1$ for $\alpha \in \bar{V}$ and $f_{V,U}(\alpha) = 0$ for $\alpha \notin U$ (cf. V.2.1). Then the set of functions $\{f_{V,U} \mid (V,U) \in \mathcal{U}^*\}$ separates the points of X. Hence, by an application of the Stone-Weierstrass theorem, the complex algebra A generated by these functions is dense in $C_0(X)$. Let A_0 be the set of all finite linear combinations of finite products of functions $f_{V,U}((V,U) \in \mathcal{U}^*)$ with coefficients of the form $q_1 + iq_2$ (q_1,q_2 rational). Then A_0 is countable and dense in A. Thus $C_0(X)$ contains a countable dense subset. □

<u>LEMMA 2.9</u>. *Let* X *be a lcsc. space. Then* X *is metrizable and every Borel measure on* X *is regular.*

<u>PROOF</u>. The first statement follows from Lemma 2.8, Lemma 2.7 and Theorem 2.4. A lcsc. space is clearly σ-compact. Now the second statement follows from V.2.5. □

3. REPRESENTATIONS OF COMMUTATIVE C^*-ALGEBRAS

3.1. Classification of representations up to equivalence

In this section we develop the representation theory of a commutative separable C^*-algebra A. In view of Theorem 2.3, Lemma 2.7 and Lemma 2.8, we can restrict ourselves to the case that $A = C_0(X)$ for some lcsc. space X. It follows from Theorems 2.3, 2.4 and VII.4.9 that the irreducible representations of $C_0(X)$ are precisely the one-dimensional representations $f \to f(\alpha)$, where α runs over X.

We now construct more general representations of $C_0(X)$: first multiplicity free representations and next general type I representations. This part closely follows ARVESON [2,§2.2]. Let μ be a finite Borel measure on X. Then $L^2(X,\mu)$ is a separable Hilbert space, which contains $K(X)$ as a dense subspace (cf. V.3.5). Hence $C_0(X)$ is dense in $L^2(X,\mu)$. We define a representation π_μ of $C_0(X)$ on $L^2(X,\mu)$ by

$$(3.1) \qquad (\pi_\mu(f)\xi)(\alpha) := f(\alpha)\xi(\alpha), \quad f \in C_0(X),\ \xi \in L^2(X,\mu),\ \alpha \in X.$$

It is easily verified that π_μ is indeed a representation. Let us show that π_μ is nondegenerate. Suppose that $\xi \in L^2(X,\mu)$ and $\pi_\mu(f)\xi = 0$ for all $f \in C_0(X)$. Then

$$(\pi_\mu(f)\xi,\xi) = \int_X f(\alpha)|\xi(\alpha)|^2 d\mu(\alpha) = 0$$

for all $f \in C_0(X)$. Let $\{f_n\}$ be a sequence in $C_0(X)$ such that $0 \le f_n(\alpha) \uparrow 1$, $\alpha \in X$. Then the monotone convergence theorem V.1.5(ii) shows that

$$0 = (\pi_\mu(f_n)\xi,\xi) \uparrow (\xi,\xi) = 0.$$

Hence $\xi = 0$.

Let B(X) be the set of all bounded Borel functions on X. Under pointwise operations it is a commutative algebra with involution. B(X) becomes a commutative (generally nonseparable) C^*-algebra with respect to the supremum norm. Clearly formula (3.1) defines also a representation of B(X) on $L^2(X,\mu)$.

THEOREM 3.1. *Let π be a nondegenerate representation of $C_0(X)$. Then the following three statements are equivalent:*

(a) $\pi \simeq \pi_\mu$ *for some finite Borel measure* μ *on* X.

(b) π *is multiplicity free.*

(c) π *is cyclic* (cf. §VII.9).

PROOF. We will prove that (a) ⇒ (b) ⇒ (c) ⇒ (a).

(a) ⇒ (b). It is our purpose to show that $R(\pi_\mu) = \pi_\mu(B(X))$. Since $\pi_\mu(B(X))$ is a commutative algebra, this will imply that π_μ is multiplicity free (cf. Theorem VII.5.4). Clearly $\pi_\mu(B(X)) \subset R(\pi_\mu)$. Conversely assume $T \in R(\pi_\mu)$. Let $1 \in L^2(X,\mu)$ be the function which is identically one on X. For $f \in C_0(X)$ we have $Tf = T(\pi_\mu(f)1) = \pi_\mu(f)(T1)$. Let $g := T1$. Then $g \in L^2(X,\mu)$ and $(Tf)(\alpha) = g(\alpha)f(\alpha)$ a.e. $[\mu]$. We will show that g is essentially bounded. Let $\xi \in C_0(X)$, $\eta \in L^2(X,\mu)$. Then

$$(*)\qquad \left|\int_X g(\alpha)\xi(\alpha)\overline{\eta(\alpha)}\,d\mu(\alpha)\right| = \left|\int_X (T\xi)(\alpha)\overline{\eta(\alpha)}\,d\mu(\alpha)\right| =$$

$$= |(T\xi,\eta)| \le \|T\|\|\xi\|_2\|\eta\|_2,$$

where $\|\cdot\|_2$ denotes the norm in $L^2(X,\mu)$. Suppose that $|g(\alpha)| \ge C > 0$ on some Borel set E in X with $\mu(E) > 0$. We will show that $C \le \|T\|$. Choose ε, $0 < \varepsilon < \frac{1}{2}\mu(E)$. By regularity (cf. Lemma 2.9 and V.2.3) we can choose a compact set K and an open set U such that $K \subset E \subset U$ and $\mu(U\backslash K) < \varepsilon$. Now we make the following choices for ξ and η. Let $\xi \in C_0(X)$ such that $0 \le \xi(\alpha) \le 1$ if $\alpha \in X$, $\xi(\alpha) = 1$ if $\alpha \in K$ and $\xi(\alpha) = 0$ if $\alpha \notin U$. Let $\eta(\alpha) := \chi_K(\alpha)|g(\alpha)|\cdot$ $\cdot(g(\alpha))^{-1}$. Then it follows from (*) that

$$C\mu(K) \le \left|\int_X g(\alpha)\xi(\alpha)\overline{\eta(\alpha)}\,d\mu(\alpha)\right| \le \|T\|\|\xi\|_2\|\eta\|_2 \le$$

$$\le \|T\|(\mu(K)+\varepsilon)^{\frac{1}{2}}(\mu(K))^{\frac{1}{2}}.$$

Hence

$$C \le \|T\|(1+\varepsilon/\mu(K))^{\frac{1}{2}} \le \|T\|(1+2\varepsilon/\mu(E))^{\frac{1}{2}}.$$

On letting $\varepsilon \downarrow 0$ we obtain $C \le \|T\|$. Thus $|g(\alpha)| \le \|T\|$ a.e. $[\mu]$. By changing g on a set of μ-measure zero we get $g \in B(X)$. Then $Tf = \pi_\mu(g)f$ for all $f \in C_0(X)$. Since $C_0(X)$ is dense in $L^2(X,\mu)$, we conclude that $T = \pi_\mu(g)$. Hence $R(\pi_\mu) \subset \pi_\mu(B(X))$.

(b) ⇒ (c). This is a general fact about representations, cf. Prop. VII.9.2.

(c) ⇒ (a). Let π be a cyclic representation of $C_0(X)$ on H with cyclic vector v, $\|v\| = 1$. Define $\rho(f) := (\pi(f)v,v)$, $f \in C_0(X)$. Then $|\rho(f)| \le \|f\|$ and

$$\rho(|f|^2) = (\pi(f^*f)v,v) = (\pi(f)v,\pi(f)v) > 0.$$

Hence ρ is a bounded and positive linear functional on $C_0(X)$ and there is a finite Borel measure μ on X such that

$$\rho(f) = \int_X f(\alpha)d\mu(\alpha), \quad f \in C_0(X),$$

cf. the Riesz representation theorem V.2.8. Define $Uf := \pi(f)v$, $f \in C_0(X)$. Then U is a linear mapping from $C_0(X)$ onto the dense subspace $\pi(C_0(X))v$ of H. For each $f \in C_0(X)$ we have

$$\|Uf\|^2 = (\pi(|f|^2)v,v) = \rho(|f|^2) =$$

$$= \int_X |f(\alpha)|^2 d\mu(\alpha) = \|f\|_2^2.$$

Hence U has a unique extension to an isometry from $L^2(X,\mu)$ onto H. Finally, we show that $U \in R(\pi_\mu,\pi)$. Let $f,g \in C_0(X)$. Then

$$U(\pi_\mu(f)g) = U(fg) = \pi(fg)v = \pi(f)\pi(g)v = \pi(f)Ug.$$

By density this extends to $g \in L^2(X,\mu)$. Hence $\pi \simeq \pi_\mu$. □

COROLLARY 3.2. *Let G be a set with involution. Let Rep be a class of nondegenerate representations of G satisfying the conditions of Assumption VII.2.3. Let $\pi \in$ Rep be such that $\pi(x)\pi(y) = \pi(y)\pi(x)$ for all $x,y \in G$. Then*
(a) *π is multiplicity free iff π is cyclic.*
(b) *π is of type I. In particular, every nondegenerate representation of a commutative separable C^*-algebra is of type I.*

PROOF.
(a) If π is multiplicity free then π is cyclic (cf. Prop. VII.9.2). Now assume that π is cyclic with cyclic vector $v \in H(\pi)$. The von Neumann algebra $(R(\pi))'$ is generated by $\pi(G)$ (cf. Prop. VII.3.1(b)). Hence $(R(\pi))'$ is commutative. There exists a separable C^*-algebra $A \subset (R(\pi))'$ which is strongly dense in $(R(\pi))'$ (cf. ARVESON [2, Prop.1.2.3 and Exercise 1.2.E.c]). Then

A is commutative and the set $\{T v \mid T \in A\}$ is dense in $H(\pi)$. Hence the natural (clearly nondegenerate) representation ρ of A on $H(\pi)$ is cyclic, so ρ is multiplicity free (cf. Theorem 3.1). Thus $R(\rho)$ is a commutative von Neumann algebra (cf. Theorem VII.5.4). Since $R(\rho) = R(\pi)$, another application of Theorem VII.5.4 yields that π is multiplicity free.

(b) π is a direct sum of cyclic subrepresentations (cf. Prop. VII.9.1). Hence, by the first part of the corollary, π is a direct sum of multiplicity free representations. Now apply Theorem VII.8.8. □

Equivalence and disjointness for representations π_μ and π_ν of $C_0(X)$ can be characterized in terms of certain relations of the measures μ and ν. The definition of domination ($\nu << \mu$) and equivalence ($\mu \equiv \nu$) was given in V.1.8. Borel measures μ and ν on X are called *mutually singular* (notation $\mu \perp \nu$) if there is a Borel set E in X such that $\mu(E) = 0 = \nu(X\backslash E)$. If μ and ν are Borel measures on X then there are unique Borel measures μ_1 and μ_2 such that $\mu = \mu_1+\mu_2$, $\mu_1 << \nu$, $\mu_2 \perp \nu$ (cf. RUDIN [11, Theorem 6.9(a)]). The pair (μ_1,μ_2) is called the *Lebesgue decomposition* of μ relative to ν.

THEOREM 3.3. *Let* μ *and* ν *be finite Borel measures on* X. *Then*
(a) $\mu \perp \nu$ *iff* $\pi_\mu \,\overset{\downarrow}{\circ}\, \pi_\nu$.
(b) $\mu << \nu$ *iff* $\pi_\mu \lesssim \pi_\nu$.
(c) $\mu \equiv \nu$ *iff* $\pi_\mu \simeq \pi_\nu$.

PROOF.
(i) $\mu << \nu \Rightarrow \pi_\mu \lesssim \pi_\nu$.
Indeed, let $\mu << \nu$. Then (formally) $d\mu(\alpha) = h(\alpha)d\nu(\alpha)$ for some Borel function $h: X \to \mathbb{R}^+$ (cf. V.1.9). For $\xi \in L^2(X,\mu)$ define $(U\xi)(\alpha) := (h(\alpha))^{\frac{1}{2}}\xi(\alpha)$, $\alpha \in X$. Then

$$\int_X |(U\ )(\alpha)|^2 d\nu(\alpha) = \int_X |\xi(\alpha)|^2 h(\alpha)d\nu(\alpha) = \int_X |\xi(\alpha)|^2 d\mu(\alpha).$$

Furthermore,

$$(U(\pi_\mu(f)\xi))(\alpha) = (h(\alpha))^{\frac{1}{2}}f(\alpha)\xi(\alpha) =$$
$$= (\pi_\nu(f)(U\xi))(\alpha), \quad f \in C_0(X),\ \xi \in L^2(X,\mu),\ \alpha \in X.$$

Hence U is an intertwining isometry for π_μ and π_ν from $H(\mu_\pi)$ onto a closed invariant subspace of $H(\pi_\nu)$.

(ii) $\mu \equiv \nu \Rightarrow \pi_\mu \simeq \pi_\nu$.
This follows from (i) and Theorem VII.6.2(iii).

(iii) $\pi_\mu \overset{|}{\circ} \pi_\nu \Rightarrow \mu \perp \nu$.
Indeed, if not $\mu \perp \nu$ then the Lebesgue decomposition of μ relative to ν yields a nonzero Borel measure σ such that $\sigma << \mu$, $\sigma << \nu$. Then (i) shows that $\pi_\sigma \lesssim \pi_\mu$ and $\pi_\sigma \lesssim \pi_\nu$. Since $\pi_\sigma \neq 0$, π_μ and π_ν are not disjoint.

(iv) $\mu \perp \nu \Rightarrow \pi_\mu \overset{|}{\circ} \pi_\nu$.
This is proved as follows. If $\mu \perp \nu$ then $\mu(X\backslash E) = 0 = \nu(E)$ for some Borel set E. Now the mapping $\xi \to (\chi_E\xi, \chi_{X\backslash E}\xi)$ is an intertwining isometry for $\pi_{\mu+\nu}$ and $\pi_\mu \oplus \pi_\nu$ from $L^2(X,\mu+\nu)$ onto $L^2(X,\mu) \oplus L^2(X,\nu)$. Hence $\pi_\mu \oplus \pi_\nu$, being equivalent to $\pi_{\mu+\nu}$, is multiplicity free (cf. Theorem 3.1). Now it follows from Theorem VII.5.4 and Prop. VII.4.5 that $\pi_\mu \overset{|}{\circ} \pi_\nu$.

(v) $\pi_\mu \lesssim \pi_\nu \Rightarrow \mu << \nu$.
Indeed, if not $\mu << \nu$ then the Lebesgue decomposition for μ relative to ν shows that $\sigma << \mu$, $\sigma \perp \nu$ for some $\sigma \neq 0$. Now (i) and (iv) imply that $\pi_\sigma \lesssim \pi_\mu$, $\pi_\sigma \overset{|}{\circ} \pi_\nu$. Since $\pi_\sigma \neq 0$, we can't have $\pi_\mu \lesssim \pi_\nu$.

(vi) $\pi_\mu \simeq \pi_\nu \Rightarrow \mu \equiv \nu$.
This follows from (v). □

Combination of the canonical decomposition (VII.8.2) for type I representations with Corollary 3.2(b) and Theorems 3.1 and 3.3 now yields a complete description of the representations of $C_0(X)$.

<u>THEOREM 3.4</u>. *Let π be a nondegenerate representation of $C_0(X)$. Then there are mutually singular, finite Borel measures $\mu_1, \mu_2, \ldots, \mu_\infty$ on X, uniquely determined by π up to equivalence, such that*

$$\pi \simeq \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus 3\pi_{\mu_3} \oplus \ldots \oplus \infty\pi_{\mu_\infty}. \tag{3.2}$$

Note that the measures μ_i can be renormalized such that $\mu := \sum_i \mu_i$ is a finite Borel measure on X.

3.2. The relationship of representations with projection-valued measures

In this subsection we will derive a one-to-one correspondence between nondegenerate representations of $C_0(X)$ and projection-valued measures on X. Projection-valued measures were introduced in Definition 1.1. It was observed in Example VII. 2.4(f) that the class of all projection-valued measures on X can be considered as a class of representations of $\mathcal{B}$ (the collection of Borel sets in X) which satisfies the conditions of Assumption VII.2.3. Hence we can freely use the concepts of representation theory in the context of projection-valued measures.

Let P be a projection-valued measure on a lcsc. space X acting in a Hilbert space $\mathcal{H}$. In §1 we defined

$$P_{v,w}(E) := (P_E v,w), \quad E \in \mathcal{B},\ v,w \in \mathcal{H}. \tag{3.3}$$

For all $v \in \mathcal{H}$, $P_{v,v}$ is a finite positive Borel measure on X. A straightforward calculation yields that

$$P_{v,w} = P_{v+w,v+w} - P_{v-w,v-w} + iP_{v+iw,v+iw} - iP_{v-iw,v-iw}, \quad v,w \in \mathcal{H}. \tag{3.4}$$

Thus $P_{v,w}$ is a complex linear combination of four finite (positive) Borel measures. Hence

$$P_{v,w}(E) = \sum_{i=1}^{\infty} P_{v,w}(E_i)$$

if E is the union of mutually disjoint Borel sets E_i, where the sum at the right-hand side is absolutely converging. Thus $P_{v,w}$ is a complex Borel measure on X (cf. V.1.13).

If $T \in \mathcal{L}(\mathcal{H})$ and $f \in B(X)$ then the formal identity

$$T = \int_X f dP = \int_X f(\alpha) dP(\alpha) \tag{3.5}$$

will mean that

$$(Tv,w) = \int_X f(\alpha) d\, P_{v,w}(\alpha) \tag{3.6}$$

for all $v,w \in \mathcal{H}$ (cf. V.1.13 for the definition of an integral with respect to a complex measure). Note that (3.6), and hence (3.5), are already implied, if the identity

$$(3.7) \qquad (Tv,v) = \int_X f(\alpha)\, dP_{v,v}(\alpha)$$

holds for all $v \in \mathcal{H}$.

PROPOSITION 3.5. *Let P be a projection-valued measure on X acting on $\mathcal{H} = \mathcal{H}(P)$. Then, for each $f \in B(X)$,*

$$(3.8) \qquad \pi_P(f) := \int_X f\, dP$$

is a well-defined element of $L(\mathcal{H})$. Furthermore, π_P is a representation on $\mathcal{H}$ of the C^*-algebra $B(X)$. Denote the restriction of π_P to $C_0(X)$ by π_P^0. Then π_P^0 is a nondegenerate representation of $C_0(X)$. Finally, if Q is another projection-valued measure on X acting on $\mathcal{H}(Q)$ then $R(P,Q) = R(\pi_P,\pi_Q) = R(\pi_P^0,\pi_Q^0)$.

PROOF. We follow the proof given in RUDIN [12, Theorem 12.21]. First we show that

$$(3.9) \qquad \left|\int_X f(\alpha)\, dP_{v,w}(\alpha)\right| \leq \|f\|\,\|v\|\,\|w\|, \qquad f \in B(X),\ v,w \in \mathcal{H}.$$

This is clear in the case of step functions $f = \sum_{i=1}^n c_i\, \chi_{E_i}$, where the E_i's are mutually disjoint Borel sets, since

$$\left|\int_X f(\alpha)\, dP_{v,w}(\alpha)\right|^2 = \left|\sum_{i=1}^n c_i\, P_{v,w}(E_i)\right|^2 =$$

$$= \left|\left(\sum_{i=1}^n c_i\, P_{E_i} v,w\right)\right|^2 \leq \left\|\sum_{i=1}^n c_i\, P_{E_i} v\right\|^2 \|w\|^2 =$$

$$= \sum_{i,j=1}^n c_i\, \bar{c}_j (P_{E_i} v, P_{E_j} v)\ \|w\|^2 =$$

$$= \sum_{i=1}^n |c_i|^2\ (P_{E_i} v,v)\|w\|^2 \leq$$

$$\leq \left(\max_{1\leq i\leq n} |c_i|\right)^2 \left(\sum_{i=1}^n P_{E_i} v,v\right)\|w\|^2 \leq \|f\|^2\|v\|^2\|w\|^2.$$

Here we used that $P_{E_i} P_{E_j} = P_{E_i \cap E_j} = 0$ if $i \neq j$. Since the collection $Step(X)$ of step functions on X is a dense subset of $B(X)$, inequality (3.9) holds for all $f \in B(X)$.

Formula (3.9) implies that for each $f \in B(X)$ the expression

$$\int_X f(\alpha)\, dP_{v,w}(\alpha)$$

is a continuous sesquilinear form in $v,w \in \mathcal{H}$. Hence there is a unique bounded linear operator $\pi_P(f)$ on $\mathcal{H}$ such that this sesquilinear form equals $(\pi_P(f)v,w)$. Then $\pi_P(f)$ satisfies (3.8) by definition. It also follows from (3.9) that

$$\|\pi_P(f)\| \le \|f\|, \quad f \in B(X). \tag{3.10}$$

Note that $\mathit{Step}(X)$ is a dense $*$-subalgebra of $B(X)$. Hence, if π_P restricted to $\mathit{Step}(X)$ is a $*$-homomorphism from $\mathit{Step}(X)$ to $L(\mathcal{H})$ then π_P will be a representation of $B(X)$ in view of (3.10). Now the proof that $\pi_P\colon \mathit{Step}(X) \to L(\mathcal{H})$ is a $*$-homomorphism is almost immediate, since

$$\pi_P\left(\sum_{i=1}^{n} c_i\, \chi_{E_i}\right) = \sum_{i=1}^{n} c_i\, P_{E_i}.$$

Next we prove that π_P^0 is nondegenerate. Let $v \in \mathcal{H}$. Suppose that $\pi_P(f)v = 0$ for all $f \in C_0(X)$. Then

$$0 = (\pi_P(f)v,v) = \int_X f\, dP_{v,v} \quad \text{for all } f \in C_0(X).$$

Hence $P_{v,v} = 0$. In particular, $P_{v,v}(X) = (P_X v,v) = \|v\|^2 = 0$. Thus π_P^0 is nondegenerate.

Finally we show that

$$R(P,Q) \subset R(\pi_P,\pi_Q) \subset R(\pi_P^0,\pi_Q^0) \subset R(P,Q).$$

We have that $A \in R(P,Q)$ iff $P_{v,A^*w} = Q_{Av,w}$ for all $v \in \mathcal{H}(P)$, $w \in \mathcal{H}(Q)$. Hence, if $A \in R(P,Q)$ then

$$(A\pi_P(f)v,w) = (\pi_P(f)v,A^*w) = (\pi_Q(f)Av,w)$$

for all $v \in \mathcal{H}(P)$, $w \in \mathcal{H}(Q)$, so $A \in R(\pi_P,\pi_Q)$. It is trivial that $R(\pi_P,\pi_Q) \subset R(\pi_P^0,\pi_Q^0)$. Finally, if $A \in R(\pi_P^0,\pi_Q^0)$ then

$$\int_X f\, dP_{v,A^*w} = \int_X f\, dQ_{Av,w}$$

for all $f \in C_0(X)$, $v \in \mathcal{H}(P)$, $w \in \mathcal{H}(Q)$. The uniqueness part of the Riesz representation theorem for complex Borel measures (cf. (V.2.13)) implies that $P_{v,A^*w} = Q_{Av,w}$ for all $v \in H(P)$, $w \in H(Q)$. Hence $A \in R(P,Q)$. □

Let μ be a finite Borel measure on X. It is rather obvious that the formula

(3.11) $$(P_E f)(\alpha) := \chi_E(\alpha) f(\alpha), \quad f \in L^2(X,\mu),\ E \in \mathcal{B},\ \alpha \in X,$$

defines a projection-valued measure P on X acting in $L^2(X,\mu)$. We will show that the corresponding representation π_P of B(X) is just the representation π_μ defined by (3.1):

(3.12) $$\pi_\mu(f) = \int_X f\, dP, \quad f \in B(X).$$

Indeed, for $\xi \in L^2(X,\mu)$, $E \in \mathcal{B}$ we have $P_{\xi,\xi}(E) = (P_E\xi,\xi) = \int_E |\xi|^2 d\mu$. Hence $dP_{\xi\xi}(\alpha) = |\xi(\alpha)|^2 d\mu(\alpha)$, formally, and

$$(\pi_\mu(f)\xi,\xi) = \int_X f(\alpha)|\xi(\alpha)|^2 d\mu(\alpha) = \int_X f\, dP_{\xi,\xi}, \quad f \in B(X).$$

This settles (3.12).

THEOREM 3.6. *There is a one-to-one correspondence between projection-valued measures P on X and nondegenerate representations π of $C_0(X)$ such that $H(\pi) = H(P)$ and*

(3.13) $$\pi(f) = \int_X f\, dP, \quad f \in C_0(X).$$

If π_i corresponds to P_i, $i = 1,2$, then $R(\pi_1,\pi_2) = R(P_1,P_2)$.

PROOF. For each P we can take $\pi = \pi_P^0$ as in Prop. 3.5. Let us show that the correspondence $P \to \pi$ is one-to-one. Let P and Q be projection-valued measures on X acting in H such that $\int_X f\, dP = \int_X f\, dQ$ for all $f \in C_0(X)$. Then $\int_X f\, dP_{v,v} = \int_X f\, dQ_{v,v}$ for all $f \in C_0(X)$ and $v \in H$. Hence $P_{v,v} = Q_{v,v}$ for all $v \in H$ (cf. V.2.7). This implies that $P = Q$.

The last statement of the theorem was already proved in Prop. 3.5. Thus Proposition VII.2.5 applies to the mapping $P \to \pi$. We now show that to each nondegenerate representation π of $C_0(X)$ there corresponds a projection-valued measure P on X such that (3.13) holds. It follows from Theorem 3.4 that π is equivalent under some intertwining isometry A to a direct sum

$\pi_{\mu_1} \oplus \pi_{\mu_2} \oplus \ldots$, where the μ_i's are certain Borel measures on X. Let P_i be defined by (3.11) with $\mu = \mu_i$. Then π_{μ_i} corresponds to P_i according to (3.12). Let $P := \sum_i^{\oplus} P_i$ and let Q be equivalent to P under the intertwining isometry A. Then it follows from Theorem VII.2.5 that π corresponds to Q. □

4. THE RELATIONSHIP BETWEEN REPRESENTATIONS OF G AND $L^1(G)$

In this section we assume that G is an arbitrary lcsc. group, not necessarily abelian. Let dx (or sometimes $d\nu(x)$) be a left Haar measure on G and denote the modular function on G by Δ (cf. V.3.7).

First we derive some results about the convolution product $f*g$, where $f \in L^1(G)$ and $g \in L^p(G)$, $1 \le p < \infty$. As a reference we mention LOOMIS [8,§31]. We only need the results in the cases p = 1 and 2, but the general case does not involve any additional difficulties. We always consider $L^p(G)$ with respect to the left Haar measure dx.

If f is any function on G and $x \in G$ then $\lambda(x)f$ is the function on G defined by

$$(4.1) \qquad (\lambda(x)f)(y) := f(x^{-1}y), \quad y \in G.$$

λ is a homomorphism: $\lambda(xy)f = \lambda(x)(\lambda(y)f)$, $x,y \in G$, and $\lambda(e)f = f$. If $f \in L^p(G)$ then $\lambda(x)f \in L^p(G)$ for all $x \in G$ and

$$(4.2) \qquad \|\lambda(x)f\|_p = \|f\|_p$$

by the left invariance of the Haar measure. Here

$$\|f\|_p := \left(\int_G |f(x)|^p \, dx\right)^{1/p}$$

denotes the norm on $L^p(G)$.

<u>LEMMA 4.1</u>. *For each* $f \in L^p(G)$ *the mapping* $x \to \lambda(x)f$ *is continuous from* G *to* $L^p(G)$.

<u>PROOF</u>. Since $\lambda(x)f = \lambda(xx_0^{-1})(\lambda(x_0)f)$, continuity at x_0 follows from continuity at e. First suppose that $f \in \mathcal{K}(G)$. Then f is uniformly continuous on G. Let V be a compact neighbourhood of e. Then the set $K := V\cdot\text{supp}(f)$ is compact. Let $\varepsilon > 0$. By uniform continuity there is a neighbourhood $U \subset V$ of e such that $|f(x^{-1}y) - f(y)| < \varepsilon$ if $x \in U$, $y \in G$. Hence, if $x \in U$ then

$$\|\lambda(x)f-f\|_p = (\int_G |f(x^{-1}y)-f(y)|^p dy)^{1/p} \le \varepsilon(\nu(K))^{1/p}.$$

This shows the continuity of $x \to \lambda(x)f$ at e. Now let $f \in L^p(G)$. Let $\varepsilon > 0$. By density of $K(G)$ in $L^p(G)$ (cf. V.2.10) we can find $g \in K(G)$ such that $\|f-g\| < \frac{1}{3}\varepsilon$ for x in some neighbourhood U of e. Then, using (4.2), we find $\|\lambda(x)f-f\|_p < \varepsilon$ if $x \in U$. □

Thus λ is a strongly continuous homomorphism from G into the group of linear isometries of $L^p(G)$. It is called the *left regular representation* of G on $L^p(G)$. If $p = 2$ then λ is a unitary representation.

Now let $f \in L^1(G)$, $g \in L^p(G)$. Define the *convolution product*

$$f*g := \int_G f(x)\ \lambda(x)g\ dx, \tag{4.3}$$

where the right-hand side is a $L^p(G)$-valued integral. We claim that the function $x \to f(x)\ \lambda(x)g : G \to L^p(G)$ is Bochner integrable, hence also weakly integrable (cf. V.1.14). Thus $f*g$ is well-defined by (4.3). For the proof of the Bochner integrability first suppose that $f \in K(G)$. Then $x \to f(x)\ \lambda(x)g : G \to L^p(G)$ is continuous with compact support (cf. Lemma 4.1), hence it is Bochner integrable (cf. V.2.14). Now assume that $f \in L^1(G)$. Let $\{f_n\}$ be a sequence in $K(G)$ such that $\|f-f_n\|_1 \to 0$ as $n \to \infty$. Now

$$\|f(x)\ \lambda(x)g - f_n(x)\ \lambda(x)g\|_p = |f(x)-f_n(x)|\,\|\lambda(x)g\|_p =$$

$$= |f(x)-f_n(x)|\,\|g\|_p$$

is a Borel function of x and

$$\int_G \|f(x)\ \lambda(x)g - f_n(x)\ \lambda(x)g\|_p\ dx = \|f-f_n\|_1\|g\|_p \to 0 \quad \text{as } n \to \infty.$$

Hence it follows from V.1.14 that $x \to f(x)\ \lambda(x)g$ is Bochner integrable.

By an application of inequality (V.1.9) we obtain

$$\|f*g\|_p \le \int_G \|f(x)\ \lambda(x)g\|_p\ dx = \|f\|_1\|g\|_p, \quad f \in L^1(G),\ g \in L^p(G). \tag{4.4}$$

Next we prove that

$$(f*g)(y) = \int_G f(x)g(x^{-1}y)dx \tag{4.5}$$

for almost all $y \in G$ if $f \in L^1(G)$, $g \in L^p(G)$. We use the weak integrability of $x \to f(x)\ \lambda(x)g$. Indeed, (4.3) is equivalent to

$$\langle f*g,h\rangle = \int_G \langle f(x)\ \lambda(x)g,h\rangle\ dx$$

for all $h \in L^q(G)$ where $p^{-1} + q^{-1} = 1$ and

$$\langle \phi,h\rangle := \int_G \phi(x)\ h(x)\ dx, \quad \phi \in L^p(G),\ h \in L^q(G).$$

Thus

$$\int_G (f*g)(y)\ h(y)\ dy = \int_G\left(\int_G f(x)\ g(x^{-1}y)\ h(y)\ dy\right) dx$$
$$= \int_G\left(\int_G f(x)\ g(x^{-1}y)\ dx\right) h(y)\ dy,$$

where we used Fubini's theorem V.1.12. Part of the statement is that

$$\int_G |f(x)\ g(x^{-1}y)\ h(y)|\ dx < \infty$$

for almost all $y \in G$. Since the above results hold for all $h \in L^q(G)$, we conclude that for almost all $y \in G$

$$\int_G |f(x)\ g(x^{-1}y)|\ dx < \infty$$

and (4.5) holds.

Now we have settled these things, it is an elementary exercise to prove the following two theorems. (Use formula (V.3.19) for the proofs of the statements involving the involution.)

THEOREM 4.2. $L^1(G)$ *is a separable involutive Banach algebra with respect to the convolution product* (4.5) *and the involution*

(4.6) $$f^*(x) := \overline{f(x^{-1})}\ \Delta(x^{-1}), \quad f \in L^1(G),\ x \in G.$$

THEOREM 4.3. *The formula*

(4.7) $$\tilde{\lambda}(f)g := f*g, \quad f \in L^1(G),\ g \in L^2(G),$$

defines a nondegenerate representation $\tilde{\lambda}$ *of* $L^1(G)$ *on* $L^2(G)$. *Furthermore,*

$\tilde{\lambda}: L^1(G) \to L(L^2(G))$ *is one-to-one.*

PROOF. We only show nondegeneracy and injectivity of $\tilde{\lambda}$. First suppose that for some $g \in L^2(G)$ we have $\tilde{\lambda}(f)g = 0$ for all $f \in L^1(G)$. This means in particular that

$$0 = (\tilde{\lambda}(f)g,g) = (\int_G f(x)\ \lambda(x)g\ dx,g) =$$

$$= \int_G f(x)(\lambda(x)g,g)dx \quad \text{for all } f \in K(G).$$

Since $(\lambda(x)g,g)$ is continuous in x (cf. Lemma 4.1), it must be identically zero. Thus $(g,g) = 0$, i.e., $g = 0$.

Next suppose for some $f \in L^1(G)$ we have $\tilde{\lambda}(f)g = 0$ for all $g \in L^2(G)$. If, in particular, $g \in K(g)$ then

$$(\tilde{\lambda}(f)g)(y) = \int_G f(x)g(x^{-1}y)dx$$

is continuous in y, because g is uniformly continuous on G. Hence, for all $g \in K(G)$ we have

$$0 = \int_G f(x)g(x^{-1})dx.$$

This implies that $f = 0$ (apply V.2.13). □

We need one further technical result, before we can discuss the main theorem of this section. An *approximate identity* in a separable Banach algebra A is a sequence $\{u_n\}$ in A such that $\|u_n\| \le 1$ for all n and $\|u_n x - x\| \to 0$ and $\|xu_n - x\| \to 0$ as $n \to \infty$ for all $x \in A$.

LEMMA 4.4. *Let G be a lcsc. group. Then $L^1(G)$ has an approximate identity $\{u_n\}$. The functions u_n can be chosen as elements of $K(G)$ such that $u_n \ge 0$, $u_n^* = u_n$, $\|u_n\| = 1$.*

PROOF. Let $U_1 \supset U_2 \supset U_3 \supset \ldots$ be a sequence of compact neighbourhoods of e such that $\{U_n\}$ forms a base of neighbourhoods for e. For each n choose $w_n \in K(G)$ such that $w_n \ge 0$, $w_n(e) > 0$, $\text{supp}(w_n) \subset U_n$ and $\text{supp}(w_n^*) \subset U_n$. Let $v_n := \frac{1}{2}(w_n + w_n^*)$ and $u_n := v_n / \int_G v_n(x)dx$. Then $u_n \in K(G)$, $u_n \ge 0$, $u_n^* = u_n$, $\|u_n\| = 1$. Let $f \in L^1(G)$. Then

$$\|u_n*f-f\| = \|\int_G u_n(x)(\lambda(x)f-f)dx\| \le$$

$$\le \sup_{x\in U_n} \|\lambda(x)f-f\| \to 0 \quad \text{as } n\to\infty,$$

where we used Lemma 4.1. Also

$$\|f*u_n-f\| = \|u_n^**f^*-f^*\| = \|u_n*f^*-f^*\| \to 0. \qquad \square$$

We can give the following interpretation to Theorem 4.3: we associated with the left regular representation λ of G on $L^2(G)$ a representation $\tilde{\lambda}$ of $L^1(G)$ on $L^2(G)$ by means of the vector-valued integral

$$\tilde{\lambda}(f)g = \int_G f(x)\ \lambda(x)g\ dx, \quad f\in L^1(G),\ g\in L^2(G),$$

or, equivalently, by the operator-valued integral

$$\tilde{\lambda}(f) = \int_G f(x)\ \lambda(x)\ dx, \quad f\in L^1(G).$$

In a similar way we can associate with any unitary representation π of G a representation $\tilde{\pi}$ of $L^1(G)$ on $H = H(\pi)$:

$$(4.8) \qquad \tilde{\pi}(f) = \int_G f(x)\ \pi(x)\ dx, \quad f\in L^1(G).$$

Here $\tilde{\pi}(f)$ is the unique element of $L(H)$ such that

$$(4.9) \qquad (\tilde{\pi}(f)v,w) = \int_G f(x)(\pi(x)v,w)dx$$

for all $v,w \in H$. The existence of $\tilde{\pi}(f)$ is guaranteed by V.1.16, since $x \to f(x)(\pi(x)v,w)$ is a Borel function on G for all $v,w \in H$ and since $x \to \|f(x)\ \pi(x)\| = |f(x)|$ is in $L^1(G)$.

It will be shown in the proof of the theorem below that $\tilde{\pi}$ is actually a representation of $L^1(G)$. Here we make the preliminary observation that

$$(4.10) \qquad \pi(x)\tilde{\pi}(f) = \tilde{\pi}(\lambda(x)f), \quad f\in L^1(G),\ x\in G.$$

Indeed, if $v,w \in H$ then

$$(\pi(x)\tilde{\pi}(f)v,w) = (\tilde{\pi}(f)v,\pi(x^{-1})w) = \int_G f(y)(\pi(y)v,\pi(x^{-1})w)\,dy =$$

$$= \int_G f(y)(\pi(xy)v,w)\,dy =$$

$$= \int_G f(x^{-1}y)(\pi(y)v,w)\,dy = (\tilde{\pi}(\lambda(x)f)v,w).$$

<u>THEOREM 4.5</u> (cf. LOOMIS [8,§32], DIXMIER [4,§13.3]). *Formula* (4.8) *establishes a one-to-one correspondence between the unitary representations* π *of* G *and the nondegenerate representations* $\tilde{\pi}$ *of* $L^1(G)$.

<u>PROOF</u>. First we assume that π is a unitary representation of G on H and we show that $\tilde{\pi}$, defined by (4.8), is a nondegenerate representation of $L^1(G)$. Linearity of $\tilde{\pi}$ is obvious. If $f,g \in L^1(G)$, $v,w \in H$, then

$$(\tilde{\pi}(f*g)v,w) = \int_G\left(\int_G f(x)g(x^{-1}y)(\pi(y)v,w)\,dx\right)dy =$$

$$= \int_G f(x)\left(\int_G f(x^{-1}y)(\pi(y)v,w)\,dy\right)dx =$$

$$= \int_G f(x)(\tilde{\pi}(\lambda(x)g)v,w)\,dx =$$

$$= \int_G f(x)(\pi(x)\ \tilde{\pi}(g)v,w)\,dx = (\tilde{\pi}(f)\tilde{\pi}(g)v,w),$$

where we used Fubini's theorem and formula (4.10). Hence $\tilde{\pi}$ is a multiplicative homomorphism. Next, if $f \in L^1(G)$, $v,w \in H$, then

$$(\tilde{\pi}(f^*)v,w) = \int_G \overline{f(x^{-1})}\ (\pi(x)v,w)\ \Delta(x^{-1})\,dx =$$

$$= \int \overline{f(x)}\quad (\pi(x^{-1})v,w)\,dx =$$

$$= \int_G \overline{f(x)}\ \overline{(\pi(x)w,v)}\,dx =$$

$$= \overline{(\tilde{\pi}(f)w,v)} = ((\tilde{\pi}(f))^*v,w).$$

This shows that $\tilde{\pi}$ is a $*$-homomorphism. Now we prove that $\tilde{\pi}$ is nondegenerate. Let $v \in H$ and suppose that $\tilde{\pi}(f)v = 0$ for all $f \in L^1(G)$. We have to show that $v = 0$. Let $\varepsilon > 0$. Let U be a neighbourhood of e such that $\|\pi(x)v-v\| < \varepsilon$ if $x \in U$. Choose $f \in L^1(G)$ such that $f \geq 0$, $\mathrm{supp}(f) \subset U$ and $\|f\| = 1$. Let

$w \in H$. Then

$$|(v,w)| = |(\tilde{\pi}(f)v-v,w)| =$$

$$= |\int_G f(x)(\pi(x)v-v,w)dx| \le$$

$$\le (\int_G f(x)dx)(\sup_{x \in U} \|\pi(x)v-v\|)\|w\| \le \varepsilon\|w\|.$$

Hence $\|v\| \le \varepsilon$ for all $\varepsilon > 0$, i.e., $v = 0$.

We have shown that, for given π, $\tilde{\pi}$ defined by (4.8) is a nondegenerate representation of $L^1(G)$. Let H_0 be the linear span of $\{\tilde{\pi}(f)v \mid f \in L^1(G), v \in H\}$. Then H_0 is dense in H. Thus, for $x \in G$, $\pi(x)$ is completely determined by its restriction to H_0. Formula (4.10) implies that $\pi(x)|_{H_0}$ can be recovered from $\tilde{\pi}$:

$$\pi(x)(\sum_{i=1}^{m} \tilde{\pi}(f_i)v_i) = \sum_{i=1}^{m} \tilde{\pi}(\lambda(x)f_i)v_i.$$

Let now ρ be a homomorphism (but not necessarily a $*$-homomorphism) from $L^1(G)$ into $L(H)$ such that $\|\rho(f)\| \le \|f\|$ for all $f \in L^1(G)$ and the linear span H_0 of $\{\rho(f)v \mid f \in L^1(G), v \in H\}$ is dense in H. If $\sum_{i=1}^{m} \rho(f_i)v_i$ is an element of H_0 then, with the use of the approximate identity $\{u_n\}$ (cf. Lemma 4.4), we have for $x \in G$:

$$\sum_{i=1}^{m} \rho(\lambda(x)f_i)v_i =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{m} \rho(\lambda(x)(u_n*f_i))v_i =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{m} \rho((\lambda(x)u_n)*f_i)v_i =$$

$$= \lim_{n\to\infty} \rho(\lambda(x)u_n) \sum_{i=1}^{m} \rho(f_i)v_i.$$

Since $\|\rho(\lambda(x)u_n)\| \le \|\lambda(x)u_n\| = \|u_n\| = 1$, it follows that

$$\|\sum_{i=1}^{m} \rho(\lambda(x)f_i)v_i\| \le \|\sum_{i=1}^{m} \rho(f_i)v_i\|.$$

We conclude that, for a given ρ, the formula

(4.11) $$\pi(x)\left(\sum_{i=1}^{m} \rho(f_i)v_i\right) = \sum_{i=1}^{m} \rho(\lambda(x)f_i)v_i$$

unambiguously defines, for each $x \in G$, a linear mapping $\pi(x): H_0 \to H_0$ such that $\|\pi(x)v\| \le \|v\|$ for each $v \in H_0$. Hence $\pi(x)$ uniquely extends to a bounded linear operator on H with $\|\pi(x)\| \le 1$.

We conlude the proof of the theorem by showing that the mapping $\pi: G \to L(H)$, defined by (4.11), is a unitary representation of G and that the representation $\tilde{\pi}$ of $L^1(G)$, defined by (4.8), equals the original homomorphism ρ. Let $f \in L^1(G)$, $v \in H$, $x,y \in G$. Then

$$\pi(xy)\rho(f)v = \rho(\lambda(xy)f)v = \rho(\lambda(x)\lambda(y)f)v =$$

$$= \pi(x)\rho(\lambda(y)f)v = \pi(x)\pi(y)\rho(f)v.$$

Hence $\pi(xy)w = \pi(x)\pi(y)w$ for all $w \in H_0$ and thus for all $w \in H$. Clearly (4.11) implies that $\pi(e) = I$. Thus $\pi(x^{-1}) = (\pi(x))^{-1}$ for all $x \in G$. If $x \in G$, $v \in H$ then

$$\|\pi(x)v\| \le \|v\| = \|\pi(x^{-1})\pi(x)v\| \le \|\pi(x)v\|.$$

Hence $\|\pi(x)v\| = \|v\|$, i.e., $\pi(x)$ is a unitary transformation. In order to prove strong continuity of π note that $x \to \pi(x)\rho(f)v = \rho(\lambda(x)f)v$ is continuous from G to H for all $f \in L^1(G)$, $v \in H$ (cf. Lemma 4.1). Hence $x \to \pi(x)w$ is continuous for all $w \in H_0$ and thus for all $w \in H$. We have shown now that π is a unitary representation of G. Finally we will show that $\tilde{\pi} = \rho$. Let $f,g \in L^1(G)$, $v,w \in H$. Since

$$\int_G f(x)\ \lambda(x)g\ dx = f*g,$$

we have

$$\int_G f(x)\ \rho(\lambda(x)g)\ dx = \rho(f*g) = \rho(f)\rho(g),$$

cf. (V.1.8). Hence

$$(\tilde{\pi}(f)\rho(g)v,w) = \int_G f(x)\,(\pi(x)\rho(g)v,w)\,dx =$$

$$= \int_G f(x)\,(\rho(\lambda(x)g)v,w)\,dx = (\rho(f)\rho(g)v,w).$$

This shows that $\tilde{\pi}(f)\rho(g)v = \rho(f)\rho(g)v$, and finally $\tilde{\pi}(f) = \rho(f)$. □

As a corollary to the proof of the preceding theorem we obtain

LEMMA 4.6. *Let* ρ *be a homomorphism from* $L^1(G)$ *into* $L(H)$ *such that* $\|\rho\| \leq 1$ *and the linear span of* $\{\rho(f)v \mid f \in L^1(G),\ v \in H\}$ *is dense in* H. *Then* ρ *is a* $*$*-homomorphism, so* ρ *is a nondegenerate representation of* $L^1(G)$ *on* H.

We conclude this section with

THEOREM 4.7. *If* π_1 *and* π_2 *are unitary representations of* G *and* $\tilde{\pi}_1$, $\tilde{\pi}_2$ *are the corresponding representations of* $L^1(G)$, *defined by* (4.8), *then* $R(\pi_1,\pi_2) = R(\tilde{\pi}_1,\tilde{\pi}_2)$.

PROOF. Let $A \in R(\pi_1,\pi_2)$, $f \in L^1(G)$, $v \in H(\pi_1)$, $w \in H(\pi_2)$. Then

$$(A\tilde{\pi}_1(f)v,w) = (\tilde{\pi}_1(f)v,A^*w) = \int_G f(x)(\pi_1(x)v,A^*w)dx =$$

$$= \int_G f(x)(\pi_2(x)Av,w)dx = (\tilde{\pi}_2(f)Av,w).$$

Hence $A\tilde{\pi}_1(f) = \tilde{\pi}_2(f)A$. Conversely, let $A \in R(\tilde{\pi}_1,\tilde{\pi}_2)$, $f \in L^1(G)$, $v \in H$, $x \in G$. Then

$$A\pi_1(x)\tilde{\pi}_1(f)v = A\tilde{\pi}_1(\lambda(x)f)v = \tilde{\pi}_2(\lambda(x)f)Av =$$

$$= \pi_2(x)\tilde{\pi}_2(f)Av = \pi_2(x)A\tilde{\pi}_1(f)v.$$

Hence $A\pi_1(x)w = \pi_2(x)Aw$ for w in the closure H of the linear span of $\{\tilde{\pi}(f)v \mid f \in L^1(G),\ v \in H\}$. □

Thus the mapping $\pi \to \tilde{\pi}$ satisfies the conditions of Prop. VII. 2.5 and subrepresentations, equivalences and direct sums in the case of representations of G nicely correspond to similar relationships between representations of $L^1(G)$. It follows, for instance, that π is irreducible, multiplicity free or of type I iff $\tilde{\pi}$ has these properties, respectively.

5. PRELIMINARIES ABOUT LCSC. ABELIAN GROUPS

Reference for this section is LOOMIS [8, § 34]. Let G be a lcsc. abelian group and let $\hat{G}$ denote the set of all irreducible unitary representations of G. Then $\hat{G}$ is just the set of all continuous homomorphisms from G into the multiplicative group $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$, cf. Theorem VII. 4.9. In a natural way, $\hat{G}$ becomes an abelian group with respect to the product

$$(5.1) \qquad (\alpha\beta)(x) := \alpha(x)\beta(x), \quad \alpha,\beta \in \hat{G},\ x \in G.$$

We call $\hat{G}$ the *dual group* to G.

Since G is abelian, the involutive Banach algebra $L^1(G)$ is commutative. It follows from Lemma 4.6 that each multiplicative linear functional α on $L^1(G)$ satisfies $\alpha(f^*) = \overline{\alpha(f)}$, for all $f \in L^1(G)$. Thus the structure space X of $L^1(G)$ consists of all irreducible representations of $L^1(G)$. Application of Theorem 4.5 and Theorem 4.7 yields a one-to-one correspondence $\alpha \leftrightarrow \tilde{\alpha}$ between the irreducible representations of G and $L^1(G)$, respectively. Therefore we can state the following theorem.

THEOREM 5.1. *The formula*

$$\tilde{\alpha}(f) = \int_G f(x)\alpha(x)dx, \quad f \in L^1(G),$$

establishes a one-to-one correspondence $\alpha \leftrightarrow \tilde{\alpha}$ *between* $\hat{G}$ *and the structure space of* $L^1(G)$.

Using the above correspondence we identify the structure space of $L^1(G)$ with $\hat{G}$. Thus $\hat{G}$ becomes a lcsc. space under the Gelfand topology (cf. Lemma 2.7) and the Gelfand transform for $L^1(G)$ takes the form

$$(5.2) \qquad \hat{f}(\alpha) = \int_G f(x)\alpha(x)dx, \quad f \in L^1(G),\ \alpha \in \hat{G}.$$

The function $\hat{f}$ on $\hat{G}$ is called the *Fourier transform* of $f \in L^1(G)$. (In literature $\hat{f}(\alpha)$ is often defined with $\alpha(x)$ at the right hand side of (5.2) being replaced by $\alpha(x^{-1})$.)

Next we prove some results which culminate into the statement that the group structure and the topology on $\hat{G}$ are compatible.

PROPOSITION 5.2. *The function* $(x,\alpha) \to \alpha(x)$ *is continuous on* $G \times \hat{G}$.

PROOF. Let $\alpha_0 \in \hat{G}$ and choose $f \in L^1(G)$ such that $\hat{f}(\alpha_0) \neq 0$. (This is possible because the algebra $\{\hat{f} \mid f \in L^1(G)\}$ separates the points of $\hat{G}$, cf. Theorem 2.2(b).) If $x \in G$, $\alpha \in \hat{G}$ then $\alpha(x)\tilde{\alpha}(f) = \tilde{\alpha}(\lambda(x)f)$, cf. (4.10). Hence, in some neighbourhood U of α_0 where $\hat{f}(\alpha) \neq 0$, we have

$$\alpha(x) = \frac{(\lambda(x)f)^\wedge(\alpha)}{\hat{f}(\alpha)}. \tag{5.3}$$

The denominator at the right hand side of (5.3) is continuous in α at α_0. The numerator is continuous in (x,α) at (x_0,α_0) for each $x_0 \in G$, since

$$|(\lambda(x)f)^\wedge(\alpha) - (\lambda(x_0)f)^\wedge(\alpha_0)| \leq$$

$$|(\lambda(x)f)^\wedge(\alpha) - (\lambda(x_0)f)^\wedge(\alpha)| + |(\lambda(x_0)f)^\wedge(\alpha) - (\lambda(x_0)f)^\wedge(\alpha_0)| \leq$$

$$\leq \|\lambda(x)f - \lambda(x_0)f\| + |(\lambda(x_0)f)^\wedge(\alpha) - (\lambda(x_0)f)^\wedge(\alpha_0)|.$$

Now use Lemma 4.1. □

In the next proposition we characterize the topology of $\hat{G}$ in terms of G rather than of $L^1(G)$.

PROPOSITION 5.3. *The sets*

$$U(K,\varepsilon,\alpha_0) := \{\alpha \in \hat{G} \mid |\alpha(x) - \alpha_0(x)| < \varepsilon \text{ if } x \in K\}, \tag{5.4}$$

where $K \subset G$ *is compact,* $\varepsilon > 0$ *and* $\alpha_0 \in \hat{G}$, *form a base for the open sets of* $\hat{G}$.

PROOF. First observe that if $\alpha_0 \in U(K_1,\varepsilon_1,\alpha_1) \cap U(K_2,\varepsilon_2,\alpha_2)$ then $U(K_1 \cup K_2,\varepsilon,\alpha_0) \subset U(K_1,\varepsilon_1,\alpha_1) \cap U(K_2,\varepsilon_2,\alpha_2)$ for ε small enough, and that the sets $U(K,\varepsilon,\alpha_0)$ cover $\hat{G}$. Hence the sets $U(K,\varepsilon,\alpha_0)$ form a base of open sets for some topology on $\hat{G}$.

Next we show that any set $U(K,\varepsilon,\alpha_0)$ is open with repect to the Gelfand topology on $\hat{G}$. It is sufficient to prove that α_0 is an interior point of $U(K,\varepsilon,\alpha_0)$, because, if $\alpha \in U(K,\varepsilon,\alpha_0)$ then $U(K,\varepsilon_1,\alpha) \subset U(K,\varepsilon,\alpha_0)$ for some $\varepsilon_1 > 0$. Now, by continuity of the function $(x,\alpha) \to \alpha(x)$ (cf. Prop. 5.2) and by compactness of K we can find a neighbourhood V (with respect to the Gelfand topology) of α_0 in $\hat{G}$ such that $|\alpha(x) - \alpha_0(x)| < \varepsilon$ if $\alpha \in V$, $x \in K$. Hence $V \subset U(K,\varepsilon,\alpha_0)$.

Finally we show that any neighbourhood $\{\alpha \in \hat{G} \mid |\hat{f}(\alpha) - \hat{f}(\alpha_0)| < \delta\}$ of α_0 in $\hat{G}$ ($f \in L^1(G)$, $\alpha_0 \in \hat{G}$, $\delta > 0$) includes a set $U(K,\varepsilon,\alpha_0)$ for some compact $K \subset G$ and $\varepsilon > 0$. This will imply that the topology generated by the sets $U(K,\varepsilon,\alpha_0)$ is finer than the Gelfand topology. Fix $f \in L^1(G)$, $\alpha_0 \in \hat{G}$, $\delta > 0$. Then for each compact $K \subset G$, $\varepsilon > 0$ and $\alpha \in U(K,\varepsilon,\alpha_0)$ we have:

$$|\hat{f}(\alpha) - \hat{f}(\alpha_0)| \le \Big(\int_K + \int_{G\setminus K}\Big) |f(x)|\,|\alpha(x) - \alpha_0(x)|\,dx$$

$$\le \varepsilon\|f\| + 2\int_{G\setminus K} |f(x)|\,dx$$

Now put $\varepsilon := \frac{1}{2}\delta/\|f\|$ and choose K such that $\int_{G\setminus K} |f(x)|\,dx \le \frac{1}{4}\delta$. Then $|\hat{f}(\alpha) - \hat{f}(\alpha_0)| < \delta$ if $\alpha \in U(K,\varepsilon,\alpha_0)$. □

It follows from Prop. 5.3 that a sequence $\{\alpha_n\}$ on $\hat{G}$ converges to $\alpha_0 \in \hat{G}$ iff $\alpha_n(x) \to \alpha_0(x)$ uniformly on compact subsets of $\hat{G}$.

<u>THEOREM 5.4</u>. *The group operations on* $\hat{G}$ *are continuous, so* $\hat{G}$ *is a lcsc. abelian group.*

<u>PROOF</u>. Consider again the open sets $U(K,\varepsilon,\alpha_0)$ defined by (5.4), which form a base for the topology of $\hat{G}$. Since $U(K,\varepsilon,\alpha_0) = U(K^{-1},\varepsilon,\alpha_0^{-1})$, the mapping $\alpha \to \alpha^{-1}: \hat{G} \to \hat{G}$ is continuous. Next observe that

$$|\alpha(x)\beta(x) - \alpha_0(x)\beta_0(x)| \le |\alpha(x) - \alpha_0(x)| + |\beta(x) - \beta_0(x)|.$$

Hence $\alpha\beta \in U(K,2\varepsilon,\alpha_0\beta_0)$ if $\alpha \in U(K,\varepsilon,\alpha_0)$ and $\beta \in U(K,\varepsilon,\beta_0)$. This shows the continuity of the mapping $(\alpha,\beta) \to \alpha\beta: \hat{G} \times \hat{G} \to \hat{G}$. □

<u>EXAMPLE 5.5</u>. Let $G = \mathbb{R}$, the additive group of real numbers. We will investigate the dual group $\hat{G}$.

First we show that each continuous homomorphism α from $\mathbb{R}$ into the multiplicative group $\mathbb{C}\setminus\{0\}$ has the form $\alpha(x) = e^{cx}$ for some $c \in \mathbb{C}$. Define

$$f(x) := x^{-1} \log \alpha(x), \quad x > 0,$$

where the logarithm is chosen such that the function $x \to \log \alpha(x)$ is continuous on $\mathbb{R}$ and takes the value 0 for $x = 0$. Let $n \in \mathbb{N}$. For each $x > 0$ we have

$$f(nx) = \frac{\log \alpha(nx)}{nx} = \frac{\log(\alpha(x))^n}{nx} = \frac{n \log \alpha(x) + 2k_x \pi i}{nx} = f(x) + \frac{2k_x \pi i}{nx}$$

for some $k_x \in \mathbb{Z}$ depending on x. Thus $x \to k_x$ is continuous and integer-valued, hence constant. Since $k_x = 0$ for small $x > 0$, we have $f(nx) = f(x)$ for all $n < \mathbb{N}$, $x > 0$. It follows that $f(q) = f(1)$ for all positive rational numbers q. By continuity of f, $f(x) = x$ on $\mathbb{R}^+$ for some constant $c \in \mathbb{C}$. Hence $\alpha(x) = e^{cx}$ for $x > 0$, and also for other real values of x, since α is a homomorphism.

If $\alpha \in \hat{G}$ and $\alpha(x) = e^{cx}$ then the condition $|\alpha(x)| = 1$ forces c to be purely imaginary, i.e. $\alpha(x) = e^{i\lambda x}$ for some $\lambda \in \mathbb{R}$. On the other hand, any function $x \to e^{i\lambda x}$ $(\lambda \in \mathbb{R})$ is a continuous homomorphism from $\mathbb{R}$ into $\mathbb{T}$. We identify $\hat{G}$ with $\mathbb{R}$ such that $\lambda(x) := e^{i\lambda x}$ if $\lambda \in \mathbb{R}$.

Finally we prove that the topology of $\hat{G}$ coincides with the usual topology of $\mathbb{R}$. Let $\lambda_0 \in \mathbb{R}$. It follows from Prop. 5.3 that the sets $U([-n,n],\varepsilon,\lambda_0)$, $n \in \mathbb{N}$, $\varepsilon > 0$, form a base of neighbourhoods of λ_0 in the topology of $\hat{G}$. Now:

$$|e^{i\lambda x} - e^{i\lambda_0 x}| < \varepsilon \text{ if } |x| \le n \iff$$

$$\iff 4\sin^2 \tfrac{1}{2}(\lambda-\lambda_0)x < \varepsilon^2 \text{ if } |x| \le n \iff$$

$$\iff |\lambda-\lambda_0| < 2n^{-1} \arcsin(\tfrac{1}{2}\varepsilon),$$

where we assumed that $\varepsilon < \pi$. Hence the above base of neighbourhoods of λ_0 is also a base of the neighbourhoods of λ_0 in the ordinary topology of $\mathbb{R}$.

The following theorem, in particular the statement about the density, is of great importance in the next section.

THEOREM 5.6. *The Fourier transform* $f \to \hat{f}$, *defined by* (5.2), *is a one-to-one* $*$*-homomorphism from* $L^1(G)$ *onto a dense* $*$*-subalgebra of* $C_o(\hat{G})$.

PROOF. We already know that $f \to \hat{f}$ is a $*$-homomorphism. It follows from Lemma 2.5 that its image is dense in $C_0(\hat{G})$. In order to prove injectivity of the Fourier transform, suppose that $f \in L^1(G)$, $\hat{f} = 0$. Let $\tilde{\lambda}$ be the representation of $L^1(G)$ defined by (4.7). It follows from (2.9) that $\|\tilde{\lambda}(f)\| \le \|\hat{f}\| = 0$, hence $\tilde{\lambda}(f) = 0$. Now injectivity of $\tilde{\lambda}$ (cf. Theorem 4.3) shows that $f = 0$. □

6. PUTTING THE PIECES TOGETHER: UNITARY REPRESENTATIONS OF LCSC. ABELIAN GROUPS

Let G be a lcsc. abelian group. In this section we will classify the unitary representations of G up to equivalence and we will prove the SNAG theorem. The first step will be to connect the representations of $L^1(G)$ with those of $C_0(\hat{G})$.

THEOREM 6.1. *Let* G *be a lcsc. abelian group. Then the formula*

$$(6.1) \qquad \tilde{\tilde{\pi}}(f) = \tilde{\pi}(f), \quad f \in L^1(G),$$

establishes a one-to-one correspondence between the nondegenerate representations $\tilde{\pi}$ *of* $L^1(G)$ *and the nondegenerate representations* $\tilde{\tilde{\pi}}$ *of* $C_0(\hat{G})$ *such that* $H(\tilde{\pi}) = H(\tilde{\tilde{\pi}})$ *and* $R(\tilde{\pi}_1,\tilde{\pi}_2) = R(\tilde{\tilde{\pi}}_1,\tilde{\tilde{\pi}}_2)$.

PROOF. The mapping $f \to \hat{f}$ is a $*$-isomorphism from $L^1(G)$ onto a dense $*$-subalgebra of $C_0(\hat{G})$ (cf. Theorem 5.6) and $\|\tilde{\pi}(f)\| \le \|\hat{f}\|$, $f \in L^1(G)$, for each representation $\tilde{\pi}$ of $L^1(G)$. Hence $\tilde{\tilde{\pi}}$, defined by (6.1), uniquely extends to a representation of $C_0(\hat{G})$ on $H(\tilde{\pi})$. Conversely, if $\tilde{\tilde{\pi}}$ is a representation of $C_0(\hat{G})$ then $f \to \tilde{\tilde{\pi}}(\hat{f})$ is a representation of $L^1(G)$ on $H(\tilde{\tilde{\pi}})$. Clearly $\tilde{\pi}$ is nondegenerate iff $\tilde{\tilde{\pi}}$ is nondegenerate. Finally, the statements $R(\tilde{\pi}_1,\tilde{\pi}_2) \subset R(\tilde{\tilde{\pi}}_1,\tilde{\tilde{\pi}}_2)$ and $R(\tilde{\tilde{\pi}}_1,\tilde{\tilde{\pi}}_2) \subset R(\tilde{\pi}_1,\tilde{\pi}_2)$ follow by density and by restriction, respectively. □

Combination of Theorems 4.5 and 6.1 shows that the formulas (4.8) and (6.1) establish a one-to-one correspondence between the unitary representations π of G and the nondegenerate representations $\tilde{\tilde{\pi}}$ of $C_0(G)$. The correspondence is also given by the formula

$$(6.2) \qquad \tilde{\tilde{\pi}}(\hat{f}) = \int_G f(x)\pi(x)dx, \quad f \in L^1(G),$$

and we have $H(\pi) = H(\tilde{\tilde{\pi}})$, $R(\pi_1,\pi_2) = R(\tilde{\tilde{\pi}}_1,\tilde{\tilde{\pi}}_2)$. In view of Prop. VII.2.5 any property of a representation which can be defined in terms of direct sums, subrepresentations and equivalence, holds for π iff it holds for $\tilde{\tilde{\pi}}$.

Let μ be a finite Borel measure on $\hat{G}$ and let P be a projection-valued measure on $\hat{G}$ acting in H. Then $\tilde{\tilde{\pi}}_\mu$ and $\tilde{\tilde{\pi}}$, defined by

$$(6.3) \qquad (\tilde{\tilde{\pi}}_\mu(f)\xi)(\alpha) = f(\alpha)\xi(\alpha), \quad f \in C_0(\hat{G}),\ \xi \in L^2(\hat{G},\mu),\ \alpha \in \hat{G},$$

and

$$(6.4) \qquad \tilde{\tilde{\pi}}(f) = \int_{\hat{G}} f\, dP, \quad f \in C_0(\hat{G}),$$

are nondegenerate representations of $C_0(\hat{G})$ (cf. (3.1) and Prop. 3.5). We will show that the corresponding representations of G are given by

$$(6.5) \qquad (\pi_\mu(x)\xi)(\alpha) = \alpha(x)\xi(\alpha), \quad \xi \in L^2(\hat{G},\mu),\ x \in G,\ \alpha \in \hat{G},$$

and

$$(6.6) \qquad \pi(x) = \int_{\hat{G}} \hat{x}\, dP, \quad x \in G,$$

where $\hat{x}(\alpha) := \alpha(x)$, $\alpha \in \hat{G}$. First we show that π_μ and π, defined by (6.5) and (6.6), are indeed unitary representations of G.

Clearly π_μ is a homomorphism from G into the group of unitary transformations of $L^2(\hat{G},\mu)$. In order to prove weak continuity of π_μ at e let $\xi,\eta \in L^2(X,\mu)$ and let K be a compact subset of $\hat{G}$. Then

$$|(\pi_\mu(x)\xi,\eta) - (\xi,\eta)| \le \left(\int_K + \int_{\hat{G}\setminus K}\right) |\alpha(x) - \alpha(e)|\,|\xi(\alpha)|\,|\eta(\alpha)|\,d\mu(\alpha) \le$$

$$\le \|\xi\|\,\|\eta\| \sup_{\alpha\in K} |\alpha(x) - \alpha(e)| + 2\int_{\hat{G}\setminus K} |\xi(\alpha)|\,|\eta(\alpha)|\,d\mu(\alpha).$$

Let $\varepsilon > 0$. Choose K such that the second term becomes less than $\frac{1}{2}\varepsilon$. Choose a neighbourhood V of e such that $|\alpha(x) - \alpha(e)| < \frac{\varepsilon}{2\|\xi\|\|\eta\|}$ if $x \in V$, $\alpha \in K$. (cf. Prop. 5.2). Then

$$|(\pi_\mu(x)\xi,\eta) - (\xi,\eta)| < \varepsilon \quad \text{if} \quad x \in V.$$

Next consider π defined by (6.6). Since $\hat{x} \in B(\hat{G})$ (cf. Prop. 5.2) and $(xy)\hat{} = \hat{x}\hat{y}$ for $x,y \in G$, π is a well-defined homomorphism from G into $L(H(P))$ (cf. Prop. 3.5). Clearly $\pi(e) = I$. Since $(x^{-1})\hat{}(\alpha) = \overline{\hat{x}(\alpha)}$, $x \in G$, $\alpha \in \hat{G}$, we have $(\pi(x))^{-1} = \pi(x^{-1}) = (\pi(x))^*$ (cf. again Prop. 3.5). Thus $\pi(x)$ is a unitary operator for all $x \in G$. In order to prove weak continuity of π let $v,w \in H(P)$, $x \in G$. Then

$$|(\pi(x)v,w) - (v,w)| = |\int_{\hat{G}} (\alpha(x) - \alpha(e))dP_{v,w}(\alpha)| \leq$$

$$\leq \int_{\hat{G}} |\alpha(x) - \alpha(e)|d|P_{v,w}|(\alpha).$$

Now proceed as in the weak continuity proof for π_μ

Let us next show that π_μ and $\tilde{\tilde{\pi}}_\mu$, defined by (6.5) and (6.3), are actually connected with each other by (6.2). Indeed, let $f \in L^1(G)$, $\xi,\eta \in L^2(X,\mu)$. Then

$$\int_G f(x)(\pi_\mu(x)\xi,\eta)dx = \int_G \left(\int_{\hat{G}} f(x)\alpha(x)\xi(\alpha)\overline{\eta(\alpha)}du(\alpha)\right)dx =$$

$$= \int_{\hat{G}} \left(\int_G f(x)\alpha(x)dx\right)\xi(\alpha)\overline{\eta(\alpha)}d\mu(\alpha) = \int_{\hat{G}} \hat{f}(\alpha)\xi(\alpha)\overline{\eta(\alpha)}d\mu(\alpha) =$$

$$= (\tilde{\tilde{\pi}}_\mu(\hat{f})\xi,\eta),$$

where we used Fubini's theorem.

Similarly, the representations π and $\tilde{\tilde{\pi}}$, defined by (6.6) and (6.4), are connected with each other by (6.2), because, for $f \in L^1(G)$, $v,w \in \mathcal{H}(P)$, we have

$$\int_G f(x)(\pi(x)v,w)dx = \int_G \left(\int_{\hat{G}} f(x)\alpha(x)dP_{v,w}(\alpha)\right)dx =$$

$$= \int_{\hat{G}} \left(\int_G f(x)\alpha(x)dx\right)dP_{v,w}(\alpha) = \int_{\hat{G}} \hat{f}(\alpha)dP_{v,w}(\alpha) = (\tilde{\tilde{\pi}}(\hat{f})v,w).$$

Now we can conclude from Theorem 3.1, Cor. 3.2(b), Theorem 3.3 and Theorem 3.4 that:

<u>THEOREM 6.2</u>. *Let* G *be a lcsc. abelian group.*

(a) *A unitary representation of* G *is multiplicity free if and only if it is equivalent to some representation* π_μ *of* G *of the form* (6.5).

(b) $\mu \perp \nu$ *iff* $\pi_\mu \;\dot{\circ}\; \pi_\nu$;

$\mu << \nu$ *iff* $\pi_\mu \leq \pi_\nu$;

$\mu \equiv \nu$ *iff* $\pi_\mu \simeq \pi_\nu$.

(c) *Each unitary representation* π *of* G *is of type* I *and there are mutually singular, finite Borel measures* $\mu_1,\mu_2,\ldots,\mu_\infty$ *as on* G, *uniquely determined by* π *up to equivalence, such that*

$$\pi \approx \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus 3\pi_{\mu_3} \oplus \ldots \oplus \infty\pi_{\mu_\infty}.$$

Similarly we conclude from Theorem 3.6 that:

THEOREM 6.3. (SNAG theorem). *Let* G *be a lcsc. abelian group. Then formula* (6.6) *establishes a one-to-one correspondence between the projection-valued measures* P *on* $\hat{G}$ *and the unitary representations* π *of* G. *We have* $H(P) = H(\pi)$ *and* $R(P_1,P_2) = R(\pi_1,\pi_2)$.

7. DIRECT INTEGRALS

In this final section we shortly discuss direct integrals of representations. The results naturally fit into this chapter, since they are connected with the representation theory of commutative C^*-algebras, given in §3. A standard reference for direct integrals is DIXMIER [3, Ch. II, §1, §2], [4, §8, §18.7]. See also KIRILLOV [7, §4.5, §8.4], ARVESON [2, §4.2], VARADARAJAN [15, Ch. IX, §2].

7.1. Direct integrals of Hilbert spaces

Let X be a lcsc. space and let μ be a finite*) Borel measure on X. For each $\alpha \in X$ let H_α be a Hilbert space. First we consider the situation that each H_α is a copy of some fixed separable Hilbert space H^0. Remember (cf. V.1.17), that the Hilbert space $L^2(X,\mu;H^0)$ consists of all functions $f\colon X \to H^0$ which are weakly Borel and which satisfy

$$\text{(7.1)} \qquad \|f\|^2 := \int_X \|f(\alpha)\|^2_{H^0}\, d\mu(\alpha) < \infty.$$

Functions f_1 and f_2 for which $f_1(\alpha) = f_2(\alpha)$ a.e. [μ] (or, equivalently $\|f_1 - f_2\| = 0$) are identified with each other. The inner product is given by

*) Finiteness of μ is technically convenient. It means no loss of generality, since each Borel measure on a lcsc. space is equivalent to some finite measure.

$$(7.2)\qquad (f,g) := \int_X (f(\alpha),g(\alpha))_{H^0} d\mu(\alpha), \qquad f,g \in L^2(X,\mu;H^0).$$

We write

$$(7.3)\qquad \int_X^{\oplus} H_\alpha d\mu(\alpha) := L^2(X,\mu;H^0) \quad \text{if } H_\alpha = H^0 \quad \text{for all } \alpha \in X,$$

and we call this Hilbert space the *direct integral* of the Hilbert spaces H_α with respect to the measure μ. (In literature one usually takes for X a measurable space which is not necessarily topological and locally compact.) In the special case $\dim H^0 = 1$ the direct integral (7.3) is just the familiar L^2-space $L^2(X,\mu)$. If $\dim H^0 = n$ ($n \in \{1,2,3,\ldots,\infty\}$) then (7.3) can be considered as the n-fold direct sum of $L^2(X,\mu)$ (cf. V.1.17). This shows that (7.3) is a separable Hilbert space.

More generally we can define the direct integral of Hilbert spaces H_α of different dimensions. Let μ be a finite Borel measure on the lcsc.space X and let m be a so-called multiplicity function, i.e. a Borel measurable function on X taking values $1,2,3,\ldots,\infty$. For each $i \in \{1,2,3,\ldots,\infty\}$ fix a Hilbert space H^i of dimension i. Then we write

$$(7.4)\qquad H_{\mu,m} = \int_X^{\oplus} H_\alpha d\mu(\alpha) := \sum_{i\in\{1,2,\ldots,\infty\}}^{\oplus} L^2(X,\mu_i;H^i)$$

$$\text{if } H_\alpha = H^{m(\alpha)} \text{ for } \alpha \in X, \text{ where } \mu_i(E) := \mu(E \cap m^{-1}(i)) \text{ for } E \in \mathcal{B},$$

and we call $H_{\mu,m}$ the *direct integral* of the Hilbert spaces H_α with respect to the measure μ. The Hilbert space $H_{\mu,m}$ is again separable. Let $X_i := m^{-1}(i)$, $i \in \{1,2,\ldots,\infty\}$. The sets X_i are mutually disjoint Borel sets and their union is X. If $f = (f_1,f_2,\ldots,f_\infty) \in H_{\mu,m}$ where $f_i \in L^2(X,\mu_i;H^i)$ for $i \in \{1,2,\ldots,\infty\}$, then we can represent f by a function $f\colon X \to \bigcup_{i\in\{1,\ldots,\infty\}} H^i$ such that $f(\alpha) = f_i(\alpha) \in H^i$ if $\alpha \in X_i$, $i \in \{1,2,\ldots,\infty\}$. Two such functions f, g are identified with each other as elements of $H_{\mu,m}$ if $f(\alpha) = g(\alpha)$ a.e. $[\mu]$. The inner product on $H_{\mu,m}$ is given by

$$(7.5)\qquad (f,g) = \int_X (f(\alpha),g(\alpha))_{H_\alpha} d\mu(\alpha), \qquad f,g \in H_{\mu,m}.$$

Direct sums of Hilbert spaces are special cases of direct integrals: choose the measure μ in (7.4) discrete.

7.2. Decomposable and diagonal operators

It follows from formula (3.1) that the Hilbert space $\mathcal{H}_{\mu,m}$, defined by (7.4), is the representation space for the representation

$$\pi_0 := \pi_{\mu_1} \oplus 2\pi_{\mu_2} \oplus \ldots \oplus \infty\pi_{\mu_\infty}$$

of the commutative C^*-algebra $C_0(X)$ and that

$$(7.6) \qquad (\pi_0(f)\xi)(\alpha) := f(\alpha)\xi(\alpha), \quad f \in C_0(X),\ \xi \in \mathcal{H}_{\mu,m},\ \alpha \in X.$$

In the same way we can define a representation π of the commutative C^*-algebra $B(X)$ (consisting of all bounded Borel functions) on $\mathcal{H}_{\mu,m}$:

$$(7.7) \qquad (\pi(f)\xi)(\alpha) := f(\alpha)\xi(\alpha), \quad f \in B(X),\ \xi \in \mathcal{H}_{\mu,m},\ \alpha \in X.$$

A bounded linear operator A on $\mathcal{H}_{\mu,m}$ is called a *diagonal operator* if $A = \pi(f)$ for some $f \in B(X)$. The C^*-algebra $\pi(B(X))$ of all diagonal operators on $\mathcal{H}_{\mu,m}$ is denoted by $\mathcal{Z}$.

The formula

$$(7.8) \qquad (P_E\xi)(\alpha) := \chi_E(\alpha)\xi(\alpha), \quad E \in \mathcal{B},\ \xi \in \mathcal{H}_{\mu,m},\ \alpha \in X,$$

defines a projection-valued measure P on X acting in $\mathcal{H}_{\mu,m}$, as can be easily verified. Then

$$(7.9) \qquad \pi(f) = \int_X f\, dP, \quad f \in B(X).$$

Indeed, let $\xi \in \mathcal{H}_{\mu,m}$. Then

$$P_{\xi,\xi}(E) = \int_E \|\xi(\alpha)\|^2_{\mathcal{H}_\alpha}\, d\mu(\alpha), \quad E \in \mathcal{B},$$

i.e.,

$$dP_{\xi,\xi}(\alpha) = \|\xi(\alpha)\|^2_{\mathcal{H}_\alpha}\, d\mu(\alpha).$$

Hence

$$(\pi(f)\xi,\xi) = \int_X f(\alpha)\|\xi(\alpha)\|^2_{H_\alpha} d\mu(\alpha) = \int_X f(\alpha)dP_{\xi,\xi}(\alpha).$$

This settles (7.9). We conclude that π_0, π and P, defined by (7.6), (7.7), (7.8), are related to each other as in Prop. 3.5. Hence π_0, π and P have the same commuting algebra Z'.

REMARK 7.1. As a side result observe that for each projection-valued measure Q on X there is a finite Borel measure μ on X and a multiplicity function m on X such that Q is equivalent to the projection-valued measure P defined by (7.8), cf. Theorems 3.4 and 3.6.

Next we introduce decomposable operators on $H_{\mu,m}$. Consider the space $B_m(X,L)$ of all functions $t: X \to \bigcup_{i\in\{1,2,\ldots,\infty\}} L(H^i)$ such that

(i) $t(\alpha) \in L(H^i)$ if $m(\alpha) = i$;

(ii) t is weakly Borel, i.e., for each $i \in \{1,2,\ldots,\infty\}$, $v,w \in H^i$, the function $\alpha \to (t(\alpha)v,w)$ is a Borel function on X_i;

(iii) $\|t\| := \sup_{\alpha\in X} \|t(\alpha)\|_{L(H^{m(\alpha)})} < \infty$.

If $m(\alpha)$ is equal to a fixed i for all $\alpha \in X$ then we write $B(X,L(H^i))$ instead of $B_m(X,L)$. It is an easy exercise to show that $B_m(X,L)$ becomes a (generally noncommutative) C^*-algebra with respect to pointwise linear operations, multiplication and involution and with respect to the norm $\|\cdot\|$ defined above. For $t \in B_m(X,L)$, $\xi \in H_{\mu,m}$, let the function $T\xi$ on X be defined by

(7.10) $\qquad (T\xi)(\alpha) := t(\alpha)\xi(\alpha), \quad \alpha \in X.$

Since, for each $v \in H^i$, $i \in \{1,2,\ldots,\infty\}$, the function

$$\alpha \to (t(\alpha)\xi(\alpha),v)_{H^i} = (\xi(\alpha),(t(\alpha))^*v)_{H^i}$$

is Borel on X_i (expand $\xi(\alpha)$ and $(t(\alpha))^*v$ with respect to some orthonormal basis for H^i), the function $T\xi$ is weakly Borel on X. Also

$$\|T\xi\|^2 = \int_X \|(T\xi)(\alpha)\|^2_{H_\alpha} d\mu(\alpha) = \int_X |t(\alpha)|^2\|\xi(\alpha)\|^2_{H_\alpha} d\mu(\alpha) \le \|t\|^2\|\xi\|^2.$$

Hence (7.10) defines a bounded linear operator T on $H_{\mu,m}$ with norm $\|T\| \le \|t\|$. One verifies easily that the mapping $t \to T$ is a representation of the C^*-algebra $B_m(X,L)$ on $H_{\mu,m}$. A bounded linear operator T is called a *decomposable operator* if T satisfies (7.10) for some $t \in B_m(X,L)$. Then T is called the *direct integral* of the operators $t(\alpha)$ and we write

$$(7.11)\qquad T = \int_X^{\oplus} T(\alpha)\,d\mu(\alpha).$$

We denote the C^*-algebra of all decomposable operators on $\mathcal{H}_{\mu,m}$ by $\mathcal{R}$. Note that diagonal operators are special decomposable operators: if $t(\alpha) = f(\alpha)I_{\mathcal{H}_\alpha}$, $\alpha \in X$, for some $f \in B(X)$ then $T = \pi(f)$.

THEOREM 7.2. *Let $\mathcal{H}_{\mu,m}$ be a direct integral of Hilbert spaces $\mathcal{H}_\alpha$ given by (7.4). Let $\mathcal{Z}$ be the class of diagonal operators and $\mathcal{R}$ the class of decomposable operators on $\mathcal{H}_{\mu,m}$. Then:*
(a) $\mathcal{R} = \mathcal{Z}'$, *hence $\mathcal{R}$ is a von Neumann algebra.*
(b) $\mathcal{R}' = \mathcal{Z}$, *hence $\mathcal{Z}$ is a von Neumann algebra.*

PROOF.
(a) Clearly $\mathcal{R} \subset \mathcal{Z}'$. Conversely assume that $T \in \mathcal{Z}'$. Then also $T \in R(P)$. For $i \in \{1,2,\ldots,\infty\}$, P_{X_i} is the projection operator mapping $\mathcal{H}_{\mu,m}$ onto $L^2(X,\mu_i;\mathcal{H}^i)$ and $TP_{X_i} = P_{X_i}T$. Hence $T = \Sigma^{\oplus}\, T_i$, where $T_i \in \mathcal{L}(L^2(X,\mu_i;\mathcal{H}^i))$ and T_i commutes with all projections P_E, $E \in \mathcal{B}(X_i)$. If we can show that all T_i's are decomposable then we have also shown that T is decomposable. Therefore, without loss of generality we can restrict ourselves to a direct integral $\mathcal{H}$ of the form (7.3) with $\mathcal{H}_\alpha = \mathcal{H}^0$ for all $\alpha \in X$. Let $T \in \mathcal{L}(\mathcal{H})$ commute with all diagonal operators on $\mathcal{H}$. We have to show that T is decomposable.

For $v \in \mathcal{H}^0$ define $\underline{v} \in \mathcal{H}$ by $\underline{v}(\alpha) := v$, $\alpha \in X$. Choose an orthonormal basis $\{e_1,e_2,\ldots\}$ for $\mathcal{H}^0$. Let $\mathcal{H}^0_0$ be the dense linear subspace of $\mathcal{H}^0$ consisting of all finite linear combinations of these basis vectors.

For each basis vector e_k choose a weakly Borel function $f_{e_k}\colon X \to \mathcal{H}^0$ which is a representative of the element $T\underline{e}_k \in L^2(X,\mu;\mathcal{H}^0)$. If $v = \Sigma_k\, c_k e_k \in \mathcal{H}^0_0$ then define $f_v\colon X \to \mathcal{H}^0$ by

$$f_v(\alpha) := \sum_k c_k f_{e_k}(\alpha), \qquad \alpha \in X.$$

For each $\alpha \in X$ the mapping $v \to f_v(\alpha)$ is linear from $\mathcal{H}^0_0$ into $\mathcal{H}^0$ and for each $v \in \mathcal{H}^0_0$ the weakly Borel function f_v is a representative of $T\underline{v} \in L^2(X,\mu;\mathcal{H}^0)$. Furthermore, for each $v \in \mathcal{H}^0_0$ there is a Borel set N_v in X of μ-measure zero such that

(7.12) $\|f_v(\alpha)\|_{H^0} \le \|T\|\,\|v\|_{H^0}$

if $\alpha \in X\backslash N_v$, since for each Borel set E in X we have

$$\int_E \|f_v(\alpha)\|^2_{H^0} d\mu(\alpha) = \int_X \|\chi_E(\alpha)(T\underline{v})(\alpha)\|^2_{H^0} d\mu(\alpha) =$$

$$= \|P_E T\underline{v}\|^2 = \|TP_E\underline{v}\|^2 \le \|T\|^2 \|P_E\underline{v}\|^2 = \|T\|^2 \int_E \|v\|^2_{H^0} d\mu(\alpha).$$

Choose a countable dense subset $\mathbb{C}_0$ of $\mathbb{C}$ such that $0 \in \mathbb{C}_0$. Then the set $H^0_{0,0}$ of all $v = \Sigma_k c_k e_k \in H^0_0$ with coefficients $c_k \in \mathbb{C}_0$ is countable. Hence $N := \bigcup_{v\in H^0_{0,0}} N^k_v$ is a Borel set of μ-measure zero in X and (7.12) holds for all $v \in H^0_{0,0}$, $\alpha \in X\backslash N$. If $v \in H^0_0$ then there is a sequence $\{v_n\}$ in $H^0_{0,0}$ such that $v_n \to v$ in H^0 and $f_{v_n}(\alpha) \to f_v(\alpha)$ in H^0 for each $\alpha \in X$. Hence (7.12) holds for all $v \in H^0$, $\alpha \in X\backslash N$.

Now redefine the functions f_v, $v \in H^0_0$, by putting $f_v(\alpha) := 0$ if $\alpha \in N$. Then the mapping $v \to f_v(\alpha): H^0_0 \to H^0$ is still linear for all $\alpha \in X$, f_v is a representative of $T\underline{v}$ for all $v \in H^0_0$ and (7.12) holds for all $v \in H^0_0$, $\alpha \in X$. Put

$$s(\alpha)v := f_v(\alpha), \quad \alpha \in X,\ v \in H^0_0.$$

Then, for each $\alpha \in X$, $s(\alpha)$ extends to a bounded linear mapping on H^0 with $\|s(\alpha)\| \le \|T\|$. The mapping $s: X \to L(H^0)$ is weakly Borel since the function $\alpha \to (s(\alpha)v,w)_{H^0} = (f_v(\alpha),w)_{H^0}$ is Borel for all $v,w \in H^0$, Hence $s \in B(X,L(H^0))$. Let $S \in R$ be defined in terms of s by (7.10). Then

$$(S\underline{v})(\alpha) = s(\alpha)v = f_v(\alpha) = (T\underline{v})(\alpha) \quad \text{a.e. } [\mu]$$

for all $v \in H^0_0$.

Let $A := S - T$. Then $A \in Z'$ and $A\underline{v} = 0$ for all $v \in H^0_0$. We have to show that $A = 0$. If e_k is a basis vector and $g_k \in B(X)$ then $A\pi(g_k)\underline{e_k} = \pi(g_k)A\underline{e_k} = 0$. Hence $Ag = 0$ if

$$g = \sum_k \pi(g_k)\underline{e_k}, \quad g_k \in B(X), \quad g_k \ne 0 \text{ for only finite many } k.$$

Note that $g(\alpha) = \Sigma_k g_k(\alpha)e_k$, $\alpha \in X$. Hence the set of all such functions g is dense in H. so $A = 0$.

(b) Since $\mathcal{Z} \subset \mathcal{R}$, we have $\mathcal{R}' \subset \mathcal{Z}' = \mathcal{R}$. Hence, if $T \in \mathcal{R}'$ then T is decomposable. If $T \in \mathcal{R}'$ then $P_{X_i}T \in \mathcal{R}'$ for each i and T will be in $\mathcal{Z}$ if $P_{X_i}T$ is in $\mathcal{Z}$ for each i. Hence, without loss of generality we may assume that we are in the situation of (7.3), where $\mathcal{H}_\alpha = \mathcal{H}^0$ for all $\alpha \in X$. Let $T \in \mathcal{R}'$ and let $t \in B(X, \mathcal{L}(\mathcal{H}^0))$ such that (7.10) holds. Choose an orthonormal basis $\{e_1, e_2, \ldots\}$ for $\mathcal{H}^0$. For each pair (k,ℓ) define $a_{k,\ell} \in B(X, \mathcal{L}(\mathcal{H}^0))$ by

$$(a_{k,\ell}(\alpha)e_p, e_q)_{\mathcal{H}^0} = \delta_{k,p}\delta_{\ell,q}$$

and define $A_{k,\ell} \in \mathcal{R}$ in terms of $a_{k,\ell}$ by (7.10). Then $TA_{k,\ell} = A_{k,\ell}T$, hence for each p, q:

$$(7.13) \qquad (t(\alpha)a_{k,\ell}(\alpha)e_p, e_q)_{\mathcal{H}^0} = (a_{k,\ell}(\alpha)t(\alpha)e_p, e_q)_{\mathcal{H}^0} \quad \text{a.e. } [\mu].$$

Hence, there is a Borel set N of μ-measure zero such that (7.13) holds for all k, ℓ, p, q if $\alpha \in X\backslash N$. We may put $t(\alpha) := 0$ for $\alpha \in N$ without affecting T. Then (7.13) holds for all $\alpha \in X$. It can be rewritten as

$$\delta_{k,p}(t(\alpha)e_\ell, e_q)_{\mathcal{H}^0} = \delta_{\ell,q}(t(\alpha)e_p, e_k)_{\mathcal{H}^0}.$$

It follows that $t_0(\alpha) := (t(\alpha)e_k, e_k)_{\mathcal{H}^0}$ is independent of k and that $(t(\alpha)e_k, e_\ell)_{\mathcal{H}^0} = 0$ if $k \neq \ell$.

Let $f \in \mathcal{H}$. Then

$$(Tf)(\alpha) = t(\alpha)f(\alpha) = \sum_{k,\ell} (f(\alpha), e_k)_{\mathcal{H}^0}(t(\alpha)e_k, e_\ell)_{\mathcal{H}^0}e_\ell =$$

$$= \sum_k (f(\alpha), e_k)_{\mathcal{H}^0}t_0(\alpha)e_k = t_0(\alpha)f(\alpha).$$

Hence $T \in \mathcal{Z}$. □

7.3. Direct integrals of representations

Let G be a set on which an involution is defined and let $\mathcal{Rep}$ be a class of representations of G satisfying the conditions of Assumption VII 2.3. Let $\mathcal{H}_{\mu,m}$ be a direct integral of Hilbert spaces $\mathcal{H}_\alpha$ as given by (7.4). For each $\alpha \in X$ let π_α be a representation of G on $\mathcal{H}_\alpha$, belonging to $\mathcal{Rep}$, such that for each $x \in G$ the mapping $\alpha \to \pi_\alpha(x)$ belongs to $B_m(X, \mathcal{L})$. A representation $\pi \in \mathcal{Rep}$ of G on $\mathcal{H}_{\mu,m}$ is said to be the *direct integral* of the representations π_α if

$$\pi(x) = \int_X^{\oplus} \pi_\alpha(x)\,d\mu(\alpha)$$

(cf. (7.11)) for all $x \in G$. Then we write

$$\pi = \int_X^{\oplus} \pi_\alpha\,d\mu(\alpha).$$

As an example consider the standard form (6.5) for a multiplicity free representation of a lcsc. abelian group G. In that case

$$L^2(\hat{G},\mu) = \int_{\hat{G}}^{\oplus} H_\alpha\,d\mu(\alpha),$$

where $H_\alpha = \mathbb{C}$ for each $\alpha \in \hat{G}$, and

$$\pi_\mu(x) = \int_{\hat{G}}^{\oplus} \alpha(x)\,d\mu(\alpha), \quad x \in G.$$

Hence

$$\pi_\mu = \int_{\hat{G}}^{\oplus} \alpha\,d\mu(\alpha).$$

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IX

INDUCED REPRESENTATIONS OF LOCALLY COMPACT GROUPS

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CONTENTS

1. INTRODUCTION

Let G be a finite group with subgroup H, and let τ be a representation of H on a finite-dimensional vector space V. In chapter VI we considered the space $F_\tau(G,V)$ of all functions f: $G \to V$ satisfying

$$(1.1) \qquad f(xh) = \tau(h^{-1})f(x), \qquad x \in G,\ h \in H.$$

On this finite-dimensional vector space we defined a representation τ^G of G by

$$(\tau^G(y)f)(x) = f(y^{-1}x), \qquad x,y \in G,$$

the so-called induced representation of τ. In this chapter we plan to do the same for unitary representations of closed subgroups of locally compact second countable (lcsc.) groups. Thus, from now on, let G denote a lcsc. group, H a closed subgroup of G and τ a unitary representation of H. Consider again the space $F_\tau(G,H)$ of functions f: $G \to H$ (where H is the separable representation space of τ) satisfying (1.1). Define operators $\hat{\tau}(y)$, $y \in G$, on this space by

$$(1.2) \qquad (\hat{\tau}(y)f)(x) := f(y^{-1}x), \qquad x \in G.$$

Since we only consider unitary representations on separable Hilbert spaces, the space $F_\tau(G,H)$ is not very useful; in particular, it will be too large. Hence we must solve the following problem: Find a nontrivial $\hat{\tau}$-invariant linear subspace of $F_\tau(G,H)$ which can be equipped with a positive-definite sesquilinear form respected by $\hat{\tau}$. In general this will not be possible without a modification of the definition (1.2) of $\hat{\tau}$. A detailed discussion of the solution of the above problem will be presented in section 4. However, in order to provide some motivation for the contents of the preliminary sections 2 and 3, we will now already give a sketch of the results.

First, suppose that H is compact. Define a linear subspace $K_\tau(G,H)$ of $F_\tau(G,H)$ by taking all continuous compactly supported functions in $F_\tau(G,H)$. Clearly $K_\tau(G,H)$ is a $\hat{\tau}$-invariant subspace. Moreover, we can provide it with an inner product by setting

$$(1.3) \qquad (f,g) := \int_G (f(x),g(x))_H dx, \qquad f,g \in K_\tau(G,H),$$

and by virtue of the left invariance of the Haar measure dx, we have

$$(\hat{\tau}(y)f,\hat{\tau}(y)g) = (f,g), \qquad y \in G.$$

Thus, $\hat{\tau}(x)$ can be extended to a unitary operator $\tau^G(x)$ on the completion of $K_\tau(G,H)$ and it can be shown (section 4) that $x \longrightarrow \tau^G(x)$ defines a nonzero representation of G. However, if H is noncompact, then it is obvious that $K_\tau(G,H)$ will contain only the zero function. Indeed, (1.1) implies that if $f \in F_\tau(G,H)$ is zero somewhere on a coset, it will be identically zero on this coset.

Since the integrand at the right hand side of (1.3) is constant on left cosets modulo H (by unitarity of τ), it defines a unique continuous function on the coset space G/H. Therefore we may as well integrate over G/H instead of G. In doing so we can relax the assumption that the functions f and g be compactly supported into the weaker condition of being compactly supported "modulo H". This means that their support has the form KH, where K is a compact subset of G depending on each function separately. Defined in this way, $K_\tau(G,H)$ will have nonzero dimension, even if H is noncompact. (This is a nontrivial fact, which will be dealt with in §4.2.) Furthermore the integral in

$$(1.4) \qquad (f,g) := \int_{G/H} (f(x),g(x))d\mu(\bar{x}) \quad (\bar{x} := xH), \qquad f,g \in K_\tau(G,H),$$

replacing (1.3), is well-defined for any Borel measure μ on G/H. But now we meet another obstacle: The inner product (1.4) will only be respected by $\hat{\tau}$ if μ is left invariant under the action of G on G/H, i.e., if $d\mu(y\bar{x}) = d\mu(\bar{x})$ for all y in G ($y\bar{x} := \overline{yx}$). Unfortunately, such an invariant measure on G/H will not exist in general, as is shown by the example

$$G = SL(2,\mathbb{R}), \qquad H = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix} \;\middle|\; a > 0,\ c \in \mathbb{R}\right\}$$

(Example 3.16). In fact, it will be demonstrated in section 3 that a necessary and sufficient condition for the existence of an invariant measure is given by the condition $\Delta_G|_H = \Delta_H$, where Δ_G and Δ_H denote the modular functions on G and H, respectively. This condition holds, for instance, if H is compact.

An invariant measure on G/H may not always exist, but an invariant measure class does always exist and is even unique. [A measure class is an equivalence class of Borel measures under the equivalence relation of "having the same null-sets". We denote the measure class containing μ by $\{\mu\}$. A measure class $\{\mu\}$ on G/H is said to be invariant if for each $\tilde{\mu} \in \{\mu\}$ and $y \in G$, the measure $\tilde{\mu}_y$ defined by $d\tilde{\mu}_y(\bar{x}) := d\tilde{\mu}(y\bar{x})$ belongs to $\{\mu\}$ as well.] The measures in the invariant measure class on G/H are called quasi-invariant. It will turn out that we can always find a quasi-invariant measure μ on G/H and a strictly positive continuous function R on G/H × G such that

$$\int_{G/H} f(y^{-1}\bar{x})\,d\mu(\bar{x}) = \int_{G/H} f(\bar{x})R(\bar{x},y)\,d\mu(\bar{x}),$$

for all continuous compactly supported functions f on G/H. If we have such μ and R and if $f,g \in K_\tau(G,H)$ (continuous functions in $F_\tau(G,H)$ with compact support modulo H) then

$$(\hat{\tau}(y)f,\hat{\tau}(y)g) = \int_{G/H} (f(x),g(x))_H R(\bar{x},y)\,d\mu(\bar{x}).$$

Hence, if we redefine $\hat{\tau}$ by

$$(\hat{\tau}(y)f)(x) := \frac{f(y^{-1}\bar{x})}{(R(y^{-1}\bar{x},y))^{\frac{1}{2}}}, \qquad f \in K_\tau(G,H),$$

then $(\hat{\tau}(y)f,\hat{\tau}(y)g) = (f,g)$, and it can again be proved that the extension of $\hat{\tau}$ to the completion of $K_\tau(G,H)$ defines a representation of G. This will be our induced representation.

Convention. All Hilbert spaces considered in this chapter are assumed to be separable. By lcsc. space (or group) we mean a locally compact Hausdorff space (or group) which satisfies the second axiom of countability.

2. HOMOGENEOUS SPACES

Let Γ be a lcsc. space and let G be a lcsc. group. Then Γ is called a *continuous G-space* if (i) Γ is a G-space (as defined in §VI.5) and (ii) G acts continuously on Γ, that is, the mapping $(x,\gamma) \mapsto x(\gamma)$ from $G \times \Gamma$ onto Γ is continuous. Note that this implies that each mapping $\gamma \mapsto x(\gamma)$ is a homeomorphism from Γ onto itself. If the G-action is both continuous and transitive then Γ is called a *homogeneous space* of G. Two continuous

G-spaces Γ and Δ are said to be G-*homeomorphic* if there exists a homeomorphism ϕ from Γ onto Δ which respects the G-action, that is, $\phi(x(\gamma)) = x(\phi(\gamma))$ for all x in G and all γ in Γ.

Let H be a closed subgroup of G, and consider the left coset space G/H. We write $\bar{x} := xH$ $(x \in G)$ for its elements. We endow G/H with a topology, the so-called quotient topology, by calling a subset $O \subset G/H$ open if its inverse image under the quotient mapping $\pi: x \mapsto \bar{x}$ is open. Then π is continuous by definition, and since $\pi^{-1}(\pi(S)) = SH$ for any subset $S \subset G$, π is also an open mapping. This implies that G/H, being the continuous, open image of a locally compact group, is itself locally compact. It is easily verified that G/H is second countable and Hausdorff. Finally, the natural action of G on G/H, defined by $x\bar{y} := \overline{xy}$, is continuous and transitive. Hence, G/H is a homogeneous space of G. In fact we have:

THEOREM 2.1. *Each homogeneous space of a lcsc. group* G *is* G-*homeomeomorphic with a coset space* G/H *for some closed subgroup* H *of* G.

PROOF. Let Γ be a homogeneous space of G, fix a point γ_0 of Γ, and set $H := \{x \in G \mid x(\gamma_0) = \gamma_0\}$. Then H is a closed subgroup of G, the so-called *stabilizer* (or *little group*) of γ_0. Consider the mapping $\beta: \bar{x} \longmapsto x(\gamma_0)$ from G/H onto Γ. Obviously, β is continuous and bijective, and $\beta(y\bar{x}) = y\beta(\bar{x})$ for all $\bar{x}$ in G/H and all y in G. By means of the Baire category theorem (RUDIN [10, §2.2]) we show that β is open. Since the natural mapping $\pi: G \to G/H$ is continuous, it suffices to show that $\beta \circ \pi: x \mapsto x(\gamma_0)$ is open. For this purpose, we prove that $\beta \circ \pi$ maps any neighbourhood of the identity $e \in G$ onto some neighbourhood of γ_0. Let V be any neighbourhood of e, and choose another open neighbourhood W of e such that (i) $W = W^{-1}$, (ii) $W^2 \subset V$ and (iii) the closure of W is compact. One readily checks that this is possible.) Since G is second countable, there exists a countable sequence $x_1, x_2, \ldots$ of elements of G such that $G = \bigcup_{i=1}^{\infty} x_i W$. Hence, Γ is the union of the countable sequence of compact subsets $\{x_i\bar{W}(\gamma_0)\}_{i=1}^{\infty}$. Since Γ is locally compact and Hausdorff, we can apply the Baire theorem, which asserts that in such a space the countable union of nowhere dense subsets has no interior points, and we conclude that for some x_{i_0} the set $x_{i_0}\bar{W}(\gamma_0)$ has a nonvoid interior. Let $x_{i_0}w(\gamma_0)$ be an interior point of $x_{i_0}\bar{W}(\gamma_0)$. (We can take $w \in W$ since $x_{i_0}\bar{W}(\gamma_0) = (x_{i_0}W(\gamma_0))^-$, by compactness of $\bar{W}$ and continuity of $x \longmapsto x(\gamma_0)$.) Then we have

$$\gamma_0 \in w^{-1}x_{i_0}^{-1}(x_{i_0}W)(\gamma_0) = w^{-1}W(\gamma_0) \subset V(\gamma_0).$$

Consequently, γ_0 is an interior point of $V(\gamma_0) = (\beta\circ\pi)(V)$, which ends our demonstration. (This proof is taken from BOURBAKI [2].) □

EXAMPLE 2.2. Consider the unit sphere S^{n-1} in $\mathbb{R}^n$. The special orthogonal group SO(n) acts continuously and transitively on S^{n-1} by rotations. The stabilizer of the pole $(1,0,\ldots,0) \in S^{n-1}$ consists of all matrices

$$\begin{pmatrix} 1 & 0 \ldots\ldots 0 \\ 0 & \\ \vdots & R \\ 0 & \end{pmatrix} \qquad \text{with } R \in SO(n-1),$$

and it is therefore naturally isomorphic with SO(n-1). Hence, S^{n-1} is homeomorphic with SO(n)/SO(n-1).

We proceed to state two lemmata which will be used in the next section.

LEMMA 2.3 (Urysohn). *Let X be a locally compact Hausdorff space, and let K and O be subsets of X, with K compact and O open, such that $K \subset O$. Then there exists a continuous function f on X with compact support, such that*

(i) $0 \le f(x) \le 1$, $\forall x \in X$;

(ii) $f(x) = 1$, $\forall x \in K$;

(iii) $f(x) = 0$, $\forall x \in X\backslash O$.

For a proof we refer to RUDIN [9, §2.12].

LEMMA 2.4. *Let $K \subset G/H$ be a compact subset. Then there exists a compact subset $K' \subset G$ such that K' is mapped onto K by the natural mapping $\pi: G \to G/H$.*

PROOF. Choose an open neighbourhood U of the identity $e \in G$, such that the closure of U is compact. Then $K \subset \bigcup_{i=1}^n \pi(x_iU)$ for certain elements $x_1,\ldots,x_n$ in G. If we set $K' := \left(\bigcup_{i=1}^n x_i\bar{U}\right) \cap \pi^{-1}(K)$, then K' is compact and $\pi(K') = K$. □

Finally, we state without proof an interesting result, due to Mackey. By a *Borel cross-section* we will mean a Borel mapping $s: G/H \to G$ which satisfies $\pi \circ s = id_{G/H}$.

LEMMA 2.5 (Mackey). *If* G *is a lcsc. group and* H *is a closed subgroup of* G, *then there always exists a Borel cross-section* $s: G/H \to G$.

In fact a more general result is true. The proof is based on a classical theorem of Morse and Federer, and can be found in MACKEY [6] or VARADARAJAN [11, thm. 8.11].

It is important to observe that the projection π generally does not admit a continuous cross-section. For instance, set $G = \mathbb{R}$ and $H = \mathbb{Z}$. Then $G/H = \mathbb{T}$, the circle group, and $\pi(x) = e^{2\pi i x}$. It can easily be shown that no mapping $s: \mathbb{T} \to \mathbb{R}$ exists which is continuous and satisfies $\pi \circ s = id_{\mathbb{T}}$.

3. QUASI-INVARIANT MEASURES ON COSET SPACES

The main references for this section are REITER [8] and VARADARAJAN [10].

Throughout this section, all measures will be assumed to be positive nonzero Borel measures. Let G be a lcsc. group, H a closed subgroup of G, and consider the homgeneous space G/H. Elements of this space are denoted by $\bar{x}$, where $\pi: x \mapsto \pi(x) = \bar{x}$ is the quotient mapping from G onto G/H. For $S \subset G/H$ and $x \in G$ we write $x[S] := \{x\bar{y} \mid \bar{y} \in S\}$.

A measure μ on G/H is said to be G-*invariant* (or *invariant*) if $\mu = \mu_x$ for all x in G. Here μ_x denotes the translated measure, defined by $\mu_x(B) := \mu(x[B])$, for Borel sets B in G/H. Thus, μ is invariant if and only if

$$(3.1) \qquad \mu(B) = \mu(x[B]), \qquad \forall x \in G,\ \forall B \in \mathcal{B}(G/H).$$

(We write $\mathcal{B}(X)$ for the collection of all Borel subsets of a Borel space X.)

For instance, if H is an invariant subgroup of G, then the space G/H becomes a lcsc. group in its own right, with respect to the quotient topology, if we define a product by $\bar{x}\bar{y} := \overline{xy}$. Since $x\bar{y} = \bar{x}\bar{y}$ for all x,y in G, we see that the left Haar measure on G/H satisfies (3.1). Hence, in this case an invariant measure always exists, and, moreover, it is unique up to a constant factor.

Returning to the general case, let ν be a left Haar measure on G, and set $\mu(B) := \nu(\pi^{-1}(B))$, $B \in \mathcal{B}(G/H)$. Then μ is a positive σ-additive function on $\mathcal{B}(G/H)$, and $\mu(\emptyset) = 0$. Hence, μ is a measure in the ordinary sense on the Borel subsets of G/H, and, since $\pi^{-1}(x[B]) = x\pi^{-1}(B)$ for all x in G and all B in $\mathcal{B}(G/H)$, it satisfies (3.1). However, if C is a compact subset of G/H, then $\pi^{-1}(C)$ is not necessarily compact in G, and $\nu(\pi^{-1}(C))$ can be infinite (and it will be, in certain cases). Hence, μ fails in general to be finite on compact sets, which is a requirement for Borel measures. Notice that, if H is compact, $\pi^{-1}(C)$ is compact for each compact subset C of G/H. Consequently, μ is a G-invariant measure in this case. Apparently, what would seem a natural way to obtain invariant measures on coset spaces does not work in general. As we will show later on in this section, there are homogeneous spaces on which no invariant measure exists at all. Therefore, we will focus on measures with a weaker invariance property than (3.1). Recall that a measure μ is said to be *absolutely continuous* with respect to another measure ν on the same space, if each null-set for ν is also a null-set for μ; notation: $\mu \ll \nu$ (cf. V.1.8). Two measures μ and ν on the same space are called *equivalent* (notation $\mu \equiv \nu$) if $\mu \ll \nu$ and $\nu \ll \mu$.

DEFINITION 3.1. A measure μ on the coset space G/H is called *quasi-invariant* if it is equivalent to each of its translates, i.e. if $\mu \equiv \mu_x$ for all x in G.

The classes of measures corresponding to the equivalence relation $\equiv$ are called *measure classes,* and the measure class containing μ is denoted by $[\mu]$. A measure class $[\mu]$ on G/H is called *invariant* if $\mu'_x \in [\mu]$ for all $\mu' \in [\mu]$ and $x \in G$. We can now restate the above definition as follows: A measure μ on G/H is called quasi-invariant if it belongs to an invariant measure class (notice that $\mu \equiv \mu' \Rightarrow \mu_x \equiv \mu'_x$).

We can give still another characterization of quasi-invariant measures by utilizing the well-known Radon-Nykodym theorem (cf. V.1.9), which gives a necessary and sufficient condition for two measures to be equivalent. It turns out (cf. V.2.11) that a measure μ on G/H is quasi-invariant if and only if for each y in G there exists a strictly positive Borel function $\bar{x} \to R(\bar{x},y)$ on G/H such that

$$\int_{G/H} f(y^{-1}\bar{x})\,d\mu(\bar{x}) = \int_{G/H} f(\bar{x})R(\bar{x},y)\,d\mu(\bar{x}) \tag{3.2}$$

for all f in $\mathcal{K}(G/H)$ (the space of continuous complex-valued functions on G/H with compact support).

In this section we will prove that there always exists a unique invariant measure class on G/H. Moreover, we will show that this class always contains a measure μ for which the function R occuring in (3.2) can be taken to be continuous in both variables (considered as a function on $G/H \times G$). As a corollary of a certain stage of the existence proof we will obtain a necessary and sufficient condition for the existence of an invariant measure on G/H.

We start with the discussion of a very useful relationship between the spaces $K(G)$ and $K(G/H)$. We fix Haar measures ν_G and ν_H on G and H, respectively. If f belongs to $K(G)$, then consider the expression $\int_H f(xh)d\nu_H(h), x \in G$. The value of this integral remains constant if we let x run through a left H-coset. Hence, if we set

$$\text{(3.3)} \qquad \tilde{f}(x) := \int_H f(xh)d\nu_H(h),$$

then we obtain a function $\tilde{f}$ on the coset space G/H.

<u>LEMMA 3.2</u>. *The assignment* $f \to \tilde{f}$ *maps* $K(G)$ *onto* $K(G/H)$. *Furthermore,* $f \geq 0$ *implies* $\tilde{f} \geq 0$.

<u>PROOF</u>. Let $f \in K(G)$. Clearly, the support of $\tilde{f}$ is contained in $\pi(\text{supp}(f))$. Continuity of f can be verified by simple standard arguments, by exploiting the fact that f is uniformly continuous. Hence, $\tilde{f} \in K(G/H)$.

Next, let $g_1 \in K(G/H)$, and set $K := \text{supp}(g_1)$. Then we can choose a compact subset K' of G such that $\pi(K') = K$ (Lemma 2.4). There exists a positive function $g_2 \in K(G)$ with $g_2(x) = 1$ for all $x \in K'$ (Lemma 2.3). If $x \in \pi^{-1}(K)$, then there exists an element $h \in H$ with $xh \in K'$. Hence, $\tilde{g}_2(\bar{x}) > 0$ for all $\bar{x} \in K$. Define a function f on G by $f(x) := 0$ if $x \notin \pi^{-1}(K)$ and by

$$f(x) := g_1(\bar{x})g_2(x)/\tilde{g}_2(x) \quad \text{if } x \in \pi^{-1}(K).$$

Clearly f is compactly supported. Continuity of f follows from the fact that $K = \text{supp}(g_1)$ and from the continuity of g_1, g_2 and $\tilde{g}_2$. Furthermore,

$$\tilde{f}(\bar{x}) = \frac{g_1(\bar{x})}{\tilde{g}_2(\bar{x})} \int_H g_2(xh)d\nu_H(h) = g_1(\bar{x}),$$

so $\tilde{f} = g$. The second assertion of the lemma is obvious. □

Let μ be a measure on G/H and define a measure $\mu^{\#}$ on G by

$$(3.4) \qquad \mu^{\#}(f) := \mu(\tilde{f}), \qquad f \in K(G).$$

From the preceding lemma and the obvious linearity of $f \mapsto \tilde{f}$ it follows that $\mu^{\#}$ is indeed a measure on G, uniquely determined by μ (cf. the Riesz representation theorem V.2.8). Hence, we have obtained a mapping $\mu \mapsto \mu^{\#}$ from the set of measures on G/H into the set of measures on G. (This mapping may be considered as the dual of the mapping $f \mapsto \tilde{f}$.)

Before we discuss this important mapping in detail, we prove the following useful extension of Lemma 3.2.

LEMMA 3.3. *Let μ be a Borel measure on G/H and let f be a Borel function on G.*

(i) *If f is nonnegative then formula* (3.3) *defines a Borel function $\tilde{f}$ on G/H with values in $[0,\infty]$, and $\mu(\tilde{f}) = \mu^{\#}(f)$.*
(ii) *If $f \in L^1(G,\mu^{\#})$ (not necessarily nonnegative) then $\tilde{f}$ is well-defined a.e. $[\mu]$ by* (3.3), *$\tilde{f} \in L^1(G/H,\mu)$ and $\mu(\tilde{f}) = \mu^{\#}(f)$.*

PROOF. If f is a nonnegative Borel function on G, then $(x,h) \longmapsto f(xh)$ is a nonnegative Borel function on the product space $G \times H$. From an argument used in the proof of the Fubini theorem V.1.12 (sometimes separately stated as the Tonelli theorem) it follows that $x \longmapsto \int_H f(xh)d\nu_H(h)$ is a Borel function on G. Moreover, this function is constant on left cosets modulo H, so there exists a unique Borel function $\tilde{f}$ on G/H as defined in the lemma. (Note that $\tilde{f}$ can be infinite.)

In order to prove the identity $\mu(\tilde{f}) = \mu^{\#}(f)$ for such functions f, we first assume that f is the characteristic function of a compact subset K of G. Then we can find a sequence of positive functions f_n in $K(G)$ descending to f. By Lebesgue's dominated convergence theorem (cf. V.1.5) and by Lemma 3.2 we have:

$$\int_G f d\mu^{\#} = \lim_{n\to\infty} \int_G f_n d\mu^{\#} = \lim_{n\to\infty} \int_{G/H} \tilde{f}_n d\mu = \int_{G/H} \tilde{f} d\mu.$$

The function $B \longmapsto \int_{G/H} \chi_B d\mu^{\#}$ on the Borel sets B contained in K is easily verified to define a Borel measure on K. Since this measure coincides with $\mu^{\#}$ on the compact subsets of K, it must coincide with $\mu^{\#}$ on all Borel subsets of K. Hence, $\mu^{\#}(\chi_B) = \mu(\tilde{\chi}_B)$ for any Borel set B in G which lies in a compact subset of G. But then, by linearity of the mapping $f \mapsto \tilde{f}$, we know that $\mu^{\#}(f) = \mu(\tilde{f})$ for any compactly supported step function f on G. For any nonnegative Borel function f on G we can find a sequence f of nonnegative compactly supported step functions on G such that $f_n \uparrow f$. Applying the monotone convergence theorem (cf. V.1.5), we conclude that $\mu^{\#}(f)$ equals $\mu(\tilde{f})$.

Next, for f in $L^1(G,\mu^{\#})$, write $f = f_+ - f_-$, where $f_+ := \sup\{f,0\}$. Then $f_+, f_- \in L^1(G,\mu^{\#})$, and

$$\int_{G/H}\left(\int_H f_{\pm}(xh)\,d\nu_H(h)\right)d\mu(\bar{x}) = \int_G f_{\pm}(x)\,d\mu^{\#}(x) < \infty.$$

Hence

$$\tilde{f}_{\pm}(\bar{x}) = \int_H f_{\pm}(xh)\,d\nu_H(h) < \infty$$

for $\bar{x}$ on the complement of a certain μ-null set in G/H. This proves that $\tilde{f}(\bar{x}) = \tilde{f}_+(\bar{x}) - \tilde{f}_-(\bar{x})$ is well-defined a.e. $[\mu]$. Furthermore, $\mu(|\tilde{f}|) < \infty$ and

$$\mu(\tilde{f}) = \mu(\tilde{f}_+) - \mu(\tilde{f}_-) = \mu^{\#}(f_+) - \mu^{\#}(f_-) = \mu^{\#}(f). \quad \square$$

<u>COROLLARY 3.4</u>. *Let* μ *be a Borel measure on* G/H *and* B *a Borel set in* G/H. *Then* $\mu(B) = 0$ *if and only if* $\mu^{\#}(\pi^{-1}(B)) = 0$.

<u>PROOF</u>. By virtue of the preceding lemma we have the two identities

$$\left(\chi_{\pi^{-1}(B)}\right)^{\sim}(\bar{x}) = \int_H \chi_{\pi^{-1}(B)}(xh)\,d\nu_H(h), \qquad x \in G,$$

and

$$\mu^{\#}(\pi^{-1}(B)) = \int_{G/H}\left(\chi_{\pi^{-1}(B)}\right)^{\sim}(\bar{x})\,d\mu(\bar{x}).$$

Since $(\chi_{\pi^{-1}(B)})^{\sim}$ vanishes outside B, $\mu(B) = 0$ implies $\mu^{\#}(\pi^{-1}(B)) = 0$. Conversely, $\mu^{\#}(\pi^{-1}(B)) = 0$ implies that $(\chi_{\pi^{-1}(B)})^{\sim}$ is zero a.e. $[\mu]$. If, in this case, $\mu(\beta)$ would be nonzero, then, in particular, $(\chi_{\pi^{-1}(B)})^{\sim}(\bar{x}) = 0$ for some $\bar{x} \in B$. But then

$$0 = \left(\chi_{\pi^{-1}(B)}\right)^{\sim}(\bar{x}) = \int_H d\nu_H(h) \neq 0;$$

a contradiction. □

<u>COROLLARY 3.5</u>. *Let* μ_1 *and* μ_2 *be Borel measures on* G/H. *Then* $\mu_1 << \mu_2$ *if and only if* $\mu_1^{\#} << \mu_2^{\#}$. *Furthermore, if* $\mu_1 << \mu_2$, *then*

$$\frac{d\mu_1}{d\mu_2}(\bar{x}) = \frac{d\mu_1^{\#}}{d\mu_2^{\#}}(x), \qquad x \in G.$$

<u>PROOF</u>. The "if" part of the first statement immediately follows from Corollary 3.4. As to the other assertions, let μ_1 and μ_2 be measures on G/H with $\mu_1 << \mu_2$. By virtue of the Radon-Nikodym theory there exists a positive Borel function ϕ on G/H such that

$$\mu_1(f) = \int_{G/H} f(\bar{x})\phi(\bar{x})d\mu_2(\bar{x})$$

for all Borel functions f on G/H. If $g \in K(G)$ then one readily verifies that $(g(\phi\circ\pi))^{\sim} = \tilde{g}\phi$. But then, by Lemma 3.3, it follows that

$$\int_G g(x)\phi(\bar{x})d\mu_2^{\#}(x) = \int_{G/H} \tilde{g}(\bar{x})\phi(\bar{x})d\mu_2(\bar{x}) =$$

$$= \int_{G/H} \tilde{g}(\bar{x})d\mu_1(\bar{x}) = \int_G g(x)d\mu_1^{\#}(x).$$

Hence $\mu_1^{\#} << \mu_2^{\#}$, and, in particular, the Radon-Nikodym derivative $d\mu_1^{\#}/d\mu_2^{\#}$ equals $\phi \circ \pi$. □

<u>THEOREM 3.6</u>. *The mapping* $\mu \mapsto \mu^{\#}$ *satisfies the following properties*:

(i) $\mu_1 = \mu_2$ *iff* $\mu_1^{\#} = \mu_2^{\#}$;

(ii) $\mu_1 \equiv \mu_2$ *iff* $\mu_1^{\#} \equiv \mu_2^{\#}$;

(iii) μ *is (quasi-) invariant iff* $\mu^{\#}$ *is (quasi-) invariant.*

<u>PROOF</u>. The first statement follows from the surjectivity of the mapping $f \mapsto \tilde{f}$ onto $K(G/H)$, the second one follows immediately from Corollary 3.5, and the last one follows from (ii) and the obvious observation that $(\mu_x)^{\#} = (\mu^{\#})_x$, $x \in G$. □

The following lemma, in combination with the statements of Theorem 3.6, establishes the uniqueness of an invariant measure class in G/H (if it exists).

LEMMA 3.7. *Each quasi-invariant measure on G is equivalent to the Haar measures on G.*

PROOF. Let μ be a quasi-invariant measure on G, and let $B \in \mathcal{B}(G)$. Then we have

$$\int_G \int_G \chi_{B^{-1}}(x)\,d\nu_G(x)\,d\mu(y) = \int_G \int_G \chi_{B^{-1}}(y^{-1}x)\,d\nu_G(x)\,d\mu(y) =$$

$$= \int_G \int_G \chi_{B^{-1}}(y^{-1}x)\,d\mu(y)\,d\nu_G(x) = \int_G \int_G \chi_{x[B]}(y)\,d\mu(y)\,d\nu_G(x) =$$

$$= \int \mu(x[B])\,d\nu_G(x).$$

Elementary considerations show that these steps are all legitimate. Now, if $\mu(B) = 0$, then $\mu(x[B]) = 0$, and hence $\nu_G(B^{-1}) = 0$. But B^{-1} has Haar measure zero if and only if B has Haar measure zero. Hence, $\nu_G(B) = 0$. Clearly this argument can be reversed, so $\mu \in [\nu_G]$. □

If we can show that the image of the mapping $\mu \to \mu^{\#}$ contains a quasi-invariant measure, then the existence of an invariant measure class on G/H follows at once from theorem 3.6 (iii). For this purpose, we first determine this image. Let Δ_H and Δ_G denote the Haar moduli of H and G respectively.

LEMMA 3.8. *Let ν be a measure on G. Then there exists a measure μ on G/H with $\nu = \mu^{\#}$ if and only if*

$$\int_G f(xh)\,d\nu(x) = \Delta_H(h^{-1}) \int_G f(x)\,d\nu(x), \quad \forall f \in \mathcal{K}(G),\ \forall h \in H. \tag{3.5}$$

PROOF. Suppose that ν is equal to $\mu^{\#}$, for a certain measure μ on G/H. Then $\nu(f) = \mu(\tilde{f})$ for all f in $\mathcal{K}(G)$. Fix $h_0 \in H$ and let $f_{h_0}(x) := f(xh_0)$, $x \in G$. Then:

$$\int_G f(xh_0)\,d\nu(x) = \int f_{h_0}(x)\,d\nu(x) = \int_{G/H} \int_H f_{h_0}(xh)\,d\nu_H(h)\,d\mu(\bar{x}) =$$

$$= \int_{G/H} \int_H \Delta_H(h_0^{-1}) f(xh)\,d\nu_H(h)\,d\mu(\bar{x}) = \Delta_H(h_0^{-1}) \int_G f(x)\,d\nu(x).$$

Next, let ν be a measure on G which satisfies (3.5). Then, for $\tilde{f} \in K(G/H)$, we set $\mu(\tilde{f}) := \nu(f)$. We first show that this definition is legitimate, by proving that $\tilde{f}_1 = \tilde{f}_2$ implies $\nu(f_1) = \nu(f_2)$. This property of ν follows primarily from (3.5).

Let f belong to $K(G)$. By virtue of the lemmata 2.3 and 3.2, we can choose a function g in $K(G)$ such that $\tilde{g}(\bar{x}) = 1$ for all $\bar{x}$ in $\pi(\mathrm{supp}(f))$. Utilizing formula (3.5) and applying the Fubini theorem, we can make the following computation:

$$\begin{aligned}
\int_G f(x)d\nu(x) &= \int_G f(x)\left(\int_H g(xh)d\nu_H(h)\right)d\nu(x) \\
&= \int_H\int_G f(x)g(xh)d\nu(x)d\nu_H(h) \\
&= \int_H \Delta_H(h^{-1})\left(\int_G f(xh^{-1})g(x)d\nu(x)\right)d\nu_H(h) \\
&= \int_G g(x)\left(\int_H \Delta_H(h^{-1})f(xh^{-1})d\nu_H(h)\right)d\nu(x) \\
&= \int_G g(x)\left(\int_H f(xh)d\nu_H(h)\right)d\nu(x) \\
&= \int_G g(x)\tilde{f}(\bar{x})d\nu(x).
\end{aligned}$$

But then, if $\tilde{f}(\bar{x}) = 0$ for all $\bar{x}$ in G/H, we have $\nu(f) = 0$. By linearity of the mapping $f \to \tilde{f}$ it follows that the number $\mu(\tilde{f}) = \nu(f)$ is well-defined.

Clearly, μ is a linear functional. Furthermore, by means of the proof of lemma 3.2, it can be easily verified that for each $g \in K(G/H)$ with $g \geq 0$, a function $f \in K(G)$ can be chosen such that $f \geq 0$ and $\tilde{f} = g$. This shows that μ is positive. Now it follows from the Riesz representation theorem (cf. V.2.8) that μ is a measure. This finishes our proof, since $\mu^{\#} = \nu$ by definition. □

If we set $\nu = \nu_G$, the Haar measure on G, then the idendity (3.5) reduces to $d\nu_G(x) = (\Delta_G(h)/\Delta_H(h))d\nu_G(x)$, $\forall h \in H$. Clearly, this will only be true if

Δ_G restricted to H is equal to Δ_H. It follows from Theorem 3.6 (*iii*) that G/H admits an invariant measure if and only if the Haar measure on G lies in the image of the mapping $\mu \mapsto \mu^\#$. Hence, Lemma 3.8 yields the following criterion for the existence of an invariant measure on G/H:

COROLLARY 3.9. *The coset space* G/H *admits an invariant measure if and only if* $\Delta_G(h) = \Delta_H(h)$ *for all* $h \in H$.

Next we consider the question of the existence of quasi-invariant measures on G/H. By virtue of Lemma 3.8 we can look as well for quasi-invariant measures on G which satisfy (3.5). This last problem can be solved in a very nice way by the use of the following crucial lemma:

LEMMA 3.10. *There exists a continuous, strictly positive solution of the functional equation*

$$\rho(x) = \frac{\Delta_G(h)}{\Delta_H(h)}\,\rho(xh), \qquad \forall x \in G,\ \forall h \in H. \tag{3.6}$$

Before we give the rather technical proof of this lemma, we will clarify its significance by stating and proving the following theorem.

THEOREM 3.11.

(a) *Let* ρ *be a continuous, strictly positive function on* G *satisfying* (3.6) *and define a measure* ν *on* G *by*

$$d\nu(x) = \rho(x)\,d\nu_G(x). \tag{3.7}$$

Then there exists a quasi-invariant measure μ *on* G/H *with* $\mu^\# = \nu$, *and the strictly positive continuous function* R *on* $G/H \times G$ *given by*

$$R(\bar{x},y) = \frac{\rho(yx)}{\rho(x)} \tag{3.8}$$

satisfies

$$\int_{G/H} f(y^{-1}\bar{x})\,d\mu(\bar{x}) = \int_{G/H} f(\bar{x})R(\bar{x},y)\,d\mu(\bar{x}), \qquad y \in G,\ f \in K(G/H). \tag{3.9}$$

(b) *Conversely, if* μ *is a quasi-invariant measure on* G/H *such that* (3.9) *holds with some continuous strictly positive function* R *on* G/H × G, *then the measure* ν *given by* (3.7) *with*

$$(3.10) \qquad \rho(x) := R(\bar{e},x), \quad x \in G,$$

satisfies $\nu = C\mu^{\#}$ (C>0) *and* ρ *is a strictly positive continuous solution of* (3.6).

PROOF.
(a) Let ρ be a continuous strictly positive solution of (3.6) and let the measure ν on G be defined by (3.7). Then

$$\int_G f(xh)d\nu(x) = \Delta_G(h^{-1}) \int_G f(x)\rho(xh^{-1})d\nu_G(x)$$

$$= \Delta_H(h^{-1}) \int_G f(x)d\nu(x), \quad f \in K(G),$$

so there exists a measure μ on G/H with $\mu^{\#} = \nu$. Furthermore, we have

$$\int_G f(y^{-1}x)d\nu(x) = \int_G f(x)\rho(yx)d\nu_G(x)$$

$$= \int_G f(x) \frac{\rho(yx)}{\rho(x)} d\nu(x), \quad f \in K(G).$$

Comparing this to the characterization of quasi-invariant measures we gave by means of the Radon-Nikodym theorem, we can conclude that ν is quasi-invariant invariant.

Furthermore, note that the Radon-Nikodym derivative $d\nu_y(x)/d\nu(x)$ is given by $\rho(yx)/\rho(x)$. By virtue of Corollary 3.5 we have

$$\frac{d\mu_1}{d\mu_2}(\bar{x}) = \frac{d\mu_1^{\#}}{d\mu_2^{\#}}(x),$$

if μ_1 and μ_2 are equivalent measures on G/H. Hence, for $\mu^{\#} = \nu$ we have

$$\frac{d\mu_y}{d\mu}(\bar{x}) = \frac{d\nu_y}{d\nu}(x) = \frac{\rho(yx)}{\rho(x)},$$

so (3.9) holds with R given by (3.8). From (3.8) we infer the fact that R, considered as a function on G/H × G, is strictly positive and continuous in both variables. We emphasize that in this case R is uniquely determined by μ.

(b) Let μ be a quasi-invariant measure on G/H such that (3.9) holds with R continuous and strictly positive on G/H × G. It follows from (3.9) that for each $f \in K(G/H)$ and $y,z \in G$ we have

$$\begin{aligned}\int_{G/H} f(\bar{x})R(\bar{x},yz)d\mu(\bar{x}) &= \int_{G/H} f(z^{-1}y^{-1}\bar{x})d\mu(\bar{x}) \\ &= \int_{G/H} f(z^{-1}\bar{x})R(\bar{x},y)d\mu(\bar{x}) \\ &= \int_{G/H} f(\bar{x})R(z\bar{x},y)R(\bar{x},z)d\mu(\bar{x}).\end{aligned}$$

Hence

$$(3.11) \qquad R(\bar{x},yz) = R(z\bar{x},y)R(\bar{x},z), \quad x,y,z \in G,$$

so $R(\bar{e},yx) = R(\bar{x},y)R(\bar{e},x)$. If we define a function ρ on G by (3.10) then ρ is continuous and strictly positive on G and R can be recovered from ρ by (3.8).

Next consider the measure $\mu^{\#}$ on G. We have

$$R(\bar{x},y) = \frac{d\mu_y}{d\mu}(\bar{x}) = \frac{d\mu_y^{\#}}{d\mu^{\#}}(x),$$

by virtue of (3.9) and Corollary 3.5. By Lemma 3.7, $\mu^{\#}$ is equivalent to the Haar measures on G. Let ϕ be a version of the Radon-Nikodym derivative $d\mu^{\#}/d\nu_G$. Then

$$\frac{d\mu_y^{\#}}{d\mu^{\#}}(x) = \frac{d\mu^{\#}(yx)}{d\mu^{\#}(x)} = \frac{\phi(yx)}{\Phi(x)}\frac{d\nu_G(yx)}{d\nu_G(x)} = \frac{\phi(yx)}{\phi(x)}.$$

Combining (3.8) with the last two equalities we find that for each y in G $\rho(yx)/\rho(x) = \phi(yx)/\phi(x)$ for almost all x.
Note that

$$F(x,y) := |\rho(yx)/\rho(x)-\phi(yx)/\phi(x)|$$

is a nonnegative Borel function on G. Hence, applying Fubini's theorem, we obtain:

$$0 = \int_G\left(\int_G F(x,y)d\nu_G(x)\right)d\nu_G(y) = \int_G\left(\int_G F(x,y)d\nu_G(y)\right)d\nu_G(x).$$

Hence $\int_G F(x,y)d\nu_G(y) = 0$ for almost all $x \in G$. Therefore we can choose $x \in G$ such that $F(x,y) = 0$ for almost all y. We conclude that, for some $c > 0$, $\rho(z) = c\phi(z)$ for almost all $z \in G$. Thus the measure ν on G defined by (3.7) equals $c\mu^{\#}$. Application of Lemma 3.6 gives for $h \in H$:

$$d\nu(xh) = \Delta_H(h)d\nu(x) = \Delta_H(h)\rho(x)/d\nu_G(x).$$

On the other hand

$$d\nu(xh) = \rho(xh)d\nu_G(xh) = \Delta_G(h)\rho(xh)d\nu_G(x).$$

Hence ρ satisfies (3.6). □

For future reference we state some properties of continuous R-functions corresponding with quasi-invariant measures by (3.9). These properties can be verified either by direct computation or by using the preceding theorem.

(3.11) $R(\bar{x},yz) = R(z\bar{x},y)R(\bar{x},z)$, $\bar{x} \in G/H$, $y,z \in G$;

(3.12) $R(\bar{x},e) = 1$, $\bar{x} \in G/H$;

(3.13) $(R(\bar{x},y))^{-1} = R(y\bar{x},y^{-1})$, $\bar{x} \in G/H$, $y \in G$;

(3.14) $R(\bar{e},h) = \dfrac{\Delta_H(h)}{\Delta_G(h)}$, $h \in H$.

Suppose that we are given a strictly positive continuous function R on $G/H \times G$ which satisfies (3.11) and (3.14). Then, if we define ρ by (3.10), we find

$$\rho(yh) = R(\bar{e},yh) = R(\bar{e},y).R(\bar{e},h) = \rho(y)\cdot\frac{\Delta_H(h)}{\Delta_G(h)}, \quad y \in G,\ h \in H.$$

In this way we obtain a quasi-invariant measure on G/H, corresponding with R via (3.9) and with ρ via $d\mu^{\#}(x) = \rho(x)d\nu_G(x)$.

REMARK. Let ρ_1 and ρ_2 be continuous, strictly positive solutions of (3.6), and let μ_1 and μ_2 be the corresponding quasi-invariant measures on G/H. Then

$$\frac{d\mu_1}{d\mu_2}(\bar{x}) = \frac{\rho_1(x)}{\rho_2(x)} =: F(\bar{x}).$$

The function F, thus defined, is continuous and strictly positive on G/H and it satisfies

$$\text{(3.15)} \qquad F(\bar{x})R_1(\bar{x},y) = F(y\bar{x})R_2(\bar{x},y),$$

where $R_i(\bar{x},y) = \rho_i(yx)/\rho_i(x), \quad i = 1,2.$

Proof of Lemma 3.10. Let f be a function in $K(G)$ with $f \geq 0$, and set

$$\rho_f(x) := \int_H f(xh)\,\frac{\Delta_G(h)}{\Delta_H(h)}\,d\nu_H(h), \quad x \in G.$$

Then ρ_f defines a positive function on G, which is continuous since f is uniformly continuous. Moreover,

$$\frac{\Delta_G(h_0)}{\Delta_H(h_0)}\,\rho_f(xh_0) = \int_H f(xh_0h)\frac{\Delta_G(h_0h)}{\Delta_H(h_0h)}\,d\nu_H(h) = \rho_f(x), \quad h_0 \in H,$$

so ρ satisfies (3.6). However, ρ_f will fail to be strictly positive in general. This can be repaired as follows. Let f_y denote the function $x \to f(xy^{-1})$. Furthermore, suppose that we are given a subset X of G such that

(i) for each compact subset $K \subset G$ the equality $\rho_{f_y} = 0$ holds on K for all but finitely many y in X;

(ii) for each x in G, $\rho_{f_y} \neq 0$ for some y in X.

Then it is clear that

$$\rho(x) := \sum_{y \in X} \rho_{f_y}(x), \quad x \in G,$$

defines a strictly positive continuous function on G satisfying (3.6).

Let $S := \{x \in G \mid f(x) > 0\}$ and suppose that $\rho \in S$ and $S = S^{-1}$ (which is, of course, legal). In the next lemma we will prove the existence of a set $X(S) \subset G$ which satisfies the following properties:

(a) For each x in G, $xH \cap Sy \neq \emptyset$ for some y in $X(S)$;

(b) for each compact subset $K \subset G$ we have $KH \cap Sy = \emptyset$ for all but finitely many y in $X(S)$.

It is obvious that this set $X(S)$ satisfies the conditions (i) and (ii) stated above. This concludes the proof of Lemma 3.10.

LEMMA. *Let* S *be an open symmetric neighbourhood of the identity in* G, *with compact closure. Then there exists a subset* $X(S)$ *of* G *which satisfies the above properties* (a) *and* (b).

PROOF. Consider the family of subsets X of G which satisfy the following symmetric condition: If $x,y \in X$ and $x \neq y$ then $x \notin SyH$. Note that this family is nonempty. It is partially ordered by inclusion, with each chain having an upper bound. Hence, we can apply Zorn's lemma and choose a maximal set, say $X(S)$. We contend that $X(S)$ meets the qualifications stated in the lemma. First, suppose $xH \cap Sy = \emptyset$ for a certain x in G and all y in $X(S)$. Clearly this contradicts the maximality of $X(S)$. As to (b), suppose that there are a compact subset K of G and countably many distinct elements $y_1, y_2, \ldots$ in $X(S)$ such that $KH \cap Sy_i \neq \emptyset$ for all i. Then there are elements $h_1, h_2, \ldots$ in H with $y_i h_i \in SK$ for all i. Since the closure of SK is compact, the sequence $\{y_i h_i\}$ has a cluster point. Hence, by passing to a convergent subsequence, we must have $y_m h_m \in Sy_n h_n$ for m,n large enough. Hence, for sufficiently large m,n we have $y_m \in Sy_n H$, and therefore $y_m = y_n$, which yields a contradiction. □

REMARK. If G is a Lie group then, in the above proof of Lemma 3.10, the function f can be chosen as a nonnegative C^∞-function with compact support.

In that case it follows that, by construction, ρ_f and ρ are also C^∞-functions on G. Thus the function

$$(\bar{x},y) \mapsto R(\bar{x},y) = \rho(yx)/\rho(x)$$

is a C^∞-function on $G/H \times G$. It is not known to the authors whether, on a Lie group G, there always exists a strictly positive analytic solution ρ of the functional equation (3.6).

EXAMPLE 3.12. In example 2.2 we showed that the homogeneous space $SO(n)/SO(n-1)$ is homeomorphic with the unit sphere S^{n-1} of dimension n-1. Since $SO(n-1)$ is a compact subgroup of $SO(n)$ for all $n = 1,2,\ldots$, there exists an invariant measure on S^{n-1}. This is the well-known rotation invariant measure.

EXAMPLE 3.13 (cf. V.3.4(e),(f)). Consider the subgroup H of $GL(2,\mathbb{R})$ consisting of all real matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$. The group $GL(2,\mathbb{R})$ can be identified in a natural way with a subset of $\mathbb{R}^4$. Let λ^4 be the Lebesgue measure on $\mathbb{R}^4$ and set

$$d\nu_{GL(2,\mathbb{R})}(x) := |\det(x)|^{-2} d\lambda^4(x), \quad x \in GL(2,\mathbb{R}).$$

Then one readily verifies that ν is a left and right invariant measure on $GL(2,\mathbb{R})$, and therefore this group is unimodular. However, if we let λ denote the Lebesgue measure on $\mathbb{R}$, and if we set

$$d\nu_H\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) := a^{-2} d\lambda(a) d\lambda(b),$$

then ν_H defines a left Haar measure on H, which is obviously not right invariant. The modular function on H is given by

$$\Delta_H\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\right) = a^{-1} = \det\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}\right),$$

cf. V.3.11(c). Define a function ρ on $GL(2,\mathbb{R})$ by

$$\rho(x) := |\det(x)|^{-1}.$$

Then ρ is a strictly positive continuous solution of equation (2.4). Hence, the measure ν on $GL(2,\mathbb{R})$ defined by

$$d\nu(x) := \rho(x)d\nu_{GL(2,\mathbb{R})}(x)$$

is quasi-invariant on $GL(2,\mathbb{R})$ and it lies in the image of the mapping $\mu \mapsto \mu^{\#}$. The quasi-invariant measure μ on $GL(2,\mathbb{R})/H$ with $\mu^{\#} = \nu$ can now be expressed in terms of $\nu_{GL(2,\mathbb{R})}$ and ρ. The corresponding R-function on $GL(2,\mathbb{R})/H \times GL(2,\mathbb{R})$ is given by

$$R(\bar{x},y) = \frac{\rho(yx)}{\rho(x)} = |\det(y)|^{-1}.$$

Notice that this function is independent of $\bar{x}$. This means that the Radon-Nikodym derivative $d\mu_y/d\mu$ is constant for all y in $GL(2,\mathbb{R})$. Quasi-invariant measures with this property are called *relatively invariant*. One easily proves the following criterion for the existence of relatively invariant measures on a coset space G/H:

<u>THEOREM 3.14</u>. *There exist relatively invariant measures on* G/H *if and only if the function* ρ *of Lemma* 3.10 *can be chosen such that* $\rho(x)\rho(y) = \rho(xy)$ *for all* x,y *in* G.

<u>EXAMPLE 3.15</u>. Consider the case where G is the product of two closed subgroups K and H, with $K \cap H = \{e\}$, and with the mapping $kh \to (k,h)$ from G onto $K \times H$ being continuous. Then an explicit expression for a quasi-invariant measure on G/H can be rather easily found as follows. Observe that G and $K \times H$ are homeomorphic. This implies $G/H \approx (K\times H)/H \approx K$, where the homeomorphism from G/H onto K is given by sending $\bar{x} = xH$ to the projection of x on K. We denote the projection of G on K and H by π_1 and π_2, respectively, that is,

$$(3.17) \qquad \pi_1(kh) := k, \quad \pi_2(kh) := h, \quad k \in K,\ h \in H.$$

Define a function ρ on G by

$$(3.18) \qquad \rho(x) := \frac{\Delta_H(\pi_2(x))}{\Delta_G(\pi_2(x))}.$$

Then ρ is single-valued, continuous and strictly positive. Moreover, it satisfies (3.6). Denote by μ the corresponding quasi-invariant measure on G/H. For the R-function we find

$$R(\bar{x},y) = \frac{\Delta_H(\pi_2(yx)(\pi_2(x))^{-1})}{\Delta_G(\pi_2(yx)(\pi_2(x))^{-1})} .$$

If we identify the homeomorphic spaces G/H and K, this expression reduces to

$$(3.19) \qquad R(k,y) = \frac{\Delta_H(\pi_2(yk))}{\Delta_G(\pi_2(yk))} .$$

In particular, $R(k,y) = 1$ if $y \in K$. Therefore, μ is invariant for the G-action on G/H restricted to K, so μ is, under the above identification, equal to the left Haar measure on K. (Note that the K-action on G/H reduces to left multiplication under this identification.)

We are now able to express the Haar measure on G in terms of those on K and H. Let $f \in \mathcal{K}(G)$. Then

$$\int_G f(x)\rho(x)d\nu_G(x) = \mu^{\#}(f) = \mu(\tilde{f}) = \nu_K(\tilde{f}) =$$

$$= \int_K \left(\int_H f(Kh)d\nu_H(h) \right) d\nu_K(k) .$$

Hence,

$$\int_G f(x)d\nu_G(x) = \int_K \int_H f(kh)(\rho(h))^{-1}d\nu_H(h)d\nu_K(k) .$$

In combination with (3.18) this yields

$$(3.20) \qquad d\nu_G(x) = \frac{\Delta_G(h)}{\Delta_H(H)} \; d\nu_K(k)d\nu_H(h) \qquad (x=kh) .$$

In the case where G is unimodular, (3.20) implies

$$(3.21)\qquad d\nu_G(x) = d\nu_K(k)\,d\nu_H^{(r)}(h) \qquad (x=kh),$$

where $\nu_H^{(r)}$ is a right Haar measure on H. (Recall that $d\nu_H^{(\ell)}(x) = \Delta_H(x)d\nu_H^{(r)}(x)$ relates the left and right Haar measures on H, cf. (V.3.23).)

For instance, the situation sketched above is encountered in the case of semi-simple Lie groups which are non-compact and connected and have finite center. Indeed, these groups admit a so-called Iwasawa decomposition G = KAN, where K is compact, A is abelian and closed, and N is nilpotent and closed. Moreover, it is known that the mapping $(k,a,n) \to kan$ from $K\times A\times N$ onto G is an analytic diffeomorphism (see HELGASON [3.thm. VI.5.1]). If we set H = AN, we obtain the situation above.

EXAMPLE 3.16. Consider the case that $G = SL(2,\mathbb{R})$, the group of real 2×2 matrices with determinant 1. Then K = SO(2), the special orthogonal group in two dimensions, and

$$A = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\middle|\, a > 0\right\},\quad N = \left\{\begin{pmatrix} 1 & 0 \\ c & 1\end{pmatrix}\middle|\, c \in \mathbb{R}\right\}.$$

Hence

$$H = \left\{\begin{pmatrix} a & 0 \\ c & a^{-1}\end{pmatrix}\middle|\, a > 0,\ c \in \mathbb{R}\right\}.$$

It is rather tedious to compute explicit expressions for π_1 and π_2 in this case, and therefore we use another method. The group $SL(2,\mathbb{R})$ acts on the one-dimensional real projective space $\mathbb{P}_1(\mathbb{R})$. This space can be obtained by identifying nonzero vectors in $\mathbb{R}^2$ which are scalar multiples of each other. By choosing so-called inhomogeneous coordinates, we can identify $\mathbb{P}_1(\mathbb{R})$ with the extended real line $\mathbb{R} \cup \{\infty\}$. Indeed, let $[x,y]$ denote an equivalence class in $\mathbb{R}^2$, and set $[x,y] \mapsto t = \frac{x}{y}$, $y \neq 0$, and $[x,0] \mapsto \{\infty\}$. The corresponding action of $SL(2,\mathbb{R})$ on $\mathbb{R} \cup \{\infty\}$ reads

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}(t) = \frac{at+b}{ct+d}.$$

The expression on the right-hand side becomes $\frac{a}{c}$ if $t = \{\infty\}$, and $\{\infty\}$ if $t = -\frac{d}{c}$. The stabilizer of $t = 0$ consists of all real matrices

$$\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, \quad a \neq 0,$$

and is thus equal to $H \times \mathbb{Z}_2$. Define a measure μ on $\mathbb{R} \cup \{\infty\}$ such that $\mu(\{\infty\}) := 0$ and $d\mu(t) := (1+t^2)^{-1} d\lambda(t)$ on $\mathbb{R}$. Then μ is SO(2)-invariant and quasi-invariant for the action of $SL(2,\mathbb{R})$. We can compute the corresponding R-function directly:

$$d\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}.t\right) = \frac{1}{1+(at+b)^2(ct+d)^{-2}}\, d\lambda\left(\frac{at+b}{ct+d}\right) =$$

$$= \frac{(ad-cb)\,d\lambda(t)}{(at+b)^2+(ct+d)^2} = \frac{d\lambda(t)}{(at+b)^2+(ct+d)^2} =$$

$$= \frac{1+t^2}{(at+b)^2+(ct+d)^2}\, d\mu(t) = R\left(t, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) d\mu(t).$$

4. INDUCED REPRESENTATIONS

In section 1 we already defined induced representations in an informal way. Here we repeat this more formally, proving each step and starting at once with the most general case. References are, for instance, MACKEY [4], [5], [6,§3.2], [7,Ch.1] and BARUT & RĄCZKA [1,Ch.16].

Let G be a lcsc. group, let H be a closed subgroup of G and let τ be a unitary representation of H on a Hilbert space $\mathcal{H}$. We consider the linear space $\mathcal{F}_\tau(G,\mathcal{H})$ consisting of all functions $f : G \to \mathcal{H}$ that satisfy

$$(4.1) \qquad f(xh) = \tau(h^{-1})f(x), \quad x \in G,\ h \in H.$$

Choose a nonzero quasi-invariant measure μ on G/H such that the corresponding function R_μ, defined by (3.2) is continuous and strictly positive on $G/H \times G$. Such a measure always exists in view of Lemma 3.10 and Theorem 3.11. For each $f \in \mathcal{F}_\tau(G,\mathcal{H})$ and $y \in G$ the $\mathcal{H}$-valued function $\hat{\tau}(y)f$ defined by

$$(4.2) \qquad (\hat{\tau}(y)f)(x) := (R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}} f(y^{-1}x)$$

again satisfies (4.1). Furthermore, for $f \in F_\tau(G,H)$ and $y,z \in G$ we have:

$$(4.3)\qquad \begin{cases} \hat{\tau}(yz)f = \hat{\tau}(y)(\hat{\tau}(z)f), \\ \hat{\tau}(e)f = f \end{cases}$$

The second equality follows from (3.12) . As to the first equality we have

$$(\hat{\tau}(yz)f)(x) = (R_\mu(\bar{x},z^{-1}y^{-1}))^{\frac{1}{2}}f(z^{-1}y^{-1}x) =$$

$$= (R_\mu(y^{-1}\bar{x},z^{-1}))^{\frac{1}{2}}(R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}}f(z^{-1}y^{-1}x) =$$

$$= (R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}}(\hat{\tau}(z)f)(y^{-1}x) = (\hat{\tau}(y)(\hat{\tau}(z)f)(x),$$

where we used (3.11). Thus $\hat{\tau}$ is a homomorphism from G into the group of invertible linear transformations of $F_\tau(G,H)$.

4.1. Continuous functions with compact support modulo H : the space $K_\tau(G,H)$

Let $K_\tau(G,H)$ be the linear space of all continuous functions $f : G \to H$ such that (i) f satisfies (4.1) and (ii) the support of f is contained in a set KH, where K is some compact subset of G. Then $K_\tau(G,H)$ is a $\hat{\tau}$-invariant linear subspace of $F_\tau(G,H)$. (Use the continuity of R_μ.)

If $f_1,f_2 \in K_\tau(G,H)$ then the complex-valued function $x \mapsto (f_1(x),f_2(x))_H$ is constant on left H-cosets of G because of (4.1) and unitarity of τ, and it is continuous on G, with support contained in a set KH for some compact $K \subset G$. It follows that $\bar{x} \mapsto (f_1(x),f_2(x))$ is a well-defined function on G/H which is continuous and has compact support.

For $f_1,f_2 \in K_\tau(G,H)$ define

$$(4.4)\qquad (f_1,f_2) := \int_{G/H} (f_1(x),f_2(x))_H \, d\mu(\bar{x}).$$

In view of the previous paragraph the integral at the right hand side of (4.4) is well-defined. It follows easily that (.,.) has all the properties of an inner product on $K_\tau(G,H)$. Moreover, this inner product is $\hat{\tau}$-invariant:

$$(4.5) \qquad (\hat{\tau}(y)f_1, \hat{\tau}(y)f_2) = (f_1, f_2), \quad y \in G,$$

where $f_1, f_2 \in K_\tau(G,H)$. Indeed:

$$(\hat{\tau}(y)f_1, \hat{\tau}(y)f_2) = \int_{G/H} (f_1(y^{-1}x), f_2(y^{-1}x))_H R_\mu(\bar{x}, y^{-1}) d\mu(\bar{x}) =$$

$$= \int_{G/H} (f_1(x), f_2(x))_H d\mu(\bar{x}) = (f_1, f_2).$$

Here we used (3.2).

Note that, for H compact, $K_\tau(G,H)$ is included in $K(G,H)$ (the space of all continuous functions $f : G \to H$ with compact support), the measure μ is invariant (i.e., $R_\mu \equiv 1$) and (4.4) simplifies to (1.3).

Now we show that the representation $\hat{\tau}$ of G on the inner product space $K_\tau(G,H)$ is weakly continuous.

<u>LEMMA 4.1.</u> *If* $f_1, f_2 \in K_\tau(G,H)$ *then the function* $y \to (\hat{\tau}(y)f_1, f_2)$ *is continuous on* G.

<u>PROOF</u>. First we show that all functions f in $K_\tau(G,H)$ are uniformly continuous on G. Let $\mathrm{supp}(f) \subset KH$, where $K \subset G$ is compact. Let $\varepsilon > 0$. For each $x \in K$ let $V_{\varepsilon,x}$ be a symmetric neighbourhood of e in G such that $\| f(y^{-1}x) - f(x) \|_H < \frac{1}{2}\varepsilon$ if $y \in (V_{\varepsilon,x})^2$. The sets $V_{\varepsilon,x}\bar{x}$, $x \in K$, form an open cover of the compact subset $\pi(K)$ of G/H. Hence, there are finitely many points $x_1, \ldots, x_n \in K$ such that $KH \subset \bigcup_{i=1}^n V_{\varepsilon,x_i} x_i H$. Let $V_\varepsilon := \bigcap_{i=1}^n V_{\varepsilon,x_i}$. Let $z_1, z_2 \in G$, $z_2^{-1}z_1 \in V_\varepsilon$. We will show that $\| f(z_1) - f(z_2) \|_H < \varepsilon$. This inequality clearly holds if $z_1, z_2 \notin KH$. Suppose that $z_1 \in KH$. Then $z_1 h x_k^{-1} \in V_{\varepsilon,x_k}$ for some $k \in \{1,\ldots,n\}$ and some $h \in H$. Hence $z_2 h x_k^{-1} = (z_2 z_1^{-1})(z_1 h x_k^{-1}) \in V_\varepsilon V_{\varepsilon,x_k} \subset (V_{\varepsilon,x_k})^2$. Now

$$\| f(z_1) - f(z_2) \|_H = \| \tau(h^{-1}) f(z_1) - \tau(h^{-1}) f(z_2) \|_H =$$

$$= \| f(z_1 h) - f(z_2 h) \|_H \le \| f(z_1 h) - f(x_k) \|_H + \| f(z_2 h) - f(x_k) \|_H <$$

$$< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus f is uniformly continuous on G.

Let $f_1, f_2 \in K_\tau(G,H)$. Since $\hat{\tau}$ is a homomorphism, it is sufficient to prove continuity of $y \to (\hat{\tau}(y)f_1, f_2)$ at e. We have:

$$(\hat{\tau}(y)f_1,f_2)-(f_1,f_2) =$$

$$= \int_{G/H} ((R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}}f_1(y^{-1}x)-f_1(x),f_2(x))_H \, d\mu(\bar{x}) =$$

$$= \int_{G/H} ((R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}}-1)(f_1(y^{-1}x),f_2(x))_H \, d\mu(\bar{x}) +$$

$$+ \int_{G/H} (f_1(y^{-1}x)-f_1(x),f_2(x))_H \, d\mu(\bar{x}).$$

Let M_1, M_2, M_3 be positive numbers such that $\|f_i(x)\|_H \le M_i$ on G, i = 1,2, and $\mu(\overline{\text{supp}(f_2)}) \le M_3$. Let $\varepsilon > 0$. There is a neighbourhood V of e such that

$$|(R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}}-1| < \frac{\varepsilon}{2M_1M_2M_3} \quad \text{if } \bar{x} \in \overline{\text{supp}(f_2)},\ y \in V$$

(here we used (3.12) and the continuity of R_μ), and

$$\|f_1(y^{-1}x)-f_1(x)\|_H < \frac{\varepsilon}{2M_2M_3} \quad \text{if } y \in V,\ x \in G.$$

Hence, if $y \in V$ then $|(\hat{\tau}(y)f_1,f_2)-(f_1,f_2)| < \varepsilon$. □

So $\hat{\tau}$ is a weakly continuous homomorphism from G into the group of unitary transformations of the inner product space $K_\tau(G,H)$. Let $\overline{K_\tau(G,H)}$ be the Hilbert space completion of $K_\tau(G,H)$. Then, for each $x \in G$, the operator $\hat{\tau}(x)$ has a unique extension to a unitary operator on this Hilbert space. The homomorphism property (4.3) and the weak continuity of $\hat{\tau}$ are preserved under this extension. Indeed, it can easily be proved that the following holds:

LEMMA 4.2. *Let σ_0 be a weakly continuous homomorphism from a lcsc. group G into the group of unitary operators on a pre-Hilbert space V_0. Let $\sigma(x)$ $(x\in G)$ be the unique extension of $\sigma_0(x)$ to a bounded linear operator on the Hilbert space completion V of V_0. Then σ is a unitary representation of G on V.*

The extension of $\hat{\tau}$ to $\overline{K_\tau(G,H)}$ (again denoted by $\hat{\tau}$) is called the representation of G *induced* by the unitary representation τ of H. In fact, $\hat{\tau}$ also depends on the choice of the quasi-invariant measure μ. However, we have:

LEMMA 4.3. *Let μ and ν be quasi-invariant measures on G/H with corresponding continuous R-functions R_μ and R_ν, respectively. Let $\hat{\tau}_\mu$ and $\hat{\tau}_\nu$ be the representations of G induced by τ with respect to μ and ν, respectively. Then $\hat{\tau}_\mu$ and $\hat{\tau}_\nu$ are equivalent.*

PROOF. Let $F_{\mu,\nu}$ be the strictly positive continuous function on G/H such that $d\nu(\bar{x}) = F_{\mu,\nu}(\bar{x})d\mu(\bar{x})$. Then the linear operator A defined by

$$(AF)(x) := (F_{\mu,\nu}(\bar{x}))^{-\frac{1}{2}}f(x), \quad x \in G,\ f \in K_\tau(G,H),$$

maps $K_\tau(G,H)$ onto itself and

$$\int_{G/H} ((Af_1)(x),(Af_2)(x))_H \, d\nu(\bar{x}) = \int_{G/H} (f_1(x),f_2(x))_H \, d\mu(\bar{x}).$$

Hence A extends to an isometry from $H(\hat{\tau}_\mu)$ onto $H(\hat{\tau}_\nu)$. It follows from (3.15) that

$$(A(\hat{\tau}_\mu(y)f))(x) = (F_{\mu,\nu}(\bar{x}))^{-\frac{1}{2}}(R_\mu(\bar{x},y^{-1}))^{\frac{1}{2}}f(y^{-1}x) =$$

$$= (R_\nu(\bar{x},y^{-1}))^{\frac{1}{2}}(F_{\mu,\nu}(y^{-1}\bar{x}))^{-\frac{1}{2}}f(y^{-1}x) = (\hat{\tau}_\nu(y)(Af))(x),$$

where $x,y \in G$, $f \in K_\tau(G,H)$. Hence A is an intertwining operator for $\hat{\tau}_\mu$ and $\hat{\tau}_\nu$. □

We conclude that the equivalence class of the representation $\hat{\tau}$ induced by τ is independent of the choice of the quasi-invariant measure on G/H.

A straightforward proof also shows that equivalent representations of H induce equivalent representations of G. Hence we may speak about the equivalence class of representations of G induced by an equivalence class of representations of H. We will often write τ^G instead of $\hat{\tau}$.

4.2. $K_\tau(G,H)$ has nonzero dimension

One important question remained unanswered in the previous subsection: Is the induced representation τ^G nontrivial in the sense that its representation space $\overline{K_\tau(G,H)}$ has nonzero dimension? Fortunately, the answer is positive. It is based on the following relationship between $K_\tau(G,H)$ and the linear space $K(G,H)$ consisting of all H-valued continuous functions with compact support on G:

LEMMA 4.4. *The linear mapping* $f \to \tilde{f}$ *defined by*

$$(4.6) \qquad \tilde{f}(x) := \int_H \tau(h)f(xh)dh$$

is a surjection from $K(G,H)$ *onto* $K_\tau(G,H)$.

PROOF. For each $x \in G$ the function $h \to \tau(h)f(xh)$ is continuous with compact support from H to H. Hence the right hand side of (4.6) is well-defined as a vector-valued integral (cf. V.1.16; in the following we will use some of the properties of vector-valued integrals mentioned there).

First we prove that $\tilde{f} \in K_\tau(G,H)$. We have $\mathrm{supp}(\tilde{f}) = \mathrm{supp}(f).H$. Hence $\tilde{f}$ has compact support modulo H. If $x \in G$, $h_0 \in H$ then

$$\tilde{f}(xh_0) = \int_H \tau(h)f(xh_0h)dh = \int_H \tau(h_0^{-1}h)f(xh)dh =$$

$$= \tau(h_0^{-1}) \int_H \tau(h)f(xh)dh = \tau(h_0^{-1})\tilde{f}(x).$$

Hence $\tilde{f}$ satisfies (4.1). Now we prove continuity of $\tilde{f}$. Let V be a compact neighbourhood of e in G. Then $K := V.\mathrm{supp}(f)$ is compact. Let $\varepsilon > 0$. Since $f \in K(G,H)$ is uniformly continuous, there is a symmetric neighbourhood $U \subset V$ of e such that $\|f(x) - f(y)\|_H < \varepsilon$ if $xy^{-1} \in U$. Fix $x_0 \in G$. Let $x \in G$ such that $xx_0^{-1} \in U$. Then

$$\|\tilde{f}(x) - \tilde{f}(x_0)\|_H \le \int_H \|\tau(h)\| \; \|f(xh) - f(x_0h)\|_H \, dh =$$

$$= \int_H \|f(xh) - f(x_0h)\|_H \, dh.$$

In the last expression the integrand has support in $(x^{-1}\mathrm{supp}(f) \cup x_0^{-1}\mathrm{supp}(f)) \cap H$. Observe that $x_0^{-1}\mathrm{supp}(f) \subset x_0^{-1}K$ and $x^{-1}\mathrm{supp}(f) = x_0^{-1}(x_0x^{-1})\mathrm{supp}(f) \subset x_0^{-1}K$. Hence, since $(xh)(x_0h)^{-1} = xx_0^{-1} \in U$:

$$\|\tilde{f}(x) - \tilde{f}(x_0)\|_H \le \varepsilon \, \nu_H((x_0^{-1}K) \cap H).$$

This completes the proof that $\tilde{f} \in K_\tau(G,H)$.

We now show surjectivity of the mapping $f \to \tilde{f}$. In the case that $\dim H = 1$ and τ is the trivial representation of H this was proved in Lemma 3.2. We will reduce the general case to this lemma. Let $p \in K_\tau(G,H)$. Choose a function $g \in K(G/H)$ such that $g(\bar{x}) = 1$ if $x \in \mathrm{supp}(p)$. By Lemma 3.2 there is a function $g_1 \in K(G)$ such that $g(\bar{x}) = \int_H g_1(xh)dh$. Let $f(x) := g_1(x)p(x)$. Then $f \in K(G,H)$ and

$$\tilde{f}(x) = \int_H \tau(h)(g_1(xh)p(xh))dh = \int_H g_1(xh)p(x)dh =$$

$$= g(\bar{x})p(x) = p(x). \quad \square$$

PROPOSITION 4.5. *For each* $x \in G$ *the set* $\{f(x) \mid f \in K_\tau(G,H)\}$ *is dense in* H.

PROOF. Fix $x \in G$. Let $v \in H$. In view of the previous lemma it is sufficient to show that $v = 0$ if $(\tilde{f}(x),v)_H = 0$ for all $f \in K(G,H)$. Let $\alpha \in K(G)$ and $f(y) := \alpha(y)v$, $y \in G$. Then $0 = (\tilde{f}(x),v)_H = \int_H(\tau(h)f(xh),v)_H\, dh = \int_H \alpha(xh)(\tau(h)v,v)_H\, dh$. Suppose that $v \neq 0$. There is a neighbourhood V of e in G such that $\mathrm{Re}(\tau(h)v,v)_H > 0$ for $h \in V \cap H$. Choose $\alpha \in K(G)$ such that α is nonnegative with support in $x\,V$ and such that $\alpha(x) > 0$. Then

$$0 = \int_H \alpha(xh)\mathrm{Re}(\tau(h)v,v)_H\, dh > 0.$$

This is a contradiction. $\square$

COROLLARY 4.6. *If* $\dim H \neq 0$ *then* $\dim K_\tau(G,H) \neq 0$.

PROOF. Let $f \in K_\tau(G,H)$ with $f(e) \neq 0$. (The existence of such f is ensured by the preceding proposition.) Then

$$\|f\|^2 = \int_{G/H} \|f(x)\|_H^2\, d\mu(\bar{x}) > 0. \quad \square$$

4.3. A realization of τ^G on the space $L^2_\tau(G,H)$

Remember that $L^2(G)$ is a model for the Hilbert space completion of $K(G)$, cf. V.3.5. Similarly, we will realize the completion of K_τ in terms of certain H-valued L^2-functions.

First consider the linear space $\mathcal{B}_\tau(G,\mathcal{H})$ which consists of all $\mathcal{H}$-valued functions f on G satisfying (4.1) which are weakly Haar measurable, that is, for each $v \in \mathcal{H}$ the function $x \mapsto (f(x),v)_{\mathcal{H}}$ is a Borel function on G (with respect to the Haar measure). Note that $\mathcal{K}_\tau(G,\mathcal{H}) \subset \mathcal{B}_\tau(G,\mathcal{H}) \subset \mathcal{F}_\tau(G,\mathcal{H})$ and that $\mathcal{B}_\tau(G,\mathcal{H})$ is invariant under $\hat{\tau}$.

If $f_1,f_2 \in \mathcal{B}_\tau(G,\mathcal{H})$ then the complex-valued function $x \to (f_1(x),f_2(x))_{\mathcal{H}}$ is constant on left H-cosets of G because of (4.1). Furthermore, if $\{e_1,e_2,\ldots\}$ is an orthonormal basis of $\mathcal{H}$ then

$$(f_1(x),f_2(x))_{\mathcal{H}} = \sum_i (f_1(x),e_i)_{\mathcal{H}}\overline{(f_2(x),e_i)_{\mathcal{H}}}.$$

Hence the function $x \mapsto (f_1(x),f_2(x))_{\mathcal{H}}$, being a countable sum of Borel functions, is a Borel function on G itself. It follows that $\bar{x} \mapsto (f_1(x),f_2(x))_{\mathcal{H}}$ is a well-defined Borel function on G/H. Thus, for $f \in \mathcal{B}_\tau(G,\mathcal{H})$ the definition

$$(4.7) \qquad \|f\|^2 := \int_{G/H} (f(x),f(x))_{\mathcal{H}}\, d\mu(\bar{x})$$

makes sense, and $0 \le \|f\| \le \infty$. Analogous to the proof of (4.5) we can verify that

$$(4.8) \qquad \|\hat{\tau}(y)f\| = \|f\|, \quad f \in \mathcal{B}_\tau(G,\mathcal{H}),\ y \in G.$$

Next, let $\mathcal{L}^2_\tau(G,\mathcal{H})$ be the linear space consisting of all $f \in \mathcal{B}_\tau(G,\mathcal{H})$ such that $\|f\| < \infty$. Then $\mathcal{K}_\tau(G,\mathcal{H}) \subset \mathcal{L}^2_\tau(G,\mathcal{H})$ and (4.8) shows that $\mathcal{L}^2_\tau(G,\mathcal{H})$ is invariant under $\hat{\tau}$. Now (f_1,f_2) is well defined by (4.4) if $f_1,f_2 \in \mathcal{L}^2_\tau(G,\mathcal{H})$ and it has all the properties of an inner product for $\mathcal{L}^2_\tau(G,\mathcal{H})$ except that it is usually not positive definite. However, this can easily be repaired by identifying f_1 and $f_2 \in \mathcal{L}^2_\tau(G,\mathcal{H})$ if $\|f_1-f_2\| = 0$ or, equivalently, if $f_1(x) = f_2(x)$ almost everywhere on G. The resulting inner product space is denoted by $L^2_\tau(G,\mathcal{H})$.

Now we will show that $L^2_\tau(G,\mathcal{H})$ is a Hilbert space and that $\mathcal{K}_\tau(G,\mathcal{H})$ is dense in $L^2_\tau(G,H)$. The proof of the first statement is analogous to the proof in the case of $L^2(G)$ (cf. RUDIN [9, Theorem 3.11]).

<u>PROPOSITION 4.7</u>. *$L^2_\tau(G,\mathcal{H})$ is a Hilbert space.*

PROOF. We have to show convergence of any Cauchy sequence $\{f_n\}$ in $L^2_\tau(G,H)$. The sequence $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ with $\| f_{n_{i+1}} - f_n \| < 2^{-i}$ and convergence of $\{f_{n_i}\}$ implies convergence of $\{f_n\}$. Hence, without loss of generality, we may assume that $\| f_{n+1} - f_n \| < 2^{-n}$. By application of the monotone convergence theorem (cf. V.1.5(ii)) we obtain

$$\left(\int_{G/H} \left(\sum_{n=1}^{\infty} \| f_{n+1}(x) - f_n(x) \|_H \right)^2 d\mu(\bar{x}) \right)^{\frac{1}{2}} =$$

$$= \lim_{N\to\infty} \left(\int_{G/H} \left(\sum_{n=1}^{N} \| f_{n+1}(x) - f_n(x) \|_H \right)^2 d\mu(\bar{x}) \right)^{\frac{1}{2}} \le$$

$$\le \lim_{N\to\infty} \sum_{n=1}^{N} \left(\int_{G/H} \| f_{n+1}(x) - f_n(x) \|_H^2 \, d\mu(\bar{x}) \right)^{\frac{1}{2}} =$$

$$= \sum_{n=1}^{\infty} \| f_{n+1} - f_n \| < 1.$$

Hence $\sum_{n=1}^{\infty} \| f_{n+1}(x) - f_n(x) \|_H < \infty$ except for $\bar{x}$ being in some Borel set of measure zero in G/H. Put $f_n(x) = 0$ for $\bar{x}$ in this exceptional set. This modified f_n still belongs to $\mathcal{B}_\tau(G,H)$ and equals the original f_n as an element of $L^2_\tau(G,H)$. Now, for each $x \in G$,

$$f(x) := f_1(x) + \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x))$$

is a well-defined element of H.

For each $v \in H$ the function $x \to (f(x),v)_H$ is a countable, absolutely converging sum of Borel functions on G. Hence f is weakly Borel. Also, for $h \in H$, $x \in G$:

$$f(xh) = f_1(xh) + \sum_{n=1}^{\infty} (f_{n+1}(xh) - f_n(xh)) =$$

$$= \tau(h^{-1}) f_1(x) + \sum_{n=1}^{\infty} \tau(h^{-1}) (f_{n+1}(x) - f_n(x))$$

$$= \tau(h^{-1}) (f_1(x) + \sum_{n=1}^{\infty} (f_{n+1}(x) - f_n(x))) = \tau(h^{-1}) f(x).$$

Hence f satisfies (4.1) and is in $\mathcal{B}_\tau(G,\mathcal{H})$.

Finally we prove, simultaneously, that $f \in L^2_\tau(G,\mathcal{H})$ and that f is the limit in this space of the sequence $\{f_n\}$. Using

$$\|f(x)-f_n(x)\|_{\mathcal{H}} \le \sum_{m=n}^{\infty} \|f_{m+1}(x)-f_m(x)\|_{\mathcal{H}}$$

we find, by a similar argument as in the beginning of the proof, that

$$\Big(\int_{G/H} \|f(x)-f_n(x)\|^2_{\mathcal{H}}\, d\mu(\bar{x})\Big)^{\frac{1}{2}} < 2^{-n+1}.$$

Hence $f-f_n \in L^2_\tau(G,\mathcal{H})$ and $\|f-f_n\| < 2^{-n+1}$. □

Before proving the density of $\mathcal{K}_\tau(G,\mathcal{H})$ we need two preliminary lemmas.

LEMMA 4.8. *For each compact* $K \subset G$ *there is a constant* $C \in (0,\infty)$ *such that* $\int_K \|f(x)\|_{\mathcal{H}}\, dx \le C\,\|f\|$ *for all* $f \in L^2_\tau(G,\mathcal{H})$.

PROOF. Let $\rho(x)$ be defined by (3.10). Choose $\alpha \in \mathcal{K}(G)$ such that $\alpha \ge 0$ and $\alpha(x)\rho(x) = 1$ if $x \in K$. Put $\beta(\bar{x}) := \int_H \alpha(xh)dh$. By using Theorem 3.11 we obtain:

$$\int_K \|f(x)\|_{\mathcal{H}}\, dx \le \int_G \|f(x)\|_{\mathcal{H}}\, \alpha(x)\rho(x)dx =$$

$$= \int_{G/H} \Big(\int_H \|f(xh)\|_{\mathcal{H}}\, \alpha(xh)dh\Big) d\mu(\bar{x}) =$$

$$= \int_{\mathrm{supp}(\beta)} \|f(x)\|_{\mathcal{H}}\, \beta(\bar{x})d\mu(\bar{x}) \le$$

$$\le \Big(\sup_{x\in G/H} \beta(\bar{x})\Big)\Big(\mu(\mathrm{supp}(\beta))\Big)^{\frac{1}{2}} \cdot \Big(\int_{G/H} \|f(x)\|^2_{\mathcal{H}}\, d\mu(\bar{x})\Big)^{\frac{1}{2}} = C\|f\|.\quad □$$

LEMMA 4.9. *Let* $f \in L^2_\tau(G,\mathcal{H})$, $g \in \mathcal{K}(G,\mathcal{H})$. *Then*

$$(4.9)\qquad (f,\tilde{g}) = \int_G (f(x),g(x))_{\mathcal{H}}\, \rho(x)dx.$$

PROOF. The integrand is absolutely integrable because of Lemma 4.8. Hence we may apply Lemma 3.3:

$$\int_G (f(x),g(x))_H \rho(x)dx = \int_{G/H}\left(\int_H (f(xh),g(xh))_H\, dh\right)d\mu(\bar{x}) =$$

$$= \int_{G/H}\left(\int_H (f(x),\tau(h)g(xh))_H dh\right)d\mu(\bar{x}) =$$

$$= \int_{G/H} (f(x),\textstyle\int_H \tau(h)g(xh)dh)_H\, d\mu(\bar{x}) = \displaystyle\int_{G/H} (f(x),\tilde{g}(x))_H\, d\mu(\bar{x}) =$$

$$= (f,\tilde{g}).\quad \square$$

PROPOSITION 4.10. *$K_\tau(G,H)$ is dense in $L^2_\tau(G,H)$.*

PROOF. Let $f \in L^2_\tau(G,H)$. Suppose that $(f,\tilde{g}) = 0$ for all $g \in K(G,H)$. In view of Lemma 4.4 we have to show that $f = 0$. Let $\{e_1,e_2,\ldots\}$ be an orthonormal basis of H and put $g(x) := \alpha(x)e_i$ for some i and some $\alpha \in K(G)$. Application of the previous lemma yields:

$$0 = (f,\tilde{g}) = \int_G (f(x),e_i)_H\, \overline{\alpha(x)}\, \rho(x)dx.$$

We conclude that, for each i, $(f(x),e_i) = 0$ for almost all $x \in G$. Hence $f(x) = 0$ almost everywhere on G. $\square$

Finally observe that for each $y \in G$ the mapping $\hat{\tau}(y)$ defined by (4.2) is a unitary transformation of $L^2_\tau(G,H)$. Hence it is the unique continuous extension of $\tilde{\tau}(y) : K_\tau(G,H) \to K_\tau(G,H)$.

5. FURTHER PROPERTIES OF INDUCED REPRESENTATIONS

5.1. Some standard properties

Proposition VI.3.7 and Corollary VI.3.5 (first part) can be generalized:

THEOREM 5.1. (Induction in stages). *If H and K are closed subgroups of a lcsc. group G such that $H \subset K$, and if τ is a unitary representation of H, then $(\tau^K)^G \simeq \tau^G$.*

THEOREM 5.2. *If* $\tau = \int_X^{\oplus} \tau_\alpha \, d\mu(\alpha)$ *is a direct integral (cf.* §VIII.7.3) *of unitary representations* τ_α *of a closed subgroup* H *of a locally compact group* G *then*

$$\tau^G \simeq \int_X^{\oplus} (\tau_\alpha)^G d\mu(\alpha).$$

In particular, if $\tau = \sum_i^{\oplus} \tau_i$ *then* $\tau^G \simeq \sum_i^{\oplus} (\tau_i)^G$

The proofs of these theorems are long and difficult, see MACKEY [4].

5.2. The Gårding space associated with an induced representation

The results of these subsection will be needed in the proof of Theorem X.1.4. Let σ be a unitary representation of a lcsc. group G on a Hilbert space H. Then a nondegenerate representation $\tilde{\sigma}$ of $L^1(G)$ on H is defined by the operator-valued integral

$$(5.1) \qquad \tilde{\sigma}(\alpha) := \int_G \alpha(x)\sigma(x)dx, \qquad \alpha \in L^1(G),$$

cf. Theorem VIII.4.5. The *Gårding space* D_σ of the representation σ is defined as the linear subspace of H which is spanned by the set $\{\tilde{\sigma}(\alpha)v \mid \alpha \in K(G),\ v \in H\}$.

PROPOSITION 5.3.

(i) D_σ *lies dense in* H.

(ii) $\sigma(x)D_\sigma \subset D_\sigma$, $\quad x \in G$.

(iii) *If* σ_1, σ_2 *are unitary representations of* G *and* $A \in R(\sigma_1,\sigma_2)$ *then* $AD_{\sigma_1} \subset D_{\sigma_2}$.

PROOF.

(i) Since $\tilde{\sigma}$ is nondegenerate, the linear span of the set $\{\tilde{\sigma}(\alpha)v \mid \alpha \in L^1(G),\ v \in H\}$ lies dense in H. Since $K(G)$ lies dense in $L^1(G)$, each element $\tilde{\sigma}(\alpha)v$ ($\alpha \in L^1(G)$, $v \in H$) can be approximated by elements in D_σ.

(ii) Let $\alpha \in K(G)$, $x \in G$, $v \in H$. Then

$$\sigma(x)\tilde{\sigma}(\alpha)v = \sigma(x)\int_G \alpha(y)\sigma(y)v\,dy = \int_G \alpha(y)\sigma(xy)v\,dy =$$

$$= \int_G \alpha(x^{-1}y)\sigma(y)v\,dy = \tilde{\sigma}(\lambda(x)\alpha)v,$$

where $y \mapsto (\lambda(x)\alpha)(y) := \alpha(x^{-1}y)$ is in $K(G)$.

(iii) Let $\alpha \in K(G)$, $v \in H(\sigma_1)$, $A \in R(\sigma_1,\sigma_2)$. Then

$$A\tilde{\sigma}_1(\alpha)v = A\int_G \alpha(x)\sigma_1(x)v\,dx = \int_G \alpha(x)A\sigma_1(x)v\,dx =$$

$$= \int_G \alpha(x)\sigma_2(x)Av\,dx = \tilde{\sigma}_2(\alpha)Av. \quad \square$$

PROPOSITION 5.4. *Let $\sigma = \tau^G$, where τ is a unitary representation of a closed subgroup H of G on a Hilbert space H. Then the following holds:*

(i) *If $\alpha \in K(G)$ and $f \in L^2_\tau(G,H)$ then*

$$(5.2) \qquad (\tilde{\sigma}(\alpha)f)(y) = \int_G \alpha(x)(R_\mu(\bar{y},x^{-1}))^{\frac{1}{2}}f(x^{-1}y)dx =$$

$$\int_G \alpha(yx^{-1})R_\mu(\bar{x},yx^{-1})^{-\frac{1}{2}}f(x)\Delta_G(x^{-1})dx$$

for almost all y.

(ii) *If $\alpha \in K(G)$, $f \in L^2_\tau(G,H)$ then $\tilde{\sigma}(\alpha)f : G \to H$ is continuous.*

(iii) *For each $y \in G$ the set $\{f(y) \mid f \in D_\sigma\}$ is dense in H.*

PROOF.

(i) Let $\alpha \in K(G)$, $f \in L^2_\tau(G,H)$, $g \in K(G,H)$. An application of Lemma 4.9 shows:

$$\int_G ((\tilde{\sigma}(\alpha)f)(y),g(y))_H\,\rho(y)dy = (\tilde{\sigma}(\alpha)f,\tilde{g}) = \int_G \alpha(x)(\sigma(x)f,\tilde{g})dx =$$

$$= \int_G \alpha(x)\left(\int_G ((\sigma(x)f)(y),g(y))_H\,\rho(y)dy\right)dx =$$

$$= \int_G \alpha(x)\Big(\int_G (R_\mu(\bar{y},x^{-1}))^{\frac{1}{2}}(f(x^{-1}y),g(y))_H\ \rho(y)dy\Big)dx =$$

$$= \int_G \Big(\int_G \alpha(x)(R_\mu(\bar{y},x^{-1}))^{\frac{1}{2}}(f(x^{-1}y),g(y))_H\ dx\Big)\ \rho(y)dy.$$

(Since both α and g have compact support, we can use Lemma 4.8 and Fubini's theorem in order to prove the last equality.) Let $\{e_1,e_2,\dots\}$ be an orthonormal basis of H and put $g(y) := \beta(y)e_i$ for some i and some $\beta \in K(G)$. Then

$$\int_G ((\tilde{\sigma}(\alpha)f)(y),e_i)_H\ \overline{\beta(y)}\ \rho(y)dy =$$

$$= \int_G (\int_G \alpha(x)(R_\mu(\bar{y},x^{-1}))^{\frac{1}{2}}f(x^{-1}y)dx,e_i)_H\ \overline{\beta(y)}\ \rho(y)dy.$$

Thus the first equality in (5.2) holds for almost all y. The second equality follows by making the substitution $x \to yx^{-1}$.

(ii) Let $\alpha \in K(G)$, $f \in L^2_\tau(G,H)$. Fix $y_0 \in G$. Then, for $y \in G$:

$$(\tilde{\sigma}(\alpha)f)(y)-(\tilde{\sigma}(\alpha)f)(y_0) =$$

$$= \int_G (\alpha(yx^{-1})-\alpha(y_0x^{-1}))((R_\mu(\bar{x},yx^{-1}))^{-\frac{1}{2}}-(R_\mu(\bar{x},y_0x^{-1}))^{-\frac{1}{2}})f(x)\Delta_G(x^{-1})dx +$$

$$+ \int_G \alpha(y_0x^{-1})((R_\mu(\bar{x},yx^{-1}))^{-\frac{1}{2}}-(R_\mu(\bar{x},y_0x^{-1}))^{-\frac{1}{2}})f(x)\Delta_G(x^{-1})dx +$$

$$+ \int_G (\alpha(yx^{-1})-\alpha(y_0x^{-1}))(R_\mu(\bar{x},y_0x^{-1}))^{-\frac{1}{2}}f(x)\Delta_G(x^{-1})dx.$$

Let V be a compact neighbourhood of e in G and let $K := (\mathrm{supp}(\alpha))^{-1}Vy_0$. Let $y \in Vy_0$. Then K is compact and in each of the three terms above the integrand has support in K. Let $\varepsilon > 0$. Then there is a symmetric neighbourhood $W \subset V$ of e in G such that for $y \in Wy_0$, $x \in K$:

$$|\alpha(yx^{-1})-\alpha y_0x^{-1})| \le \min\{\varepsilon^{\frac{1}{2}},\varepsilon(\sup_{z\in K}(R(\bar{z},y_0z^{-1}))^{-\frac{1}{2}})^{-1}\},$$

$$|(R(\bar{x},yx^{-1}))^{-\frac{1}{2}}-(R(\bar{x},y_0x^{-1}))^{-\frac{1}{2}}| \le \min\{\varepsilon^{\frac{1}{2}},\varepsilon(\sup_{z\in K}|\alpha(y_0z^{-1})|)^{-1}\}.$$

Let C be the constant associated with K according to Lemma 4.8. Then, for $y \in Wy_0$:

$$\|(\tilde{\sigma}\ (\alpha)f)(y)-(\tilde{\sigma}(\alpha)f)(y_0)\|_H \le 3C\ \|f\|(\sup_{x\in K}\Delta_G(x^{-1}))\varepsilon. \quad \square$$

(iii) Fix $y \in G$. Let $v \in H$. Suppose that $((\tilde{\sigma}(\alpha)f)(y),v)_H = 0$ for all $\alpha \in K(G)$, $f \in L^2_\tau(G,H)$. We have to prove that $v = 0$. Put

$$\beta(x) := (\rho(x))^{-1}(R(\bar{x},yx^{-1}))^{-\frac{1}{2}}\Delta(x^{-1})\alpha(yx^{-1}).$$

Then

$$0 = \int_G \alpha(yx^{-1})(R(\bar{x},yx^{-1}))^{-\frac{1}{2}}(f(x),v)_H\cdot\Delta_G(x^{-1})dx$$

$$= \int_G \beta(x)(f(x),v)_H\ \rho(x)dx$$

for all $f \in L^2_\tau(G,H)$, $\beta \in K(G)$. Let $g(x) := \beta(x)v$. Then $g \in K(G,H)$ and $0 = (f,\tilde{g})$ (by Lemma 4.9) for all $f \in L^2_\tau(G,H)$. Hence

$$0 = \tilde{g}\ (x) = \int_H \beta(xh)\tau(h)v\ dh$$

for all $\beta \in K(G)$ and for all x. Now proceeding as in the proof of Proposition 4.5 we obtain that $v = 0$. $\square$

5.3. A realization of τ^G on the space $L^2(G/H,H,\mu)$

Let the situation be as in §4.3, and consider again the space $L^2_\tau(G,H)$. Let $L^2(G/H,H,\mu)$ denote the Hilbert space of (equivalence classes of) H-valued functions on G/H which are weakly Borel and square integrable with respect to μ (cf. V.1.17). Furthermore, choose a Borel cross-section $s : G/H \to G$ (cf. Lemma 2.5). For each f in $B_\tau(G,H)$ we define a function g_f on G/H by

(5.2) $$g_f(\bar{x}) := f(s(\bar{x})), \quad \bar{x} \in G/H.$$

Then g_f is weakly Borel on G/H, and

$$\int_{G/H} \|g_f(\bar{x})\|^2 \, d\mu(\bar{x}) = \int_{G/H} \|f(s(\bar{x})\|^2 \, d\mu(\bar{x}) = \int_{G/H} \|f(x)\|^2 \, d\mu(\bar{x}).$$

Hence, we may consider the mapping $f \mapsto g_f$ as an isometric isomorphism from the space $L^2_\tau(G,\mathcal{H})$ into the space $L^2(G/H),\mathcal{H},\mu)$. This isomorphism is surjective. Indeed, let $g \in L^2(G/H,\mathcal{H},\mu)$. Define a function f on G by

(5.3) $$f(x) := \tau(x^{-1}s(\bar{x}))g(\bar{x}), \quad x \in G.$$

Then $f \in \mathcal{B}_\tau(G,\mathcal{H})$ and $g = g_f$. Hence $\|f\| = \|g\|$, i.e., $f \in L^2_\tau(G,\mathcal{H})$.

The isomorphism $f \to g_f$ can be extended to the corresponding algebras of bounded linear operators by setting

$$(\tilde{T}g_f)(\bar{x}) := g_{Tf}(\bar{x}),$$

for any T in $\mathcal{L}(L^2_\tau(G,\mathcal{H}))$. Next we ask ourselves what the induced representation τ^G will look like, when lifted to $L^2(G/H,\mathcal{H},\mu)$. We have

$$(\tilde{\tau}^G(y)g_f)(\bar{x}) = g_{\tau^G(y)f}(\bar{x}) = f(y^{-1}s(\bar{x}))(R(s(\bar{x}),y^{-1}))^{\frac{1}{2}}.$$

The element $y^{-1}s(\bar{x})$ belongs to the left H-coset $(y^{-1}x)H$, so there is a unique element h of H such that

$$y^{-1}s(\bar{x}) = s(y^{-1}\bar{x})h.$$

Hence the expression above can be rewritten as

$$f(s(y^{-1}\bar{x})h)(R(\bar{x},y^{-1}))^{\frac{1}{2}} = \tau(h^{-1})g_f(y^{-1}\bar{x})(R(\bar{x},y^{-1}))^{\frac{1}{2}},$$

by (4.1). We define an operator-valued function $A\colon G/H \times G \to \mathcal{U}(\mathcal{H})$ by

(5.4) $$A(\bar{x},y) := \tau(s(\overline{yx})^{-1}ys(\bar{x})).$$

This function satisfies

(5.5)(i) $A(\bar{x},e) = I, \quad \forall\bar{x} \in X,$

(5.5)(ii) $A(\bar{x},yz) = A(\overline{zx},y)A(\bar{x},z), \quad \forall x \in X, \forall y,z \in G,$

cf. (3.11). Clearly, A is weakly measurable. The representation of G on $L^2(G/H,\mathcal{H},\mu)$ which is equivalent to τ^G by the isomorphism (5.2), is thus given by

$$(5.6) \qquad (\tilde{\tau}^G(y)g)(\bar{x}) = A(\overline{y^{-1}x},y)g(y^{-1}\bar{x})(R(\bar{x},y^{-1}))^{\frac{1}{2}}.$$

These considerations lead to an alternative approach to induction. Indeed, let X be a homogeneous space of G and let μ be a quasi-invariant measure on X, with $R\colon X \times G \to (0,\infty)$ denoting a corresponding continuous R-function. Furthermore, suppose we are given a weakly measurable operator-valued function

$$A\colon X \times G \to \mathcal{U}(\mathcal{H}),$$

where $\mathcal{H}$ is a certain Hilbert space. Then we can define operators T(y) on $L^2(X,\mathcal{H},\mu)$ for each y in G by

$$(5.7) \qquad (T(y)f)(\bar{x}) := (R(\bar{x},y^{-1}))^{\frac{1}{2}}A(\overline{y^{-1}x},y)f(y^{-1}\bar{x}).$$

Clearly these operators are well-defined, and, moreover, unitary. Furthermore if A satisfies (5.5), then $y \mapsto T(y)$ is a homomorphism. In this case, it can be shown that A is of the form (5.4) for a certain Borel cross section $s\colon X \to G$ and a certain unitary representation τ of a closed subgroup H of G with $G/H \approx X$. As a matter of fact, this assertion forms an important stage in the proof of the infinite version of the imprimitivity theorem, a sketch of which will be given in §X.2 (cf. also VARADARAJAN [11,thm.9.7]).
Another important observation in this context is the following. Let A_1 and A_2 be weakly measurable operator-valued functions from $X \times G$ into $\mathcal{U}(\mathcal{H})$ which satisfy (5.5) and let T_1,T_2 be the representations defined by A_1 and A_2 through (5.7). Then the original representations τ_1 and τ_2 with $\tau_i^G \simeq T_i$, $i = 1,2$, are equivalent if and only if an operator-valued function $C\colon X \to \mathcal{U}(\mathcal{H})$ exists with

(5.8) $A_1(\bar{x},y) = C(\overline{yx})^{-1}A_2(\bar{x},y)C(\bar{x}), \qquad x,y \in G.$

See §X.2 for details.

6. EXAMPLES OF INDUCED REPRESENTATIONS

6.1. Representations of semidirect products

Let G be a lcsc. group with a closed abelian normal subgroup N and a closed subgroup H such that $N \cap H = \{e\}$, $G = NH$ and G is homeomorphic with $N \times H$. Then G is called the *semidirect product* of N and H. Let ϕ_0 be a character on N (cf. §VIII.5) and let H_0 be a closed subgroup of H such that

$$\phi_0(h^{-1}n\,h) = \phi_0(n), \quad n \in N,\ h \in H_0.$$

Let σ be a unitary representation of H_0. Then

(6.1) $\tau(n\,h) := \phi_0(n)\sigma(h), \quad n \in N,\ h \in H_0,$

defines a unitary representation τ of the group NH_0 on $H(\sigma)$. We will consider the induced representation τ^G of G.

First observe that G/NH_0 can be identified with H/H_0. Indeed, the formula

$$nh(\bar{h}_1) := \overline{hh_1}, \quad n \in N,\ h \in H,\ \bar{h}_1 \in H_1/H_0,$$

defines a continuous and transitive action of G on H/H_0 and the stabilizer of $\bar{e} \in H/H_0$ in G is NH_0. Hence H/H_0 is G-homeomorphic with G/NH_0 (cf. Theorem 2.1). Choose a quasi-invariant measure μ on H/H_0 such that the corresponding function R_μ is continuous and strictly positive. Then μ is also quasi-invariant with respect to the action of G on H/H_0 and the corresponding R-function is given by $(\bar{h}_1,nh) \mapsto R_\mu(\bar{h}_1,h)$.

Let $f \in K_\tau(G,H(\sigma))$. Then

$$(\tau^G(nh)f)(n_1h_1) = (R_\mu(\bar{h}_1,h^{-1}))^{\frac{1}{2}}f(h^{-1}n^{-1}n_1h_1), \quad n,n_1 \in N,\ h,h_1 \in H, \text{ cf. (4.2).}$$

Hence

$$(6.2)\qquad (\tau^G(nh)f)(h_1) = (R_\mu(\bar{h}_1,h^{-1}))^{\frac{1}{2}} f(h^{-1}n^{-1}h_1) =$$

$$= (R_\mu(\bar{h}_1,h^{-1}))^{\frac{1}{2}} f(h^{-1}h_1(h_1{}^{-1}nh_1)^{-1}) =$$

$$= \phi_0(h_1^{-1}n\ h_1)(R_\mu(\bar{h}_1,h^{-1}))^{\frac{1}{2}} f(h^{-1}h_1),$$

cf. (4.1). Note that the formula

$$(6.3)\qquad g = f|_H$$

defines a one-to-one linear mapping $f \mapsto g$ from $K_\tau(G,H(\sigma))$ onto $K_\sigma(H,H(\sigma))$ with inversion formula

$$f(nh) = \phi_0(h^{-1}n^{-1})g(h), \quad n \in N,\ h \in H.$$

This mapping is an isometry:

$$\|f\|^2 = \int_{H/H_0} \|f(nh)\|^2_{H(\sigma)} d\mu(nh(\bar{e})) = \int_{H/H_0} \|g(h)\|^2_{H(\sigma)} d\mu(\bar{h}) = \|g\|^2.$$

It follows from (6.2), (6.3) and (4.2) that

$$(\tau^G(nh)f)(h_1) = \phi_0(h_1^{-1}nh_1)(\sigma^H(h)g)h_1), \quad n \in N,\ h,h_1 \in H.$$

Hence, on putting

$$(\pi(nh)g)(h_1) := (\tau^G(nh)f)(h_1), \quad n \in N,\ h,h_1 \in H,$$

where f and g are related by (6.3), we obtain a unitary representation π on the pre-Hilbert space $K_\sigma(H,H(\sigma))$ which is equivalent to τ^G and which is explicitly given by

$$(6.4)\qquad (\pi(nh)g)(h_1) = \phi_0(h_1^{-1}nh_1)(\sigma^H(h)g)(h_1), \quad g \in K_\sigma(H,H(\sigma)),\ n \in N,\ h,h_1 \in H.$$

It is an easy exercise to prove that the extension of π to the Hilbert space completion $L^2_\sigma(H,\mathcal{H}(\sigma))$ of $\mathcal{K}_\sigma(H,\mathcal{H}(\sigma))$ is still given by (6.4). Formula (6.4) is the starting point for Mackey's classification of the irreducible representations of a semidirect product with abelian normal subgroup; cf. §XI.3. See sections 4, 5 and 6 of Ch. XI for further specializations of formula (6.4).

6.2. Principal series representations

Let G be a unimodular lcsc. group with a compact subgroup K and a closed subgroup H such that $K \cap H = \{e\}$, $G = KH$ and G is homeomorphic with $K \times H$. Let M be a closed subgroup of K with the property that M normalizes H, i.e., $mHm^{-1} = H$ for all $m \in M$. Let ϕ_0 be a one-dimensional unitary representation of H such that

$$\phi_0(m^{-1}hm) = \phi_0(h), \quad m \in M,\ h \in H.$$

Let σ be a unitary representation of M. Now MH is a closed subgroup of G and

$$\tau(mh) := \phi_0(h)\sigma(m), \quad m \in M,\ h \in H, \tag{6.5}$$

defines a unitary representation τ of MH on $\mathcal{H}(\sigma)$. Similarly as in §6.1 we will consider the induced representation τ^G of G.

First we make a few preliminary remarks. Let the projections π_1 and π_2 be defined by (3.17). Let $x \in G$, $m \in M$. Then

$$xm = \pi_1(x)\pi_2(x)m = (\pi_1(x)m)(m^{-1}\pi_2(x)m).$$

Hence

$$\pi_1(xm) = \pi_1(x)m, \tag{6.6}$$

$$\pi_2(xm) = m^{-1}\pi_2(x)m, \quad x \in G,\ m \in M. \tag{6.6}$$

It is easily derived that a left and a right Haar measure on MH are given by

$$d\nu_{MH}(mh) = d\nu_M(m)d\nu_H(h),$$

$$d\nu^{(r)}_{MH}(mh) = d\nu_M(m)d\nu^{(r)}_H(h),$$

respectively. (Use the fact that M is compact and normalizes H.) Hence

$$(6.8)\qquad \Delta_{MH}(mh) = \Delta_H(h), \quad m \in M,\ h \in H.$$

Now we look for a ρ-function corresponding to the pair (G,MH). Let

$$(6.9)\qquad \rho(x) := \Delta_H(\pi_2(x)), \quad x \in G.$$

Then ρ is continuous and strictly positive on G. Furthermore, using (6.8) and (6.7) we have for $x \in G$, $m \in M$, $h \in H$:

$$\begin{aligned}\rho(xmh) &= \Delta_{MH}(\pi_2(xmh)) = \Delta_{MH}(\pi_2(xm)h) = \\ &= \Delta_{MH}(\pi_2(xm))\Delta_{MH}(h) = \Delta_{MH}(m^{-1}\pi_2(x)m)\Delta_{MH}(mh) = \\ &= \Delta_{MH}(\pi_2(x))\Delta_{MH}(mh) = \rho(x)\Delta_{MH}(mh).\end{aligned}$$

Hence ρ satisfies the conditions of Lemma 3.10 for the pair (G,MH).

Next observe that we can identify G/MH with K/M. Indeed, the formula

$$(6.10)\qquad (kh)(\bar{k}_1) := \overline{\pi_1(khk_1)}, \quad k \in K,\ h \in H,\ \bar{k}_1 \in K/M,$$

defines a continuous and transitive action of G on K/M. (Note that, in view of (6.6), the right hand side of (6.10) is independent of the choice of the representative $k_1 \in K$ for $\bar{k}_1$.) The stabilizer of $\bar{e} \in K/M$ in G under the action (6.10) is MH. Hence K/M considered as a G-space is G-homeomorphic with G/MH (cf. Theorem 2.1).

Let $f \in \mathcal{K}_\tau(G,\mathcal{H}(\sigma))$. Then

$$(\tau^G(kh)f)(x) = \left(\frac{\rho(h^{-1}k^{-1}x)}{\rho(x)}\right)^{\frac{1}{2}} f(h^{-1}k^{-1}), \quad k \in K,\ h \in H,\ x \in G,$$

cf. (4.2) and (3.8).

Specializing x to K and substituting (6.9) we find:

$$\begin{aligned}(6.11)\qquad (\tau^G(kh)f)(x) &= (\Delta_H(\pi_2(h^{-1}k^{-1}x)))^{\frac{1}{2}} f(h^{-1}k^{-1}x) = \\ &= (\Delta_H(\pi_2(h^{-1}k^{-1}x)))^{\frac{1}{2}}\phi_0((\pi_2(h^{-1}k^{-1}x))^{-1})f(\pi_1(h^{-1}k^{-1}x)).\end{aligned}$$

According to Theorem 3.11 there corresponds to the function ρ defined by (6.9) a quasi-invariant measure μ on G/MH and a R-function given by

$$R_\mu(\bar{x},y) := \frac{\Delta_H(\pi_2(yx))}{\Delta_H(\pi_2(x))}, \quad x,y \in G.$$

If $x,y \in K$ then $R_\mu(\bar{x},y) = 1$. Hence, if G/MH is identified with K/M as above then μ becomes a K-invariant measure on K/M.

The formula

$$g = f|_K$$

defines one-to-one linear mapping $f \to g$ from $K_\tau(G,H(\sigma))$ onto $K_\sigma(K,H(\sigma))$ with inversion formula

$$f(kh) = \phi_0(h^{-1})g(k), \quad k \in K,\ h \in H.$$

This mapping is an isometry:

$$\|f\|^2 = \int_{K/M} \|f(kh)\|^2_{H(\sigma)} d\mu(kh(\bar{e})) = \int_{K/M} \|g(k)\|^2_{H(\sigma)} d\mu(\bar{k}) = \|g\|^2.$$

It follows from (6.11) that the formula

$$(6.12) \qquad (\pi(y)g)(k) := (\Delta_H(\pi_2(y^{-1}k)))^{\frac{1}{2}}\phi_0((\pi_2(y^{-1}k))^{-1})g(\pi_1(y^{-1}k)),$$

$$g \in K_\sigma(K,H(\sigma)),\ y \in G,\ k \in K,$$

defines a unitary representation π on $K_\sigma(K,H(\sigma))$ which is equivalent to τ^G. For g in the completion $L^2_\sigma(K,H(\sigma))$ of $K_\sigma(K,H(\sigma))$, π is still given by (6.12).

Note that (6.12) has the form

$$(6.13) \qquad (\pi(y)g)(k) = \phi(\pi_2(y^{-1}k))g(\pi_1(y^{-1}k)),\ g \in L^2_\sigma(K,H(\sigma)),\ y \in G,\ k \in K,$$

where ϕ is a continuous homomorphism from H into the multiplicative group $\mathbb{C} \setminus \{0\}$ such that $\phi(m^{-1}hm) = \phi(h)$, $m \in M$, $h \in H$. In fact, for any ϕ with these properties formula (6.13) defines a strongly continuous but not necessarily unitary representation π of G on $L^2_\sigma(K,H(\sigma))$. It follows from (6.13) that the restriction of π to K is the unitary representation σ^K.

Finally assume that G is a noncompact connected semisimple Lie group with finite center and Iwasawa decomposition G = KAN. Then the subgroups K, A, N are compact, abelian and nilpotent, respectively. Furthermore, A normalizes N. Let M be the centralizer of A in K. Then M normalizes N. Put H := AN. It is easily verified that $\Delta_H(an) := \delta(a)$, $a \in A$, $n \in N$, for some (nonunitary) character δ on A. Put $\phi_0(an) := \alpha(a)$ for some (not necessarily unitary) character on A. Then (6.12) defines a so-called *principal series* representation of G. If α is a unitary character on A then π is a unitary representation. In §XIII.2 this will be worked out in the case G = SL(2,$\mathbb{R}$).

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X

INFINITE IMPRIMITIVITY AND LOCALIZABILITY IN QUANTUM MECHANICS

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CONTENTS

1. THE IMPRIMITIVITY THEOREM FOR LOCALLY COMPACT SECOND COUNTABLE GROUPS

In this section we will state the analogue of Theorem VI. 5.2 for locally compact second countable (lcsc.) groups. This generalization, which is due to Mackey, will play an essential role in the next chapter, together with theorem 1.4 below.

Let τ be a representation of a finite group G. Recall (cf. §VI. 5) that a system of imprimitivity (s.o.i.) for τ was defined to be a family of subspaces $\{V_\gamma\}_{\gamma\in\Gamma}$ of the representation space $V(\tau)$, indexed by a G-space Γ, such that

(i) $V(\tau) = \sum_{\gamma\in\Gamma}^{\oplus} V_\gamma$ as a vector space direct sum;

(ii) $\tau(x)V_\gamma = V_{x(\gamma)}$, $\forall x \in G, \forall\gamma \in \Gamma$.

We will adjust this definition such as to enable a canonical extension to general topological groups with possibly infinite-dimensional representations.

Consider a family of projections $\{P_\gamma\}_{\gamma\in\Gamma}$ in $V(\tau)$ such that

$$P_\gamma(V(\tau)) = V_\gamma \quad , \quad \forall\gamma \in \Gamma.$$

Note that Γ as a finite set has a trivial Borel structure, generated by its discrete topology. In other words, the Borel sets of Γ are just its subsets.

If we define a projection P_E in $V(\tau)$ for each subset E of Γ by

$$P_E := \sum_{\gamma\in E} P_\gamma \quad ,$$

(in particular $P_{\{\gamma\}} = P_\gamma$) then it is clear that the mappping P: $E \to P_E$ satisfies

$$\begin{cases} P_\Gamma = I, \\ P_{E\cap F} = P_E P_F \quad , \quad \forall E,F \subset \Gamma \\ \Sigma P_{E_i} = P_{\cup E_i}, \quad \forall E_1,E_2,\ldots \subset \Gamma \text{ with } E_i \cap E_j = \emptyset \text{ for } i \neq j. \end{cases}$$

Hence, P is a projection-valued measure on Γ, acting in $V(\tau)$ (cf. Def. VIII. 1.1). Let $\xi = \sum_{\gamma\in\Gamma} \xi_\gamma$ be the decomposition of an element ξ of $V(\tau)$ into V_γ-components. Then, by virtue of property (ii) above, we have

$$\tau(x)P_{\gamma_0}\tau(x^{-1})\xi = \tau(x)P_{\gamma_0}\sum_{\gamma\in\Gamma}(\tau(x^{-1})\xi)_{x^{-1}(\gamma)}$$

$$= \tau(x)(\tau(x^{-1})\xi)_{\gamma_0} = \xi_{x(\gamma_0)} = P_{x(\gamma_0)}\xi.$$

By linearity, this implies

$$(1.1) \qquad \tau(x)P_E\tau(x^{-1}) = P_{x[E]},$$

for all x in G and all subsets E of Γ, where $x[E] := \{x(\gamma); \gamma \in E\}$.

Conversely, if we are given a finite group G with a representation τ, a G-space Γ and a projection-valued measure P, based on Γ and acting in $V(\tau)$, such that Γ and P are related by (1.1), then it is clear that the collection $\{P_{\{\gamma\}}(V(\tau))\}_{\gamma\in\Gamma}$ forms a s.o.i. for τ.

These considerations lead us to the following definition.

DEFINITION 1.1. Let G be a topological group. A *system of imprimitivity* (s.o.i.) for G acting in a Hilbert space H, is a triple (Γ,τ,P), where

(i) Γ is a continuous G-space;

(ii) τ is a unitary representation of G on H;

(iii) P is a projection-valued measure on Γ, acting in H, such that for all Borel subsets E of Γ and all x in G:

$$\tau(x)P_E\tau(x)^{-1} = P_{x[E]}.$$

As in the finite case, the system is said to be *transitive (trivial)* according to Γ being a transitive (trivial) G-space.

Many properties of representations may be formulated in terms of imprimitivity systems as well (cf. Example VII. 2.4(g)). Instead of the intertwining space $R(\tau,\sigma)$ of two representations τ and σ of G, we can consider the intertwining space of two s.o.i. based on G-homeomorphic G-spaces, say (Γ,τ,P) and (Δ,σ,Q), denoted by $R((\tau,P),(\sigma,Q))$. This space is defined to consist of all operators $T: H(\tau) \to H(\sigma)$, which satisfy

$$(1.2)(i) \qquad T\tau(x) = \sigma(x)T, \qquad \forall x \in G;$$

$$(1.2)(ii) \qquad TP_E = Q_{\Phi(E)}T, \qquad \text{for all Borel sets E in } \Gamma.$$

Here $\Phi: \Gamma \to \Delta$ denotes the G-homeomorphism of Γ onto Δ. Thus, if we denote by $R(P,Q)$ the space of operators T which satisfy (1.2)(ii), we have

$$R((\tau,P),(\sigma,Q)) := R(\tau,\sigma) \cap R(P,Q).$$

Two systems (Γ,τ,P) and (Δ,σ,Q) for G are said to be *equivalent* if

(i) Γ and Δ are G-homeomorphic;

(ii) $R((\tau,P),(\sigma,Q))$ contains an isometrical isomorphism.

Finally we shall say that a s.o.i. (Γ,τ,P) is *irreducible* if the sets of operators $\{\tau(x);\ x \in G\}$ and $\{P_E;\ E$ a Borel set in $\Gamma\}$ have no common non-triival invariant subspaces. This is equivalent to the condition:

$$(1.3)\qquad R(\tau,P)\ \ (:= R((\tau,P),(\tau,P))) = \{\lambda I;\ \lambda \in C\}.$$

EXAMPLE 1.2. Let λ be the regular representation of a lcsc. group G on $L^2(G)$. Define a projection P_E for each Borel set E in G by

$$(1.4)\qquad (P_E(f))(x) := \chi_E(x)f(x), \qquad f \in L^2(G),$$

where χ_E denotes as usual the characteristic function of E. Obviously, relation (1.1) holds with τ replaced by λ. Therefore (G,λ,P) is a (transitive) s.o.i. for G, where G is considered as a continuous G-space by left translation (cf. Example VI. 5.1).

EXAMPLE 1.3. Suppose that τ is a unitary representation of a locally compact second countable group G, which is induced on G by a unitary representation σ of a certain closed subgroup H of G. Thus, the space $\mathcal{H}(\sigma)$ consists of $\mathcal{H}(\sigma)$-valued functions on G (cf. §IX. 4.3). We define a projection-valued measure P, based on the coset space G/H, and acting in $\mathcal{H}(\tau)$, by

$$(1.5)\qquad (P_E(f))(x) := \chi_E(\bar{x})f(x), \qquad E \text{ Borel set in } G/H.$$

Using the definition of an induced representation (cf. IX. 4.2)) one finds:

$$\begin{aligned}(\tau(x)P_E\tau(x)^{-1}f)(y) &= (P_E\tau(x)^{-1}f)(x^{-1}y)(R(\bar{y},x^{-1}))^{\frac{1}{2}}\\ &= \chi_E(\overline{x^{-1}y})(\tau(x)^{-1}f)(x^{-1}y)(R(\bar{y},x^{-1}))^{\frac{1}{2}}\\ &= \chi_{x[E]}(\bar{y})f(y)(R(\overline{x^{-1}\ y},x)R(\bar{y},x^{-1}))^{\frac{1}{2}}\end{aligned}$$

$$= \chi_{x[E]}(\bar{y})f(y) = (P_{x[E]}f)(y).$$

Consequently, $(G/H,\tau,P)$ is a (transitive) s.o.i. admitted by τ. We shall call this system *canonically associated* with the induced representation $\sigma^G = \tau$, or, shortly, the *canonical system* of σ^G.

Let σ and τ be unitary representations of H and let $T \in R(\sigma,\tau)$. We will demonstrate a both interesting and important relationship between $R(\sigma,\tau)$ and the intertwining space for the canonical systems of σ^G and τ^G. For f in $\mathcal{H}(\sigma^G)$ we define a function $\hat{T}f\colon G \to \mathcal{H}(\tau)$ by

$$(1.6) \qquad (\hat{T}f)(x) := Tf(x),$$

If x and h are elements of G and H, respectively, we have

$$(\hat{T}f)(xh^{-1}) = T\sigma(h)f(x) = \tau(h)Tf(x) = \tau(h)(\hat{T}f)(x),$$

since T belongs to $R(\sigma,\tau)$. Furthermore, it is clear that

$$\|\hat{T}f\| \le \|T\| \; \|f\|, \qquad f \in \mathcal{H}(\sigma^G).$$

(Note that $\hat{T}f$ is weakly measurable.) Hence, $\hat{T}$ is a bounded operator from $\mathcal{H}(\sigma^G)$ into $\mathcal{H}(\tau^G)$. Now, let P and Q be the projection-valued measures corresponding to σ and τ, respectively, as defined by (1.5). We have the following theorem:

THEOREM 1.4. *The mapping* $T \to \hat{T}$ *maps the intertwining space* $R(\sigma,\tau)$ *isomorphically onto the intertwining space* $R((\sigma^G,P),(\tau^G,Q))$ *of the canonical systems of* σ^G *and* τ^G.

PROOF. For T in $R(\sigma,\tau)$ and for each Borel set E in G, we have

$$(\hat{T}(\chi_E.f))(x) = T(\chi_E(x).f(x))$$
$$= \chi_E(x)Tf(x) = \chi_E(x)(\hat{T}f)(x), \qquad f \in \mathcal{H}(\sigma^G).$$

Since $\chi_F(\bar{x}) = \chi_{FH}(x)$ if F is a Borel set in G/H, $\hat{T} \in R(P,Q)$. Moreover,

$$(\hat{T}\sigma^G(y)f)(x) = T(f(y^{-1}x)(R(\bar{x},y^{-1}))^{\frac{1}{2}})$$

$$= (R(\bar{x},y^{-1}))^{\frac{1}{2}}(\hat{T}f)(y^{-1}x) = (\tau^G(y)(\hat{T}f))(x), \quad f \in H(\sigma^G),$$

since R is a real-valued function. (Without damaging generality we can assume that $R = R_\sigma = R_\tau$.) Thus, $\hat{T}$ belongs to $R(\sigma^G,\tau^G)$ as well.

As to injectivity, this follows at once from Proposition IX. 4.5, which stated that for any x in G the subset $\{f(x);\ f \in K_\sigma\}$ lies dense in $H(\sigma)$. Indeed, if $Tf(x) = Sf(x)$ for two bounded operators T and S from $H(\sigma)$ into $H(\tau)$ and all f in K_σ, then T = S. Since K_σ can be considered as a dense subspace of $H(\sigma^G)$ this proves injectivity of $T \to \hat{T}$.

Finally, we prove surjectivity onto $R((\sigma^G,P),(\tau^G,Q))$. Consider the Gårding spaces D_{σ^G} and D_{τ^G}. (For the definition of Gårding spaces, see § IX. 5.2.) For S in $R((\sigma^G,P),(\tau^G,Q))$ we have

(1.7) $$SD_{\sigma^G} \subset D_{\tau^G},$$

by virtue of proposition IX. 5.3(iii). Furthermore, for each f in $H(\sigma^G)$ and each Borel set B in G/H, we have

$$\int_B \|(Sf)(x)\|^2 d\mu(\bar{x}) = \int_{G/H} \|\chi_B(x)(Sf)(x)\|^2 d\mu(\bar{x})$$

$$= \|(Q_B S)f\|^2 = \|(SP_B)f\|^2 \le \|S\|^2 \|P_B f\|^2$$

$$= \|S\|^2 \int_B \|f(x)\|^2 d\mu(\bar{x}).$$

For f in D_{σ^G} this means

(1.8) $$\|(Sf)(x)\| \le \|S\|\ \|f(x)\|, \quad \forall x \in G,$$

since $x \to \|(Sf)(x)\|^2$ and $x \to \|f(x)\|^2$ are both continuous, by Proposition IX. 5.4(ii) and (1.7), and since nonvoid open subsets of G/H have positive μ-measure. By virtue of (1.8) we can legitimately define an operator S_0 from the dense subspace $\{f(e);\ f \in D_{\sigma^G}\}$ of $H(\sigma)$ (cf. Proposition IX. 5.4(iii)) into $H(\tau)$ by

$$S_0(f(e)) := (Sf)(e), \qquad f \in D_{\sigma^G}.$$

Clearly S_0 is bounded. Therefore it has a unique extension to $H(\sigma)$, which we denote by S_0 as well. We have

$$\begin{aligned}(Sf)(x) &= (R(\bar{e},x))^{-\frac{1}{2}}(\tau^G(x^{-1})Sf)(e)\\ &= (R(\bar{e},x))^{-\frac{1}{2}}(S\sigma^G(x^{-1})f)(e)\\ &= S_0(R(\bar{e},x))^{-\frac{1}{2}}(\sigma^G(x^{-1})f)(e)\\ &= S_0 f(x), \qquad \forall f \in D_{\sigma^G}, \quad \forall x \in G.\end{aligned}$$

Furthermore,

$$\begin{aligned}\tau(h)S_0 f(e) &= \tau(h)(Sf)(e) = (Sf)(h^{-1})\\ &= S_0 f(h^{-1}) = S_0\sigma(h)f(e).\end{aligned}$$

Hence, $\tau(h)S_0 v = S_0\sigma(h)v$ for v in a dense subspace of $H(\sigma)$, and by continuity we can conclude that S_0 belongs to $R(\sigma,\tau)$, which proves our theorem, since $\hat{S}_0 = S$. □

COROLLARY 1.5.

(i) *The canonical systems of induced representations are equivalent if and only if the original representations are equivalent.*

(ii) *The canonical system of an induced representation is irreducible if and only if the original representation is irreducible.* □

Next, we state the general imprimitivity theorem.

THEOREM 1.6. (MACKEY). *Let* τ *be a unitary representation of a lcsc. group* G, *and let* H *be an arbitrary closed subgroup of* G. *Then the existence of a transitive system of imprimitivity* $(G/H,\tau,P)$ *implies the existence of a unitary representation* σ *of* H *such that* $(G/H,\tau,P)$ *is equivalent to the system canonically associated with* σ^G. *In particular,* τ *is equivalent to* σ^G. *Moreover, the equivalence class of* σ *is completely determined by the system* $(G/H,\tau,P)$.

The proof of this theorem is rather complicated. There exist several variants, of which the most recent ones (BARUT & RĄCZKA [1], KIRILLOV [4]) are based purely on functional analysis. The original proofs of MACKEY (see [5], [6] or [7]) are maybe not very accessible, in that they leave a lot to the imagination. However, they are based on some essential ideas, which play a fundamental (though not very perceptible) role in the work of Mackey on induction for locally compact groups. The ideas are connected with the "classical" cohomology theory of groups (Eilenberg/MacLane).

In the next section we will try and sketch these ideas, following VARADARAJAN [9], and show how the imprimitivity theorem can be derived from them.

The theorem has proved to be amenable to generalizations in many directions. One has to start with extending the concept of induction to larger classes of groups on the one hand, and larger classes of representations on the other hand. For instance, second countability of G and separability of the representation space (which we use as a convention) can be omitted from the theorem. Furthermore, after a suitable reformulation, the theorem keeps its validity for so-called projective or multiplier representations (see MACKEY [6]).

2. ON A PROOF OF THE IMPRIMITIVITY THEOREM

Let G be a lcsc. group, and let $H \subset G$ be a closed subgroup, fixed throughout this section. We set $X := G/H$. M will denote a second countable Hausdorff group, until further specifications.

DEFINITION 2.1. A Borel map $f: X \times G \to M$ is called a (X,G,M)-*cocycle* if it satisfies

(2.1)(i) $\quad f(\bar{x},e) = I, \quad \forall x \in G;$

(2.1)(ii) $\quad f(\bar{x},yz) = f(\overline{zx},y)f(\bar{x},z), \quad \forall x,y,z \in G.$

(Here I denotes the identity in M.)

DEFINITION 2.2. Two (X,G,M)-cocyles f_1 and f_2 are said to be *cohomologous* ($f_1 \simeq f_2$) if there exists a Borel map $b: X \to M$ such that

(2.2) $\quad f_1(\bar{x},y) = b(\overline{yx})^{-1} f_2(\bar{x},y)b(\bar{x}), \quad \forall x,y \in G.$

Obviously, (2.2) defines an equivalence relation in the set of all (X,G,M)-cocycles. Equivalence classes are called (X,G,M)-*cohomology classes*.

REMARK. In VARADARAJAN [9, p.27], the functions of definition 2.1 are called *strict cocycles*, and the relation between f_1 and f_2 given by (3.8) is expressed by calling them *stricly cohomologous*. His definitions of cocycles and cohomologous admit deviations on null-sets in the identities (2.1) (i), (ii) and (2.2) (that is, null-sets in X, $X \times G \times G$ and $X \times G$, respectively, w.r.t. Haar measure on G and quasi-invariant measure on X). In view of lemma 8.26 and part of theorem 8.27 in [9] we feel justified to circumvent measure theoretical details and use definitions 2.1 and 3.3. The contents of the lemma and the theorem we are referring to, or, rather, the portions of it we need, amount to the following statements:

- Each cocycle (in the sense of [9]) is a.e. (on $X \times G$) equal to a strict cocycle, which is unique up to strict cohomology;
- each cohomology class (sic) contains a unique, nonvoid, strict cohomology class.

[However, in a certain part of the proof of the imprimitivity theorem, the use of cocycles in the sense of [9] can not be avoided. We will omit this part.]

Let f be a (X,G,M)-cocycle. Then the map τ: H → M, defined by $\tau(h) := f(\bar{e},h)$, is clearly a Borel homomorphism. We will call τ the *homomorphism associated with* f. Two homomorphisms τ_i: H → M (i = 1,2) are called equivalent ($\tau_1 \simeq \tau_2$) if there exists an element T of M with

$$T\tau_1(h) = \tau_2(h)T, \qquad \forall h \in H.$$

By virtue of lemma 8.29 in [9], any Borel homomorphism from a lcsc. group into a second countable Hausdorff group is automatically continuous. The reader should keep this in mind, since we are going to use this fact later on, when we take M to be the unitary group of a separable Hilbert space and call τ a representation of H.

The following theorem relates (X,G,M)-cohomology classes to equivalence classes of Borel homomorphisms from H into M, and constitutes the first important step towards the proof of the imprimitivity theorem. If γ is a (X,G,M)-cohomology class, we let $\tilde{\gamma}$ denote the set of continuous homomorphisms associated with the elements of γ.

THEOREM 2.3. *The assignment* $\gamma \to \tilde{\gamma}$ *establishes a one-to-one correspondence between the set of all* (X,G,M)*-cohomology classes and the set of all equivalence classes of continuous homomorphisms from* H *into* M.

PROOF. We can split the proof into two parts:

(i) Let f_1 and f_2 be (X,G,M)-cocycles and let τ_1 and τ_2 be the associated homomorphisms. Then $f_1 \simeq f_2$ iff $\tau_1 \simeq \tau_2$.

(ii) Each continuous homomorphism τ: H $\to$ M is associated with a certain (X,G,M)-cocycle.

Let f_1 and f_2 be two (X,G,M)-cocycles and let τ_1 and τ_2 be the associated homomorphisms. We define two Borel maps b_i: G $\to$ M by

$$b_i(y) := f_i(\bar{e},y), \qquad y \in G.$$

From this definition, we have

$$b_i(xh) = b_i(x)\tau_i(h), \qquad \forall x \in G,\ h \in H, \tag{2.3}$$

and

$$f_i(\bar{x},y) = b_i(yx)b_i(x)^{-1}, \qquad \forall x,y \in G, \tag{2.4}$$

as can be easily checked.

Now, suppose that for some T $\in$ M, we have

$$T\tau_1(h) = \tau_2(h)T, \qquad \forall h \in H.$$

Then, from (2.3) it follows that

$$b_2(x)Tb_1(x)^{-1} = b_2(xh)Tb_1(xh)^{-1}, \qquad \forall x \in G,\ h \in H.$$

Hence, a unique Borel map b: X $\to$ M exists such that

$$b(\bar{x}) = b_2(x)Tb_1(x)^{-1}, \qquad x \in G.$$

Using the properties of cocycles and identity (2.4), an easy calculation yields

$$f_2(\bar{x},y) = b(\overline{yx})f_1(\bar{x},y)b(\bar{x})^{-1}, \qquad \forall x,y \in G. \tag{2.5}$$

Thus, $f_1 \simeq f_2$.

Conversely, suppose $f_1 \simeq f_2$, and let this equivalence be established by a Borel map $b\colon X \to M$. Then (2.2) yields

$$f_1(\bar{e},h) = b(\bar{e})^{-1} f_2(\bar{e},h) b(\bar{e})^{-1}, \qquad \forall h \in H.$$

Thus, setting $T := b(\bar{e})$, we obtain

$$T\tau_1(h) = \tau_2(h)T, \qquad \forall h \in H.$$

This finishes the first part of the proof. As to surjectivity of $\gamma \to \tilde{\gamma}$, let τ be a continuous homomorphism from H into M. We choose a Borel cross-section $s\colon X \to G$, with $s(\bar{e}) = e$ (this is legal, because of Lemma IX.2.5 and since we can, for any Borel cross-section s, define a new Borel cross-section s' by setting $s'(\bar{x}) := s(\bar{x}).\, s(\bar{e})^{-1}$). Next, we define a Borel map $f\colon X \times G \to M$, by

$$f(\bar{x},y) := \tau(s(\overline{yx})^{-1} y s(\bar{x})).$$

A brief calculation shows that f is a well-defined (X,G,M)-cocycle, and, moreover, since $s(\bar{e}) = e$, we have

$$f(\bar{e},h) = \tau(h). \qquad \square$$

The next theorem implies the imprimitivity theorem, and explains at the same time the idea behind the discussion in this section. First we introduce a few notations, which will be sustained till the end of this subsection. $\mathcal{H}_n$ will denote a fixed Hilbert space of dimension $n = \infty, 1, 2, \ldots$, and M_n will denote its unitary group, provided with the weak topology. Furthermore, we fix a quasi-invariant measure μ on X with continuous R-function, and set $\mathcal{K}_n := L^2(X, \mathcal{H}_n, \mu)$. In $\mathcal{K}_n$ we define a projection-valued measure P^n, based on X, by

$$P^n_E f = \chi_E f, \qquad E \in \mathcal{B}(X).$$

Note that the equivalence class of P^n is independent of our choice of a quasi-invariant measure on X. The proof of this assertion is similar to the one of Lemma IX.4.3. [Recall that two projection-valued measures P and Q acting in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, respectively, and based on a Borel space B, are said to be equivalent if there exists an isometric isomorphism $T: \mathcal{H} \to \mathcal{H}'$ such that $TP_E = Q_E T$, $\forall E \in \mathcal{B}(B)$.]

THEOREM 2.4.

(i) *Any system of imprimitivity* (X,τ,P) *for* G *is equivalent to a system of the form* (X,τ',P^n), *for a unique* $n \in \{\infty,1,2,\ldots\}$.

(ii) *There exists a one-to-one map* $\gamma \to \Sigma(\gamma)$ *from the set of* (X,G,M_n)-*cohomology classes onto the set of equivalence classes of systems of imprimitivity of the form* (X,τ,P^n).

(iii) *The map* $\gamma \to \Sigma(\gamma)$ *enjoys the following property: Let* τ *be the homomorphism determined by an element of* γ. *Then any system in* $\Sigma(\gamma)$ *is equivalent to the canonical system associated with the induced representation* τ^G.

Before giving the proof of this theorem, we will state an important result from spectral multiplicity theory, and deduce a lemma from it which applies to our situation. This is done in order to obtain part (i) of the above theorem. This result can be found in e.g. HALMOS [2, chapter III, particularly §67 and §68], see also Remark VIII.7.1.

Let Y be a second countable Hausdorff space. For any finite Borel measure ν on Y set $\mathcal{K}_{n,\nu} := L^2(Y,\mathcal{H}_n,\nu)$. Furthermore, let $P^{n,\nu}$ denote the projection-valued measure based on Y and acting in $\mathcal{K}_{n,\nu}$ by

$$P_E^{n,\nu} f = \chi_E f, \qquad E \in \mathcal{B}(Y).$$

It will turn out that the measures $P^{n,\nu}$ are the canonical building blocks for arbitrary projection-valued measures on Y. Indeed, let $\nu_\infty,\nu_1,\nu_2,\ldots$ be a sequence of mutually singular finite Borel measures on Y. (Recall that two measures μ and ν on Y are called mutually singular, notation $\mu \perp \nu$, if $\mu(B) = \nu(Y-B) = 0$ for some Borel set B in Y.) We set

$$\mathcal{K} = \bigoplus_n \mathcal{K}_{n,\nu_n},$$

and define a projection-valued measure $P = P(\{\mathcal{H}_n\},\{\nu_n\})$ acting in $\mathcal{K}$ and based on Y, by

$$P_E[(f_\infty,f_1,f_2,\ldots)] := (P_E^{\infty,\nu_\infty} f_\infty, P_E^{1,\nu_1} f_1,\ldots), \qquad E \in \mathcal{B}(Y).$$

THEOREM 2.5. *Any projection-valued measure* P *on* Y *determines a unique sequence* $[\mu_\infty], [\mu_1], [\mu_2], \ldots$ *of mutually singular measure classes on* Y *such that*

$$\nu_n \in [\mu_n],\ n = \infty,1,2,\ldots \Longleftrightarrow P \simeq P(\{\mathcal{H}_n\},\{\nu_n\}).$$

In particular $P(\{\mathcal{H}_n\},\{\nu_n\}) \simeq P(\{\mathcal{H}_n\},\{\nu'_n\})$ *implies* $\nu_n \equiv \nu'_n$, $n = \infty,1,2,\ldots$.

Now, we consider the consequences of this theorem in the present situation, that is, with Y = X, and P being part of a system of imprimitivity (X,τ,P) for G. We call a projection-valued measure *homogeneous* if all but one of the measure classes it determines are zero.

LEMMA 2.6. *Let* (X,τ,P) *be a system of imprimitivity for* G *acting in a Hilbert space* $\mathcal{H}$. *Then* P *is homogeneous. Moreover, the only nonzero measure class associated with* P *is the class of quasi-invariant measures on* X.

PROOF. Suppose $P \simeq P(\{\mathcal{H}_n\},\{\nu_n\})$ for some sequence $\nu_\infty,\nu_1,\nu_2,\ldots$ of mutually singular Borel measures on X. Let x be any element of G, and set $Q_E := P_{x[E]}$, $E \in \mathcal{B}(X)$. Then $Q\colon E \to Q_E$ is a projection-valued measure on X, which is equivalent to P, since

$$\tau(x)P_E\tau(x)^{-1} = P_{x[E]} = Q_E, \qquad \forall E \in \mathcal{B}(X).$$

On the other hand, we have

$$Q \simeq P(\{\mathcal{H}_n\},\{(\nu_n)_x\}),$$

where $(\nu_n)_x$ denotes the translated measure $(\nu_n)_x(E) := \nu_n(x[E])$, $E \in \mathcal{B}(X)$. (Note that $\nu_n \perp \nu_m$, $n \neq m$, implies $(\nu_n)_x \perp (\nu_m)_x$.) This can be seen as follows. Define a linear map

$$U := \oplus_n \mathcal{K}_{n,\nu_n} \longrightarrow \oplus_n \mathcal{K}_{n,(\nu_n)_x},$$

by

$$U[(f_\infty,f_1,\ldots)] := ((f_\infty)_x,(f_1)_x,\ldots),$$

where $(f_n)_x(y) := f_n(xy)$. Obviously, U establishes an isometric isomorphism. Write $P^* := P(\{\mathcal{H}_n\},\{\nu_n\})$ and $P^{*,x} := P(\{\mathcal{H}_n\},\{(\nu_n)_x\})$. There exists an isometric isomorphism.

$$T\colon \mathcal{H} \longrightarrow \bigoplus_n \mathcal{K}_{n,\nu_n}$$

with

$$TP_E := P_E^{\star}T, \qquad \forall E \in \mathcal{B}(X).$$

We have

$$UTQ_E = P_E^{\star,x}UT, \qquad \forall E \in \mathcal{B}(X),$$

as can be readily verified. By virtue of the preceding theorem, $\nu_n \equiv (\nu_n)_x$, $n = \infty,1,2,\ldots$, which implies, x being arbitrary, that all measure classes determined by P are invariant. But then P must be homogeneous, since quasi-invariant measures can not be mutually singular. □

We now proceed to the proof of Theorem 2.4. Let (X,τ,P) be a system of imprimitivity for G acting in $\mathcal{H}$. Then $P \simeq P^n$, for a certain $n \in \{\infty,1,2,\ldots\}$, on account of the preceding lemma. Thus, there exists an isometric isomorphism $T\colon \mathcal{H} \to \mathcal{K}_n$ such that

$$TP_E = P_E^n T, \qquad \forall E \in \mathcal{B}(X).$$

Define a new representation τ' of G, on $\mathcal{K}_n$, by

$$\tau'(x) := T\tau(x)T^{-1}, \qquad x \in G.$$

Then (X,τ',P^n) is a system of imprimitivity, and equivalent to (X,τ,P). This proves part (i) of Theorem 2.4.

Next, let ϕ be a (X,G,M)-cocycle and define a representation τ of G on $\mathcal{K}_n$ by

$$(\tau(x)f)(\bar{y}) = (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi(\overline{x^{-1}y},x)f(\overline{x^{-1}y}),$$

where R is a continuous R-function corresponding to μ. It can be easily verified that τ is a well-defined unitary representation of G. Furthermore

$$\begin{aligned}(\tau(x)P_E^n\tau(x^{-1})f)(y) &= (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi(\overline{x^{-1}y},x)(P_E^n\tau(x^{-1})f)(x^{-1}y)\\ &= \chi_E(x^{-1}y)(R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi(\overline{x^{-1}y},x)(\tau(x^{-1})f)(x^{-1}y)\\ &= \chi_E(x^{-1}y)(R(\bar{y},x^{-1})R(x^{-1}y,x))^{\frac{1}{2}}\phi(\overline{x^{-1}y},x)\phi(\bar{y},x^{-1})f(\bar{y})\end{aligned}$$

$$= \chi_{x[E]}(\bar{y})\phi(\bar{y},e)f(\bar{y})$$

$$= \chi_{x[E]}(\bar{y})f(\bar{y}) = (P^n_{x[E]}f)(\bar{y}).$$

Hence, (X,τ,P^n) is a system of imprimitivity for G. The equivalence class of this system does not depend on our choice of μ, as can be proved along the same lines as Lemma IX.4.3. It is also not affected if we choose another cocycle in the cohomology class of ϕ. Indeed suppose $\phi' \simeq \phi$, and let $b: X \to M_n$ be a Borel map with

$$\phi(\bar{x},y) = b(\overline{yx})^{-1}\phi'(\bar{x},y)b(\bar{x}), \qquad \forall x,y \in G.$$

If we set

$$(Bf)(\bar{x}) := b(\bar{x})f(\bar{x}), \qquad f \in K_n,$$

then it is clear that B defines a unitary operator on K_n. Moreover, we have

(2.6)(i) $\quad BP^n_E = P^n_E B, \qquad \forall E \in \mathcal{B}(X);$

(2.6)(ii) $\quad B\tau(x) = \tau'(x)B, \qquad \forall x \in G,$

where τ' is the representation on K_n defined by ϕ'. Formula (2.6)(i) is trivial, and (2.6)(ii) follows from the following easy computation:

$$(B\tau(x)B^{-1}f)(\bar{y}) = (R(\bar{y},x^{-1}))^{\frac{1}{2}}b(\bar{y})\phi(\overline{x^{-1}y},x)f(\overline{x^{-1}y})$$

$$= (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi'(\overline{x^{-1}y},x)f(\overline{x^{-1}y})$$

$$= (\tau'(x)f)(\bar{y}).$$

Consequently, $(X,\tau,P^n) \simeq (X,\tau',P^n)$. Thus, we have constructed a map $\gamma \to \Sigma(\gamma)$ from the set of (X,G,M_n)-cohomology classes into the set of equivalence classes of systems of impritivity of the form (X,τ,P^n). Proving surjectivity of this map requires some technicalities which have no direct relationship to our subject matter, and are therefore omitted. We refer the reader to [9, thm.9.11].

As to part (iii) of Theorem 2.4, let γ be a (G,X,M_n)-cohomology class, and let ϕ belong to γ. The representation τ of H on H_n associated with ϕ is given by $\tau(h) = \phi(\bar{e},h)$, $h \in H$. Let $s: X \to G$ be a Borel cross-section with $s(\bar{e}) = e$, and set

$$\phi'(\bar{x},y) = \tau(s(\overline{yx})^{-1}ys(\bar{x})), \qquad x,y \in G.$$

Then ϕ' is a (G,X,M_n)-cocycle which is cohomologous to ϕ (Theorem 2.3). In §IX.5.3 we have seen that the formula

$$(\tilde{\tau}^G(x)f)(\bar{y}) = (R(\bar{y},x^{-1}))^{\frac{1}{2}}\phi'(\overline{x^{-1}y},x)f(\overline{x^{-1}y})$$

defines a realization τ^G on $L^2(X,H_n,\mu)$ of the induced representation $\tilde{\tau}^G$. Let (X,τ^G,P) be the canonical system associated with τ^G. Then the isometric isomorphism $f \to g_f$ from $H(\tau^G)$ onto $L^2(X,H_n,\mu)$ given by

$$g_f(\bar{x}) := f(s(\bar{x})),$$

is easily seen to establish equivalence between (X,τ^G,P) and $(X,\tilde{\tau}^G,P^n)$. Indeed, we have

$$g_{\tau^G(x)f} = \tilde{\tau}^G g_f, \qquad x \in G,$$

by definition of $\tilde{\tau}^G$, and

$$g_{P_E f}(\bar{x}) = \chi_E(\overline{s(\bar{x})})f(s(\bar{x})) = \chi_E(\bar{x})f(s(\bar{x}))$$

$$= P^n_E g_f(\bar{x}), \qquad x \in G, \quad E \in \mathcal{B}(X),$$

by definition of the projection-valued measure in the canonical system. On the other hand, since $\phi \simeq \phi'$, we have also $(X,\tilde{\tau}^G,P^n) \simeq (X,\sigma,P^n)$, where (X,σ,P^n) denotes the system defined by ϕ. This finishes the proof of Theorem 2.4.

3. LOCALIZABILITY IN QUANTUM MECHANICS

It was discovered independently by Mackey and the physicist Wightman that the notion of imprimitivity can be employed in giving a mathematically rigorous description of the difficult physical concept of localizability.

The physical background of this result can be traced back to a paper written by NEWTON & WIGNER [8], in which the localizability concept was approached from a rather heuristic point of view. By coincidence this paper was published in the same year (1949) as was the first paper of MACKEY [5] on imprimitivity, which provided the tools to repair the mathematical shorthcomings of [8]. We will try to sketch the relationship between imprimitivity and localizability, and in doing so we will more or less follow the exposition by WIGHTMAN [10]. The reader is assumed to be familiar with some of the basic principles explained in the chapters II and III.

It has to be understood that the concept of localizability as we will develop it, is of a rather academical nature, and can serve only as a basis for a physically consistent theory of "measurement of position".

Consider a relativistic system in the Minkowski space-time M. We denote this system by S. There is associated with S a unitary representation of the continuous Poincaré group $P_+^\uparrow$ on the space of states of S. This representation is possibly a projective representation with phase-factor -1. This can be remedied by considering the covering group $\tilde{P}_+^\uparrow$ of $P_+^\uparrow$, but we will assume that we are dealing with a proper unitary representation of $P_+^\uparrow$, and at the end of our discussion we will make a remark on the projective case. Denote this representation by $U: x \to U(x)$, and let $H = H(U)$ be the space of states of S.

We assume that the system S is localizable somewhere in the space $\mathbb{R}^3 \subset M$ at a fixed time. That is, there exist well-defined observables corresponding with the measurement of the position of S in any state in the various parts of $\mathbb{R}^3$. If B is a Borel subset of $\mathbb{R}^3$, we denote the self-adjoint operator corresponding to the observable measuring the position of S in B by E(B). Then $E(B)\psi = \psi$ if S, being in the state ψ, is localized inside B, and $E(S)\psi = 0$ if it is not. Clearly, the only eigenvalues of E(B) are zero and one. Together with its self-adjointness this implies that E(B) is a projection in H. We now give a set of axioms for the family

$$\{E(B);\ B \in \mathcal{B}(\mathbb{R}^3)\},$$

where $\mathcal{B}(\mathbb{R}^3)$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^3$. These axioms express just the reasonable expectations one would have from a well-grounded notion of localization. That we choose the family of Borel sets as our point of departure is not as strange as it maybe seems to be; this is explained in appendix I of WIGHTMAN [10].

AXIOMS

I. *For each* B *in* $\mathcal{B}(\mathbb{R}^3)$, *the operator* E(B) *on* H *is well-defined.*

II. *If* B_1 *and* B_2 *are disjoint Borel sets, then the system can be localized only in one of* B_1 *and* B_2, *i.e.,*

$$B_1 \cap B_2 = \emptyset \Rightarrow E(B_1)E(B_2) = 0.$$

III. *The set of states in which the system is localized in a union* $\cup_i B_i$ *of Borel sets is the linear span of the states in which the system is localized in one of the sets* B_i. *Together with* II *this means:*

$$B_i \cap B_j = \emptyset \Rightarrow E(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} E(B_i).$$

IV. *In each state the system can be localized in* $\mathbb{R}^3$, *i.e.,* $E(\mathbb{R}^3) = I$.

V. *The measurement of position is in a sense invariant (or, rather, covariant) under Euclidean transformations of* $\mathbb{R}^3$. We explain this below.

Notice that II and III imply

$$E(B_1)E(B_2) = E(B_1 \cap B) = E(B_2)E(B_1), \qquad \forall B_1, B_2 \in \mathcal{B}(\mathbb{R}^3).$$

In this set-up the number p(B), defined by

$$p(B) = \frac{\| E(B)\psi \|^2}{\| \psi \|^2}.$$

is equal to the probability of finding S inside B, if it is in the state ψ.

The first four axioms imply that the mapping E: B $\to$ E(B) from $\mathcal{B}(\mathbb{R}^3)$ into $L(H)$ is a projection-valued measure on $\mathbb{R}^3$. The fifth one provides us with a relationship between this measure and the representation U of $P_+^\uparrow$, associated with S. Indeed, let E(3) denote the group of Euclidean motions of $\mathbb{R}^3$, then E(3) is a subgroup of $P_+^\uparrow$, which is to be interpreted as the pointwise stabilizing subgroup of the time-axis in M. By V: x $\to$ V(x) we denote the unitary representation of E(3) obtained by restricting U to E(3). Thus, the operators V(x) give the symmetries of H corresponding to the Euclidean transformations of $\mathbb{R}^3$. Now axiom V expresses that if ψ is a state in which S is localized inside a Borel set B, then the state V(x)ψ in which S is after transformation of the space by x, is a state for which S is localized in the transformed set x[B]. In formula, this means:

$$E(B)\psi = \psi \Longleftrightarrow E(x[B])V(x)\psi = V(x)\psi.$$

Since this equivalence is valid for all states, we infer to the identity

$$V(x)E(B)V(x)^{-1} = E(x[B]), \qquad \forall x \in E(3),\ \forall B \in \mathcal{B}(\mathbb{R}^3).$$

But this expression implies exactly that the triple $(\mathbb{R}^3, V, E)$ is a system of imprimitivity for E(3). Moreover, this system is transitive, since $\mathbb{R}^3$ is a homogeneous space of E(3). Notice that the stabilizer in E(3) at any point of $\mathbb{R}^3$ is isomorphic with SO(3).

Application of the imprimitivity theorem yields the following results:

(i) V is induced on E(3) by a certain representation τ of SO(3),

(ii) The space $\mathcal{H}$ can be identified with a space $L^2(\mathbb{R}^3, \mathcal{H}(\tau))$, such that V acts in this space by

$$(V((y,R))f(x) = \tau(R)f(R^{-1}(x-y)), \qquad x,y \in \mathbb{R}^3,\ R \in SO(3).$$

(iii) E is equivalent to the projection-valued measure in $L^2(\mathbb{R}^3, \mathcal{H}(\tau))$ defined by multiplication with characteristic functions.

From these facts, the first one is perhaps the most striking. Indeed, the restriction of U to E(3) being induoed from SO(3) is a stringent condition on U. Since the only assumption we have made, is localizability of the system S, we can derive from (i) a criterion of localizability:

CRITERION. *A relativistic system in the Minkowski space-time is localizable in* $\mathbb{R}^3$ *if and only if the corresponding unitary representation of the continuous Poincaré group on the space of states, restricted to the Euclidean group* E(3)*, is induced on* E(3) *from its subgroup* SO(3).

The problem now arises of determining which representations of $P_+^\uparrow$ enjoy the property described in the criterion. The solution to this problem is highly nontrivial. If S consists only of one particle, the representation associated with it has to be irreducible. In this case we know that it can be interpreted as being induced on $P_+^\uparrow$ from a certain proper subgroup. There is a theorem of Mackey's which gives the decomposition of the restriction of an induced representation to a subgroup, the so-called *subgroup theorem* (cf. MACKEY [8], BARUT & RĄCZKA [1]). Applying this theorem one arrives at the following result: A relativistic particle is localizable in $\mathbb{R}^3$ if and only if it has real nonzero mass, or if its mass and spin are both zero. Among other things, it follows from this observation that a single photon will not be localizable (at least not in $\mathbb{R}^3$). For a proof, see WIGHTMAN [10] or BARUT & RĄCZKA [1, prop.20.1].

It is possible to change axiom V in such a way as to enable localization of massless particles with nonzero spin and of particles of imaginary mass, in other homogeneous spaces of E(3) than $\mathbb{R}^3$ (cf. BARUT & RĄCZKA [1, ch.20]).

The above described method can also be applied to nonrelativistic systems in space-time. Then the Poincaré group has to be replaced by the Galilei group. For details, see WIGHTMAN [10].

Next we will show how to derive a set of position operators for the system S, from the projection-valued measure E. For each ψ in the space of states $\mathcal{H}$, we define a finite, positive Borel measure μ_ψ on $\mathbb{R}^3$ by

$$\mu_\psi(B) := (\psi, E(B)\psi), \qquad B \in \mathcal{B}(\mathbb{R}^3).$$

By means of these measures, we can define three generally unbounded operators Q_i, $i = 1,2,3$, on $\mathcal{H}$ by

$$(\psi, Q_i\psi) := \int_{\mathbb{R}^3} x_i \, d\mu_\psi(x).$$

It can be verified straightforwardly, that these definitions are legitimate, and, moreover, that the Q_i are self-adjoint. By virtue of the axioms II and III, we have

$$(3.1) \qquad [Q_i, Q_j] = 0, \qquad 1 \le i,j \le 3,$$

i.e., the operators Q_i commute with each other. We denote them symbolically by

$$Q_i = \int_{\mathbb{R}^3} x_i \, dE(x).$$

Let (y,R) denote an element of E(3), with $y \in \mathbb{R}^3$ and $R \in SO(3)$, such that

$$(y,R)[x] = R(x) + y, \qquad x \in \mathbb{R}^3,$$

and set $R = (r_{ij})$, $R^{-1} = (r_{ij}^{-1})$. Then we have

$$\begin{aligned} V(y,R)Q_iV(y,R)^{-1} &= \int_{\mathbb{R}^3} x_i d(V(y,R)E(x)V(y,R)^{-1}) \\ &= \int_{\mathbb{R}^3} x_i dE((y,R)[x]) = \int_{\mathbb{R}^3} ((y,R)^{-1}[x])_i dE(x) \\ &= \int_{\mathbb{R}^3} ((-R^{-1}(y),R^{-1})[x])_i dE(x) \\ &= \int_{\mathbb{R}^3} \sum_j r_{ij}^{-1}(x_j - y_j) dE(x) \\ &= \sum_j r_{ij}^{-1}\left[\int_{\mathbb{R}^3} x_j dE(x) - \int_{\mathbb{R}^3} y_j dE(x)\right] \\ &= \sum_j r_{ij}^{-1}(Q_j - y_j I). \end{aligned}$$

This identity expresses the transformation property of the "position vector" (Q_1,Q_2,Q_3) under symmetries implied by E(3). It comes up to the expectations one would have from a rightly defined set of position operators. Moreover, from this identity we can derive the Heisenberg commutation relations. Indeed, by a theorem of Stone there exist three commuting selfadjoint operators P_k, $k = 1,2,3$, on $\mathcal{H}$ such that

$$V(y,I) = \exp i(y_1P_1+y_2P_2+y_3P_3), \qquad \forall y \in \mathbb{R}^3.$$

These operators are called the momentum operators of S; they are the generators of the three-parameter translation group

$$\{V(y,I);\ \ y \in \mathbb{R}^3\}.$$

From the transformation rule given above for the three-vector (Q_1,Q_2,Q_3) one derives readily (for instance, by formal differentiation) the following relations:

$$[Q_j,P_k] = -i\delta_{jk}I, \qquad 1 \le j,k \le 3.$$

Together with (3.1) and

$$[P_j,P_k] = 0, \qquad 1 \le j,k \le 3,$$

we have the *Heisenberg commutation relations*, which therefore fit perfectly in our model.

Here we arrive at the more general problem of finding the representations of an algebra generated by 2n formal elements $Q_1,\ldots,Q_n$, $P_1,\ldots,P_n$, which satisfy

$$\left.\begin{aligned}[P_j,P_k] &= [Q_j,Q_k] = 0\\ [Q_j,P_k] &= -i\delta_{jk}I\end{aligned}\right\}, \quad i \le j,k \le n.$$

This problem can be solved in a very rigorous (and nice) way by application of the imprimitivity theorem (or, rather, a corollary of the imprimitivity theorem). Indeed, it can be shown that the well-known Schrödinger representation on $L^2(\mathbb{R}^n)$, defined by

$$\left.\begin{aligned}(Q_jf)(x) &= x_jf(x)\\ (P_jf)(x) &= -i\left.\frac{\partial f}{\partial x_j}\right|_x\end{aligned}\right\}, \quad f \in L^2(\mathbb{R}^n),$$

is in a sense unique. This can be found in, for instance, MACKEY [7], JAUCH [3], BARUT & RĄCZKA [1].

<u>REMARK</u>. After the proof of Theorem 1.6 we mentioned that the notions of induction and imprimitivity can be generalized to projective representations such as to enable an extension of the imprimitivity theorem to these representations. This can be used in the case that the representation U is projective, and the procedure described above can be followed without any substantial modification.

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XI

REPRESENTATIONS OF SEMIDIRECT PRODUCTS

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CONTENTS

1. SEMIDIRECT PRODUCTS

Let G be a lcsc. (i.e. locally compact second countable) group, and suppose that we are given two closed subgroups N and H of G, such that

(i) N is invariant;

(ii) $G = N.H$ and $G \approx N\times H$;

(iii) $N \cap H = \{e\}$.

Then we shall call G the *semidirect product* of N and H. Note that each element h of H defines an automorphism of N, by

$$(1.1) \qquad h : n \to hnh^{-1} =: \alpha_h(n), \quad n \in N.$$

Since the group operations in G are continuous, the mapping $(h,n) \to \alpha_h(n)$ is continuous on H×N. The multiplication in G may be written as

$$nhmk = nhmh^{-1}\, hk = n\alpha_h(m)\, hk, \quad n,m \in N,\ h,k \in H.$$

Conversely, if we are given two groups N and H such that there exists a one-to-one homomorphism $\alpha: h \to \alpha_h$ from H into the automorphism group of N, then we can provide the Cartesian product N×H with a group structure by defining

$$(n,h)(m,k) := (n\alpha_h(m),hk), \quad n,m \in N,\ h,k \in H.$$

Note that the homomorphism property of $h \to \alpha_h$ is needed in order to ensure associativity of this structure. The group obtained in this manner is called the semidirect product of N and H relative to α, and usually denoted by N Ⓢ H. It will be a lcsc. group in the product topology if N and H are lcsc. groups and if the mapping $(h,n) \to \alpha_h(n)$ is continuous.

Throughout the remainder of this subsection we will assume that G *is a lcsc. group, and that* G = N Ⓢ H, *and, moreover, we will take* N *to be abelian.*

Let $\hat{N}$ be the family of all irreducible characters of N, that is, $\hat{N}$ consists of all continuous homomorphism $\phi: N \to \mathbb{T}$. If $\phi \in \hat{N}$, then we define a new element $h[\phi]$ of $\hat{N}$ for each h in H by

$$(1.2) \qquad (h[\phi])(n) := \phi(\alpha_{h^{-1}})(n)).$$

It is easily verified that $h[\phi]$ is indeed a member of $\hat{N}$. The character $h[\phi]$ is said to be *conjugate* to ϕ. The action of H on $\hat{N}$ defined by $h: \phi \to h[\phi]$

makes $\hat{N}$ into a H-space. Indeed, we have

$$((h_1h_2)[\phi])(n) = \phi(\alpha_{(h_1h_2)^{-1}}(n)) = \phi(\alpha_{h_2^{-1}}\alpha_{h_1^{-1}}(n)) =$$
$$= (h_2[\phi])(\alpha_{h_1^{-1}}(n)) = (h_1[h_2[\phi]])(n).$$

This property of the H-action is, of course, the reason of introducing inversion into the definition given by formula (1.2).

It is well known that the dual of a lcsc. abelian group can be made into a lcsc. group in its own right, by taking multiplication of characters as composition, and providing it with the so-called topology of uniform convergence on compacta (cf. §VIII. 6 or KIRILLOV [5, 7.3]). A basis for the open sets of this topology is formed by the family of sets $U(C,\varepsilon,\phi_0) \subset \hat{N}$ defined by

$$U(C,\varepsilon,\phi_0) := \{\phi \in \hat{N};\ |\phi(x)-\phi_0(x)| < \varepsilon,\ \forall x \in C\},$$

where C is a compact subset of N, ε a positive nonzero real number, and ϕ_0 an element of $\hat{N}$. The continuity of the mappings $(h,n) \to \alpha_h(n)$ and $n \to \phi(n)$ $(h \in H, n \in N, \phi \in \hat{N})$ leads via the inequality

$$|(h[\phi])(x) - \phi_0(x)| \le |\phi(\alpha_{h^{-1}}(x)) - \phi_0(\alpha_{h^{-1}}(x))| + |\phi_0(\alpha_{h^{-1}}(x)) - \phi_0(x)|,$$

and some simple standard arguments, to the conclusion that the mapping $(h,\phi) \to h[\phi]$ from $H\times\hat{N}$ into $\hat{N}$ is continuous. Hence, $\hat{N}$ is a continuous H-space.

The orbit structure in $\hat{N}$ will play an important role in section 3. Let $\omega \subset \hat{N}$ be an H-orbit and let H_0 be the stabilizer in H of a fixed point $\phi_0 \in \omega$. Then H_0 is a closed subgroup of H and we consider the mapping $\beta\colon hH_0 \to h[\phi_0]$ from the homogeneous space H/H_0 onto ω. This mapping is clearly one-to-one. Furthermore, if we consider the Borel structure on ω generated by the relative topology which ω inherits from $\hat{N}$, and the Borel structure on H/H_0 generated by the quotient topology, then it can be shown that the mappings β and β^{-1} are both Borel mappings, which is expressed by calling β a Borel isomorphism (see VARADARAJAN [11, thm. 8.11]). It can also be verified, by a simple argument, that ω is a Borel subset of $\hat{N}$ (ibidem, p.12). Notice that these observations enable us to identify the Borel measures (or projection-valued measures) on $\hat{N}$, restricted to ω, with those on H/H_0.

Since $\hat{N}$ is a continuous H-space, the mapping β is continuous. However, the inverse mapping $\beta^{-1}\colon \omega \to H/H_0$ need not be continuous, so β is generally not a homeomorphism. It is known that a sufficient condition for β to be a homeomorphism is that ω is a lcsc. space with respect to its relative topology (cf. VARADARAJAN [11, thm. 8.11]). One verifies readily that this condition is satisfied if, for instance, H_0 is compact.

The discussion of examples is postponed to the end of the treatment of the representation theory.

2. THE REPRESENTATIONS OF FINITE SEMIDIRECT PRODUCTS

The author thinks that a good understanding of the representation theory of general lcsc. semidirect products benefits from a preliminary discussion of the finite case. For, the arguments which we will use to derive a classification of the representations of finite semidirect products can be extended to infinite groups with only standard adjustments of a measure theoretical kind. This will be shown in the next section. We are aware of the fact that our treatment of the finite case is amenable to substantial simplifications and generalizations, but our strategy is attuned to the infinite case.

Thus, let G denote the finite semidirect product of an abelian invariant subgroup N and a subgroup H. Consider a representation τ of N (or any finite abelian group). It can be decomposed into a linear combination of characters

$$\tau = \sum_{\phi\in\hat{N}}^{\oplus} n_\phi \phi,$$

where $\{n_\phi\}_{\phi\in\hat{N}}$ is a set of natural numbers, uniquely determined by τ. This decomposition corresponds to a decomposition of the representation space $\mathcal{H}(\tau)$, which is also unique:

$$\mathcal{H}(\tau) = \sum_{\phi\in\hat{N}}^{\oplus} \mathcal{H}_\phi, \qquad \mathrm{Dim}(\mathcal{H}_\phi) = n_\phi.$$

Let P_ϕ be the projection operator of $\mathcal{H}(\tau)$ which has $\mathcal{H}_\phi$ as its range. Then

$$\tau(n) = \sum_{\phi\in\hat{N}} \phi(n)\, P_\phi, \qquad n \in N. \tag{2.1}$$

(Note that this decomposition is the finite counterpart of the spectral decomposition of representations of locally compact abelian groups, provided by the SNAG theorem (§VIII. 6).) As explained in §X. 1 we can view upon $P: \phi \to P_\phi$ as a projection-valued measure based on $\hat{N}$ by setting

$$P_E := \sum_{\phi\in E} P_\phi,$$

for any subset E of $\hat{N}$.

If σ is a representation of G, then $\tau := \sigma_{|N}$ is a representation of N. For any representation τ of N we will denote the corresponding projection-valued measure on $\hat{N}$ by $P^\tau: E \to P^\tau_E$. We can now state the following lemmata on the relationship between σ and $P^{\sigma|N}$.

<u>LEMMA 2.1</u>. *Let τ and ρ be representations of* N *and* H *on the same Hilbert space. Then the following assertions are equivalent:*

(i) *There exists a representation σ of* G *such that $\sigma_{|N} = \tau$ and $\sigma_{|H} = \rho$.*
(ii) *The triple $(\hat{N},\rho,P^\tau)$ is a system of imprimitivity for* H *(cf.* Def. X.1.1*).*

<u>PROOF</u>. (i) $\Rightarrow$ (ii). Condition (i) implies (and is implied by) the following identity:

$$\rho(h)\tau(n)\rho(h)^{-1} = \tau(hnh^{-1}), \quad \forall n \in N, \forall h \in H. \tag{2.2}$$

Using the decomposition (2.1) of τ we obtain

$$\rho(h)\Big(\sum_{\phi\in\hat{N}} \phi(n)P^\tau_\phi\Big)\rho(h)^{-1} = \sum_{\phi\in\hat{N}} \phi(hnh^{-1})P^\tau_\phi.$$

The right hand side can be rewritten as:

$$\sum_{\phi\in\hat{N}} \phi(hnh^{-1})P^\tau_\phi = \sum_{\phi\in\hat{N}} (h^{-1}[\phi])(n)P^\tau_\phi = \sum_{\phi\in\hat{N}} \phi(n)P^\tau_{h[\phi]}.$$

Hence, we have

$$\sum_{\phi\in\hat{N}} \phi(n)\,(\rho(h)P^\tau_\phi\rho(h)^{-1}) = \sum_{\phi\in\hat{N}} \phi(n)P^\tau_{h[\phi]}. \tag{2.3}$$

By uniqueness of decomposition it follows that

$$\rho(h)P^\tau_\phi\rho(h)^{-1} = P^\tau_{h[\phi]}, \quad \forall h \in H, \forall\phi \in \hat{N}. \tag{2.4}$$

But this implies (by linearity) that condition (ii) is satisfied, so we are through.

(ii) $\Rightarrow$ (i). Obviously the above argument can be reversed, that is, (2.4) implies (2.2). But then $\sigma(nh) := \tau(n)\rho(h)$ is a representation of G. □

LEMMA 2.2. *Let* σ_1 *and* σ_2 *be representations of* G. *Then the intertwining space* $R(\sigma_1,\sigma_2)$ *is equal to the intertwining space* $R((\sigma_{1|N},P^{\sigma_1|N}),(\sigma_{2|H},P^{\sigma_2|N}))$ *of the corresponding imprimitivity systems for* H. *In particular one has:*

(i) $\sigma_1 \simeq \sigma_2$ *if and only if the corresponding systems are equivalent.*

(ii) *A representation of* G *is irreducible if and only if the corresponding system is irreducible.*

PROOF. It is clear that

$$R(\sigma_1,\sigma_2) = R(\sigma_{1|N},\sigma_{2|N}) \cap R(\sigma_{1|H},\sigma_{2|H}).$$

Furthermore we have: $T \in R(\sigma_{1|N},\sigma_{2|N})$ iff

$$T\sigma_1(n) = \sigma_2(n)T, \quad \forall n \in N$$

iff

$$\sum_{\phi\in\hat{N}} \phi(n)\ TP_\phi^{\sigma_1|N} = \sum_{\phi\in\hat{N}} \phi(n)\ P_\phi^{\sigma_2|N}\ T, \quad \forall n \in N,$$

iff, for all $\xi \in H(\sigma_1)$ and $\eta \in H(\sigma_2)$:

$$\sum_{\phi\in\hat{N}} \phi(n)\,(TP_\phi^{\sigma_1|N}\xi,\eta) = \sum_{\phi\in\hat{N}} \phi(n)\,(P_\phi^{\sigma_2|N}T\xi,\eta), \quad \forall n \in N.$$

Since the elements of $\hat{N}$ form an orthonormal basis for the space of all complex-valued functions on N (cf. §VI.2), the last identity is equivalent to

$$(TP_\phi^{\sigma_1|N}\xi,\eta) = (P_\phi^{\sigma_2|N}T\xi,\eta),\ \forall\xi \in H(\sigma_1),\ \forall\eta \in H(\sigma_2),\ \forall\phi \in \hat{N}.$$

But this is true if and only if

$$TP_E^{\sigma_1|N} = P_E^{\sigma_2|N}\ T, \quad \forall E \subset \hat{N}.$$

Therefore, we have

$$T \in R(\sigma_{1|N},\sigma_{2|N}) \iff T \in R(P^{\sigma_1|N},P^{\sigma_2|N}),$$

which implies

$$R(\sigma_1,\sigma_2) = R(P^{\sigma_1|N},P^{\sigma_2|N}) \cap R(\sigma_{1|H},\sigma_{2|H}) = R((\sigma_{1|H},P^{\sigma_1|N}),(\sigma_{2|H},P^{\sigma_2|N})).$$

This proves the first statement of the lemma. (i) and (ii) are immediate consequences. □

Next we show how to construct a number of irreducible representations of G. Fix a point ϕ_0 in $\hat{N}$, and let ω_0 denote the orbit of ϕ_0 in $\hat{N}$ under the action of H, i.e.,

$$\omega_0 := \{h[\phi_0];h \in H\}.$$

Then ω_0 is H-homeomorphic with H/H_0, where

$$H_0 := \{h \in H;h[\phi_0] = \phi_0\}$$

denotes the stabilizer in H at ϕ_0.

Now, let ρ be an irreducible representation of H_0, and, for each element nh of G, define an operator $\sigma(nh)$ in the induced representation space $H(\rho^H)$, by

$$(\sigma(nh)f)(x) := (x[\phi_0])(n)(\rho^H(h)f)(x) = (x[\phi_0])(n)f(h^{-1}x), \; x \in H.$$

It is obvious that $\sigma(nh)f$ does belong to $H(\rho^H)$. We show that σ is a representation of G:

$$\begin{aligned}(\sigma(nh)\sigma(mk)f)(x) &= (x[\phi_0])(n)(\sigma(mk)f)(h^{-1}x) = \\ &= (x[\phi_0])(n)(h^{-1}x[\phi_0])(m)f((hk)^{-1}x) = \\ &= \phi_0(x^{-1}nx)\phi_0(x^{-1}hmh^{-1}x)f((hk)^{-1}x) = \\ &= \phi_0(x^{-1}nhmh^{-1}x)f((hk)^{-1}x) = \\ &= (x[\phi_0])(nhmh^{-1})f((hk)^{-1}x) = \\ &= (\sigma(nhmh^{-1}hk)f)(x) = \\ &= (\sigma(nhmk)f)(x).\end{aligned}$$

In fact, σ is an induced representation itself (cf. §IX.6.1). Let $\tau := \sigma|_N$. By virtue of Lemma 2.1, $(\hat{N}, \sigma|_H, P^\tau)$ is a s.o.i. for H, and Lemma 2.2(i) implies that σ is determined up to equivalence by this system. We now determine P^τ:

$$(\tau(n)f)(x) = (x[\phi_0])(n)f(x) = (\sum_{y \in H/H_0} (y[\phi_0])(n)(\chi_{\{yH_0\}} \cdot f)(x).$$

The second step is legitimate since $x \in H$ belongs to exactly one H_0-coset, say y_0H_0, and then $y_0[\phi_0] = x[\phi_0]$, since H_0 stabilizes ϕ_0. Next we define a projection-valued measure based on $\hat{N}$ and acting in $H(\rho^h)$, by

$$P_E := \sum_{\phi \in E} P_\phi, \quad E \subset \hat{N},$$

where

$$(2.5) \qquad (P_\phi(f))(x) := \begin{cases} 0 & \text{if } \phi \in \hat{N}\backslash\omega_0 \\ \chi_{\{yH_0\}}(x) \cdot f(x) & \text{if } \phi = y[\phi_0],\ y \in H. \end{cases}$$

Then we may write

$$(\tau(n)f)(x) = (\sum_{\phi \in \hat{N}} \phi(n)P_\phi(f))(x), \qquad f \in H(\rho^H),$$

so P is the projection-valued measure associated with τ. Moreover, P is based on ω_0, actually, since it vanishes on $N\backslash\omega_0$. We express this fact by saying that P is *concentrated on one orbit* (ω_0). Besides, we know that ω_0 is H-homeomorphic with H/H_0, and that the homeomorphism is given by

$$\Phi : y\,H_0 \to y[\phi_0], \qquad y \in H.$$

Consequently, we may consider P as a projection-valued measure on H/H_0, by defining:

$$P_E := P_{\Phi(E)}, \qquad E \subset H/H_0.$$

Then we find (by (2.5)):

$$(2.6) \qquad (P_E(f))(x) = (P_{\Phi(E)}(f))(x) = \chi_E(\bar{x})f(x), \qquad E \subset H/H_0,\ f \in H(\rho^H).$$

But formula (2.6) defines a projection-valued measure equal to the one occuring in the canonical imprimitivity system of ρ^H (cf. Example X.1.3). Hence, by Corollary X.1.5(ii), the irreducibility of ρ implies irredicibility of the system $(N,\sigma|_H,P^T) = (H/H_0,\rho^H,P)$ (note that $\sigma|_H = \rho^H$ by definition), which in its turn results in irreducibility of σ, by virtue of Lemma 2.2(ii). Finally, Lemma 2,2(i) together with Corollary X.1.5(i) yields that σ is determined up to equivalence by ρ.

In the construction of σ we have chosen a fixed point ϕ_0 in ω_0, but it will turn out in the next theorem that the collection of representations, obtained by letting ρ run through $\hat{H}_0$ is independent (up to equivalence) of the choice of ϕ_0 in ω_0. This fact can also easily be verified straightforwardly.

We shall call the above constructed representation σ of G *associated* with the orbit ω_0 and the representation $\rho \in \hat{H}_0$, and we shall denote it by $\sigma^{(\omega_0,\rho)}$. The following theorem concludes the discussion of representation of finite semidirect products.

<u>THEOREM 2.3</u>. (Mackey) *Let* $\{\omega\}_{\omega\in\Omega}$ *be the collection of* H*-orbits in* $\hat{N}$ *and let* H_ω *be the stabilizer in* H *at a fixed point of* ω. *Then*:

(i) $\sigma^{(\omega,\rho)}$ *is an irreducible representation of* G *for all pairs* (ω,ρ), *in which* ρ *is an irreducible representation of* H_ω.

(ii) $\sigma^{(\omega,\rho)} \simeq \sigma^{(\omega',\rho')}$ *if and only if* $\omega = \omega'$ *and* $\rho \simeq \rho'$.

(iii) *Each member of* $\hat{G}$ *is of the form* $\sigma^{(\omega,\rho)}$, *for some* $\omega \in \Omega$ *and* $\rho \in \hat{H}_\omega$.

<u>PROOF</u>. (i) was already proved above. There we also showed that $\sigma^{(\omega,\rho)} \simeq \sigma^{(\omega,\rho')}$ if and only if $\rho \simeq \rho'$. As to the role of the orbit ω in determining the equivalence class of $\sigma^{(\omega,\rho)}$, it suffices to make the obvious observation that the restrictions to N of $\sigma^{(\omega,\rho)}$ and $\sigma^{(\omega',\rho')}$ will not be equivalent if $\omega \neq \omega'$. This proves (ii).

(iii): Let σ be an irreducible representation of G, and consider the projection-valued measure $P := P^{\sigma|N}$. We contend that P is concentrated on one orbit. Indeed, let ω be an orbit, then by virtue of the identity

$$\sigma|_H(h)P_\omega\, \sigma|_H(h)^{-1} = P_{h[\omega]} = P_\omega, \qquad \forall h \in H,$$

It follows that P_ω commutes with all operators $\sigma|_H(h)$, $h \in H$. Furthermore, P_ω commutes with $\sigma|_N$ as well, so $P_\omega = 0$ or I, by the irreducibility of σ.

Suppose that $P_\omega = 0$ for all orbits ω. Then we would have

$$I = P_{\hat{N}} = \sum_{\omega\in\Omega} P_{\omega} = 0;$$

a contradiction. On the other hand, it is obvious that $P_{\omega_1} = P_{\omega_2} = I$ is not allowed, unless $\omega_1 = \omega_2$. Hence, there is exactly one orbit, say ω_0, with $P_{\omega_0} = I$ and $P_{\hat{N}\setminus\omega_0} = 0$ *).

But then we may view P as a projection-valued measure on $H/H_{\omega_0} \approx \omega_0$, and therefore we see that $(H/H_{\omega_0}, \sigma|_H, P^{\sigma|N})$ is a transitive s.o.i. for H (Lemma 2.1). By the imprimitivity theorem it follows that (i) $\sigma|_H$ is induced on H by a certain representation ρ of H_{ω_0} and (ii) the system is equivalent to the canonical system of ρ^H. Since σ is irreducible, ρ is also irreducible, by Lemma 2.2(ii) and Corollary X.1.5, and this fact together with Lemma 2.2 (i) yields that σ is equivalent to $\sigma^{(\omega_0,\rho)}$. □

<u>EXAMPLE 2.4</u>. In §VI.2 we discussed the permutation group S_3. Let $N = A_3$, the alternating subgroup and $H = \{(1),(12)\}$, a cyclic subgroup of order 2. Then it is readily verified that S_3 is the semidirect product of N and H. The characters of A_3 were denoted by ψ_1, ψ_2, ψ_3 in Example VI.2.11. The group H acts on $\hat{A}_3$ by

$$(1)[\psi_i] = \psi_i, \quad i = 1,2,3;$$

$$(12)[\psi_1] = \psi_1; \quad (12)[\psi_2] = \psi_3; \quad (12)[\psi_3] = \psi_2.$$

Hence there are two orbits:

$$\omega_1 = \{\psi_1\}, \quad \text{stabilizer: } H_1 = H;$$

$$\omega_2 = \{\psi_2,\psi_3\}, \quad \text{stabilizer: } H_2 = \{(1)\}.$$

Note that the character table of H is

	(1)	(12)
ρ_1	1	1
ρ_2	1	-1

*) This paragraph marks the main difference between the representation theories of finite and general lcsc. semidirect products. In fact, let P be a projection-valued measure based on a continuous G-space, where G is a lcsc. group. Then the fact that $P_\omega = 0$ for each orbit ω does not necessarily imply that $\Sigma_{\omega\in\Omega} P_\omega = 0$, since this "sum" may be continuous. By laying a condition of a measure theoretical kind on the orbit structure of the G-space, this defect can be repaired. This will be shown in the next section.

By some elementary computations we find:

$$\sigma^{(\omega_1,\rho_1)} = \chi_1; \quad \sigma^{(\omega_1,\rho_2)} = \chi_2; \quad \sigma^{(\omega_2,1)} \cong \tau_3,$$

where we use the notation of Example VI.2.11.

In SERRE [9] the reader can find a shorter proof of Theorem 2.3. In REYES [8] an analogous approach to finite semidirect products $G = N \circledS H$ with N not abelian is given. It proceeds by admitting irreducible projective representations of the little groups in the construction of irreducible representations of G.

3. THE REPRESENTATIONS OF LCSC. SEMIDIRECT PRODUCTS

Throughout this section G denotes a lcsc. group which is the semidirect product of two of its subgroups N and H, with N abelian and invariant.

First we recall the statements of the SNAG theorem (cf. Theorem VIII. 6.3 or BARUT & RAÇZKA [2, 6.2]):

(i) *If τ is a unitary representation of a lcsc. abelian group* A, *then there exists a unique projection-valued measure* $P: E \to P_E$, *based on* $\hat{A}$ *and acting in the representation space of* τ, *such that*

$$(\tau(a)\xi,\eta) = \int_{\hat{A}} \phi(a)d\mu_{\xi,\eta}(\phi), \quad \forall\xi,\eta \in H(\tau), \quad \forall a \in A,$$

where the complex Borel measure $\mu_{\xi,\eta}$ *on* $\hat{A}$ *is defined by*

$$\mu_{\xi,\eta}(E) := (P_E\xi,\eta) \quad \textit{for Borel sets } E \textit{ in } \hat{A}.$$

We write as usual:

$$\tau(a) = \int_{\hat{A}} \phi(a)dP_\phi, \quad a \in A. \tag{3.1}$$

(ii) *Conversely, if* $P: E \to P_E$ *is a projection-valued measure on* $\hat{A}$, *acting in a certain seperable Hilbert space* H, *then* (3.1) *defines a unitary representation of* A *on* H.

(iii) *If* τ_i *and* P_i (i=1,2) *are related by* (3.1), *then* $R(\tau_1,\tau_2) = R(P_1,P_2)$.

If τ is a representation of N, then we denote by P^{τ} the projection-valued measure which corresponds to τ by virtue of this theorem.

Let τ and ρ be unitary representations of N and H respectively, with the same representation space. Suppose that they satisfy

$$\rho(h)\tau(n)\rho(h)^{-1} = \tau(hnh^{-1}), \quad \forall n \in N, \forall h \in H.$$

By decomposing τ on both sides as in (3.1), we obtain

$$\rho(h)\left(\int_{\hat{N}} \phi(n)dP_{\phi}^{\tau}\right)\rho(h)^{-1} = \int_{\hat{N}} \phi(hnh^{-1})dP_{\phi}^{\tau}.$$

This yields

$$\int_{\hat{N}} \phi(n)d(\rho(h)P_{\phi}^{\tau}\rho(h)^{-1}) = \int_{\hat{N}} (h^{-1}[\phi](n)dP_{\phi}^{\tau} = \int_{\hat{N}} \phi(n)dP_{h[\phi]}^{\tau}.$$

This formula is obviously the infinite counterpart of formula (2.3). By the uniqueness granted in statement (i) above, we conclude that

$$\rho(h)P_{E}^{\tau}\rho(h)^{-1} = P_{h[E]}^{\tau},$$

for all h in H and all Borel subsets E of $\hat{N}$. This proves Lemma 2.1 in the case of lcsc. groups.

Next, let ρ_1 and ρ_2 be unitary representations of G, and set $\tau_1 := \rho_1|_N$, $\tau_2 := \rho_2|_N$. Then, by virtue of Theorem VIII.6.3, we have

$$R(\tau_1,\tau_2) = R(P^{\tau_1},P^{\tau_2}).$$

This proves Lemma 2.2 in the general case.

We will now repeat the construction of representations of G. Fix an orbit ω_0 in $\hat{N}$, a point ϕ_0 in ω_0, and denote by H_0 the stabilizer in H at ϕ_0. Let ρ be an irreducible unitary representation of H_0 and let f be a function in the induced representation space $\mathcal{H}(\rho^H)$, considered as a space of L_2-functions. For each element nh $\in$ G we define a new function σ(nh)f on H by

$$(3.2) \qquad (\sigma(nh)f)(x) := (x[\phi_0])(n)(\rho^H(h)f)(x).$$

We have

(i) $(\sigma(nh)f)(xh_0) = \rho(h_0^{-1})(\sigma(nh)f)(x), \quad h_0 \in H_0;$

(ii) $x \mapsto ((\sigma(nh)f)(x),\xi) = (x[\phi_0])(n)((\rho^H(h)f)(x),\xi)$
is a Borel function for each $\xi \in H(\rho)$, since it is the product of two Borel functions;

(iii) $$\int_{H/H_0} \|(x[\phi_0])(n)(\rho^H(h)f)(x)\|^2 \, d\mu(\bar{x})$$
$$= \int_{H/H_0} \|(\rho^H(h)f)(x)\|^2 d\mu(\bar{x}) = \|f\|^2.$$

These properties imply that $\sigma(nh)$ belongs to $H(\rho^H)$ and (iii) implies also $\|\sigma(nh)f\| = \|f\|$. Furthermore, we have

$$(\sigma(nh)\sigma(mk)f)(x) = (x[\phi_0])(n)(\sigma(mk)f)(h^{-1}x)(R(\bar{x},h^{-1}))^{\frac{1}{2}} =$$
$$= (x[\phi_0])(n)(h^{-1}x[\phi_0])(m)f((hk)^{-1}x)(R(\bar{x},h^{-1})(R(\overline{h^{-1}x},k^{-1}))^{\frac{1}{2}}$$
$$= (x[\phi_0])(n)(nhmk^{-1})f((hk)^{-1}x)(R(\bar{x},(hk)^{-1}))^{\frac{1}{2}} = (\sigma(nhmh^{-1}hk)f)(x)$$
$$= (\sigma(nhmk)f)(x), \qquad n,m \in N,\ h,k \in H.$$

Here R is a continuous strictly positive function on $H/H_0 \times H$ corresponding to the quasi-invariant measure μ on H/H_0.

Putting the pieces together we see that G defines a homomorphism *) from G into the algebra of unitary operators on $H(\rho^H)$. As to continuity, this follows from the equality $\sigma(nh) = \sigma(n)\sigma(h)$, $n \in N$, $h \in H$, and the obvious observation that $\sigma|_N$ and $\sigma|_H$ are both continuous functions on G.

If we set $\tau = \sigma|_N$, then

$$(\tau(n)f)(x) = (x[\phi_0])(n)f(x), \qquad f \in H(\rho^H).$$

We contend that this identity implies that the projection-valued measure P^τ on $\hat{N}$, associated with τ, is concentrated on the orbit ω_0. To prove this, we define a projection-valued measure $P: E \to P_E$ on $\hat{N}$, which acts in $H(\rho^H)$, by

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*) In fact, σ can be considered as an induced representation, cf. §IX.6.1.

$$(P_E f)(x) = \chi_E(x[\phi_0])f(x), \qquad E \text{ a Borel set in } \hat{N}.$$

For each $f,g \in H(\rho^H)$, P yields a complex Borel measure on $\hat{N}$:

$$\mu_{f,g}(E) = (P_E f,g) = \int_{H/H_0} \chi_E(x[\phi_0])(f(x),g(x))d\mu(\bar{x}).$$

This can be rewritten (by abuse of notation) as follows

$$d\mu_{f,g}(\phi) = \begin{cases} 0 \quad \text{if} \quad \phi \in \hat{N}\backslash\omega_0 \\ (f(x),g(x))d\mu(\bar{x}) \quad \text{if } \phi = x[\phi_0]. \end{cases}$$

Hence we find

$$(\tau(n)f,g) = \int_{H/H_0} (x[\phi_0])(n)(f(x),g(x))d\mu(\bar{x})$$

$$= \int_{\hat{N}} \phi(n)d\mu_{f,g}(\phi),$$

which proves our assertion, by virtue of the uniqueness of the projection-valued measure associated with τ.

The set of representations σ obtained by letting ρ run through $(\text{stab}(\phi_0))\hat{}$ is independent (up to equivalence) of the choice of ϕ_0. Verification of this assertion can be done by straightforward manipulation of the definition of induced representations.

Before we state the analogue of Theorem 2.3 we have to consider what happens to the third statement of this theorem. As we have pointed out after the proof of this theorem, one of the arguments used in proving the third statement, does not apply to general lcsc. semidirect products. Besides, it is possible to give counterexamples of lcsc. semidirect products having a lot of irreducible unitary representations which can not be constructed in the above described manner. Hence we must look for a more restricted class of groups, such that Theorem 2.3 carries over completely.

DEFINITION 3.1. A continuous G-space X for a lcsc. group G is said to be *countably separated* (or to have a *smooth orbit structure*) if there exists a countable sequence $B_1, B_2, \ldots$ of Borel subsets in X, such that:

(i) Each B_i is a union of G-orbits.

(ii) Each orbit in X is the intersection of those B_i that contain it.

If G is a lcsc. semidirect product of N and H such that the H-orbit structure in $\hat{N}$ is countably seperated, then we shall call G a *regular semidirect product*.

DEFINITION 3.2. Let P be a projection-valued measure on a continuous G-space X for a lcsc. group G. Then we say that P is *almost transitive* (or *ergodic*) if $P_B = 0$ or I for each G-invariant Borel subset $B \subset X$ (i.e., $x[B] = B$, $\forall x \in G$).

Note that this definition implies two possibilities: (i) $P_\omega = 0$ for all orbits or (ii) $P_\omega = I$ and $P_{X\setminus\omega} = 0$, for a certain orbit. In the last case we call P *transitive*, or *concentrated in one orbit*.

LEMMA 3.3. *Let P be a projection-valued measure on a continuous G-space with countably seperated orbit structure. If P is almost transitive, then it is concentrated on one orbit.*

PROOF. Let $\{B_i\}_{i=1}^{\infty}$ be a sequence of Borel subsets of X, having the properties described in Definition 3.1. Suppose $P_\omega = 0$, for all orbits ω in X. For a fixed orbit ω_0 there exists a subsequence $\{B_{n_i}\}_{i=1}^{\infty}$ of $\{B_i\}_{i=1}^{\infty}$ such that

$$\omega_0 = \bigcap_{i=1}^{\infty} B_{n_i} \quad \text{and } B_{n_{i+1}} \subset B_{n_i} \quad \text{for } i = 1,2,\dots .$$

(Here we assume, without damaging generality, that $\{B_i\}_{i=1}^{\infty}$ is closed under finite intersection.) But then we have

$$0 = P_{\omega_0} = P_{\bigcap_{i=1}^{\infty} B_{n_i}} = \lim_{i\to\infty} P_{B_{n_i}} .$$

Since each B_i is a union of orbits, $P_{B_i} = 0$ or I for all i. Hence, the above identity implies that $P_{B_{n_i}} = 0$ for at least one B_{n_i}. Consequently, each orbit in X is contained in a Borel set B_i of P-measure zero, which in turn implies that X can be covered with a countable family of P-null-sets. Thus $P_X = 0$; a contradiction. □

For the sake of completeness we will show by means of an example that the condition of countable seperateness can not be omitted in Lemma 3.3.

EXAMPLE 3.4. Let $\mathbb{T}$ be the circle group, consisting of all complex numbers of modulus one, and let $\mathbb{Z}$ be the additive group of integers. We make $\mathbb{T}$ into a continuous $\mathbb{Z}$-space by defining a $\mathbb{Z}$-action on $\mathbb{T}$ by

$$n(z) = e^{in}z, \quad n \in \mathbb{Z}, \quad z \in \mathbb{T}.$$

Consider the projection-valued measure P on $\mathbb{T}$ which is canonically associated with the regular representation of $\mathbb{T}$ on $L^2(\mathbb{T},\alpha)$, where α is the normalized rotation invariant measure on $\mathbb{T}$. That is, for each Borel subset E of $\mathbb{T}$ we have

$$(P_E(f))(z) = \chi_E(z)f(z), \quad f \in L^2(\mathbb{T},\alpha).$$

The Fourier coefficients of χ_E are given by

$$(\chi_E)^\wedge(n) = \frac{1}{2\pi}\int_0^{2\pi} \chi_E(\phi)e^{-in\phi}\,d\phi, \quad n \in \mathbb{Z}.$$

For $\mathbb{Z}$-invariant E we get

$$\begin{aligned}(\chi_E)^\wedge(n) &= \frac{1}{2\pi}\int_0^{2\pi} \chi_E(\phi)e^{-in\phi}\,d\phi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \chi_E(\phi+1)e^{-in\phi}\,d\phi \\ &= e^{in}(\chi_E)^\wedge(n).\end{aligned}$$

Hence, $(\chi_E)^\wedge(n) = 0$ unless $n = 0$, so $\chi_E(\phi) = (\chi_E)^\wedge(0)$ (equality in the sense of L^2-functions). Consequently, χ_E is constant a.e. $[\alpha]$, which proves that $P_E = 0$ or I for any $\mathbb{Z}$-invariant subset E of $\mathbb{T}$. However, since the $\mathbb{Z}$-orbits are countable they have α-measure zero, so $P_\omega = 0$ for all orbits ω.

REMARK. In §1 we mentioned that the H-orbits in $\hat{N}$ can be provided with two topologies, the quotient topology from H/H_ω and the relative topology from $\hat{N}$. The one-to-one mapping $xH_\omega \to x[\phi]$ (where H_ω stabilizes $\phi \in \omega$) is a homeomorphism with respect to the quotient topology and a continuous mapping with respect to the relative topology, The following highly nontrivial fact can be proved (GLIMM [3]): G *is regular if and only if the mapping* $xH_\omega \to x[\phi]$ *is a homeomorphism with respect to the relative topology on* ω *from* $\hat{N}$, *for each orbit* ω. We emphasize that in general this equivalence is only valid if G is second countable. By simple standard methods one verifies that the necessary condition is satisfied if H is compact. But the Poincaré group, for instance, is a regular semidirect product, as will be shown in section 6, and in this case H is equal to the restricted Lorentz group, which is not compact.

Let us assume that our semidirect product G is regular. Let σ be an irreducible unitary representation of G, and set $\tau = \sigma_{|N}$. For each G-invariant Borel subset B of $\hat{N}$, we see that P^τ_B commutes with any operator belonging to one of the sets $\{\sigma(h);\ h \in H\}$ and $\{\tau(n) = \sigma(n);\ n \in N\}$. Hence, since σ is irreducible, P^τ_B is either zero or the identity. But then, by virtue of Lemma 3.3, P^τ_B is concentrated on one orbit, say ω_0. Therefore, we may view $(\hat{N}, \sigma_{|H}, P^\tau)$ as a transitive system, based on H/H_0, where H_0 is the stabilizer at a fixed point of ω_0. This implies

(i) $\sigma_{|H}$ is induced by a certain unitary representation ρ of H_0;
(ii) $(H/H_0, \sigma_{|H}, P^\tau)$ is equivalent to the canonical system of ρ^H.

By virtue of Lemma 2.2(ii) for lcsc. semidirect products and Corollary X.1.5 we conclude that ρ is irreducible. Finally, from Lemma 2.2(i) it follows that σ is equivalent to $\sigma^{(\omega_0,\rho)}$.

THEOREM 3.5 (Mackey). *Let* G *be a lcsc. semidirect product of* N *and* H, *with* N *abelian, let* $\{\omega\}_{\omega\in\Omega}$ *be the collection of* H*-orbits in* $\hat{N}$, *and let* H_ω *denote the stabilizer in* H *at a fixed point of* $\omega \in \Omega$. *Then one has:*

(i) $\sigma^{(\omega,\rho)}$ *is irreducible for all* ω *in* Ω *and all* ρ *in* $\hat{H}_\omega$;
(ii) $\sigma^{(\omega,\rho)} \simeq \sigma^{(\omega',\rho')}$ *if and only if* $\omega = \omega'$ *and* $\rho \simeq \rho'$.

If G *is regular, then:*

(iii) *The representations* $\sigma^{(\omega,\rho)}$, $\omega \in \Omega$, $\rho \in \hat{H}_\omega$, *exhaust the set of all unitary irreducible representations of* G, *up to equivalence.* □

REMARK 3.6. Several authors use a somewhat different construction of the representations $\sigma^{(\omega,\rho)}$ (MACKEY [7], LIPSMAN [6]). They proceed as follows. Choose an element ϕ of ω, let ρ be an irreducible unitary representation of H_ω and set

$$\tau(nh) = \phi(n)\rho(h), \quad n \in N,\ h \in H_\omega.$$

It is readily verified that τ defines a unitary representation of the subgroup $N \otimes H_\omega$ of G. The next step is induction of τ on G, and it can be shown that τ^G is irreducible.

We suggest that the reader thinks out for himself how equivalence of τ^G and $\sigma^{(\omega,\rho)}$ can be proved.

REMARK 3.7. For convenience we wish to mention two special cases of the construction of $\sigma^{(\omega,\rho)}$ which do often occur.

First, consider the trivial character of N, which sends all elements of N to the identity of $\mathbb{C}$. If ϕ_0 denotes this character, then its orbit is $\omega_0 = \{\phi_0\}$, and its little group comprises all of H. Hence, we get

$$\sigma^{(\omega_0,\tau)}(n,h)x = \tau(h)x, \qquad \tau \in \hat{H},\ x \in \mathcal{H}(\tau).$$

Thus, the irreducible unitary representations associated with ω_0 are just the trivial extensions to G of the irreducible unitary representations of H. We shall call ω_0 the *trivial orbit*.

Another extreme case is the one in which the little group is trivial. Suppose that ω is an orbit with $H_\omega = \{e\}$. Then, for a fixed point ϕ of ω we obtain

$$(\sigma^{(\omega,1)}(n,h)f(x) = (x[\phi])(n)(\lambda(h)f)(x) =$$

$$= (x[\phi])(n)f(h^{-1}x), \quad f \in L^2(H),$$

where 1 denotes the unique irreducible character of H_ω, and λ the regular representation of H on $L^2(H)$.

We conclude this section with a discussion of the large amount of literature dealing with various generalizations of Theorem 3.5. In particular, one is concerned about what happens if N is no longer abelian. We give an example of the results in this case.

Let N be a closed invariant subgroup of a lcsc. group G. Then, if τ is a unitary representation of N, and if x is an element of G, the mapping $x[\tau]$ from N into $L(\mathcal{H}(\tau))$ given by $x[\tau] : n \to \tau(x^{-1}nx)$, $n \in N$, still defines a unitary representation of N, which will be irreducible if τ is irreducible. Hence, $\hat{N}$ can be made into a G-space. Moreover, if N is type I (cf. MACKEY [7, p.42] or chapter VII of these notes), then it can be shown (cf.

VARADARAJAN [11, p.10]) that $\hat{N}$ is a standard Borel space and that the mapping $(x,\tau) \to x[\tau]$ is a Borel mapping from $G \times \hat{N}$ onto $\hat{N}$. Such a G-space is called a Borel G-space instead of a continuous G- space. Clearly, the definition of countable separateness extends to these spaces without alterations. If the orbit structure in $\hat{N}$ is countably separated then we say that N is *regularly embedded* in G. We have the following theorem (MACKEY [7], LIPSMAN [6]):

THEOREM 3.8. *Let* N *be a type* I, *regularly embedded closed invariant subgroup of a lcsc. group* G *and denote by* G_ω *the stabilizer in* G *at a fixed point of* ω, *where* $\{\omega\}_{\omega\in\Omega}$ *is the collection of* G*-orbits in* $\hat{N}$. *Then*

$$\hat{G} = \bigcup_{\omega\in\Omega} \{\rho^G ; \rho \in \check{G}_\omega\},$$

where, denoting by ϕ_ω *the element of* ω *stabilized by* G_ω, *the set* $\check{G}_\omega$ *is defined by*

$$\check{G}_\omega := \{\rho \in \hat{G}_\omega ; \rho|_N \text{ is equivalent to a direct sum of } n \text{ copies of } \phi_\omega, \text{ with } n = \infty,1,2,\ldots\}.$$

4. THE "ax+b"-GROUP

Consider the semidirect product G of $N = \mathbb{R}$ and $H = \mathbb{R}_+$, the multiplicative group of nonnegative real numbers, relative to

$$\alpha_h : n \to hn, \qquad h \in \mathbb{R}_+,\ n \in \mathbb{R}.$$

Thus, $G = \{(n,h);\ n \in \mathbb{R},\ h \in \mathbb{R}_+\}$, and

$$(n,h)(m,k) = (n+hm,hk).$$

We have $\hat{N} = \mathbb{R}$, and the irreducible characters of N are given by

$$\phi_a(n) = e^{ian}, \qquad a \in \mathbb{R}.$$

H acts on $\hat{N}$ by

$$(h[\phi_a])(n) = \phi_a(\tfrac{n}{h}) = \phi_{\frac{a}{h}}(n).$$

Hence, the orbits in $\hat{N}$ are:

$$\omega_0 = \{\phi_0\}, \qquad \text{stabilizer: } H_0 = H;$$

$$\omega_+ = \{\phi_a; a>0\}, \qquad \text{stabilizer: } H_+ = \{1\};$$

$$\omega_- = \{\phi_a; a<0\}, \qquad \text{stabilizer: } H_- = \{1\}.$$

Consequently, there are no proper little groups, and we find the following irreducible unitary representations of G:

Ad ω_0: This orbit is the trivial orbit (cf. Remark 3.7); the representations associated with it are just the trivial extensions to G of the irreducible representations of H. These are given by

$$\psi_a(h) = h^{ia}, \qquad a \in \mathbb{R}.$$

Hence, we find

$$\sigma^{(\omega_0,\psi_a)}(n,h)z = h^{ia} z, \qquad z \in \mathbb{C}.$$

Ad ω_+: The little group in this case is the trivial subgroup $\{1\}$ of H. Hence, choosing ϕ_1 as a fixed point in ω_+, we obtain one irreducible unitary representation of G on $L^2(\mathbb{R}_+)$:

$$(\sigma^{(\omega_+,1)}(n,h)f)(x) = \phi_{x^{-1}}(n)f(h^{-1}x) = e^{i\frac{n}{x}} f(h^{-1}x).$$

Ad ω_-: This case is analogous to ω_+. We choose ϕ_{-1} as a fixed point, and get

$$(\sigma^{(\omega_-,1)}(n,h)f)(x) = \phi_{-x^{-1}}(n)f(h^{-1}x) = e^{-i\frac{n}{x}} f(h^{-1}x), \quad f \in L^2(\mathbb{R}_+).$$

Thus, we found a continuous family of irreducible characters, and two infinite-dimensional representations. Since the number of orbits in $\hat{N}$ is finite, G is regular and hence the above representations exhaust the set of all irreducible unitary representations of G.

The group G is usually called the "ax+b-group" (it can be interpreted as being the identity component in the group of all linear transformations of a straight line in a plane), and it is of historical interest, since it was one of the first noncompact groups to have all of its irreducible unitary representations classified (see GEL'FAND & NAIMARK [4]). Moreover, this was done before Mackey introduced his general theory.

This remark applies also to three other examples; the Euclidean groups E(2) and E(3) (§5) and the continuous Poincaré group $P_+^{\uparrow}$ (§6). The historical references for these examples are WIGNER [12] and BARGMANN [1].

5. THE EUCLIDEAN GROUPS

Let G be a lcsc. group which is the semidirect product of two of its subgroups N and H with N abelian and invariant. We say that G is a *motion group* if H is compact. Notice that this implies that G is regular (see the remark before Theorem 3.5). Well-known examples of such groups are the Euclidean motion groups E(n).

Let $N = \mathbb{R}^n$ and $H = SO(n)$ (the rotation group of $\mathbb{R}^n$), and let G be the semidirect product of N and H relative to

$$\alpha_R(x) = R(x), \qquad R \in SO(n),\ x \in \mathbb{R}^n.$$

Then G can be viewed as being the group of all rotations and translations of $\mathbb{R}^n$, and it is called the Euclidean motion group of $\mathbb{R}^n$, denoted by E(n).

The character group of N is isomorphic with $\mathbb{R}^n$, and the characters are given by

$$\phi_y(x) = e^{i(x,y)}, \qquad y \in \mathbb{R}^n,$$

where (x,y) denotes the Euclidean inner product on $\mathbb{R}^n$. For all $R \in SO(n)$, we have

$$(Rx,y) = (x,R^{-1}y), \qquad \forall x,y \in \mathbb{R}^n.$$

Hence, SO(n) acts on $\hat{N}$ by

$$R[\phi_y] = \phi_{R(y)}.$$

Consequently, the orbits in $\hat{N}$ are (n-1)-dimensional spheres, i.e.

$$\omega_r = \{\phi_x;\ (x,x) = r^2\}, \qquad r \geq 0.$$

Regularity of E(n) can be verified directly. Indeed, for each ordered pair (r_1,r_2) of rational numbers with $0 < r_1 < r_2$, let a Borel set $B(r_1,r_2)$ in $\hat{N}$ be defined by

$$B(r_1,r_2) := \bigcup_{r_1<s<r_2} \omega_s,$$

and set

$$B_0 := \omega_0.$$

Then $\{B_0\} \cup \{B(r_1,r_2) \mid 0 < r_1 < r_2;\ (r_1,r_2) \in \mathbb{Q}^2\}$ is a countable family of Borel subsets of $\hat{N}$, which satisfies the conditions of Def. 3.1.

The stabilizers of the fixed points $\phi_{(r,0,0,\ldots,0)}$ are given by

$H_0 := \mathrm{stab}(\phi_{(0,\ldots,0)}) = SO(n)$, and

$H_r := \mathrm{stab}(\phi_{(r,0,\ldots,0)}) = SO(n-1)$, $r > 0$.

Here we consider SO(n-1) as a subgroup of SO(n) by the embedding

$$R \in SO(n-1) \to \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \in SO(n).$$

In the case $n = 2$, H_0 is isomorphic with the circle group $\mathbb{T} = \{e^{i\phi};\phi \in [0,2\pi)\}$, and $H_r = \{1\}$, $r > 0$. Hence, the set of irreducible unitary representations of E(2) consists of

(i) a countable family of characters, parametrized by $n = 0, \pm 1, \pm 2,\ldots,$ which are the extensions to E(2) of the characters of $\mathbb{T}$;

(ii) a continuous family of infinite-dimensional representations, parametrized by $r > 0$, which have the form

$$(\sigma^{(\omega_r,1)}(y,R)f)(S) = e^{ir(S^{-1}y)_1} f(R^{-1}S),$$

where S belongs to SO(2), and f belongs to $L^2(SO(2),\alpha)$ with α being the rotation invariant measure on SO(2).

For $n = 3$, $H_r \simeq \mathbb{T}$, $r > 0$, and $H_0 = SO(3)$. It is well-known that the set of irreducible unitary representations of SO(3) consists of a countable family of representations, usually denoted by $D^{(s)}$, $s = 0,1,2,\dots$, where the dimension of $D^{(s)}$ equals $2s + 1$ (cf. §III.6). Note that the unitary irreducible representations of the special unitary group SU(2) are usually denoted by $D^{(s)}$ as well, for $s = 0,\frac{1}{2},1,\dots$.

This is explained as follows: The group SU(2) is the two-fold covering group of SO(3), and therefore its irreducible unitary representations give rise to irreducible unitary representations of SO(3), which are possibly projective with phasefactor -1. It can be shown that $D^{(s)}$ yields a proper representation of SO(3) for s integer, which is also denoted by $D^{(s)}$, and a projective representation of SO(3) for $s = \frac{1}{2},\frac{3}{2},\dots$.

The set of irreducible unitary representations of E(3) is given by

(i) a series $\sigma^{(s)} := \sigma^{(\omega_0,D^{(s)})}$, $s = 0,1,2,\dots$, with $\dim(\sigma^{(s)}) = 2s + 1$;

(ii) a continuous family of infinite-dimensional representations, parametrized by pairs (r,n), $r > 0$, $n = 0,\pm1,\pm2,\dots$. They can be realized on the space of square integrable functions on the sphere $S^2 \approx SO(3)/SO(2)$.

6. THE CONTINUOUS POINCARÉ GROUP

6.1. Preliminaries

We start with recollecting some general facts discussed before in Chapter III (cf. also VARADARAJAN [11, Ch.XII]). Let $M = \mathbb{R}^4$ be the Minkowski space-time. Elements of M will be denoted by $\underline{x} = (x_0{=}ct,x_1,x_2,x_3)$. The distance $(\underline{x},\underline{y})$ between two events $\underline{x}$ and $\underline{y}$ is defined by

(6.1) $$(\underline{x},\underline{y})^2 = (x_0-y_0)^2 - \sum_{i=1}^{3} (x_i-y_i)^2.$$

A nonsingular inhomogeneous transformation of M has the form

(6.2) $$\underline{x} \to T\underline{x} + \underline{y}, \quad x \in M,$$

where T is a nonsingular linear operator on M and $\underline{y}$ a fixed point in M. The *Poincaré group* P is defined to consist of those transformations (6.2) that respect the distance (6.1). Clearly, a nonsingular linear operator T belongs to P if and only if

(6.3) $$T^tFT = F,$$

where

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Notice that (6.3) is equivalent to the condition that T preserves the quadratic form

$$(\underline{x})^2 := x_0^2 - x_1^2 - x_2^2 - x_3^2, \quad x \in M.$$

This is the so-called *Minkowskian norm* on M. From (6.3) it follows that if $T = (t_{ij})_{i,j=0}^{3}$ belongs to P, then (i) $\det(T) = \pm 1$ and (ii) $|t_{00}| \geq 1$. The subgroup of P consisting of all nonsingular operators T satisfying (6.3) is called the *Lorentz group* (or the *homogeneous Poincaré group*). General elements of P are denoted by $(\underline{y},T)$. Multiplication in P is given by

$$(\underline{y},T)(\underline{z},U) = (\underline{y}+T(\underline{z}),TU).$$

Notice that L is a lcsc. group since it is a closed subgroup of $GL(4,\mathbb{R})$.The mapping $(\underline{y},T) \to T(\underline{y})$ is clearly continuous in the product topology of $\mathbb{R}^4 \times L$, and therefore P is the semidirect product of $N = \mathbb{R}^4$ (considered as a translation group) and H = L, relative to

$$\alpha_T(\underline{y}) = T(\underline{y}).$$

We are, however, at this moment merely interested in the connected component of the identity (0,I) in P. Since $\mathbb{R}^4$ is already connected it suffices to look for the connected component of the identity in L. It can be shown that L consists of four connected components (cf. VARADARAJAN [1, thm.12.1]), which are given by

$$L_+^{\uparrow} = \{T \in L;\ \det(T) = +1, \quad t_{00} \geq 1\};$$
$$L_-^{\uparrow} = \{T \in L;\ \det(T) = -1, \quad t_{00} \geq 1\};$$
$$L_+^{\downarrow} = \{T \in L;\ \det(T) = +1, \quad t_{00} \leq -1\}$$
$$L_-^{\downarrow} = \{T \in L;\ \det(T) = -1, \quad t_{00} \leq -1\}$$

$L_+^{\uparrow}$ is the connected component of the identity, and therefore a closed invariant subgroup of L. The semidirect product $\mathbb{R}^4 \circledS L_+^{\uparrow}$ is called the *continuous Poincaré group* and denoted by $P_+^{\uparrow}$. The group $L_+^{\uparrow}$ is called the *restricted Lorentz-group*. For computing the representations of $P_+^{\uparrow}$ it is rather convenient to compute those of its two-fold covering group.

It can be shown (cf. §III.3) that the unimodular Lie group $SL(2,\mathbb{C})$ is the two-fold covering group of $L_+^\uparrow$. Since $\mathbb{R}^4$ and $SL(2,\mathbb{C})$ are both simply connected, their topological product also enjoys this property. If $\Lambda: SL(2,\mathbb{C}) \to L_+^\uparrow$ denotes the two-to-one covering homomorphism, we can make the product $\mathbb{R}^4 \times SL(2,\mathbb{C})$ into a semidirect product by setting

$$(\underline{x},A)(\underline{y},B) = (\underline{x}+\Lambda(A)\underline{y},AB), \quad \underline{x},\underline{y} \in \mathbb{R}^4, \; A,B \in SL(2,\mathbb{C}).$$

The mapping $(\underline{x},A) \to (\underline{x},\Lambda(A))$ provides a two-to-one covering of $P_+^\uparrow$.

For convenience, we recall how the mapping $\Lambda: SL(2,\mathbb{C}) \to L_+^\uparrow$ is defined. Let $\sigma_0,\sigma_1,\sigma_2,\sigma_3$ denote the four Pauli matrices, defined by

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With an element $\underline{x}$ of M we associate an hermitian 2×2-matrix $\hat{\underline{x}}$ by

$$(6.4) \qquad \hat{\underline{x}} := \sum_{i=0}^{3} x_i \sigma_i = \begin{pmatrix} x_0+x_3 & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3 \end{pmatrix}.$$

It can be readily verified that the assignment $\underline{x} \to \hat{\underline{x}}$ is a linear isomorphism from $\mathbb{R}^4$ onto the space of all hermitian 2×2-matrices. Denote this space by H(2). If $A \in SL(2,\mathbb{C})$ and if A^* is the hermitian adjoint on A, then

$$X \to AXA^*, \qquad X \in H(2),$$

defines a linear one-to-one mapping from H(2) onto itself, which preserves determinants. Now we define an operator $\Lambda(A)$, $A \in SL(2,\mathbb{C})$, on M, by

$$\Lambda(A)(\underline{x}) = \underline{y}, \quad \text{where } \hat{\underline{y}} = A\hat{\underline{x}}A^*.$$

We contend that the operator $\Lambda(A)$ respects the distance (6.1). Indeed, straightforward calculation shows that the distance $(\underline{x},\underline{y})$ is equal to the square root of $\det((\underline{x}-\underline{y})^\wedge)$, and

$$\det(\Lambda(A)(\underline{x})^\wedge) = \det(\hat{\underline{x}}).$$

The character group of $\mathbb{R}^4$ is isomorphic with $\mathbb{R}^4$, and the characters are given by

$$\phi_{\underline{y}}(\underline{x}) = e^{i(\underline{x},\underline{y})}, \qquad \underline{y} \in \mathbb{R}^4,$$

where $(\underline{x},\underline{y}) = x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3$. The group $SL(2,\mathbb{C})$ acts on $\hat{\mathbb{R}}^4$ by

$$A[\phi_{\underline{y}}] = \phi_{\Lambda(A)\underline{y}}.$$

Indeed, since $\det(T) = 1$, for T in $L_+^\uparrow$, we have

$$(A[\phi_{\underline{y}}])(\underline{x}) = \phi_{\underline{y}}(\Lambda(A)^{-1}\underline{x}) = e^{i(\Lambda(A)^{-1}\underline{x},\underline{y})} = e^{i(\underline{x},\Lambda(A)\underline{y})} = \phi_{\Lambda(A)\underline{y}}(\underline{x}).$$

6.2. Orbits

The orbits of $SL(2,\mathbb{C})$ in $\mathbb{R}^4$ are characterized in the first place by the relation $(\underline{x})^2$ = constant. That is, each set $\{\underline{x} \in \mathbb{R}^4 ;\ (\underline{x})^2 = m^2\}$, $m \in \mathbb{R}$, must be a union of orbits. There are three types of such sets:

$(\underline{x})^2 = m^2$, $m > 0$: two-sheeted hyperboloid;

$(\underline{x})^2 = 0$: cone;

$(\underline{x})^2 = -m^2$, $m > 0$: one-sheeted hyperboloid.

Since $SL(2,\mathbb{C})$ is connected, its orbits in $\mathbb{R}^4$ have to be connected as well. Therefore, in the case $(\underline{x})^2 = m^2$, $m > 0$, each sheet of the hyperboloid is a union of orbits. Using concrete Lorentz transformations one can readily show that the sheets are actually orbits in their own right. As to the cone, it contains the trivial orbit ω_0, which splits it in two disconnected parts. Using straightforward arguments one proves transitivity of $SL(2,\mathbb{C})$ on the following sets:

$\tilde{\omega}_0 := \{\underline{0}\}$;

$\tilde{\omega}_m^+ := \{\underline{x} \in \mathbb{R}^4 ;\ (\underline{x})^2 = m^2,\ x_0 > 0\}$, $m > 0$;

$\tilde{\omega}_m^- := \{\underline{x} \in \mathbb{R}^4 ;\ (\underline{x})^2 = m^2,\ x_0 < 0\}$, $m > 0$;

$\tilde{\omega}_{im} := \{\underline{x} \in \mathbb{R}^4 ;\ (\underline{x})^2 = -m^2\}$, $m > 0$;

$\tilde{\omega}_0^+ := \{\underline{x} \in \mathbb{R}^4 ;\ (\underline{x})^2 = 0,\ x_0 > 0\}$;

$\tilde{\omega}_0^- := \{\underline{x} \in \mathbb{R}^4 ;\ (\underline{x})^2 = 0,\ x_0 < 0\}$.

Consequently, these are the orbits of SL(2,$\mathbb{C}$) in $\mathbb{R}^4$. Accordingly, the orbits in $\hat{\mathbb{R}}^4$ are given by $\omega_m^+ := \{\phi_{\underline{x}};\ \underline{x} \in \tilde{\omega}_m^+\}$, etc. If we keep x_3 fixed, then it is possible to make an interesting drawing of the parametrization of the orbits, see figure 1.

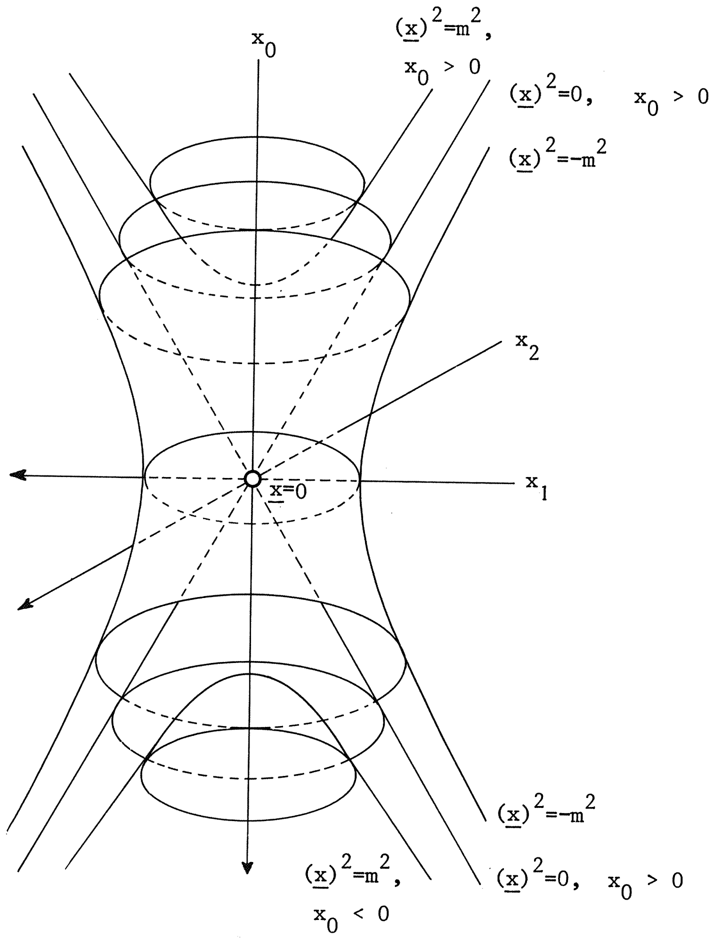

Figure 1

6.3. Stabilizers

Ad ω_0: $H_0 = SL(2,\mathbb{C})$.

Ad ω_m^+: Fix the point $\phi_{(m,0,0,0)}$, and consider the stabilizer H_m^+ in $SL(2,\mathbb{C})$ of $(m,0,0,0)$. The corresponding matrix $\hat{\underline{x}}$ in $H(2)$ defined by (6.4) is

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} .$$

An element A of $SL(2,\mathbb{C})$ belongs to H_m^+ if and only if

$$\hat{\underline{x}} = A\hat{\underline{x}}A^* . \tag{6.5}$$

This equation is equivalent to $AA^* = I$, and therefore $H_m^+ = SU(2)$, the special unitary group.

Ad ω_m^-: $H_m^- = H_m^+ = SU(2)$.

Ad ω_{im}: Consider the stabilizer H_{im} of $(0,0,m,0)$. It consists of all matrices A in $SL(2,\mathbb{C})$ which satisfy $\sigma_2 = A\sigma_2 A^*$. Since

$$(A^*)^{-1} = \sigma_2^{-1} A\sigma_2 = -\sigma_2^t A\sigma_2 = (A^t)^{-1} ,$$

this condition is equivalent to $A^* = A^t$. This is true if and only if A has real entries. Hence, $H_{im} = SL(2,\mathbb{R})$.

Ad ω_0^+: Fix $\underline{x} = (1,0,0,1)$, then

$$\hat{\underline{x}} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} .$$

Identity (6.5) in this case is readily seen to be equivalent to

$$A = \begin{pmatrix} e^{i\theta} & z \\ 0 & e^{-i\theta} \end{pmatrix} , \quad \theta \in [0,2\pi), \ z \in \mathbb{C}. \tag{6.6}$$

Hence, the stabilizer H_0^+ is the group of all matrices of the form (6.6). We define

$$(z,\theta) := \begin{pmatrix} e^{2i\theta} & e^{-i\theta}z \\ 0 & e^{-2i\theta} \end{pmatrix} .$$

Then H_0^+ can be identified with the set $\{(z,\theta);\ z \in \mathbb{C}, \theta \in [0,2\pi)\}$, and its multiplication is given by

$$(z_1,\theta_1)(z_2,\theta_2) = (z_1+e^{2i\theta_1}z_2, \theta_1+\theta_2).$$

Therefore, H_0^+ is a semidirect product of $\mathbb{C}$ and $\mathbb{T}$, the circle group, relative to

$$\alpha_\theta(z) = e^{2i\theta}z.$$

The Euclidean group $E(2)$ can also be considered as a semidirect product of $\mathbb{C}$ and $\mathbb{T}$, with multiplication given by

$$[z_1,\theta_1][z_2,\theta_2] = [z_1+e^{i\theta_1}z_2, \theta_1+\theta_2].$$

Obviously, the mapping $(z,\theta) \mapsto [z,2\theta]$ from H_0^+ onto $E(2)$ is a two-to-one homomorphism. Hence, we see that H_0^+ can be considered as a two-fold covering group of $E(2)$. This fact leads us to the notation $H_0^+ = \tilde{E}(2)$.

<u>Ad ω_0^-:</u> $H_0^- = H_0^+ = \tilde{E}(2)$.

6.4. <u>Irreducible unitary representations</u>

One shows readily that $\tilde{P}_+^\uparrow$ is regular. Indeed, for each ordered pair (r_1,r_2) of rational numbers such that $0 < r_1 < r_2$, define three Borel subsets of $\hat{\mathbb{R}}^4$ by

$$B^\pm(r_1,r_2) := \bigcup_{r_1<m<r_2} \omega_m^\pm;\ B^i(r_1,r_2) = \bigcup_{r_1<m<r_2} \omega_{im}.$$

The collection of all such sets, complemented with the Borel sets ω_0, ω_0^+ and ω_0^-, is a countable family which meets the requirements of Def. 3.1.

Consequently, the representation theory of $\tilde{P}_+^\uparrow$ (and hence that of $P_+^\uparrow$) is reduced to those of four smaller groups, $SL(2,\mathbb{C})$, $SL(2,\mathbb{R})$, $SU(2)$ and $\tilde{E}(2)$. We proceed to classify the irreducible unitary representations associated with the orbits. The irreducible unitary representations of $SL(2,\mathbb{C})$

are not discussed in these notes, and we will only state the results in this case. For details, see for instance BARUT & RACZKA [2].

Ad ω_0: The set of irreducible unitary representation of $SL(2,\mathbb{C})$ consists of two series:

(i) the so-called *principal series*, parametrized by two numbers (r,j), $r \geq 0$, $j = 0,\frac{1}{2},1,\frac{3}{2},\ldots$;

(ii) the so-called *supplementary series*, parametrized by a real number $r \in (-1,1)$, $r \neq 0$.

The extensions to $\tilde{P}_+^{\uparrow}$ are denoted by $\sigma^{(0,r,j)}$ and $\sigma^{(0,r)}$, respectively.

Ad $\omega_m^{\pm}$: As mentioned before, the set of irreducible unitary representations of $SU(2)$ consists of a series $D^{(s)}$, $s = 0,\frac{1}{2},1,\frac{3}{2},\ldots$, with $\dim(D^{(s)}) = 2s+1$. Hence, for each $m > 0$ we get two series of representations of $\tilde{P}_+^{\uparrow}$, associated with the orbits ω_m^+ and ω_m^-, respectively. We denote these series by $\sigma^{(m,\pm,s)}$.

Ad ω_{im}: The group $SL(2,\mathbb{R})$ has three series of irreducible unitary representations, which will be discussed in Chapter XIII. They are:

(i) the *principal series*, parametrized by two numbers (t,ε), $t \in \mathbb{R}$, $\varepsilon = 0$ or 1;

(ii) the *discrete series*, parametrized by integers, $n = 0,\pm1,\pm2,\ldots$

(iii) the *supplementary series*, parametrized by a real number $r \in (-1,1)$, $r \neq 0$.

The corresponding representations of $\tilde{P}_+^{\uparrow}$ are denoted by $\sigma^{(im,t,\varepsilon)}$, $\sigma^{(im,n)}$ and $\sigma^{(im,r)}$.

Ad $\omega_0^{\pm}$: We showed that $H_0^{\pm} = \tilde{E}(2)$ is the semidirect product of $\mathbb{C}$ and $\mathbb{T}$, relative to $\alpha_\theta(z) = e^{2i\theta}z$. The character group $\hat{\mathbb{C}}$ is isomorphic with $\mathbb{C}$, and the characters are given by

$$\phi_w(z) = e^{i\mathrm{Re}(z\bar{w})}, \quad w \in \mathbb{C}.$$

The circle group $\mathbb{T}$ acts on $\hat{\mathbb{C}}$ by

$$\theta[\phi_w] = \phi_{e^{2i\theta}w}, \quad \theta \in [0,2\pi).$$

Hence, the orbits are circles in $\mathbb{C}$, which we denote by $\omega_r := \{\phi_z;\ |z| = r\}$, $r \geq 0$. The irreducible unitary representations of $\tilde{E}(2)$ associated with ω_0 are those of $\mathbb{T}$, extended to $\tilde{E}(2)$. We denote them by L^j, $j = 0,\pm1,\pm2,\ldots$.

The stabilizer in $\mathbb{T}$ of ϕ_r, $r > 0$, is $\{0,\pi\}$. This cyclic group has two irreducible representations on $\mathbb{C}$:

$$\psi^0: 0,\pi \mapsto 1, \quad \psi^1: 0 \mapsto 1, \ \pi \mapsto -1.$$

We denote the corresponding representations of $\tilde{E}(2)$ by $L^{r,\varepsilon}$ with $\varepsilon = 0$ or 1.

For $\tilde{P}_+^\uparrow$ we find the following two series of irreducible unitary representations, associated with the orbits $\omega_0^\pm$:

$$\sigma^{(\omega_0^\pm, L^j)} =: \sigma^{(0,\pm,j)}, \qquad j = 0,\pm1,\pm2,\ldots;$$

$$\sigma^{(\omega_0^\pm, L^{r,\varepsilon})} =: \sigma^{(0,\pm,r,\varepsilon)}, \ r > 0, \quad \varepsilon = 0 \text{ or } 1.$$

THEOREM 6.1. *The set of irreducible unitary representations of* $\tilde{P}_+^\uparrow$ *consists (up to equivalence) of the following eight series:*

(i) $\sigma^{(0,r,j)}$, $r \ge 0$, $j = 0,\frac{1}{2},1,\frac{3}{2},\ldots;$

(ii) $\sigma^{(0,r)}$, $-1 < r < 1$, $r \ne 0;$

(iii) $\sigma^{(m,\pm,s)}$, $s = 0,\frac{1}{2},1,\frac{3}{2},\ldots;$

(iv) $\sigma^{(im,t,\varepsilon)}$, $t \in \mathbb{R}$, $\varepsilon = 0$ or $1;$

(v) $\sigma^{(im,n)}$, $n = 0,\pm1,\pm2,\ldots;$

(vi) $\sigma^{(im,r)}$, $-1 < r < 1$, $r \ne 0;$

(vii) $\sigma^{(0,\pm,j)}$, $j = 0,\pm1,\pm2,\ldots;$

(viii) $\sigma^{(0,\pm,r,\varepsilon)}$, $r > 0$, $\varepsilon = 0$ or 1.

6.5. Another realization of the representation $\sigma^{(m,+,s)}$

The representations $\sigma^{(m,+,s)}$ were already derived explicitly in Chapter III. In order to establish consistency with those results we conclude this chapter by considering this special case in more detail. We will realize $\sigma^{(m,+,s)}$ on the space $L^2(SL(2,\mathbb{C})/SU(2), \mathbb{C}^{2s+1})$, or rather on $L^2(\tilde{\omega}_m^+, \mathbb{C}^{2s+1})$, by means of a continuous cross-section $H: \tilde{\omega}_m^+ \to SL(2,\mathbb{C})$.

The natural projection π: $SL(2,\mathbb{C}) \to \tilde{\omega}_m^+$ is given by

$$\pi(A) = \Lambda(A)(m,0,0,0), \quad A \in SL(2,\mathbb{C}).$$

Hence, we must look for a continuous mapping H: $\tilde{\omega}_m^+ \to SL(2,\mathbb{C})$ such that

$$\Lambda(H(\underline{x}))(m,0,0,0) = \underline{x}, \quad \underline{x} \in \tilde{\omega}_m^+.$$

From the definition of Λ it follows that H is given by

$$(6.7) \qquad \begin{aligned} \hat{\underline{x}} &= H(\underline{x})(m,0,0,0)^{\wedge} H(\underline{x})^* \\ &= H(\underline{x})\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} H(\underline{x})^* = mH(\underline{x})H(\underline{x})^*. \end{aligned}$$

Recall that $SL(2,\mathbb{C})$ admits the decomposition $SU(2)\times\tilde{H}(2)$, where $\tilde{H}(2)$ consists of all positive definite hermitian matrices of determinant one (cf. §III.3). Therefore it is natural to assume that we can find H as a bijection from $\tilde{\omega}_m^+$ onto $\tilde{H}(2)$. Suppose $H(\underline{x}) \in \tilde{H}(2)$, say

$$H(\underline{x}) = \begin{pmatrix} h_0+h_3 & h_1-ih_2 \\ h_1+ih_2 & h_0-h_3 \end{pmatrix}$$

Then (6.7) reduces to $\hat{\underline{x}} = mH(\underline{x})^2$, and (h_0,h_1,h_2,h_3) is found by solving

$$\begin{cases} x_0 = 2m(h_0^2+h_1^2+h_2^2+h_3^2) \\ x_i = 2mh_0h_i, \quad i = 1,2,3. \end{cases}$$

One finds easily:

$$H(\underline{x}) = \frac{1}{(2m(x_0+m))^{\frac{1}{2}}}\begin{pmatrix} x_0+x_3+m & x_1-ix_2 \\ x_1+ix_2 & x_0-x_3+m \end{pmatrix} = \frac{m\sigma_0 + \sum_{i=0}^{3} x_i\sigma_i}{(2m(x_0+m))^{\frac{1}{2}}}.$$

This is formula (3.21) of Chapter III. (Note that our σ_i $(i=1,2,3)$ equals the σ^i of Chapter III.) Now we can use formulas (IX.5.6) and (3.2) to write down our realization of $\sigma^{(m,+,s)}$.

$$(\tilde{\sigma}^{(m,+,s)}(\underline{x},A)f)(\underline{y}) =$$

$$e^{i(\underline{y},\underline{x})}D^{(s)}(H(\underline{y})^{-1}AH(\Lambda(A)^{-1}\underline{y}))f(\Lambda(A)^{-1}\underline{y}), \quad f \in L^2(\tilde{\omega}_m^+, \mathbb{C}^{2s+1}).$$

This formula is essentially the same as (III.4.38).

Forced by lack of space-time we can not discuss the physical interpretations of the representations given in Theorem 6.1, nor can we go into their explicit realizations any further. Some suggestions to the reader for finding information on these aspects are: BARUT & RĄCZKA [2], SIMMS [10] and VARADARAJAN [11]. See also Chapter III of these notes.

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XII

COMPACT LIE GROUPS AND THEIR REPRESENTATIONS

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CONTENTS

In this paper we will treat the theory of compact Lie groups. Examples of these groups are the unitary, orthogonal and symplectic groups.

The structure of these groups is very well understood and it leads to a complete classification. This part of the theory is explained in section 1. In section 2 we deal with the representation theory. Because every irreducible representation of a compact group is finite-dimensional, we will discuss only finite-dimensional representations. The structure theory plays an important role in the classification of all irreducible representations.

Only a few proofs are given here. For the omitted proofs many good books are available. In section 3 one can find a guide to the literature.

1. STRUCTURE THEORY

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Because G is compact the left Haar measure dg on G is also right invariant and can be normalized by the condition $\int_G dg = 1$.

The group G acts on $\mathfrak{g}$ by the adjoint representation, cf. §IV. 2.8. For linear groups this becomes conjugation:

$$\mathrm{Ad}(g)X = gXg^{-1} \qquad (g\in G, X\in\mathfrak{g}).$$

Choose some inner product on $\mathfrak{g}$. Averaging over G gives a new inner product, say $(.,.)$, which is invariant under Ad(g) for all $g \in G$. On Lie algebra level this means that ad X is skew-symmetric for all $X \in \mathfrak{g}$.

For $X \in \mathfrak{g}$ consider the centralizer $\mathfrak{g}_X = \mathrm{Ker}(\mathrm{ad}\ X)$ of X, and let $\mathfrak{g}^X = \mathrm{Im}(\mathrm{ad}\ X)$. Because ad X is skew-symmetric, $\mathfrak{g}^X$ is the orthoplement of $\mathfrak{g}_X$, so $\mathfrak{g} = \mathfrak{g}_X \oplus \mathfrak{g}^X$. We call $X \in \mathfrak{g}$ *regular* if $\dim(\mathfrak{g}_X)$ is minimal. For regular $X \in \mathfrak{g}$, $\mathfrak{g}_X$ is called a *Cartan subalgebra* (CSA), and the dimension of a CSA is called the *rank* of G.

<u>PROPOSITION 1.1</u>. *Every* CSA *is a maximal abelian subalgebra of* $\mathfrak{g}$.

<u>PROOF</u>. Let $X \in \mathfrak{g}$ be regular. Clearly, if $\mathfrak{g}_X$ is abelian then it is maximal abelian. Suppose that $\mathfrak{g}_X$ is not abelian. Then there exist $Y,Z \in \mathfrak{g}_X$ such that $[Y,X] \neq 0$. For small $t > 0$ we have $\dim(\mathfrak{g}_{X+tY}) \le \dim(\mathfrak{g}_X)$ and equality holds because X is regular. Since ad(X+tY) and ad X commute, we have

$\mathfrak{g}_{X+tY} = \mathfrak{g}_X$ for small $t > 0$. This contradicts the existence of $Z \in \mathfrak{g}_X$ for which $[Y,Z] \neq 0$. □

LEMMA 1.2. *If $\mathfrak{t}$ is a maximal abelian subalgebra of $\mathfrak{g}$ then $\mathfrak{t} = \mathfrak{g}_X$ for some $X \in \mathfrak{t}$.*

PROOF. Choose $X \in \mathfrak{t}$ such that $\dim \mathfrak{g}_X$ is minimal. Clearly $\mathfrak{t} \subset \mathfrak{g}_X$. Suppose that $\mathfrak{t} \neq \mathfrak{g}_X$. Choose $Z \in \mathfrak{g}_X \backslash \mathfrak{t}$. Since $\mathfrak{t}$ is maximal abelian, $[Y,Z] \neq 0$ for some $Y \in \mathfrak{t}$. Just as in the proof of Proposition 1.1 we can show that $\mathfrak{g}_{X+tY} = \mathfrak{g}_X$ for small $t > 0$. This leads to a contradiction. □

THEOREM 1.3 (Conjugation theorem).
The $\mathrm{Ad}(G)$-orbit O_Y through $Y \in \mathfrak{g}$ intersects a CSA $\mathfrak{g}_X$ in at least one point.

PROOF. Consider the real-valued function $f(g) = (\mathrm{Ad}(g)Y,X)$ on G. Because of compactness f takes its minimum value in some point $g_0 \in G$, so

$$\frac{d}{dt}\{f(g_0 \exp tZ)\}_{t=0} = 0, \qquad \text{for all } Z \in \mathfrak{g}.$$

Since

$$\begin{aligned} f(g_0 \exp tZ) &= (\mathrm{Ad}(g_0)\ \mathrm{Ad}(\exp tZ)Y,X) \\ &= (\mathrm{Ad}(\exp tZ)Y,\ \mathrm{Ad}(g_0^{-1})X) \\ &= (\exp(t\ \mathrm{ad}\ Z)Y,\ \mathrm{Ad}(g_0^{-1})X) \\ &= (Y,\mathrm{Ad}(g_0^{-1})X) + t([Z,Y],\ \mathrm{Ad}(g_0^{-1})X) + \mathcal{O}(t^2), \end{aligned}$$

we have

$$\frac{d}{dt}\{f(g_0 \exp tZ)\}_{t=0} = ([Z,Y],\ \mathrm{Ad}(g_0^{-1})X) = (Z,[Y,\ \mathrm{Ad}(g_0^{-1})X]).$$

Hence

$$(Z,[Y,\ \mathrm{Ad}(g_0^{-1})X]) = 0 \qquad \text{for all } Z \in \mathfrak{g},$$

so

$$[Y,\ \mathrm{Ad}(g_0^{-1})X] = 0, \quad \text{so} \quad \mathrm{Ad}(g_0)Y \in \mathfrak{g}_X. \quad \square$$

COROLLARY 1.4. *Every two CSA's are conjugate under $\mathrm{Ad}(G)$. The CSA's are just the maximal abelian subalgebras of $\mathfrak{g}$.*

PROOF. In view of Proposition 1.1 and Lemma 1.2 it is sufficient to show that any maximal abelian subalgebra $\mathfrak{g}_X$ and any CSA $\mathfrak{g}_Y$ are conjugate to each other. According to Theorem 1.3 there exists a $g_0 \in G$ such that $\mathrm{Ad}(g_0)X \in \mathfrak{g}_Y$. Since $\mathfrak{g}_Y$ is abelian, we have

$$\mathrm{Ad}(g_0)\mathfrak{g}_X = \mathfrak{g}_{\mathrm{Ad}(g_0)X} \supset \mathfrak{g}_Y$$

and equality follows because $\mathfrak{g}_Y$ is maximal abelian. □

EXAMPLE 1.5. As a standard example we will consider the unitary group $G = U(n)$ with Lie algebra $\mathfrak{g} = \mathfrak{u}(n)$, the set of all skew-hermitian matrices. The diagonal matrices in $\mathfrak{g}$ form a CSA of dimension n, which we denote by $\mathfrak{t}$. Then $X \in \mathfrak{t}$ is regular if all eigenvalues of X are distinct. The conjugation theorem implies that every skew-hermitian matrix is conjugate under U(n) to a diagonal matrix with purely imaginary diagonal elements.

From now on we fix some CSA $\mathfrak{t} \subset \mathfrak{g}$. Let $T = \exp(\mathfrak{t}) \subset G$ be the corresponding *Cartan subgroup* (CSG). We saw that ad X is skew-symmetric for all $X \in \mathfrak{g}$, so the space $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus_{\mathbb{R}} \mathbb{C}$ splits as a direct sum of eigenspaces and the corresponding eigenvalues are purely imaginary. Because $\mathfrak{t}$ is abelian, we can diagonalize the family $\{\mathrm{ad}\,X \mid X \in \mathfrak{t}\}$ simultaneously:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{ad}(H)X = \alpha(H)X \ \forall H \in \mathfrak{t}\}$ and $\Phi = \{\alpha \in \sqrt{-1}.\mathfrak{t}^* \mid \mathfrak{g}_\alpha \neq 0\}$. The elements of Φ are called the *roots* of the pair $(\mathfrak{g},\mathfrak{t})$, and the above splitting is called the *root decomposition*.

Let W be the group of all linear transformations of $\mathfrak{t}$ which are restrictions of transformations Ad(g) ($g\in G$) such that $\mathrm{Ad}(g)\,\mathfrak{t} = \mathfrak{t}$. Then W is called the *Weyl group*. One can prove that the reflections $s_\alpha : \mathfrak{t} \to \mathfrak{t}$ in the hyperplane $\{\alpha=0\}$ do belong to W, and that they generate W. It is easily seen that the roots are invariant under W (W acts on $\mathfrak{t}^*$ by dualisation), and that the only $s \in W$ which leaves Φ pointwise fixed, is the identity. This implies that W is a *finite reflection group* (FRG). FRG's have been classified by Coxeter. The following theorem provides us with a complete classification of compact connected Lie groups.

THEOREM 1.6. *The root space Φ determines the Lie algebra g up to isomorphism.*

EXAMPLE 1.7. Let us consider the example of U(n). Let $E_{ij} \in g_C = g\ell(n,C)$ be the matrix with 1 on the place (i,j) and zero elsewhere. Then

$$\mathrm{ad}\begin{pmatrix} x_1 & & \theta \\ & \ddots & \\ \theta & & x_n \end{pmatrix}(E_{ij}) = (x_i - x_j).\,E_{ij}.$$

Hence we have n(n-1) roots and the Weyl group W is the permutation group S_n of the eigenvalues $\{x_1,\ldots,x_n\}$ of $X \in t$.

A Lie algebra g is called *simple* if g is non-abelian and g has no non-trivial ideals. We call g *semi-simple* if g is a direct sum of simple ideals. If g is the Lie algebra of a compact Lie group then for every ideal $k \subset g$ one can see that $k^{\perp}$ is also an ideal. Thus we can write $g = \sum_{i=0}^{n} \oplus\, g_i$, where g_0 is the centre of g, and $g_1, g_2, \ldots, g_n$ are simple ideals of g. This splitting gives a disjoint union $\Phi = \bigcup_{i=1}^{n} \Phi_{i_n}$ of the roots, and a direct product decomposition of the Weyl group $W = \prod_{i=1} W_i$.

THEOREM 1.8 (Classification theorem).
Every simple compact connected Lie group is locally isomorphic to one of the groups listed below.

Cartan's notation	classical notation	dim G	Weyl group
$A_\ell (\ell \geq 1)$	$SU(\ell+1)$	$\ell(\ell+2)$	$S_{\ell+1}$
$B_\ell (\ell \geq 2)$	$SO(2\ell+1)$	$\ell(2\ell+1)$	$(\mathbb{Z}_2)^{\ell} Ⓢ S_\ell$
$C_\ell (\ell \geq 3)$	$Sp(\ell)$	$\ell(2\ell-1)$	$(\mathbb{Z}_2)^{\ell-1} Ⓢ S_\ell$
$D_\ell (\ell \geq 4)$	$SO(2\ell)$	$\ell(2\ell+1)$	$(\mathbb{Z}_2)^{\ell} Ⓢ S_\ell$
E_6		78	
E_7		133	
E_8		248	
F_4		52	
G_2		14	$\mathcal{D}_6$

Table 1

The groups occurring in the four infinite sequences are called the *classical* *groups*, the remaining five groups are called the *exceptional groups*.

<u>EXAMPLE 1.9</u>. The root systems of the simple groups A_2, B_2 and G_2 of rank 2 are given by figures 1, 2 and 3, respectively, below.

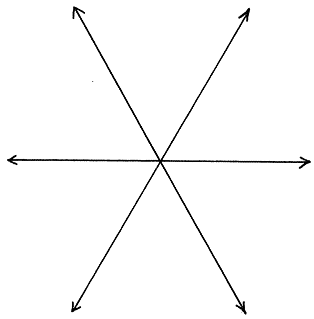


figure 1

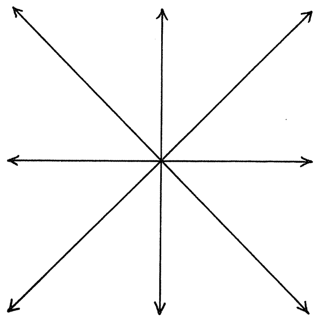


figure 2

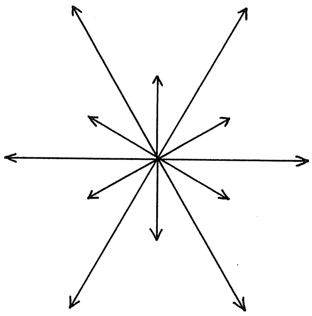


figure 3

2. REPRESENTATION THEORY

Let G again be a compact connected Lie group, and π a unitary representation of G on some finite-dimensional Hilbert space $\mathcal{H}$. The *character* of π is a function χ_π on the group G, defined by

$$\chi_\pi(g) = \mathrm{Trace}(\pi(g)).$$

The following properties hold:

a. The operator $P = \int_G \pi(g)dg$ is projection onto $\mathcal{H}^G$, where $\mathcal{H}^G = \{v \in \mathcal{H} \mid \pi(g)v = v \ \forall g \in G\}$ is the space of invariants. This implies $\int_G \chi_\pi(g)dg = \dim(\mathcal{H}^G)$.

b. $\chi_{\pi_1 \oplus \pi_2} = \chi_{\pi_1} + \chi_{\pi_2}$,

$\chi_{\pi_1 \otimes \pi_2} = \chi_{\pi_1} \cdot \chi_{\pi_2}$.

c. If π^* is the dual representation of π i.e., if $\pi^*(x) = \pi(x^{-1})^t = \overline{\pi(x)})$,
then $\chi_{\pi^*}(g) = \chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$.

d. Characters are *class functions*, i.e.
$\chi_\pi(yxy^{-1}) = \chi_\pi(x)$ for all $x,y \in G$.

COROLLARY 2.1 (Orthogonality relations).
Let π_1 *and* π_2 *be irreducible unitary representations of* G.

If π_1 *and* π_2 *are not equivalent,* $\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}dg = 0$.

If π_1 *and* π_2 *are equivalent,* $\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}dg = 1$.

PROOF. $\int_G \chi_{\pi_1}(g)\overline{\chi_{\pi_2}(g)}dg = \int_G \chi_{\pi_1 \otimes \pi_2^*}(g)dg =$

$= \int_G \chi_{\mathrm{Hom}(\pi_1,\pi_2)}(g)dg = \dim\{\mathrm{Hom}(\pi_1,\pi_2)^G\}$.

Using Schur's lemma one finds the required result. □

Let π be a finite dimensional unitary representation on a Hilbert space H. We can write π as a direct sum of irreducible representations $\pi = m_1\pi_1 \oplus \ldots \oplus m_p\pi_p$ with $m_1,\ldots,m_p \in \mathbb{N}$. Then $m_i = \int_G \chi_\pi(g)\overline{\chi_{\pi_i}(g)}dg$, so χ_π completely determines the representation π up to equivalence.

Let $\hat{G}$ be the set of equivalence classes of irreducible finite-dimensional unitary representations. Then our problem will be to classify $\hat{G}$, and to find the corresponding characters.

EXAMPLE 2.2. Let T be a torus, i.e. $T = \mathbb{R}^n/\Lambda$ where $\Lambda \subset \mathbb{R}^n$ is a lattice. According to Schur's lemma every $\pi \in \hat{T}$ is one-dimensional. Fix some $\pi \in \hat{T}$. Differentiation gives a linear map $d\pi\colon \mathbb{R}^n \to \sqrt{-1}.\,\mathbb{R}$, such that $d\pi(\Lambda) \subset 2\pi\,\mathbb{Z}\,\sqrt{-1}$. Put $\Lambda^* = \{\text{linear maps } \lambda\colon \mathbb{R}^n \to \sqrt{-1}\,\mathbb{R} \mid \lambda(X) \in 2\pi\,\mathbb{Z}\,\sqrt{-1}$ for all $X \in \Lambda\}$, then Λ^* is called the *weight lattice* of T. Every $\lambda \in \Lambda^*$ defines a representation $\chi_\lambda \in \hat{T}$ by

$$\chi_\lambda(\exp X) = e^{\lambda(X)} \quad (X \in \mathbb{R}^n).$$

Because $d\chi_\lambda = \lambda$ we have a bijection between $\hat{T}$ and Λ^*.

In the general case we choose a CSA $\mathfrak{t} \subset \mathfrak{g}$, and let $T = \exp(\mathfrak{t})$ be the corresponding CSG. Because characters are class functions, it follows from the conjugation theorem that the character χ_π of some $\pi \in \hat{G}$ is completely determined by its restriction to T:

$$\chi_\pi(\exp X) = \sum_{i=1}^{p} m_\pi(\lambda_i) e^{\lambda_i(X)} \qquad (X \in \mathfrak{t}),$$

where $\lambda_1,\ldots,\lambda_p \in \Lambda^*$, Λ^* is the weight lattice of T and $m_\pi(\lambda_1),\ldots,m_\pi(\lambda_p) \in \mathbb{N}$. The set $\mathcal{W}(\pi) = \{\lambda_1,\ldots,\lambda_p\}$ is called the set of *weights* of the representation π. Because the character is a class function, the weights of π and their multiplicities are invariant under the Weyl group W.

EXAMPLE 2.3. The weights of the adjoint representation of G on $\mathfrak{g}_C$ are just the roots of the pair $(\mathfrak{g},\mathfrak{t})$.

Let V_α be the hyperplane $\{\alpha = 0\}$ in $\mathfrak{t}$, and put $R = \mathfrak{t}\backslash\{\bigcup_{\alpha\in\Phi} V_\alpha\}$. The closure of a connected component of R is called a *Weyl chamber*. Fix some Weyl chamber $C \subset \mathfrak{t}$. Then C is a fundamental domain for the action of the Weyl group. Let $\Phi^+ = \{\alpha \in \Phi \mid \frac{\alpha(X)}{\sqrt{-1}} \geq 0 \text{ for all } X \in C\}$ be the set of *positive roots* relative to the Weyl chamber C.

By dualisation we have an inner product on $\mathfrak{t}^*$ and also on $\sqrt{-1}.\ \mathfrak{t}^*$. Let $\Lambda^+ = \{\lambda \in \Lambda^* \mid (\lambda,\alpha) \geq 0 \text{ for all } \alpha \in \Phi^+\}$ be the set of *dominant weights* (relative to C). A weight $\lambda \in \mathcal{W}(\pi)$ is called a *highest weight* (relative to C) if for no $\alpha \in \Phi^+$ we have $\lambda + \alpha \in \mathcal{W}(\pi)$.

THEOREM 2.4. (Highest weight theorem).

a. *Every* $\pi \in \hat{G}$ *has a unique highest weight* $\lambda \in \Lambda^+$.
b. *For every* $\lambda \in \Lambda^+$ *there exists a unique* (*up to equivalence*) *irreducible unitary representation* $\pi_\lambda \in \hat{G}$ *with highest weight* λ.

EXAMPLE 2.5. In figure 4 we present the *Cartan-Stiefel diagram* for A_2. The figure can be interpreted as $\sqrt{-1}.\mathfrak{t}^*$. The six vectors correspond to the six roots. The elements of the weight lattice correspond to the intersection points of the lines. The Weyl chamber is shaded and the weights which lie in the Weyl chamber are the dominant weights. For the Cartan-Stiefel diagrams of B_2 and G_2 see SAMELSON [5, pp. 121/122]. In figure 5 the *weight diagram* for the representation with highest weight $\lambda = 3\lambda_1 + 2\lambda_2$ is given. The dots indicate which weights occur. The multiplicities in this case are also indicated, increasing from one to three. (Passing from the outer "shell" to the inner "shell" the multiplicity increases steadily by one until the shells become triangles, at which point the multiplicity stabilizes.) This simple behaviour of the multiplicities is a special fact about A_2. (See ANTOINE & SPEISER [1].)

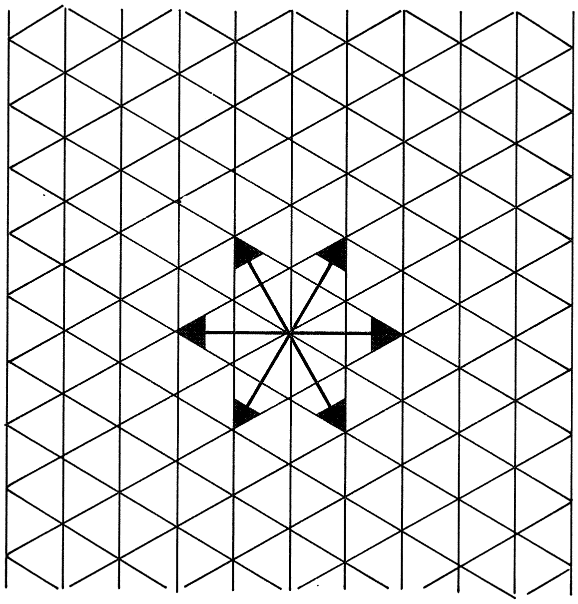
REMARK 2.6. Suppose now that the centre of $\mathfrak{g}$ is 0. One can prove that there exists a $\mathbb{Z}$-basis $\{\lambda_1,\ldots,\lambda_\ell\}$ (ℓ=rank(G)) of dominant weights for the weight lattice Λ^*, such that the coordinates of a dominant weight λ are nonnegative integers. After fixing such a basis a representation with highest weight λ is determined by ℓ nonnegative integers.

Suppose λ is a dominant weight, and π_λ the representation with highest weight λ. Weyl has derived a formula for the character of π_λ in terms of the weight λ and the Weyl group W. Calculating the value of the character at the identity (one has to take a limit because the Weyl character formula is singular at the identity) one finds the celebrated Weyl dimension formula. For more details see VARADARAJAN [7, ch.4,§14]. A formula for the multiplicities of the weights has been derived by Freudenthal (see HUMPHREYS[3,§22]). It is often a quite cumbersome job to use this formula. A computer program using this formula has been written by KRUSEMEYER [4]. A geometric realisation of representations with highest weight has been derived by Borel and Weil. They realize the representations π_λ on the space of holomorphic sections in a certain line bundle. For more details see SERRE [6].

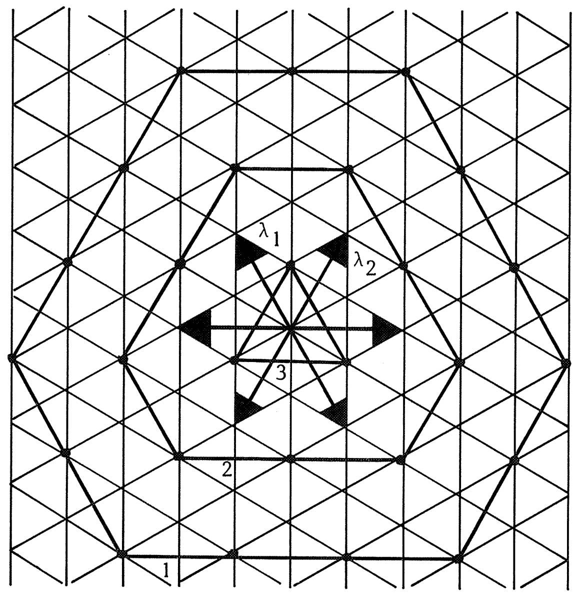
3. HISTORICAL REMARKS AND LITERATURE

The classification of compact connected Lie groups goes back to E. CARTAN's thesis [2] (1894). The theory of the highest weight is also due to him. The Weyl group plays an important role in the work of H. WEYL [9] (1925/1926), where he derives the character and the dimension formula. Weyl uses the notion of invariant integration over the group, while the work of Cartan is of a more algebraic nature.

At the moment VARADARAJAN [7] and HUMPHREYS [3] are standard textbooks. The latter is of a very algebraic nature (the Weyl character formula is derived in a purely algebraic way following the work of VERMA [8]), while the first book follows a more analytic approach. Especially the last chapter of [7] I can recommend to those who are more interested in this subject. WEYL's book [9] contains many computations about the classical groups.



Cartan-Stiefel diagram for A_2
(figure 4)



Weightdiagram with highest weight $\lambda=3\lambda_1+2\lambda_2$
(figure 5)

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XIII

THE IRREDUCIBLE UNITARY REPRESENTATIONS OF SL(2,ℝ)

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CONTENTS

In the following three lectures we treat the representation theory of SL(2,ℝ), which is a prototype of a non-compact semi-simple Lie group. The group SL(2,ℝ) appears in many branches of mathematics in different forms. We have met this group already in § XI.6 as "little group". The realization of SL(2,ℝ) as the special unitary group SU(1,1) of the indefinite hermitian form $z_1\bar{z}_1 - z_2\bar{z}_2$ is particularly useful for constructing certain series of representations. The group SL(2,ℝ) has three different series of representations: the principal series, the discrete series and the complementary series. The principal series is constructed in Part I, the discrete series and the complementary series in Part II, while Part III is concerned with the classification of the irreducible unitary representations, which is carried out by the infinitesimal method. Our treatment follows in this respect the classical paper of BARGMANN [1]. We have tried however to keep the treatment as general as possible, in order to give the reader a feeling how one could extend the theory to general semi-simple Lie groups. We hope that this effort has been worth while.

Part I. THE PRINCIPAL SERIES OF SL(2,ℝ)

1. STRUCTURE AND HAAR MEASURE OF SL(2,ℝ)

G = SL(2,ℝ) is a connected, unimodular Lie group. For $\theta \in \mathbb{R}$, $t \in \mathbb{R}$ and $z \in \mathbb{R}$, we write

$$(1.1)\qquad u_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}, \quad n_z = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

Let

$$(1.2)\qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$u_\theta = \exp\theta(X-Y),\ a_t = \exp tH,\ \ n_z = \exp zX.$$

Suppose $x \in G$, say

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1.$$

Then x is uniquely decomposable in the form

(1.3) $$x = u_\theta a_t n_z$$

with

$$t = \tfrac{1}{2} \log (a^2+c^2), \qquad e^{i\theta} = \frac{a-ic}{\sqrt{a^2+c^2}}, \qquad z = \frac{ab+cd}{a^2+c^2}.$$

Put

(1.4) $$K = \{u_\theta : 0 \le \theta < 2\pi\}, \qquad A = \{a_t : t \in \mathbb{R}\}$$

and

$$N = \{n_z : z \in \mathbb{R}\}.$$

These are subgroups of G. A normalizes N. By the above decomposition we have:

PROPOSITION 1.1. *The mapping* (k,a,n) $\mapsto$ kan *is an analytic diffeomorphism of* K × A × N *onto* G.

This is called the *Iwasawa decomposition* of G: G = KAN. Put M = {±I}. Then P = MAN is a closed subgroup of G, a so-called minimal parabolic subgroup. Note that M is the centralizer of A in K and M normalizes N.

Let du_θ be the normalized Haar measure on K:

$$\int_K f(u_\theta)du_\theta = \frac{1}{2\pi}\int_0^{2\pi} f(u_\theta)d\theta \qquad (f \in C(K)).$$

Choose standard Haar measures $da_t = dt$ and $dn_z = dz$ on A and N respectively. Then the Haar measure dx on G can be so normalized that

(1.5) $$dx = e^{2t}\, du_\theta da_t dn_z \qquad (x=u_\theta a_t n_z).$$

This follows easily from formula (IX.3.21).

Given $x \in G$ and $0 \le \theta < 2\pi$, we write

$$(1.6) \qquad x^{-1}u_\theta = u_{\psi(x,\theta)}\, a_{t(x,\theta)}\, n_{z(x,\theta)}$$

with

$$0 \le \psi(x,\theta) < 2\pi, \qquad t(x,\theta) \in \mathbb{R},\ z(x,\theta) \in \mathbb{R}.$$

We list here some properties of the maps $x \mapsto \psi(x,\theta)$ and $x \mapsto t(x,\theta)$.

(1.7) (i) $\psi(xy,\theta) = \psi(y,\psi(x,\theta))$,

(ii) $t(xy,\theta) = t(x,\theta) + t(y,\psi(x,\theta))$,

(iii) $t(u_{\theta'},\theta) = 0$ for all $\theta' \in \mathbb{R}$.

Now fix $g \in G$. Then, given $x = u_\theta a_t n_z$ in G, we have

$$g^{-1}x = g^{-1}u_\theta a_t n_z = u_{\psi(g,\theta)}\, a_{t(g,\theta)} n_{z(g,\theta)} a_t n_z$$

$$= u_{\psi(g,\theta)} a_{t+t(g,\theta)} (a_{-t} n_{z(g,\theta)} a_t) n_z.$$

We obtain

$$\int_G f(x)dx = \int_G f(g^{-1}x)dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u_{\psi(g,\theta)} a_{t+t(g,\theta)} n_z) e^{2t} d\theta dt dz$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u_{\psi(g,\theta)} a_t n_z) e^{2t} e^{-2t(g,\theta)} d\theta dt dz \quad (f \in C_c(G)).$$

This yields

$$(1.8) \qquad \frac{1}{2\pi}\int_0^{2\pi} f(u_\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(u_{\psi(g,\theta)}) e^{-2t(g,\theta)} d\theta$$

for all $f \in C(K)$ and $g \in G$.

It is also possible to give a proof of formula (1.8) by direct calculation of Jacobians.

2. DEFINITION OF THE PRINCIPAL SERIES

Let + denote the trivial representation of M, - the representation which sends $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to ± 1. Then $\hat{M} = \{+,-\}$.

Let $\eta \in \hat{M}$ and let $H = L^2(K)$. H_η is the closed subspace of all $f \in H$ such that for each $m \in M$,

$$f(km) = \eta(m)^{-1} f(k)$$

for almost all $k \in K$.

Let $\lambda \in \mathbb{C}$ and put ξ_λ for the map

$$\xi_\lambda : a_t \to e^{\lambda t} \tag{2.1}$$

of A into $\mathbb{C}^*$. ξ_λ is a complex (quasi-) character of A. The ξ_λ for $\lambda \in i\mathbb{R}$ are the unitary characters of A.

The representations $\pi_{\eta,\lambda}$ of G are now defined as follows. Let $\eta \in \hat{M}$ and H_η be defined as above. Then for any $\lambda \in \mathbb{C}$, $\pi_{\eta,\lambda}$ is a representation of G acting in H_η, defined by

$$(\pi_{\eta,\lambda}(x)f)(u_\theta) = e^{-(\lambda+1)t(x,\theta)} f(u_{\psi(x,\theta)}) \quad (0 \le \theta < 2\pi). \tag{2.2}$$

$\pi_{\eta,\lambda}$ is a strongly continuous representation of G by bounded linear operators on H_η. To prove this, apply (1.7) and (1.8). For $\lambda \in i\mathbb{R}$, it is easy to identify $\pi_{\eta,\lambda}$ with the unitary representation of G induced by the representation

$$\sigma_{\eta,\lambda} : man \to \xi_\lambda(a)\eta(m) \quad (m \in M, a \in A, n \in N) \tag{2.3}$$

of P. (cf. § IX.6.2). $\pi_{\eta,\lambda}$ is unitary if and only if $\lambda \in i\mathbb{R}$. The family

$$\{\pi_{\eta,\lambda} : \eta \in \hat{M},\ \lambda \in i\mathbb{R}\}$$

is called the *principal series of representations of* G *associated with* P.

Let ℓ be the representation of K in H_η given by

$$\ell(k)f(u_\theta) = f(k^{-1}u_\theta) \quad (k \in K, 0 \le \theta < 2\pi).$$

Then ℓ is the representation of K induced by η and $\pi_{\eta,\lambda}|_K = \ell$ (apply (1.7)(iii)).

Put $\phi_m(u_\theta) = e^{-im\theta}$ ($m\in\mathbb{Z}$). The ϕ_m with even m span H_+; those with odd m span H_-; $H = H_+ \oplus H_-$.

3. DIFFERENTIABLE AND ANALYTIC VECTORS

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We shall write $U(\mathfrak{g}_{\mathbb{C}})$ for the (associative) algebra over $\mathbb{C}$ of left G-invariant differential operators on G. Any basis of $\mathfrak{g}$ (as a real vector space) generates the algebra $U(\mathfrak{g}_{\mathbb{C}})$.

Assume G unimodular and fix a Haar measure dx on G. Let π be a strongly continuous representation of G by bounded linear operators on a Banach space V. A vector $v \in V$ is called a *differentiable* or C^∞*-vector* if the mapping

(3.1) $\quad x \mapsto \pi(x)v \quad (x\in G)$

is C^∞. Denote by V^∞ the space of all differentiable vectors in V. V^∞ is a $\pi(G)$-stable linear subspace of V. Given $f \in C_c^\infty(G)$, denote by $\pi(f)$ the bounded linear operator on V defined by

$$\pi(f) = \int_G f(x)\pi(x)dx.$$

Then for all $f \in C_c^\infty(G)$ and all $v \in V$ we have $\pi(f)\, v \in V^\infty$. The space of all finite linear combinations of such C^∞-vectors is called the *Gårding subspace* of V; it is a dense subspace of V. (Note that in § IX.5.2 a different notion of Gårding subspace was used.) More precisely, let $\{f_n\}_{n\geq 1}$ be a sequence of elements of $C_c^\infty(G)$ such that

1) $\quad f_n \geq 0, \int_G f_n(x)dx = 1 \quad (n\geq 1)$,

2) $\quad$ there exists a decreasing sequence $\{\mathcal{O}_n\}_{n\geq 1}$ of compact neighborhoods of e such that supp $(f_n) \subseteq \mathcal{O}_n$ for all n and $\bigcap_{n\geq 1} \mathcal{O}_n = \{e\}$.

Then, for each $v \in V$, $\pi(f_n)v \to v$ as $n \to \infty$. In particular, V^∞ is dense in V.

Given $v \in V^\infty$, put $\tilde{v}(x) = \pi(x)v$ ($x\in G$). So $\tilde{v} \in C^\infty(G,V)$. For any $D \in U(\mathfrak{g}_{\mathbb{C}})$ and $v \in V^\infty$, we define

(3.2) $\pi(D)v = D\tilde{v}(e)$.

In particular, if $X \in \mathfrak{g}$,

(3.3) $$\pi(X)v = \frac{d}{dt}\pi(\exp tX)v\Big|_{t=0}\ .$$

Obviously $\pi(D)V^{\infty} \subset V$. Since $\pi(x)\pi(D)v = D\tilde{v}(x)$ $(v \in V^{\infty}, x \in G)$, we even have $\pi(D)V^{\infty} \subset V^{\infty}$. In addition

$$\pi(D_1D_2)v = \pi(D_1)\pi(D_2)v \qquad (D_1, D_2 \in U(\mathfrak{g}_{\mathbb{C}});v \in V^{\infty}).$$

We have obtained a representation of $U(\mathfrak{g}_{\mathbb{C}})$ on V^{∞}, which we occasionally shall denote by π_{∞}. In particular we have a representation of $\mathfrak{g}$ on V^{∞}, which one might call the *differential of* π.

In order to get a suitable correspondence between representations of G and representations of $\mathfrak{g}$, one needs the concept of analytic vectors. Let π and V be as above. $v \in V$ is called an *analytic vector* if for each $v^* \in V^*$ the function

$$x \mapsto \langle\pi(x)v, v^*\rangle \qquad (x \in G)$$

is a (real) analytic function on G, or, equivalently, if the map $x \mapsto \pi(x)v$ is an analytic map of G into V.

We write V_{ω} for the space of analytic vectors in V. It is $\pi(G)$-stable, so we have the *analytic representation* π_{ω} of G in V_{ω}. Note that $V_{\omega} \subset V^{\infty}$, and if $D \in U(\mathfrak{g}_{\mathbb{C}})$ then $\pi_{\infty}(D)V_{\omega} \subset V_{\omega}$. The representation of $U(\mathfrak{g}_{\mathbb{C}})$ on V_{ω} is also denoted π_{ω}. This is justified by the following

PROPOSITION 3.1. *If* $v \in V_{\omega}$ *then there is a neighborhood* $\mathcal{O}$ *of* 0 *in* $\mathfrak{g}$ *such that* $\sum_{m=0}^{\infty} \frac{1}{m!}\pi_{\omega}(X)^m v$ *converges to* $\pi(\exp X)v$ *for all* $X \in \mathcal{O}$.

One studies analytic vectors because of the following

COROLLARY 3.2. *If* W *is a* $\pi_{\omega}(\mathfrak{g})$*-stable subspace of* V_{ω}, *then its closure is a* $\pi(G)$*-stable subspace of* V.

PROOF. It is enough to show $\pi(G)W \subset \bar{W}$. Choose $\mathcal{O}$ as in Proposition 3.1. Fix $v \in W$. For any $\phi \in V^*$ which vanishes on W, one has

$$<\pi_\omega(D)v,\phi> = 0$$

for all $D \in U(\mathfrak{g}_{\mathbb{C}})$. Hence

$$<\pi(\exp X)v,\phi> = 0$$

for all $X \in \mathcal{O}$. Since G is connected and $v \in V_\omega$, we obtain $<\pi(x)v,\phi> = 0$ for all $x \in G$. Consequently $\pi(x)v \in \overline{W}$ for all $x \in G$. □

Of course V_ω is only useful because of *Nelson's Theorem:* V_ω is dense in V (cf. WARNER, [7, 4.4.5.7]).

4. REDUCIBILITY PROPERTIES OF THE REPRESENTATIONS $\pi_{\eta,\lambda}$

It is not difficult to show that $C^\infty(K) \cap H_\eta$ is contained in H_η^∞ for all $\pi_{\eta,\lambda}$. Moreover, given $f \in C^\infty(K) \cap H_\eta$ and $D \in U(\mathfrak{g}_{\mathbb{C}})$, one has

$$(4.1) \qquad (\pi_{\eta,\lambda}(D)f)(u_\theta) = D_x[(\pi_{\eta,\lambda}(x)f)(u_\theta)]_{x=e}.$$

Write, as before,

$$x^{-1}u_\theta = u_{\psi(x,\theta)}\, a_{t(x,\theta)}\, n_{z(x,\theta)}.$$

Then, with $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$(4.2) \qquad e^{i\psi(x,\theta)} = \frac{(d+ic)\cos\theta + (b+ia)\sin\theta}{|(d+ic)\cos\theta + (b+ia)\sin\theta|}$$

$$t(x,\theta) = \tfrac{1}{2}\log|(d+ic)\cos\theta + (b+ia)\sin\theta|^2$$

$$= \tfrac{1}{2}\log[(d\cos\theta + b\sin\theta)^2 + (c\cos\theta + a\sin\theta)^2].$$

Note $t(1,\theta) = 0$, $\psi(1,\theta) = \theta$.
Let $x = x_s = \exp sZ$, $-\infty < s < \infty$ and $Z \in \mathfrak{g} = s\ell(2,\mathbb{R})$. If $Z = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$, then

$$(4.3) \qquad t(1;Z,\theta) := \frac{d}{ds}t(x_s,\theta)\Big|_{s=0} =$$

$$[\cos\theta(-\alpha\cos\theta+\beta\sin\theta) + \sin\theta(\gamma\cos\theta+\alpha\sin\theta)],$$

$$\psi(1;Z,\theta) := \frac{d}{dt}\,\psi(x_s,\theta)\Big|_{s=0} =$$

$$[\cos\theta(\gamma\cos\theta+\alpha\sin\theta) - \sin\theta(-\alpha\cos\theta+\beta\sin\theta)].$$

For $f \in H_\pm$ we have

$$(\pi_{\pm,\lambda}(x)f)(\theta) = e^{-(\lambda+1)t(x,\theta)}f(\psi(x,\theta))$$

(f a periodic function on $\mathbb{R}$ with period 2π).
Hence, for smooth f we get, for $Z \in \mathfrak{g}$,

$$(4.4)\qquad (\pi_{\pm,\lambda}(Z)f)(\theta) = -(\lambda+1)t(1;Z,\theta)f(\theta) + \psi(1;Z,\theta)\,\frac{d}{d\theta}\,f(\theta).$$

We thus obtain:

$$(4.5)\qquad (\pi_{\pm,\lambda}(H)f)(\theta) = (\lambda+1)\cos 2\theta\; f(\theta) + \sin 2\theta\; f'(\theta),$$

$$(\pi_{\pm,\lambda}(X)f)(\theta) = -\tfrac{1}{2}(\lambda+1)\sin 2\theta\; f(\theta) + \tfrac{1}{2}(\cos 2\theta-1)f'(\theta),$$

$$(\pi_{\pm,\lambda}(Y)f)(\theta) = -\tfrac{1}{2}(\lambda+1)\sin 2\theta\; f(\theta) + \tfrac{1}{2}(\cos 2\theta+1)f'(\theta).$$

Let $\phi_m(\theta) = e^{-im\theta}$ ($m\in\mathbb{Z}$). Then we are led to the following:

$$(4.6)\qquad \pi_{\pm,\lambda}(H)\phi_m = \tfrac{1}{2}(\lambda+1+m)\phi_{m+2} + \tfrac{1}{2}(\lambda+1-m)\phi_{m-2},$$

$$\pi_{\pm,\lambda}(X)\phi_m = \frac{1}{4i}(\lambda+1+m)\phi_{m+2} + \frac{im}{2}\,\phi_m - \frac{1}{4i}(\lambda+1-m)\phi_{m-2},$$

$$\pi_{\pm,\lambda}(Y)\phi_m = \frac{1}{4i}(\lambda+1+m)\phi_{m+2} - \frac{im}{2}\,\phi_m - \frac{1}{4i}(\lambda+1-m)\phi_{m-2}.$$

Put

$$(4.7)\qquad H' = -i(X-Y),\quad X' = H + i(X+Y),\quad Y' = H-i(X+Y).$$

Then

$$(4.8)\qquad \pi_{\pm,\lambda}(H') : \phi_m \to m\phi_m\,,$$

$$\pi_{\pm,\lambda}(X') : \phi_m \to (\lambda+1+m)\phi_{m+2}\,,$$

$$\pi_{\pm,\lambda}(Y') : \phi_m \to (\lambda+1-m)\phi_{m-2}.$$

Now observe that all ϕ_m are analytic vectors for all $\pi_{\pm,\lambda}$. Therefore, in view of Corollary 3.2, these formulae lead at once to a complete analysis of the reducibility properties of the representations $\pi_{\pm,\lambda}$. The results are as follows:

THEOREM 4.1.

a) *If* $\lambda \notin \mathbb{Z}$, $\pi_{+,\lambda}$ *and* $\pi_{-,\lambda}$ *are irreducible.*

b) *If* λ *is an even integer,* $\pi_{+,\lambda}$ *is irreducible, while* $\pi_{-,\lambda}$ *is not irreducible. For* $\pi_{-,\lambda}$ *the decomposition is as follows:*

   (i) $\lambda = 0$. *In this case,* $H_-^{(1)}$, *spanned by* $\phi_{-1},\phi_{-3},\dots$, *and* $H_-^{(2)}$, *spanned by* $\phi_1,\phi_3,\dots$, *are invariant. No other closed invariant subspaces of* H_- *exist.*

   (ii) $\lambda = 2k$, $k \geq 1$. *In this case,* $H_-^{(1)}$, *spanned by* $\phi_{-2k-1},\phi_{-2k-3},\dots$, *and* $H_-^{(2)}$, *spanned by* $\phi_{2k+1},\phi_{2k+3},\dots$, *are invariant.* $H_-^{(1)}, H_-^{(2)}$ *and* $H_-^{(1)} \oplus H_-^{(2)}$ *are the only proper closed invariant subspaces.* $H_-/H_-^{(1)} \oplus H_-^{(2)}$ *is finite-dimensional and defines the irreducible representation with highest weight* $2k-1$.

   (iii) $\lambda = -2k$. $k \geq 1$. *In this case,* $H_-^{(1)}$, *spanned by* $\phi_{2k-1},\phi_{2k-3},\dots$, *and* $H_-^{(2)}$, *spanned by* $\phi_{-(2k-1)},\phi_{-(2k-3)},\dots$, *are invariant; and these, along with* $H_-^{(1)} \cap H_-^{(2)}$ *are all the proper closed invariant subspaces.* $H_-^{(1)} \cap H_-^{(2)}$ *is finite-dimensional and defines the irreducible representation with highest weight* $2k-1$.

c) *If* λ *is an odd integer,* $\pi_{-,\lambda}$ *is irreducible, while* $\pi_{+,\lambda}$ *is reducible. For* $\pi_{+,\lambda}$, *the splitting is as follows.*

   (i) $\lambda = 2k+1$, $k \geq 0$. $H_+^{(1)}$, *spanned by* $\phi_{-2k-2},\phi_{-2k-4},\dots$, *and* $H_+^{(2)}$, *spanned by* $\phi_{2k+2},\phi_{2k+4},\dots$, *are invariant; these and their direct sum are the only proper closed invariant subspaces;* $H_+/H_+^{(1)} \oplus H_+^{(2)}$ *is finite-dimensional and defines the irreducible representation of highest weight* $2k$.

   (ii) $\lambda = -2k-1$, $k \geq 0$. $H_+^{(1)}$, *spanned by* $\phi_{2k},\phi_{2k-2},\dots$, *and* $H_+^{(2)}$, *spanned by* $\phi_{-2k},\phi_{-2k+2},\dots$, *are invariant; these, together with their intersection, exhaust all proper closed invariant subspaces.* $H_+^{(1)} \cap H_+^{(2)}$ *is finite-dimensional and defines the irreducible representation of highest weight* $2k$.

REMARK 4.2. One can show that for $\lambda \in i\mathbb{R}$, the representations $\pi_{+,\lambda}$ (resp. $\pi_{-,\lambda}$) and $\pi_{+,-\lambda}$ (resp. $\pi_{-,-\lambda}$) are *(unitarily) equivalent*. This is usually done by computing their characters.

NOTES

A good reference for this Part is the book by SUGIURA [5, Ch V, §1, 2]. To have a clear insight into the role of differentiable and analytic vectors in representation theory and the theory of the principal series for general semi-simple Lie groups, one should look at papers of Bruhat, Harish-Chandra, Kostant, Gelfand and Neumark, Wallach and others. Here the most comprehensive reference is WARNER [7]. Also the book by WALLACH [6] may be helpful for understanding the theory. For the preparation of this lecture we have benefitted by (unpublished) seminar notes of V.S. VARADARAJAN.

Part II. THE DISCRETE SERIES AND THE COMPLEMENTARY SERIES OF SL(2,ℝ)

5. THE GROUP SU(1,1)

The group $GL(2,\mathbb{C})$ acts on the *Riemann sphere* $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the one point compactification of the complex plane $\mathbb{C}$, as follows: let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{C})$, then

$$(5.1) \qquad g.z = \frac{az+b}{cz+d} \quad (z \in \bar{\mathbb{C}}).$$

The action is transitive and $\mathrm{Stab}(0) = \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a,d \neq 0\right\}$. If we let $\mathbb{C}^+$ denote the upper half-plane $\{z = x + iy \in \mathbb{C}: y > 0\}$, then, for $g \in SL(2,\mathbb{C})$ we have

$$g.\mathbb{C}^+ = \mathbb{C}^+ \iff g \in SL(2,\mathbb{R}).$$

$SL(2,\mathbb{R})$ acts transitively on $\mathbb{C}^+$; $\mathrm{Stab}(i) = K$. In fact, $\mathbb{C}^+$ is diffeomorphic to $SL(2,\mathbb{R})/K$.

Let $D = \{z \in \mathbb{C}: |z| < 1\}$ be the unit disc. The Cayley transformation

$$(5.2) \qquad c: z \mapsto cz = \frac{z-i}{z+i}$$

transforms $\mathbb{C}^+$ bijectively onto D. Let C be the 2×2 matrix given by

$$C = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Let $G = C \cdot SL(2,\mathbb{R}) \cdot C^{-1}$. Then G is the group of complex 2×2 matrices $\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}$ satisfying $|\alpha|^2 - |\beta|^2 = 1$. This is just the group SU(1,1) of complex 2×2 matrices with determinant one, which leave the Hermitean form $|z_1|^2 - |z_2|^2$ invariant. SU(1,1) acts on D by the linear fractional transformations

$$(5.3) \qquad \zeta \mapsto g \cdot \zeta = \frac{\alpha\zeta+\beta}{\bar\beta\zeta+\bar\alpha}, \qquad \text{where } g = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix}.$$

The action is transitive. $\mathrm{Stab}(0) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \le \theta < 2\pi \right\}$ and SU(1,1)/Stab(0) is diffeomorphic to D.

Let $g \in SU(1,1)$, $g' = C^{-1} g\, C$, then the following diagram is commutative:

$$(5.4) \qquad \begin{array}{ccc} D & \xrightarrow{\;g\;} & D \\ {\scriptstyle C}\uparrow & & \uparrow{\scriptstyle C} \\ \mathbb{C}^+ & \xrightarrow{\;g'\;} & \mathbb{C}^+ \end{array} .$$

By the isomorphism $h: g \to C g\, C^{-1}$ from $SL(2,\mathbb{R})$ to SU(1,1), we have

$$(5.5) \qquad u_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \xrightarrow{\;h\;} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

$$a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \xrightarrow{\;h\;} \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

$$n_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \xrightarrow{\;h\;} \begin{pmatrix} 1+i\xi/2 & -i\xi/2 \\ i\xi/2 & 1-i\xi/2 \end{pmatrix}.$$

We shall identify corresponding objects by the isomorphism $h: SL(2,\mathbb{R}) \to SU(1,1)$ and we shall write $h(u_\theta) = u_\theta$, $h(a_t) = a_t$, $h(n_\xi) = n_\xi$; $G = SU(1,1)$, $h(K) = K$, $h(A) = A$, $h(N) = N$. So, in particular, G/K is diffeomorphic to D.

6. THE INVARIANT MEASURE ON D

$D \simeq G/K$ carries a G-invariant measure (cf. § IX.3). The purpose of this section is to determine this measure. We start with $SL(2,\mathbb{R})$ acting on $\mathbb{C}^+$. The Haar measure dg on $SL(2,\mathbb{R})$ can be so normalized that

$$dg = da_t \; dn_\xi \; du_\theta$$

according to the decomposition $g = a_t \, n_\xi \, u_\theta$. Let f be a continuous function with compact support on $\mathbb{C}^+$. Put $\phi_f(g) = f(g.i)$. Then $\phi_f \in C_c(SL(2,\mathbb{R}))$ and

$$\phi_f(gk) = \phi_f(g) \qquad (g\in SL(2,\mathbb{R}), k\in K).$$

Then $\mu: f \mapsto \int_{SL(2,\mathbb{R})} \phi_f(g)dg$ defines a $SL(2,\mathbb{R})$-invariant measure on $\mathbb{C}^+$.

We have:

$$\int_{SL(2,\mathbb{R})} \phi_f(g)\, dg = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a_t . n_\xi . i)\, dt\, d\xi$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(e^{2t}(i+\xi))\, dt\, d\xi.$$

Put $z = x + iy = e^{2t}(i+\xi)$. Then we obtain

$$\int_{\mathbb{C}^+} f(z)d\mu(z) = \tfrac{1}{2}\int_{\mathbb{C}^+} f(z)\,\frac{dxdy}{y^2} \qquad (f\in C_c(\mathbb{C}^+)).$$

Now applying the Cayley transformation, and keeping in mind (5.4), we easily obtain the following

PROPOSITION 6.1. *Let* $\zeta \in D$, $\zeta = x + iy$ *and put* $dm(\zeta) = dx\, dy$. *Then* $d\nu(\zeta) = (1-|\zeta|^2)^{-2}dm(\zeta)$ *is a* G-*invariant measure on* D.

The proposition can be proved more directly, without going back to SL(2,ℝ). We need some notation.

Let $g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in G$, $\zeta \in D$ and put

(6.1) $$J(g,\zeta) = \bar{\beta}\zeta + \bar{\alpha}.$$

J is a so-called *automorphic factor,* i.e.,

(6.2) $$J(g_1g_2,\zeta) = J(g_1,g_2\zeta)J(g_2,\zeta),$$

$$J(e,\zeta) = 1.$$

In particular, $J(g^{-1},g\zeta)J(g,\zeta) = 1$, hence $J(g,\zeta) \neq 0$ for all $g \in G$, $\zeta \in D$. The following relation is easily verified: $|J(g,\zeta)|^2(1-|g\zeta|^2) = 1-|\zeta|^2$. Now put $g.\zeta = \zeta' = x' + iy'$. Then the Jacobian of $\zeta \longmapsto g.\zeta$ is

$$\begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} \end{vmatrix} = \left(\frac{\partial x'}{\partial x}\right)^2 + \left(\frac{\partial y'}{\partial x}\right)^2 =$$

(by the use of the Cauchy-Riemann equations)

$$= \left|\frac{\partial \zeta'}{\partial \zeta}\right|^2 = \frac{1}{|\bar{\beta}\zeta+\bar{\alpha}|^4} = |J(g,\zeta)|^{-4}.$$

Hence we have $dm(g\zeta) = |J(g,\zeta)|^{-4}\, dm(\zeta)$ and

$$d\nu(g\zeta) = (1-|g\zeta|^2)^{-2}\, dm(g\zeta) = (1-|g\zeta|^2)^{-2}|J(g,\zeta)|^{-4}\, dm(\zeta) =$$

$$= (1-|\zeta|^2)^{-2}\, dm(\zeta) = d\nu(\zeta).$$

7. THE DISCRETE SERIES

7.1. The representations ρ_n

We let G = SU(1,1). For $n \in \frac{1}{2}\mathbb{Z}$, $\chi_n(u_\theta) = e^{-2in\theta}$ gives a one-dimensional unitary representation of K. We consider the unitary representation ρ_n' of G induced by χ_n. The space V_n' of ρ_n' is the set of all functions f satisfying the following three conditions:

(7.1) (i) $f: G \to \mathbb{C}$ is measurable;

(ii) $f(gk) = \chi_n(k^{-1})f(g)$ a.e. $(g\in G, k\in K)$;

(iii) $\|f\|^2 = \int_{G/K} |f(g)|^2 d\nu(\bar{g}) < \infty$, where $\bar{g} = g.0$.

ρ_n' is defined by $\rho_n'(g_0)f(g) = f(g_0^{-1}g)$ $(f\in V_n')$.

We shall identify V_n' with a space of functions V_n on D. Therefore define:

$$(7.2) \qquad Af(g) = J(g,0)^{2n} f(g) \qquad (f\in V_n').$$

Then

$$Af(gk) = J(gk,0)^{2n}f(gk) = J(g,k.0)^{2n}J(k,0)^{2n}f(gk) =$$

$$= J(g,0)^{2n}f(g) = Af(g).$$

Hence Af is defined on D and one has

$$(7.3) \qquad \|f\|^2 = \int_D |Af(\zeta)|^2(1-|\zeta|^2)^{2n-2}\, dm(\zeta).$$

Let V_n denote the (Hilbert)space of all measurable functions ϕ on D such that

$$\int_D |\phi(\zeta)|^2\ (1-|\zeta|^2)^{2n-2}\, dm(\zeta) < \infty.$$

V_n is a G-space: G acts unitarily in V_n by ρ_n given by

$$(7.4) \qquad \rho_n(g)\phi(\zeta) = J(g^{-1},\zeta)^{-2n}\ \phi(g^{-1}\zeta) = (\bar{\beta}\zeta+\bar{\alpha})^{-2n}\ \phi\left(\frac{\alpha\zeta+\beta}{\bar{\beta}\zeta+\bar{\alpha}}\right),$$

where

$$g^{-1} = \begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}.$$

A is an isomorphism of V_n' onto V_n such that

$$A\rho_n'(g) = \rho_n(g)A$$

for all $g \in G$.

7.2. Definition of the discrete series

In general ρ_n is not irreducible. Let H_n be the space of all $f \in V_n$ which are *holomorphic* on D. We have the following results:

THEOREM 7.1.

(i) $H_n \neq \{0\}$ *if and only if* $n \geq 1$.

(ii) $1 \in H_n$ *for* $n \geq 1$ *and* $\|1\|^2 = (2n-1)/\pi$.

(iii) H_n *is a closed invariant subspace of* V_n $(n\geq 1)$.

For proofs we refer to SUGIURA [5, Ch.V,§3].

It is convenient to normalize the inner product on H_n such that $\|1\| = 1$. So, let, for $n \in \frac{1}{2}\mathbb{Z}$ and $n \geq 1$, $\mathcal{H}_n$ denote the space of all holomorphic functions in $L^2(D,(2n-1)\pi^{-1}(1-|\zeta|^2)^{2n-2}dm(\zeta))$ and denote by π_n the unitary representation of SU(1,1) on $\mathcal{H}_n$ given by

$$\pi_n(g)f(\zeta) = J(g^{-1},\zeta)^{-2n}f(g^{-1}\zeta) = (\bar{\beta}\zeta+\bar{\alpha})^{-2n}f\left(\frac{\alpha\zeta+\beta}{\bar{\beta}\zeta+\bar{\alpha}}\right)$$

($f \in \mathcal{H}_n$), where $g^{-1} = \begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix}$.

Moreover, let for $n \in \frac{1}{2}\mathbb{Z}$ and $n \geq 1$, $\mathcal{H}_{-n}$ be the subspace of $L^2(D,(2n-1)\pi^{-1}(1-|\zeta|^2)^{2n-2}dm(\zeta))$ of *antiholomorphic* functions. Let σ be the map from $L^2(D,(2n-1)\pi^{-1}(1-|\zeta|^2)^{2n-2}dm(\zeta))$ onto itself given by $\sigma\colon f \to \bar{f}$. The map σ is an isometric antilinear mapping and we have $\sigma(\mathcal{H}_n) = \mathcal{H}_{-n}$. Define $\pi_{-n}(g) = \sigma\,\pi_n(g)\sigma^{-1}$ for $g \in G$. Then $\mathcal{H}_{-n}$ is a closed subspace of $L^2(D,(2n-1)\pi^{-1}(1-|\zeta|^2)^{2n-2}dm(\zeta))$ and π_{-n} is a unitary representation of G on $\mathcal{H}_{-n}$.

DEFINITION 7.2. The set of unitary representations $\{\pi_n \mid n \in \frac{1}{2}\mathbb{Z}, |n| \geq 1\}$ is called the *discrete series* of G.

7.3. Irreducibility of the discrete series

THEOREM 7.3.

(i) *Every closed invariant subspace* $\mathcal{H}' \neq \{0\}$ *of* $\mathcal{H}_n$ *contains* 1.

(ii) π_n *is an irreducible unitary representation of* G *on* $\mathcal{H}_n$.

PROOF. (i) Given $f \in \mathcal{H}_n$, we can write $f(\zeta) = \sum_{m=0}^{\infty} a_m\zeta^m$ in D. We have

$$\pi_n(g)f(\zeta) = (\bar{\beta}\zeta+\bar{\alpha})^{-2n}\; f\left(\frac{\alpha\zeta+\beta}{\bar{\beta}\zeta+\bar{\alpha}}\right),$$

where

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Let $g = u_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, then

$$\pi_n(u_\theta)f(\zeta) = e^{-2in\theta} f(e^{-2i\theta}\zeta) = e^{-2in\theta} \sum_{m=0}^{\infty} a_m e^{-2im\theta}\zeta^m.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} e^{2in\theta} \pi_n(u_\theta)f(\zeta)d\theta = a_0 = f(0).$$

If $f \in H'$, then $(2\pi)^{-1} \int_0^{2\pi} e^{2in\theta} \pi_n(u_\theta)f \, d\theta \in H'$ and therefore the constant function $f(0)$ belongs to H'. If $H' \neq \{0\}$, then there is an $f \in H'$, $f \neq 0$, and there is some point $\zeta_0 \in D$ such that $f(\zeta_0) \neq 0$. Since G acts transitively on D, there is $g \in G$ such that $g^{-1}.0 = \zeta_0$. Hence

$$\pi_n(g)f(0) = \bar{\alpha}^{-2n}f(g^{-1}.0) = \bar{\alpha}^{-2n}f(\zeta_0) \neq 0,$$

where

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Since H' is an invariant subspace we have $\pi_n(g)f \in H'$; by what was said above, $\pi_n(g)f(0) \in H'$; and since $\pi_n(g)f(0) \neq 0$, we conclude that $1 \in H'$.
(ii) Let $H' \neq \{0\}$ be a closed invariant subspace of H_n. If $H' \neq H_n$, then the orthogonal complement $H'^{\perp}$ of H' is not $\{0\}$ and is a closed invariant subspace of H_n. Hence by (i), $1 \in H' \cap H'^{\perp} = \{0\}$, a contradiction. Therefore $H' = H_n$. □

Via the isometry σ it follows at once that π_{-n} $(n \in \frac{1}{2}\mathbb{Z}, n \geq 1)$ is irreducible. Therefore the discrete series of G consists of irreducible unitary representations.

7.4. <u>K-weights</u>

Let

$$\phi_m^n(\zeta) = \left\{\frac{\Gamma(2n+m)}{\Gamma(2n)\Gamma(m+1)}\right\}^{\frac{1}{2}} \zeta^m, \qquad m = 0,1,2,\ldots .$$

Then $\{\phi^n_m \mid m \in \mathbb{N}\}$ (resp. $\{\bar\phi^n_m \mid m \in \mathbb{N}\}$ is a complete orthonormal system in H_n (resp H_{-n}) for $n \in \frac{1}{2}\mathbb{Z}$, $n \geq 1$. We have the following formulae for $\pi_n|_K$.

$$(7.5) \qquad \pi_n(u_\theta)\phi^n_m = \chi_{n+m}(u_\theta)\phi^n_m ,$$

$$(7.6) \qquad \pi_{-n}(u_\theta)\bar\phi^n_m = \chi_{-n-m}(u_\theta)\bar\phi^n_m \qquad (n\geq 1, n\in\tfrac{1}{2}\mathbb{Z}, m\in \mathbb{N}).$$

8. THE COMPLEMENTARY SERIES

8.1. A new realization of the representations $\pi_{\pm,\lambda}$

We recall the definition of $\pi_{\eta,\lambda}$ from section 2. H_η is the space of measurable functions f satisfying

(i) $f(gman) = \sigma^{-1}_{\eta,\lambda+1}(man)f(g)$ $(m\in M, a\in A, n\in N)$ for almost all $g \in SL(2,\mathbb{R})$;

(ii) $\int_K |f(u_\theta)|^2 du_\theta < \infty$.

$\pi_{\eta,\lambda}$ is the representation of $SL(2,\mathbb{R})$ on H_η given by

$$\pi_{\eta,\lambda}(g_0)f(g) = f(g_0^{-1}g) \qquad (g_0, g\in SL(2,\mathbb{R})).$$

For technical purposes we shall work with $G = SU(1,1)$ here, instead of $SL(2,\mathbb{R})$. We can use the above definition, replacing just $SL(2,\mathbb{R})$ by $SU(1,1)$.

Observe that $SU(1,1)$ acts transitively on $U = \{\zeta \in \mathbb{C}: |\zeta| = 1\}$ by linear fractional transformations:

$$g.\zeta = \frac{\alpha\zeta+\beta}{\bar\beta\zeta+\bar\alpha}\ ; \qquad g = \begin{pmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{pmatrix}, \quad \zeta \in U.$$

Moreover Stab(1) = MAN.

Put

$$(8.1) \qquad \sigma'_{\eta,\lambda}(u_\theta a_t n_\xi) = e^{\frac{1}{2}i\theta(1-\eta)}\sigma_{\eta,\lambda}(a_t n_\xi) \qquad (u_\theta\in K, a_t\in A, n_\xi\in N).$$

For $f \in H_\eta$ let $Af(g) = \sigma'_{\eta,\lambda+1}(g)f(g)$ $(g\in G)$. Then $Af(gman) = Af(g)$, hence Af can be identified with a function on U. Actually, A is an isometric isomorphism of H_η onto $L^2(U)$. The representation $\pi_{\eta,\lambda}$ is transformed by this isomorphism into the following one, which we denote by $\pi_{\eta,\lambda}$ again:

$$(8.2) \qquad \pi_{\eta,\lambda}(g)f(\zeta) = |\bar\beta\zeta+\bar\alpha|^{-(\lambda+1)}\left(\frac{\bar\beta\zeta+\bar\alpha}{|\bar\beta\zeta+\bar\alpha|}\right)^{\frac{1}{2}(1-\eta)} f\left(\frac{\alpha\zeta+\beta}{\bar\beta\zeta+\bar\alpha}\right)$$

where

$$g^{-1} = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad f \in L^2(U), \ \zeta \in U.$$

All this is easily verified.

8.2. Definition of the complementary series

It is known that $\pi_{+,\lambda}$ is a unitary representation if and only if $\lambda \in i\mathbb{R}$. It turns out that for some real λ the space $L^2(U)$ can be endowed with a continuous positive definite Hermitean form

$$(8.3) \qquad (\phi,\psi)_\lambda = \int_U \int_U \Phi_\lambda(\zeta,\zeta')\phi(\zeta)\overline{\psi(\zeta')}d\zeta\, d\zeta'$$

which is invariant under $\pi_{+,\lambda}(g)$ for all $g \in G$. As already observed by BARGMANN [1, p.617], this is possible for $0 < \lambda < 1$, taking

$$(8.4) \qquad \Phi_\lambda(\zeta,\zeta') = [1-\mathrm{Re}(\zeta\overline{\zeta'})]^{\frac{1}{2}(\lambda-1)}.$$

Let H_λ $(0<\lambda<1)$ denote the closure of $L^2(U)$ with respect to the inner product defined by (8.3). The injection of $L^2(U)$ into H_λ is continuous, $L^2(U)$ being considered as a Hilbert space with the usual L^2-norm. For any $g \in G$, $\pi_{+,\lambda}(g)$ can be extended uniquely to a unitary operator on H_λ, which we denote by $\pi_\lambda(g)$ $(0<\lambda<1)$. $\pi_\lambda\colon g \to \pi_\lambda(g)$ is a unitary representation of G on H_λ.

DEFINITION 8.1. The set of representations π_λ $(0<\lambda<1)$ is called the *complementary series* of unitary representations of G.

8.3. Irreducibility of the complementary series

Since $\{\chi_m \mid m \in \mathbb{Z}\}$ is a complete, orthogonal set in H_λ, the following result is obvious:

PROPOSITION 8.2. *Let* $0 < \lambda < 1$. *Then:*

$$\pi_{+,\lambda} \mid_K = \bigoplus_{m\in\mathbb{Z}} \chi_m,$$

$$\pi_\lambda \mid_K = \bigoplus_{m\in\mathbb{Z}} \chi_m.$$

Our aim is to prove the following

THEOREM 8.3. *For any* λ *satisfying* $0 < \lambda < 1$, π_λ *is an irreducible unitary representation of* G.

PROOF. Recall that $\pi_{+,\lambda}$ is an irreducible representation of G on $L^2(U)$ (Theorem 4.1). We have embedded $L^2(U)$ into H_λ, the embedding is continuous. Let H' be a non-zero closed linear subspace of H_λ, which is invariant with respect to $\pi_\lambda(G)$. Then $H' \cap L^2(U)$ is a closed linear subspace of $L^2(U)$, invariant under $\pi_{+,\lambda}(G)$. Moreover, by looking at the representations $\pi_{+,\lambda|K}$ on $H' \cap L^2(U)$ and $\pi_{\lambda|K}$ on H', we obtain, applying Proposition 8.2, $H' \cap L^2(U) \neq \{0\}$. Hence $H' \supset L^2(U)$. Therefore we have, by taking closures in H_λ, $H' = H_\lambda$. This completes the proof of the theorem. □

NOTES

Our main reference for this Part is SUGIURA [5, Ch V, § 3,4].

Part III. CLASSIFICATION OF THE IRREDUCIBLE UNITARY REPRESENTATIONS OF SL(2,ℝ)

9. FINITE K-MULTIPLICITIES

Let $G = SL(2,\mathbb{R})$.

LEMMA 9.1. *Every element* $g \in G$ *can be written in the form* $g = u_\theta a_t u_{\theta'}$.

PROOF. Let P be the set of all positive definite real symmetric 2 × 2 matrices. Then ${}^tgg \in P$. So there exists $p \in P$ such that $p^2 = {}^tgg$. Let $u = gp^{-1}$. Then ${}^tuu = p^{-1}\,{}^tggp^{-1} = 1$, i.e., u is an orthogonal matrix. Since det g = 1, we have det p = det u = 1, so $u,p \in G$. Since $p \in G \cap P$, there exists a $\nu \in K$ such that $p = \nu^{-1}a_t\nu$ for some $t \in \mathbb{R}$. Write $u_\theta = u\nu^{-1}$ and $u_{\theta'} = \nu$. Then $g = u_\theta a_t u_{\theta'}$. □

As usual, we put for $n \in \frac{1}{2}\mathbb{Z}$,

$$\chi_n(u_\theta) = e^{-2in\theta} \quad (\theta \in \mathbb{R}).$$

χ_n defines a one-dimensional unitary representation of K. Let A_n denote the space of functions $f \in C_c(G)$ satisfying the relation

$$(9.1) \qquad f(u_\theta g u_{\theta'}) = \chi_n(u_{\theta+\theta'}) f(g)$$

for all $\theta,\theta' \in \mathbb{R}$ and $g \in G$. It is easy to verify that A_n is an algebra under convolution. A_n is an involutive subalgebra of $L^1(G)$ (cf. § VIII. 4).

PROPOSITION 9.2. *The algebra A_n is commutative.*

PROOF. For any $g \in G$, put

$$^\#g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} {}^t g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $g \to {}^\#g$ is an anti-automorphism of G, satisfying $^\#(^\#g) = g$ for all $g \in G$. Therefore $d(^\#g) = dg$, where dg is the Haar measure on G. Given $f \in A_n$, put $^\#f(g) = f(^\#g)$ $(g \in G)$. Then we easily obtain the following relation:

$$^\#(f_1 * f_2) = {}^\#f_2 * {}^\#f_1 \qquad (f_1, f_2 \in A_n).$$

Let $g \in G$ be written in the form $g = u_\theta a_t u_{\theta'}$. Then $^\#g = u_{\theta'} a_t u_\theta$. Therefore

$$^\#f(g) = f(^\#g) = \chi_n(u_{\theta'+\theta}) f(a_t) = f(g)$$

for all $f \in A_n$. Hence

$$f_1 * f_2 = {}^\#(^\#f_2 * {}^\#f_1) = {}^\#(f_2 * f_1) = f_2 * f_1$$

for all $f_1, f_2 \in A_n$. □

Let π be a unitary representation of G on a Hilbert space $\mathcal{H}$. According to Theorem VIII. 4.5 we obtain a $*$-representation of $L^1(G)$ on $\mathcal{H}$ by putting

$$\pi(f) = \int_G f(g)\pi(g)\,dg \qquad (f \in L^1(G)).$$

In particular we have a $*$-representation of A_n on $\mathcal{H}$. By restriction to K, π defines a unitary representation of K on $\mathcal{H}$. In particular $\pi(\chi_n)$ is defined and one easily verifies the relation

$$(9.2) \qquad \pi(f) = \pi(\chi_n)\pi(f)\pi(\chi_n)$$

for $f \in A_n$. Moreover $\pi(\chi_n)$ is easily seen to be the orthogonal projection on the space of vectors in $\mathcal{H}$ which transform under $\pi_{|K}$ according to χ_{-n}. Put $\mathcal{H}(n) = \pi(\chi_n)\mathcal{H}$. Then because of (9.2), we actually have a $*$-representation of A_n on $\mathcal{H}(n)$. Furthermore, if π is irreducible, the corresponding representation of A_n on $\mathcal{H}(n)$ is irreducible. Indeed, given $v \in \mathcal{H}(n)$, $v \neq 0$, we have $\pi(C_c(G))v$ dense in $\mathcal{H}$, hence $\pi(A_n)v = \pi(\chi)\pi(C_c(G))v$ dense in $\mathcal{H}(n)$. Therefore, given any non-zero closed $\pi(A_n)$-stable subspace $\mathcal{H}' \subset \mathcal{H}(n)$, we get $\mathcal{H}' = \mathcal{H}(n)$.

We are now able to state our main result of this section.

THEOREM 9.3. *Let* π *be an irreducible unitary representation of* G *on* $\mathcal{H}$. *Then each irreducible unitary representation* χ_n *of* K *is contained at most once in the restriction of* π *to* K.

PROOF. We have to show that $\dim \mathcal{H}(n) \leq 1$ for all $n \in \frac{1}{2}\mathbb{Z}$. But this is an immediate consequence of Schur's Lemma, Proposition 9.2 and the irreducibility of the $*$-representation of A_n on $\mathcal{H}(n)$, defined above. □

REMARK 9.4. Given a compact group K, let $d(\delta)$ denote the dimension of the space of $\delta \in \hat{K}$. As a generalization of Theorem 9.3, the following general theorem holds: let G be a connected semi-simple Lie group with finite center and K a maximal compact subgroup of G. Then every irreducible representation $\delta \in \hat{K}$ of K is contained at most $d(\delta)$ times in the restriction of every irreducible unitary representation of G to K ([7, Corollary 5.5.1.8],[4]).

10. TWO FUNDAMENTAL THEOREMS

In this section we state two fundamental results which are due to HARISH-CHANDRA. The results imply that classifying irreducible unitary representations of SL(2,$\mathbb{R}$) can be reduced to classifying algebraically irreducible representations of the Lie algebra sl(2,$\mathbb{C}$).

10.1. K-finite vectors

In this subsection G is a connected Lie group with Lie algebra $\mathfrak{g}$, K a connected compact Lie subgroup of G with Lie algebra $\mathfrak{k} \subset \mathfrak{g}$.

Given $\delta \in \hat{K}$, let χ_δ denote its character and put

$$(10.1) \qquad \alpha_\delta(k) = d(\delta)\bar{\chi}_\delta(k) \qquad (k \in K).$$

Let π be a strongly continuous representation of G by bounded linear operators on a Banach space V. We shall call π a Banach representation of G. As observed earlier, $\pi(\alpha_\delta)$ is defined for all $\delta \in \hat{K}$. Furthermore, the orthogonality relations for the χ_δ imply that $\pi(\alpha_\delta)$ is a continuous projection and $\pi(\alpha_\delta)\pi(\alpha_{\delta'}) = \pi(\alpha_{\delta'})\pi(\alpha_\delta) = 0$ if $\delta \neq \delta'$. We shall write $V_\delta = \pi(\alpha_\delta)V$. The V_δ's are $\pi(K)$-stable, closed and linearly independent subspaces of V.

DEFINITION 10.1. An element $v \in V$ is called K-*finite* if the dimension of the space spanned by $\pi(K)v$ is finite.

We shall write V_K for the vector space of K-finite vectors in V. The following proposition gives some insight into the structure of V_K.

PROPOSITION 10.2.

(i) $V_K = \sum_{\delta \in \hat{K}} V_\delta$ *(algebraic direct sum)*.

(ii) V_K *is dense in* V.

(iii) V_δ *is the space of vectors which transform under* $\pi_{|K}$ *according to* δ.

PROOF. Let $v \in V_\delta$. Call W the smallest closed subspace of V_δ which contains the elements $\pi(k)v$ $(k \in K)$. For $w^* \in W^*$, write

$$(10.2) \qquad c_{w^*}(k) = \langle \pi(k^{-1})v, w^* \rangle \qquad (k \in K).$$

Since $\pi(\alpha_\delta)$ commutes with $\pi(K)$ we obtain

$$\alpha_\delta * c_{w^*} = c_{w^*}.$$

Hence $c_{w^*} \in \alpha_\delta * L^2(K)$, which is a finite-dimensional space. Since $w^* \to c_{w^*}$ is an injective linear map from W^* into $\alpha_\delta * L^2(K)$, we have $\dim W^* < \infty$ and hence, by the Hahn-Banach Theorem, $\dim W < \infty$. So we have proved (iii) and the inclusion $\sum_{\delta \in \hat{K}} V_\delta \subset V_K$. Now fix $v \in V_K$. The space $\mathrm{span}(\pi(K)v)$ is finite-dimensional and splits therefore as a direct sum of $\pi(K)$-invariant subspaces, consisting of vectors which transform under $\pi_{|K}$ according to some $\delta \in \hat{K}$. By (iii) we get $V_K \subset \sum_{\delta\in\hat{K}} V_\delta$. We apply the Hahn-Banach Theorem to show that V_K is dense in V. So assume $v^* \in V^*$ to vanish on V_K. Define c_{v^*} as in (10.2). Fix $v \in V$. Then $\alpha_\delta * c_{v^*}(k^{-1}) = \langle\pi(k)\pi(\alpha_\delta)v,v^*\rangle = \langle\pi(\alpha_\delta)\pi(k)v,v^*\rangle = 0$. Hence, by the Peter-Weyl Theorem, $c_{v^*} = 0$ a.e. But c_{v^*} is continuous, so $c_{v^*} = 0$ and therefore $\langle v,v^*\rangle = c_{v^*}(e) = 0$. □

10.2. Two theorems

Let G and K be as in 10.1. Let π be, as usual, a Banach representation of G on V. We call π a *K-finite representation* if $\dim V_\delta < \infty$ for all $\delta \in \hat{K}$. Observe that any irreducible unitary representation of SL(2,ℝ) is SO(2,ℝ)-finite (Theorem 9.3).

PROPOSITION 10.3. *Let* π *be a K-finite Banach representation of* G *on* V. *Then* V_K *is contained in the space* V_ω *of analytic vectors and* V_K *is* $\pi_\omega(U(\mathfrak{g}_{\mathbb{C}}))$*-stable.*

We start with a lemma.

LEMMA 10.4. *Let* f *be an analytic function on* G *and* $h \in C(K)$. *Then*

$$\phi(x) = \int_K h(k)f(xk)dk$$

is analytic on G.

PROOF. Fix $x_0 \in G$. Since $(x,k) \mapsto f(xk)$ is an analytic function on $G \times K$, there exist a coordinate neighbourhood N of x_0 with local coordinates $x_1,\ldots,x_n$, a finite set $(U_\alpha)_{\alpha\in I}$ of open coordinate neightbourhoods of K whose union is K, and a set of power series $(p_\alpha)_{\alpha\in I}$ such that

$$f(xk) = p_\alpha(x_1,\ldots,x_n,k_1,\ldots,k_p) \qquad (x\in N, k\in U_\alpha),$$

where $(k_1,\ldots,k_p)$ is a system of local coordinates on U_α. Let $1 = \sum_{\alpha\in I}\phi_\alpha$ be a continuous partition of unity subordinate to the covering $(U_\alpha)_{\alpha\in I}$. Then

$$\phi(x) = \sum_{\alpha\in I} \int_K p_\alpha(x_1,\ldots,x_n,k)\phi_\alpha(k)h(k)dk$$

is analytic on N. □

PROOF OF PROPOSITION 10.3. We have $V_K = \sum_{\delta\in\hat{K}} V_\delta$ and $\dim V_\delta < \infty$ for all $\delta \in \hat{K}$. Fix $\delta \in \hat{K}$ and choose $v \in V_\delta$. Since V_ω is dense in V by Nelson's Theorem, there are $v_n \in V_\omega$ satisfying $v = \lim v_n$. Hence $v = \lim \pi(\alpha_\delta)v_n$. By Lemma 10.4, $\pi(\alpha_\delta)v_n \in V_\omega$, hence $V_\omega \cap V_\delta$ is dense in V_δ. Since $\dim V_\delta < \infty$ we now have $V_\delta = V_\omega \cap V_\delta$ or $V_\delta \subset V_\omega$. Consequently $V_K \subset V_\omega$. Let w be a K-finite vector and $X \in \mathfrak{g}$. Then $\pi_\omega(X)w$ is K-finite, because $\pi(k)\pi_\omega(X)w = \pi_\omega(\mathrm{Ad}\,(k)X)\pi(k)w$ $(k\in K)$ and $\dim \mathfrak{g} < \infty$. Hence V_K is $\pi_\omega(U(\mathfrak{g}_\mathbb{C}))$-stable. □

Given a K-finite Banach representation π of G on V, we write π_K for the representation of $U(\mathfrak{g}_\mathbb{C})$ on V_K, defined by π_ω. The following theorems relate π to π_K (cf. [2].

THEOREM 10.5. *Let π be a K-finite Banach representation of G. Then π is irreducible if and only if π_K is algebraically irreducible.*

The proof of this theorem is based on Corollary 3.2 and Proposition 10.3.

THEOREM 10.6 (cf. [2, Theorem 8]). *Let π and π' be two K-finite irreducible unitary representations of G. Then π is equivalent to π' if and only if π_K is equivalent to π'_K.*

It remains to classify algebraically irreducible representations of $U(\mathfrak{g}_\mathbb{C})$ and to find criteria under which they yield global representations, especially unitary representations, of G. For semi-simple Lie groups G, the reader is referred to [2, Theorem 9], [3, Theorem 4] and a generalization of the latter result by LEPOWSKY [4].

11. THE CASIMIR OPERATOR

In this section G is a connected Lie group with semi-simple Lie algebra $\mathfrak{g}$. Let κ denote the Killing form on $\mathfrak{g}_\mathbb{C}$:

$$\kappa(X,Y) = \mathrm{tr}(\mathrm{ad}X\ \mathrm{ad}Y).$$

κ is a non-degenerate, G-invariant, symmetric bilinear form on $\mathfrak{g}_{\mathbb{C}}$. Choose a basis $X_1,\dots,X_n$ of $\mathfrak{g}_{\mathbb{C}}$ ($n=\dim\mathfrak{g}$) and define $Y_1,\dots,Y_n$ by the relation

$$(11.1) \qquad \kappa(X_i,Y_j) = \delta_{ij}.$$

Put $\omega = \sum_{i=1}^{n} X_iY_i \in U(\mathfrak{g}_{\mathbb{C}})$.

PROPOSITION 11.1.

(i) ω *does not depend on the choice of the basis* $X_1,\dots,X_n$ *of* $\mathfrak{g}_{\mathbb{C}}$.

(ii) ω *is a bi-invariant differential operator on* G.

PROOF. (i) Let $X_1',\dots,X_n'$ be another basis and $Y_1',\dots,Y_n'$ be such that $\kappa(X_i',Y_j') = \delta_{ij}$.

Put

$$X_i = \sum_j \alpha_{ji}X_j', \quad Y_K = \sum_j \beta_{jk}Y_j'.$$

Then

$$\sum_j \alpha_{ji}\,\kappa(X_j',Y_k) = \sum_j \alpha_{ji}\beta_{jk} = \delta_{ik}.$$

Hence,

$$\sum_j \alpha_{ij}\beta_{kj} = \delta_{ik}$$

and therefore

$$\sum_{i=1}^{n} X_iY_i = \sum_{i,j,k} \alpha_{ji}\beta_{ki}\, X_j'Y_k' = \sum_{j=1}^{n} X_j'Y_j'.$$

(ii) It is enough to show that $\mathrm{Ad}(g)\omega = \omega$ for all $g \in G$. But this is immediate from (i), taking $X_i' = \mathrm{Ad}(g)X_i$ and $Y_i' = \mathrm{Ad}(g)Y_i$ $(i = 1,\dots,n)$. □

ω is called the *Casimir operator* of G.

Let π be a Banach representation of G on V. Then $\pi(\omega)$ is defined on the dense subspace V^∞ of V. Since ω is bi-invariant, an easy calculation implies

$$(11.2) \qquad \pi(g)\pi(\omega)v = \pi(\omega)\pi(g)v \qquad (g\in G, v\in V^{\infty}).$$

Indeed, if we denote by R_x the right-translation by $x \in G$, we have with the notations of section 3,

$$\pi(x)\pi(\omega)v = \omega\tilde{v}(x) = (R_x(\omega\tilde{v}))(e) = \omega(R_x\tilde{v})(e) = \omega(\widetilde{\pi(x)v})(e)$$

$$= \pi(\omega)\pi(x)v \qquad (v\in V^{\infty}).$$

Now assume that V is a Hilbert space. $\pi(\omega)$ is a densely defined (in general) unbounded operator. If π is unitary, we have

$$\langle\pi(X)v,w\rangle = -\langle v,\pi(X)w\rangle$$

for all $X \in \mathfrak{g}$ and all $v,w \in V^{\infty}$. Consequently $\pi(\omega)^*$ is densely defined and, applying Proposition 11.1 (i),

$$(11.3) \qquad \pi(\omega)^*v = \pi(\omega)v \qquad (v\in V^{\infty}).$$

This implies that $\pi(\omega)$ is hermitean.

We shall make use of the following facts from functional analysis about a linear operator A on the Hilbert space V with dense domain:

(1) A^* is a closed operator. A admits a closed extension if and only if $\mathrm{Dom}(A^*)$ is dense.
(2) If A is closed, $\mathrm{Dom}(A^*A)$ is dense and A^*A is selfadjoint and positive.
(3) The spectral decomposition theorem for A if A is selfadjoint.
(4) The polar decomposition of A: if A is closed, one can write A = UP where P is selfadjoint and positive, and U a partial isometry (hence bounded). Moreover $P = (A^*A)^{\frac{1}{2}}$.

PROPOSITION 11.2. *Let* π *be an irreducible unitary representation of* G *on* V. *Then* $\pi(\omega)$ *is a real scalar.*

PROOF. As observed above $\pi(\omega) = A$ is hermitean and thus has a closed extension $\bar{A}$. Obviously $\pi(g)\bar{A} \subset \bar{A}\pi(g)$ for all $g \in G$. The proposition now follows from the polar decomposition of a closed operator, the spectral theorem and the fact that $\pi(\omega)$ is hermitean. □

EXAMPLE 11.3. Let $G = SL(2,\mathbb{R})$. Define H,X,Y as in (1.2). Then $\omega = \frac{1}{8}(H^2+2H+4YX)$. Formula (4.6) implies $\pi_{\pm,\lambda}(\omega) = \frac{1}{8}(\lambda^2-1)$ for all $\lambda \in \mathbb{C}$. Observe that $\pi_{\pm,\lambda}(\omega)$ is a (complex) scalar even if $\pi_{\pm,\lambda}$ is neither irreducible nor unitary.

12. CLASSIFICATION OF $\hat{G}$ FOR $G = SU(1,1)$

12.1. Some preparation

Let $G = SU(1,1)$ and $\mathfrak{g} = su(1,1)$ its Lie algebra. $\mathfrak{g}$ consists of all the matrices of the form

$$\begin{pmatrix} ia & b \\ \bar{b} & -ia \end{pmatrix}, \qquad a \in \mathbb{R},\ b \in \mathbb{C}.$$

The three elements

$$X_0 = \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad X_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \tfrac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

form a basis of $\mathfrak{g}$. Observe that $\mathfrak{g}_{\mathbb{C}} = sl(2,\mathbb{C})$. Put $H_0 = -iX_0$, $E = X_1 + iX_2$. $F = -X_1 + iX_2$. Then

$$(12.1) \qquad H_0 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

In view of Example 11.3, the Casimir operator ω of G is given by

$$\omega = -\tfrac{1}{4}(-2H_0^2 + EF + FE).$$

We put

$$(12.2) \qquad 2\Omega = -2H_0^2 + EF + FE.$$

Hence $\omega = -\tfrac{1}{2}\Omega$.

Let π be an irreducible unitary representation of G on $\mathcal{H}$. Then $\pi(\Omega)$ is a scalar, say $\pi(\Omega) = q, q \in \mathbb{R}$. Let K be as usual, $K = \{u_\theta = \exp(2i\theta H_0) : \theta \in \mathbb{R}\}$. The restriction of π to K is decomposed into a direct sum of irreducible unitary representations $\chi_n : u_\theta \to e^{-2in\theta}$ $(n \in \tfrac{1}{2}\mathbb{Z})$. Each irreducible unitary representation χ_n is contained in $\pi_{|K}$ at most once (Theorem 9.3). Hence we have $\pi_{|K} = \bigoplus_{n\in M} \chi_n$, where M is the set of n such that χ_n does occur.

M is called the set of K-*weights* of π.

Let $v \in \mathcal{H}_K$ be chosen such that $\pi(u_\theta)v = e^{2i\mu\theta}v$, $\mu \in \frac{1}{2}\mathbb{Z}$. Then an easy calculation gives:

$$\pi(H_0)v = \mu v. \tag{12.3}$$

In the above notations, we have the following

PROPOSITION 12.1. *Put*

$$\rho_m = q + (\mu+m-1)(\mu+m) \quad \textit{and} \quad \sigma_m = q + (\mu-m+1)(\mu-m)$$

for $m \in \mathbb{N}$. *Then*:

$$\begin{aligned} \pi(H_0)\pi(E)^m v &= (\mu+m)\pi(E)^m v, \\ \pi(H_0)\pi(F)^m v &= (\mu-m)\pi(F)^m v. \end{aligned} \tag{12.4}$$

$$\begin{aligned} \pi(F)\pi(E)^m v &= \rho_m \pi(E)^{m-1} v, \\ \pi(E)\pi(F)^m v &= \sigma_m \pi(F)^{m-1} v . \end{aligned} \tag{12.5}$$

$$\begin{aligned} \|\pi(E)^{m+1}v\|^2 &= \rho_{m+1}\|\pi(E)^m v\|^2, \\ \|\pi(F)^{m+1}v\|^2 &= \sigma_{m+1}\|\pi(F)^m v\|^2. \end{aligned} \tag{12.6}$$

PROOF. We have

$$[H_0,E] = E, \quad [H_0,F] = -F, \quad [F,E] = 2H_0.$$

Making use of these relations, (12.4) is proved by induction on m. Furthermore

$$[\pi(F)\pi(E) + \pi(E)\pi(F)]v = 2(q+\mu^2)v,$$

$$[\pi(F)\pi(E) - \pi(E)\pi(F)]v = 2\mu v.$$

Hence

$$(12.7) \qquad \pi(F)\pi(E)v = (q+\mu(\mu+1))v,$$

$$(12.8) \qquad \pi(E)\pi(F)v = (q+\mu(\mu-1))v.$$

Replacing v in (12.7) by $\pi(E)^{m-1}v$ and μ by $\mu + m - 1$, we get the first equality in (12.5). Similarly we get the second equality in (12.5) by replacing v in (12.8) by $\pi(F)^{m-1}v$ and μ by $\mu - m + 1$.
We have

$$\begin{aligned}\langle\pi(E)u,v\rangle &= \langle(\pi(X_1)+i\pi(X_2))u,v\rangle \\ &= -\langle u,(\pi(X_1)-i\pi(X_2))v\rangle = \langle u,\pi(F)v\rangle\end{aligned}$$

for all $u,v \in H^{\infty}$. Hence

$$\begin{aligned}\|\pi(E)^{m+1}v\|^2 &= \langle\pi(E)^{m+1}v,\pi(E)^{m+1}v\rangle \\ &= \langle\pi(E)^{m}v,\pi(F)\pi(E)^{m+1}v\rangle \\ &= \rho_{m+1}\|\pi(E)^{m}v\|^2.\end{aligned}$$

Similarly we obtain $\|\pi(F)^{m+1}v\|^2 = \sigma_{m+1}\|\pi(F)^{m}v\|^2$.

12.2. M and q

In this subsection we determine the possibilities for the sets of K-weights M and the values of $\pi(\Omega) = q$ for any irreducible unitary representation π of $G = SU(1,1)$ on a Hilbert space $\mathcal{H}$. We shall show that the pair (M,q) determines π.

THEOREM 12.2. *Let* π *be an irreducible unitary representation of* G *on* $\mathcal{H}$ *and* M *be the set of* K-*weights of* π. *Let* $q = \pi(\Omega)$. *Then the pair* (M,q) *is equal to one of the following sets:*

1) $M = \mathbb{Z}$, $q > 0$.
2) $M = \frac{1}{2} + \mathbb{Z}$, $q > \frac{1}{4}$.
3) $M_n^+ = \{n+p : p \in \mathbb{N}\}$, $q = n(1-n)$ $\quad (n\in\frac{1}{2}\mathbb{N}, n>0)$.
4) $M_n^- = \{-(n+p) : p \in \mathbb{N}\}$, $q = n(1-n)$ $\quad (n\in\frac{1}{2}\mathbb{N}, n>0)$.
5) $M = \{0\}$, $q = 0$.

In case 5), π *is the identity representation of* G.

PROOF. Since $H \neq (0)$, M is not empty and there exist a $\mu \in M$ and a non-zero vector $v \in H$ such that $\pi(H_0)v = \mu v$. There are four possibilities for the vanishing of $\pi(E)^m v$ and $\pi(F)^m v$.

a) $\pi(E)^m v \neq 0$ and $\pi(F)^m v \neq 0$ for all $m \in \mathbb{N}$.

b) $\pi(E)^k v = 0$ for some $k \in \mathbb{N}$ and $\pi(F)^m v \neq 0$ for $m \in \mathbb{N}$.

c) $\pi(E)^m v \neq 0$ for all $m \in \mathbb{N}$ and $\pi(F)^k v = 0$ for some $k \in \mathbb{N}$.

d) $\pi(E)^k v = 0$ for some $k \in \mathbb{N}$ and $\pi(F)^n v = 0$ for some $n \in \mathbb{N}$.

In any case, the subspace of H_K spanned by $\{\pi(E)^m v, \pi(F)^m v\}_{m\in\mathbb{N}}$ is invariant under π_K and must be equal to H_K, because π_K is irreducible by Theorem 10.5. Hence $M = \{\mu + m: \pi(E)^m v \neq 0\} \cup \{\mu - m: \pi(F)^m v \neq 0\}$.

Case a) Since $\mu \in \frac{1}{2}\mathbb{Z}$ we have two cases: $M = \mathbb{Z}$ and $M = \frac{1}{2} + \mathbb{Z}$.

a_1) $M = \mathbb{Z}$. We use the notation of Proposition 12.1. Since $\rho_m \neq 0$ and $\sigma_m \neq 0$ for all $m \in \mathbb{N}$, we can replace v by $\pi(E)^{-\mu}v$ if $\mu < 0$ or by $\pi(F)^{\mu}v$ if $\mu > 0$ and assume $\mu = 0$. Then we have $\rho_m = \sigma_m = q + m(m-1) > 0$ for all $m \geq 1$. This implies $q > 0$.

a_2) $M = \frac{1}{2} + \mathbb{Z}$. Here we may assume $\mu = \frac{1}{2}$ and therefore $\sigma_m = (q-\frac{1}{4}) + (m-1)^2 > 0$ for all $m \geq 1$. So $q > \frac{1}{4}$.

Case b) Let h be the smallest integer such that $v, \pi(E)v, \ldots, \pi(E)^h v$ are not equal to zero, but $\pi(E)^{h+1}v = 0$. Then $\pi(E)^k v = 0$ for $k \geq h+1$. We have: $\rho_1, \rho_2, \ldots, \rho_h > 0$ and $\rho_{h+1} = 0$. Hence $\pi(F)^m \pi(E)^h v \neq 0$ for all $m \in \mathbb{N}$. We now replace v by $\pi(E)^h v$ and assume $\pi(E)v = 0$. We have $\rho_1 = 0$ and hence $q = -\mu(\mu+1)$. On the other hand we now get $\sigma_m = m(m-1-2\mu)$. The condition that $\sigma_m > 0$ for all $m \geq 1$ implies $\mu < 0$. Put $\mu = -n$ $(n \in \frac{1}{2}\mathbb{N}, n>0)$. Then $q = n(1-n)$ and $M = \{-(n+p): p \in \mathbb{N}\}$.

Case c) is similar to case b) and leads to : $q = n(1-n)$ $(n \in \frac{1}{2}\mathbb{N}, n>0)$, $M = \{n+p: p \in \mathbb{N}\}$.

Case d) In this case, $\dim H_K < \infty$, so $\dim H < \infty$. By Proposition I.3.1, $\dim H = 1$ and π is the identity representation of G. Hence $M = \{0\}$, $q = 0$.

The cases a_1, a_2, b, c, d correspond to the cases 1), 2), 4), 3) and 5) in the theorem, respectively. □

For the next theorem, we make the following convention: given any irreducible unitary representation π of G, we shall write $q(\pi)$, $M(\pi)$ instead of q,M respectively.

THEOREM 12.3. *Let* π *and* π' *be two irreducible unitary representations of* $G = SU(1,1)$. *Then* π *is equivalent to* π' *if and only if* $M(\pi) = M(\pi')$ *and* $q(\pi) = q(\pi')$.

PROOF. By Theorem 10.6, π is equivalent to π' if and only if π_K is (algebraically) equivalent to π'_K. So it is sufficient to describe explicitly the structure of the $\mathfrak{g}_{\mathbb{C}}$-module H_K in the cases 1)-5) of Theorem 12.2 and to observe that this structure is completely determined by the value of q. This is easily done. For instance, let us consider case 3). A basis of H_K is given by $v, \pi(E)v, \pi(E)^2v, \ldots$, where v is a non-zero vector satisfying $\pi(H_0)v = nv$. The $\mathfrak{g}_{\mathbb{C}}$-module structure is now completely given by (12.4) and (12.5) with the following convention: $\pi(F)v = 0$, $\mu = n$. Since (12.5) depends only on q (given $\mu=n$), the proof of the theorem follows for case 3). The other cases can be treated in a similar way.

12.3. The classification

Theorems 12.2 and 12.3 are the main instruments to determine the dual of SU(1,1). First we have to list M and q for the irreducible unitary representations constructed in Part I and Part II.

THEOREM 12.4. *The sets* M *of* K-*weights and the value* $q = \pi(\Omega)$ *for the irreducible unitary representations* π *of* Part I *and* Part II *are given by the following table:*

π	M	q
$\pi_{+,\lambda}$ $(\lambda\in i\mathbb{R})$	$\mathbb{Z}$	$\frac{1}{4}(1-\lambda^2)$
$\pi_{-,\lambda}$ $(\lambda\in i\mathbb{R}, \lambda\neq 0)$	$\frac{1}{2}+\mathbb{Z}$	$\frac{1}{4}(1-\lambda^2)$
$\pi_{-,0}\vert H_-^{(1)}$	$-\frac{1}{2}-\mathbb{N}$	$\frac{1}{4}$
$\pi_{-,0}\vert H_-^{(2)}$	$\frac{1}{2}+\mathbb{N}$	$\frac{1}{4}$
π_n $(n\in\frac{1}{2}\mathbb{N}, n\geq 1)$	$-n-\mathbb{N}$	$n(1-n)$
π_{-n} $(n\in\frac{1}{2}\mathbb{N}, n\geq 1)$	$n+\mathbb{N}$	$n(1-n)$
π_λ $(0<\lambda<1)$	$\mathbb{Z}$	$\frac{1}{4}(1-\lambda^2)$
I	$\{0\}$	0

PROOF. The table is easily verified by direct computation. Part of it has already been done in Theorem 4.1, section 7.7 and Example 11.3. As to $\pi_\lambda (0<\lambda<1)$, observe that $(\pi_\lambda)_K = (\pi_{+,\lambda})_K$ (Proposition 8.2) and $\pi_\lambda(\Omega) = \pi_{+,\lambda}(\Omega)$ on $H_\eta^\infty \simeq L^2(U)^\infty \subset H_\lambda^\infty$, with the notation of section 8. □

We now come to our main theorem.

THEOREM 12.5 (BARGMANN). *Any irreducible unitary representation* π *of* $G = SU(1,1)$ *(or of* $SL(2,\mathbb{R})$*) is equivalent to one and only one of the following representations*:

1) $\pi_{+,\lambda}$ $(\lambda=i\nu,\ \nu\in\mathbb{R},\nu\geq 0)$.
2) $\pi_{-,\lambda}$ $(\lambda=i\nu,\ \nu\in\mathbb{R},\nu>0)$.
3) $\pi_{-,0}|H^{(1)}$.
4) $\pi_{-,0}|H^{(2)}$.
5) π_n $(n\in\frac{1}{2}\mathbb{Z},|n|\geq 1)$.
6) π_λ $(0<\lambda<1)$.
7) I (*The identity representation*).

PROOF. Let π be an irreducible unitary representation of $G = SU(1,1)$. According to Theorem 12.2 the pair (M,q) coincides with one of the following sets:

1) $\mathbb{Z}$, $q > 0$. 2) $\frac{1}{2}+\mathbb{Z}$, $q > \frac{1}{4}$. 3) $M_n^+ = \{n+p\colon p\in\mathbb{N}\}$, $q = n(1-n)$ $(n\in\frac{1}{2}\mathbb{N}, n>0)$. 4) $M_n^- = \{-(n+p)\colon p\in\mathbb{N}\}$, $q = n(1-n)$ $(n\in\frac{1}{2}\mathbb{N}, n>0)$. 5) $M = \{0\}$, $q = 0$.

1) $M = \mathbb{Z}, q > 0$. We distinguish between two cases: $0 < q < \frac{1}{4}$ and $q \geq \frac{1}{4}$.
   If $q \geq \frac{1}{4}$, there exists a unique $\nu\in\mathbb{R}$, $\nu\geq 0$, such that $q = \frac{1}{4}(1+\nu^2)$ and π is equivalent to $\pi_{+,i\nu}$ by Theorem 12.3.
   If $0 < q < \frac{1}{4}$, there exists a unique $\lambda\in\mathbb{R}$, $0<\lambda<1$, such that $q = \frac{1}{4}(1-\lambda^2)$ and π is equivalent to π_λ.

2) $M = \frac{1}{2}+\mathbb{Z}$, $q > \frac{1}{4}$.
   There exists a unique $\nu > 0$ such that $q = \frac{1}{4}(1+\nu^2)$ and π is equivalent to $\pi_{-,i\nu}$.

3) $M = M_n^+ = \{n+p\colon p\in\mathbb{N}\}$, $q = n(1-n)$ $(n\in\frac{1}{2}\mathbb{N}, n>0)$.
   If $n = \frac{1}{2}$, then $q = \frac{1}{4}$ and π is equivalent to $\pi_{-,0}|H^{(2)}$.
   If $n > \frac{1}{2}$, then $q = n(1-n)$ and π is equivalent to π_{-n}.

4) $M = M_n^- = \{-(n+p): p \in \mathbb{N}\}$, $q = n(1-n)$ $(n\in\frac{1}{2}\mathbb{N}, n>0)$.
If $n = \frac{1}{2}$, then $q = \frac{1}{4}$ and π is equivalent to $\pi_{-,0}|H^{(1)}$.
If $n > \frac{1}{2}$, then $q = n(1-n)$ and π is equivalent to π_n^-.

5) $M = \{0\}$, $q = 0$. Then π is the identity representation by Theorem 12.2.
Since the pair (M,q) takes different values for the representations written down in the theorem, no two of them are equivalent to each other by Theorem 12.3. □

NOTES

The contents of this Part are close to the treatment of the classification in the book of SUGIURA [5, Ch. V, §5,6].

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