Special Functions Associated with Root Systems:
A First Introduction for Non-Specialists

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Abstract   A survey is given about recent developments in special functions associated with root systems. The paper addresses a general mathematical audience and it does not assume earlier knowledge of either special functions or root systems. The emphasis is on multivariable orthogonal polynomials: the Heckman-Opdam Jacobi polynomials and Maconald's q-analogues of these.

1. Introduction

In this paper I present a brief survey of the active area of Special Functions associated with Root Systems. The article is intended for a general mathematical audience. It will not suppose prerequisites on either special functions or root systems. It will also skip many technical details. Some early developments in this area took place during the seventies ([21], [40]). During the last ten years important break-throughs were made by Heckman and Opdam [14], [35], [36], [15], I. G. Macdonald [29], [30], C. F. Dunkl [10] and I. Cherednik [4], [5], [6], [7].

A lot of the motivation for the subject of this paper comes from analysis on semisimple Lie groups. Spherical functions on Riemannian symmetric spaces of the compact or non-compact type can be written as special functions depending on parameters which assume only special discrete values. In the one-variable cases these special functions were classical, also for parameter values without group theoretic interpretation, but in the more-variable cases they were new. In the case of group theoretic interpretation, many properties of these special functions, as well as associated harmonic analysis, immediately follow by group theoretic arguments. The case of more general parameter values yields the special functions associated with root systems. Properties derived in the group case can still be formulated in the general case, but now as conjectures rather than theorems. This paper describes some of the progress which has been made in proving these conjectures. For convenience, I will restrict to the polynomial (compact) case, with Bessel functions as a sole exception. (For an introduction to the non-polynomial case see [37], [16].) Neither will I discuss the recent work on commuting operators with elliptic functions as coefficients.

An important aspect of the whole theory, which will not be discussed very much in this paper, is the connection with completely integrable systems, for instance the generalized Calogero-Moser system.

Special functions associated with root systems have also been developed in the q-case, where q is a deformation parameter giving back the earlier cases when $q = 1$. Motivation and development of the theory in the q-case has been quite different from the $q = 1$ case. Except for the case of Hall polynomials [27], theory was developed [30], [29], [23] without interpretation in group theory. But afterwards quantum groups looked very promising as a natural setting for these polynomials. This had already turned out to be true in the one-variable case [24], and more recently some interpretations of more-variable cases on quantum groups were found [33], [12].

In any case, a quantum group interpretation for generic values of the parameters cannot be expected. But, by Cherednik's work [4], [5], [6] we know already another algebraic setting for special functions associated with root systems: affine and graded Hecke algebras [26]. As shown by work of Opdam [37], this new algebraic context also allows harmonic analysis.

An earlier version of this paper appeared in [25]. It is my intention to make updates of this paper regularly.

2. The one-variable case

In this section I will introduce three classical families of special functions, each depending on a real parameter $k \geq 0$, and such that the cases $k = 0$ and $k = 1$ are elementary. The three families are connected with each other by limit transitions. Later, for each of the families I will discuss generalizations which are associated with root systems.

2.1. Bessel functions. Consider *Bessel functions* in a non-standard notation:

$$\mathcal{J}_k(x) := \sum_{j=0}^{\infty} \frac{(-\frac{1}{4}x^2)^j}{(k + \frac{1}{2})_j \, j!} \quad (x \in \mathbb{R}). \tag{2.1}$$

Here we use the notation for *shifted factorial*:

$$(a)_j := a\,(a+1)\dots(a+j-1) \quad (j = 1, 2, \dots); \qquad (a)_0 := 1.$$

The function $\mathcal{J}_k$ is related to the Bessel function J_α in standard notation [11, Ch. VII] by

$$\mathcal{J}_k(x) = \frac{2^{k-\frac{1}{2}}\,\Gamma(k + \frac{1}{2})}{x^{k-\frac{1}{2}}}\,J_{k-\frac{1}{2}}(x).$$

Note that

$$\mathcal{J}_k(x) = \mathcal{J}_k(-x), \qquad \mathcal{J}_k(0) = 1. \tag{2.2}$$

The cases $k = 0$ and $k = 1$ yield elementary functions:

$$\mathcal{J}_0(x) = \cos x, \qquad \mathcal{J}_1(x) = \frac{\sin x}{x}. \tag{2.3}$$

The function $x \mapsto \mathcal{J}_k(\lambda x)$ $(\lambda \in \mathbb{R})$ is an eigenfunction of a differential operator:

$$\left(\frac{d^2}{dx^2} + \frac{2k}{x} \frac{d}{dx} \right) \mathcal{J}_k(\lambda\, x) = -\lambda^2\, \mathcal{J}_k(\lambda x).$$

It is the unique C^∞ solution of this differential equation under conditions (2.2).

2.2. Ultraspherical polynomials. Consider *ultraspherical* or *Gegenbauer polynomials* [11, §10.9], i.e. polynomials C_n^k of degree n on $\mathbb{R}$ such that

$$\int_0^\pi C_n^k(\cos x)\, C_m^k(\cos x)\, (\sin x)^{2k}\, dx = 0 \quad (n, m \in \mathbb{Z}_+,\ n \neq m). \tag{2.4}$$

Then the C_n^k are determined up to a constant factor (in general, we will not use the standard normalization for Gegenbauer polynomials). For $k = 0, 1$ we have:

$$C_n^0(\cos x) = \text{const.}\ \cos(nx), \qquad C_n^1(\cos x) = \text{const.}\ \frac{\sin((n+1)x)}{\sin x}\,. \tag{2.5}$$

The function $x \mapsto C_n^k(\cos x)$ is an eigenfunction of a differential operator:

$$\left(\frac{d^2}{dx^2} + 2k\, \cot x\, \frac{d}{dx} \right) C_n^k(\cos x) = -n(n+2k)\, C_n^k(\cos x).$$

For (n_N) being a sequence of positive integers such that $n_N/N \to \lambda$ for some $\lambda \geq 0$ as $N \to \infty$, we have the limit result

$$\lim_{n \to \infty} \frac{C_{n_N}^k(\cos(x/N))}{C_{n_N}^k(1)} = \mathcal{J}_k(\lambda x).$$

2.3. q-Ultraspherical polynomials. Let $0 < q < 1$ and define for $a \in \mathbb{C}$, $n \in \mathbb{Z}_+$:

$$(a; q)_\infty := \prod_{j=0}^\infty (1 - aq^j), \qquad (a; q)_n := \frac{(a; q)_\infty}{(q^n a; q)_\infty}\,.$$

The infinite product converges because of the condition on q. When we compare with each other the *q-binomial series* and the *binomial series* (see Gasper & Rahman, [13, §1.3])

$$\sum_{n=0}^\infty \frac{(a; q)_n}{(q; q)_n}\, z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \qquad \sum_{n=0}^\infty \frac{(a)_n}{n!}\, z^n = (1 - z)^{-a},$$

with both power series converging for $|z| < 1$, we see that, at least formally,

$$\lim_{q \uparrow 1} \frac{(q^a z; q)_\infty}{(z; q)_\infty} = (1 - z)^{-a} \tag{2.6}$$

(see Koornwinder [22, Appendix A] for more rigorous statements).

We will consider *q-ultraspherical polynomials* [1] in a non-standard notation. These are polynomials $C_n^{k,q}$ of degree n on $\mathbb{R}$ such that

$$\int_0^\pi C_n^{k,q}(\cos x)\, C_m^{k,q}(\cos x) \left| \frac{(e^{2ix};q)_\infty}{(q^k e^{2ix};q)_\infty} \right|^2 dx = 0 \qquad (n,m \in \mathbb{Z}_+,\ n \neq m). \qquad (2.7)$$

Then the $C_n^{k,q}$ are determined up to a constant factor. It follows from (2.6) that the weight function in (2.7) tends to $(2\sin x)^{2k}$ as $q \uparrow 1$, i.e. to the weight function in (2.4). In fact, we have accordingly with suitable normalization the limit relation

$$\lim_{q\uparrow 1} C_n^{k,q}(\cos x) = C_n^k(\cos x),$$

which also follows from explicit expressions for the $(q\text{-})$ultraspherical polynomials. If we put

$$P_n(e^{ix}) := C_n^{k,q}(\cos x)$$

then P_n is an eigenfunction of a q-difference operator:

$$\frac{1 - q^k e^{2ix}}{1 - e^{2ix}}\, P_n(q^{\frac{1}{2}} e^{ix}) + \frac{1 - q^k e^{-2ix}}{1 - e^{-2ix}}\, P_n(q^{-\frac{1}{2}} e^{ix}) = (q^{-\frac{1}{2}n} + q^{\frac{1}{2}n+k})\, P_n(e^{ix}).$$

Note that the P_n on the left hand side have arguments off the unit circle, while orthogonality is on the unit circle. The cases $k = 0$ and $k = 1$ are elementary as in (2.5) (not depending on q):

$$C_n^{0,q}(\cos x) = \text{const. } \cos(nx), \qquad C_n^{1,q}(\cos x) = \text{const. } \frac{\sin((n+1)x)}{\sin x}\,.$$

The q-ultraspherical polynomials form a subclass of the *Askey-Wilson polynomials* [2]: a family of orthogonal polynomials depending, apart from q, on four non-trivial parameters.

2.4. Dunkl operators in one variable. We will now generalize the elementary formulas

$$e^{i\lambda x} = \mathcal{J}_0(\lambda x) + i\lambda x\, \mathcal{J}_1(\lambda x) \quad \text{and} \quad \frac{d}{dx}\, e^{i\lambda x} = i\lambda\, e^{i\lambda x} \qquad (2.8)$$

(the first formula follows by (2.3)). Dunkl [10] generalized the operator d/dx to a mixture of a differential and a reflection operator:

$$(D^{(k)} f)(x) := f'(x) + k\, \frac{f(x) - f(-x)}{x}\,. \qquad (2.9)$$

Note that this *Dunkl operator* sends smooth functions to smooth functions. Let us define a *generalized exponential function* in terms of Bessel functions (2.1) by

$$\mathcal{E}_k(\lambda x) := \mathcal{J}_k(\lambda x) + \frac{i\lambda x}{2k+1}\, \mathcal{J}_{k+1}(\lambda x). \qquad (2.10)$$

Then it follows immediately from well-known differential recurrence formulas for Bessel functions that

$$D^{(k)} \mathcal{E}_k(\lambda x) = i\lambda \, \mathcal{E}_k(\lambda x). \tag{2.11}$$

Formulas (2.10) and (2.11) generalize the formulas in (2.8). The function $x \mapsto \mathcal{E}_k(\lambda x)$ is the unique C^∞ function which equals 1 in 0 and which is an eigenfunction with eigenvalue $i\lambda$ of $D^{(k)}$.

For $(D^{(k)})^2$ we compute

$$(D^{(k)})^2 \, f(x) = f''(x) + \frac{2k}{x} \, f'(x) - k \, \frac{f(x) - f(-x)}{x^2}.$$

Thus, on even functions f the square of the Dunkl operator acts as the differential operator $(d/dx)^2 + 2kx^{-1} \, d/dx$. In particular, its action on

$$\mathcal{J}_k(\lambda x) = \tfrac{1}{2}(\mathcal{E}_k(\lambda x) + \mathcal{E}_k(-\lambda x))$$

yields

$$(D^{(k)})^2 \, \mathcal{J}_k(\lambda x) = -\lambda^2 \, \mathcal{J}_k(\lambda x).$$

3. Preliminaries about root systems

3.1. Definition of root system.
A reference for this subsection is Humphreys [18]. Let V be a d-dimensional real vector space with inner product $\langle \, . \, , . \, \rangle$. For $\alpha \in V \backslash \{0\}$ let s_α denote the orthogonal reflection with respect to the hyperplane orthogonal to α (cf. Fig. 1):

$$s_\alpha(\beta) := \beta - \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \, \alpha \quad (\beta \in V).$$

A *root system in V* is a finite non-empty subset R of $V \backslash \{0\}$ which spans V and which satisfies for all $\alpha, \beta \in R$ the two properties that

$$s_\alpha(\beta) \in R \quad \text{and} \quad \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

The elements of R are called *roots*. By the *rank of R* we mean the dimension d of V. Clearly, if $\alpha \in R$ then $-\alpha = s_\alpha(\alpha) \in R$. It can be shown that, for a given root α, we have $R \cap \mathbb{R}\alpha = \{\pm \alpha\}$ or $\{\pm \alpha, \pm 2\alpha\}$ or $\{\pm \alpha, \pm \tfrac{1}{2}\alpha\}$. The root system R is called *reduced* if, for all $\alpha \in R$, we have $R \cap \mathbb{R}\alpha = \{\pm \alpha\}$.

The root system R is called *irreducible* if R cannot be written as the orthogonal disjoint union $R = R_1 \cup R_2$ of two root systems R_1 and R_2. Any root system can be written as a unique orthogonal disjoint union of irreducible root systems. The reduced irreducible root systems can be classified as four infinite families A_n, B_n, C_n, D_n of *classical root systems* and five *exceptional root systems* G_2, F_4, E_6, E_7, E_8. Here the subscript denotes the rank

of the root system. There is one infinite family of non-reduced irreducible root systems: of type BC_n.

An example for $d = 1$ is the set $R := \{\pm 2\} \subset \mathbb{R}$ (root system of type A_1, cf. Fig. 2). An example for $d = 2$ is the set $R = R_+ \cup (-R_+)$, where $R_+ := \{(1, -1), (2, 0), (1, 1), (0, 2)\} \subset \mathbb{R}^2$ (root system of type C_2, cf. Fig. 3). In general, when we have a root system R in V then we can write it as a disjoint union $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated from each other by a hyperplane in V through the origin. The choice for the set of *positive roota* R_+ is not unique. For a given choice of R_+ there is a unique subset $\alpha_1, \ldots, \alpha_d$ of R_+ (the so-called *simple roota*) such that each $\alpha \in R_+$ can be written as a linear combination of $\alpha_1, \ldots, \alpha_d$ with nonnegative integer coefficients. The simple roots form a basis of V.

Let $GL(V)$ be the group of invertible linear transformations of V. The *Weyl group* W of the root system R is the subgroup of $GL(V)$ which is generated by the reflections s_α ($\alpha \in R$). The group W is finite and it acts on R. It permutes the possible choices of R_+ in a simply transitive way.

If $v \in V \backslash \{0\}$ then write

$$\check{v} := \frac{2v}{\langle v, v \rangle} \,.$$

The set $R\check{} := \{\check{\alpha} \mid \alpha \in R\}$ is again a root system in V, the so-called *dual root system to* R. The root systems R and $R\check{}$ have the same Weyl group. Corresponding to a choice of R_+ there is a choice $R_+\check{} := \{\check{\alpha} \mid \alpha \in R_+\}$ of positive roots for $R_+\check{}$.

3.2. Dunkl operators associated with R. Let R be a reduced root system in V. Let $k: \alpha \mapsto k_\alpha: R \to [0, \infty)$ be a function which is W-invariant, i.e., which satisfies $k_{w\alpha} = k_\alpha$ for all $w \in W$ and all $\alpha \in R$. If R is an irreducible (reduced) root system then the Weyl group is transitive on all roots of equal length and there are at most two different root lengths. Thus k_α then assumes at most two different values. See the above examples: one root length in A_1 and two root lengths in C_2. The function k is called a *multiplicity function*. The reason for this name is that root systems naturally arise in the structure theory of real semisimple Lie algebras, where roots have an interpretation as joint eigenvalues of certain operators and the k_α then are (integer) multiplicities of such eigenvalues.

For $\xi \in V$ we will denote by ∂_ξ the corresponding directional derivative. The *Dunkl operators* [10], [19] associated with the root system R and the multiplicity function k are defined as the operators $D_\xi^{(k)}: C^\infty(V) \to C^\infty(V)$ ($\xi \in V$) given by

$$(D_\xi^{(k)} f)(x) := (\partial_\xi f)(x) + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{f(x) - f(s_\alpha x)}{\langle \alpha, x \rangle} \,. \tag{3.1}$$

This definition is easily seen to be independent of the choice of R_+, since we may replace $\sum_{\alpha \in R_+}$ by $\frac{1}{2} \sum_{\alpha \in R}$. In case of root system A_1 formula (3.1) reduces for $\xi := 1$ to formula (2.9). Note that the operator (3.1) consists of a term involving a first order derivative and terms involving reflection operators, just as we have seen in (2.9). It is an amazing fact, which can be proved in a rather straightforward way, that the operators $D_\xi^{(k)}$ commute:

$$[D_\xi^{(k)}, D_\eta^{(k)}] = 0 \quad (\xi, \eta \in V).$$

Let $\mathbb{D}^{(k)}$ be the algebra generated by the operators $D_\xi^{(k)}$. This is a commutative algebra. It can be shown that each W-invariant operator D in $\mathbb{D}^{(k)}$, when restricted in its action to the W-invariant C^∞ functions on V, coincides with a partial differential operator (so its reflection terms vanish when acting on a W-invariant function). The joint W-invariant smooth eigenfunctions of the W-invariant operators in $\mathbb{D}^{(k)}$ are called *Bessel functions associated with R*. In the example A_1 things reduce to the one-variable considerations of §2.1 and §2.4. More generally, one may study the joint eigenfunctions of the full algebra $\mathbb{D}^{(k)}$ and one may try to do harmonic analysis for these eigenfunctions. A lot of satisfactory results have been obtained, see [19] and the references given there.

3.3. Weight lattice asscociated with R. A reference for this subsection is again Humphreys [18]. We still assume a root system R in V. The *weight lattice P* of R is defined by

$$P := \{\lambda \in V \mid \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \quad \text{for all } \alpha \in R\}.$$

The subset P_+ of *dominant weights* is then defined by

$$P_+ := \{\lambda \in P \mid \frac{2\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \geq 0 \quad \text{for all } \alpha \in R_+\}.$$

Note that $R \subset P$, but it is in general not true that $R_+ \subset P_+$. Let $\alpha_1, \ldots, \alpha_d$ be the simple roots in R_+. These uniquely determine $\lambda_1, \ldots, \lambda_d \in V$ such that

$$\frac{2\langle \lambda_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{i,j} \qquad (i, j = 1, \ldots, d). \tag{3.2}$$

Then $\lambda_1, \ldots, \lambda_d$ are linear independent elements of P_+ (the so-called *fundamental dominant weights*), P is the $\mathbb{Z}$-span of $\lambda_1, \ldots, \lambda_d$, and P_+ is the $\mathbb{Z}_+$-span of $\lambda_1, \ldots, \lambda_d$.

It is easily seen that $w(P) = P$ for $w \in W$, so the Weyl group acts on P. Moreover, it can be shown that each Weyl group orbit in P has a one-point intersection with P_+:

$$\forall \lambda \in P \quad \text{Card}(W\lambda \cap P_+) = 1.$$

Thus the dominant weights can be used as a set of representatives for the W-orbits in P.

We introduce a partial ordering $\prec$ on P which is induced by the root system: for $\lambda, \mu \in P$ we say that

$$\mu \prec \lambda \quad \text{iff} \quad \lambda - \mu = \sum_{\alpha \in R_+} m_\alpha \alpha \text{ for certain nonnegative integers } m_\alpha. \tag{3.3}$$

For root system C_2 the above concepts are illustrated in Fig. 4.

3.4. Trigonometric polynomials associated with R. Let P be the weight lattice of a root system R in V. For $\lambda \in P$ define the function e^λ on V by

$$e^\lambda(x) := e^{i\langle \lambda, x \rangle} \quad (x \in V).$$

Note that $e^\lambda\, e^\mu = e^{\lambda+\mu}$. Thus the space

$$\mathcal{A} := \mathrm{Span}\{e^\lambda \mid \lambda \in P\}$$

is an algebra: the *algebra of trigonometric polynomials on V* (with respect to R). For a function f on V write $(wf)(x) := f(w^{-1}x)$ ($w \in W$, $x \in V$). Then $we^\lambda = e^{w\lambda}$ ($w \in W$, $\lambda \in P$). Put

$$m_\lambda := \sum_{\mu \in W\lambda} e^\mu \quad (\lambda \in P_+).$$

Then the functions m_λ are W-invariant and they form a basis of the space $\mathcal{A}^W$ of W-invariant elements in $\mathcal{A}$.

Let the *dual root lattice $Q^{\vee}$* be defined by

$$Q^{\vee} := \{\lambda \in V \mid \langle \lambda, \mu \rangle \in \mathbb{Z} \quad \text{for all } \mu \in P\}.$$

This lattice gives rise to a torus

$$T := V/(2\pi Q^{\vee}).$$

Let $x \mapsto \dot{x}$ be the natural mapping of V onto T. Then each function f in $\mathcal{A}$ actually lives on T: $f(x) = \widetilde{f}(\dot{x})$ for a suitable function $\widetilde{f}$ on T.

In the example A_1 we have $P = \mathbb{Z}$, $P_+ = \{0, 1, 2, \ldots\}$, the algebra $\mathcal{A}$ is spanned by the functions $x \mapsto e^{inx}$ ($n \in \mathbb{Z}$) and the subalgebra $\mathcal{A}^W$ by the functions 1 and $x \mapsto 2\cos(nx)$ ($n = 1, 2, \ldots$). The torus T equals $\mathbb{R}/(2\pi\mathbb{Z})$.

4. Jacobi polynomials associated with R

4.1. Definition of Jacobi polynomials for R. Let R be a root system in V and let $k\colon R \to [0, \infty)$ be a W-invariant multiplicity function as before. Define a weight function δ_k on T by

$$\delta_k(x) := \prod_{\alpha \in R_+} |2\sin(\langle \alpha, x \rangle)|^{2k_\alpha} = \prod_{\alpha \in \mathbb{R}} |2\sin(\tfrac{1}{2}\langle \alpha, x \rangle)|^{k_\alpha}. \tag{4.1}$$

Hence δ_k is nonnegative, W-invariant, and invariant under translations by elements of $Q^{\vee}$. Define an inner product on the linear space $\mathcal{A}$ by

$$\langle f, g \rangle_k := \int_T f(x)\, \overline{g(x)}\, \delta_k(x)\, d\dot{x} \quad (f, g \in \mathcal{A}). \tag{4.2}$$

Here $d\dot{x}$ denotes the measure on T which is induced from Lebesgue measure on V and which is normalized such that $\int_T d\dot{x} = 1$.

The *Jacobi polynomial $P_\lambda^{(k)}$* (cf. [14]) of "degree" $\lambda \in P_+$ and of "order" k is an element of $\mathcal{A}^W$ of the form

$$P_\lambda^{(k)} = \sum_{\substack{\mu \in P_+ \\ \mu \prec \lambda}} c_{\lambda,\mu}\, m_\mu$$

such that $c_{\lambda,\lambda} = 1$ and

$$\langle P_\lambda^{(k)}, m_\mu \rangle_k = 0 \quad \text{if } \mu \in P_+ \text{ and } \mu \nleq \lambda. \tag{4.3}$$

Instead of (4.3) we can equivalently require that $P_\lambda^{(k)}$ satisfies the second order differential equation

$$\left(\Delta + \sum_{\alpha \in R_+} k_\alpha \cot(\tfrac{1}{2}\langle \alpha, x \rangle) \partial_\alpha\right) P_\lambda^{(k)}(x) = -\langle \lambda, \lambda + \textstyle\sum_{\alpha \in R_+} k_\alpha \alpha \rangle P_\lambda^{(k)}(x). \tag{4.4}$$

In the example A_1 we get $k \in [0, \infty)$, $\delta_k(x) = |2\sin x|^{2k}$ $(x \in R)$, and

$$P_n^{(k)}(x) = \sum_{\substack{m=-n \\ n-m \text{ even}}}^{n} c_{n,m} e^{imx} \quad (n = 0, 1, 2, \ldots) \tag{4.5}$$

with $c_{n,m} = c_{n,-m}$ and $c_{n,n} = 1$. Equation (4.3) becomes

$$\frac{1}{2\pi} \int_0^{2\pi} P_n^{(k)}(x) \cos((n-2m)x) |\sin x|^{2k} \, dx = 0 \quad (m = 1, 2, \ldots, [\tfrac{1}{2}n])$$

and equation (4.4) takes the form

$$\left(\frac{d^2}{dx^2} + 2k \cot x \frac{d}{dx}\right) P_n^{(k)}(x) = -n(n+2k) P_n^{(k)}(x).$$

We see that $P_n^{(k)}(x) = \text{const.} \, C_n^k(\cos x)$, where C_n^k is the ultraspherical polynomial of §2.2. The case of the (non-reduced) root system BC_1 would have given us, more generally, the classical one-variable Jacobi polynomials.

4.2. Three problems and their solutions. As soon as the above definition of Jacobi polynomials associated with R is given, three highly nontrivial questions can naturally be posed:

1. It follows immediately from the definition that the orthogonality

$$\langle P_\lambda^{(k)}, P_\mu^{(k)} \rangle_k = 0 \tag{4.6}$$

   holds if $\mu \nleq \lambda$ or $\lambda \nleq \mu$. What about (4.6) if λ and μ are not related in the partial ordering?

2. Prove the existence of a commutative algebra of differential operators with d algebraically independent generators, such that the operators in this algebra have the $P_\lambda^{(k)}$ $(\lambda \in P_+)$ as joint eigenfunctions. (Note that the operator in (4.4) can be taken as one of the generators.)

3. Give an explicit expression for $\langle P_\lambda^{(k)}, P_\lambda^{(k)} \rangle_k$, or rather for its two factors

$$\frac{\langle P_\lambda^{(k)}, P_\lambda^{(k)} \rangle_k}{\langle P_0^{(k)}, P_0^{(k)} \rangle_k} \quad \text{and} \quad \int_T \delta_k(x)\, d\dot{x}. \tag{4.7}$$

In the past few years all these questions have been answered in the positive sense. Let me give some indications.

- If problem 2 can be solved then the answer to 1 follows readily, cf. [14]. Indeed, we need sufficiently many differential operators having the $P_\lambda^{(k)}$ as eigenfunctions such that the joint eigenvalues, in their dependence on λ, separate the points of P_+.

- For certain special choices of k the functions $P_\lambda^{(k)}$, renormalized such that $P_\lambda^{(k)}(0) = 1$, have an interpretation as spherical functions on compact symmetric spaces G/K, cf. [17]. (For instance, in case A_1 the ultraspherical polynomial $C_n^{\frac{1}{2}m-1}$ can be interpreted as spherical function on the $(m-1)$-dimensional sphere $SO(m)/SO(m-1)$.) Then problems 1, 2 and the first half of problem 3 can be solved by using the group theoretic interpretation. The orthogonality (4.6) for general λ, μ follows by Schur's orthogonality relations for matrix elements of irreducible unitary representations of G. The first expression in (4.7) was explicitly computed by Vretare [42] in terms of Harish-Chandra's c-function related to the spherical functions on the corresponding non-compact symmetric space. The algebra of differential operators in problem 2 can be obtained by taking the radial parts of the G-invariant differential operators on G/K.

- For the classical root systems question 2 could be answered in a positive way by giving explicit expressions for generators of the algebra, see [21] for BC_2 and A_2, and Olshanetsky & Perelomov [34], Sekiguchi [39] and Debiard [8] for the higher rank cases.

- Heckman and Opdam [14] have given positive answers to 2, and hence to 1, by use of deep transcendental arguments. This also solved part of Problem 3 (the first expression in (4.7)). In 1982 Macdonald [28] had already given conjectures for the explicit evaluation of the second expression in (4.7), which could be proved in a number of special cases.

- Problem 3 for general λ was solved by Opdam [36] by using so-called *shift operators* [35]. The most simple example, for case A_1, of such operators is the following pair of differential recurrence relations for Gegenbauer polynomials:

$$\frac{d}{dx} C_n^k(x) = \text{const.}\, C_{n-1}^{k+1}(x),$$

$$\left((1-x^2)^{-k+\frac{1}{2}} \frac{d}{dx} \circ (1-x^2)^{k+\frac{1}{2}} \right) C_{n-1}^{k+1}(x) = \text{const.}\, C_n^k(x).$$

By use of these two formulas we can write $\int_{-1}^1 (C_n^k(x))^2 (1-x^2)^{k-\frac{1}{2}}\, dx$ as an explicit constant times $\int_{-1}^1 (C_{n-1}^{k+1}(x))^2 (1-x^2)^{k+\frac{1}{2}}\, dx$. Opdam's shift operators in general have a similar structure of lowering λ and raising k, or conversely. The case of root system BC_2 was already considered in [21], [40].

4.3. Dunkl type operators. Some years after Heckman first solved the problems 1 and 2 of the previous subsection he discovered a dramatical simplification [15] for proving these results. For a given root system R in V and a given multiplicity function k he wrote down a trigonometric variant of the Dunkl operators (3.1) for $\xi \in V$:

$$(D_\xi^{(k)}f)(x) := (\partial_\xi f)(x) + \tfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \cot(\tfrac{1}{2}\langle \alpha, x \rangle)(f(x) - f(s_\alpha x))$$

$$(x \in V,\ f \in C^\infty(V)). \qquad (4.8)$$

Now the operators $D_\xi^{(k)}$ will no longer commute, in general. However, Heckman showed that the operators $\sum_{\eta \in W\xi}(D_\eta^{(k)})^j$ ($\xi \in V,\ j = 0,1,2,\ldots$), when restricted to the W-invariant C^∞ functions on V, coincide with differential operators which commute with each other and form a commutative algebra. This is the algebra looked for in Problem 2 of the previous subsection. The Jacobi polynomials $P_\lambda^{(k)}$ are the joint eigenfunctions of the operators in this algebra. By this approach, Heckman also obtained a quick existence proof for Opdam's shift operators.

Next Cherednik [4] (see also Opdam [37, Definition 2.2]) made a slight but significant variation in Heckman's Dunkl type operators (4.8). He put

$$(\widetilde{D}_\xi^{(k)}f)(x) := (\partial_\xi f)(x) + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-\alpha}(x)}(f(x) - f(s_\alpha x))$$

$$- \tfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle f(x).$$

Cherednik's operators have the nice property that they mutually commute, without the need of first restricting to W-invariant functions. On the other hand, they do not share the property $w\, D_\xi^{(k)}\, w^{-1} = D_{w\xi}^{(k)}$ of Heckman's operators. This has to be replaced by a relation coming from graded Hecke algebras. Anyhow, by means of Cherednik's operators one can draw the same conclusions as by Heckman's operators, and in a similar way.

5. Macdonald polynomials associated with R

5.1. Definition of Macdonald polynomials. The reference for this subsection is Macdonald [29], see also the summary in [31]. Let $0 < q < 1$. Assume that R is an irreducible and reduced root system. We keep the assumptions of §4.1 except that we replace the weight function δ_k in (4.1) by $\delta_{k,q}(x) := \delta_{k,q}^+(x)\,\delta_{k,q}^+(-x)$, where

$$\delta_{k,q}^+(x) := \prod_{\alpha \in R_+} \frac{(e^{i\langle \alpha, x \rangle}; q)_\infty}{(q^{k_\alpha} e^{i\langle \alpha, x \rangle}; q)_\infty}.$$

Then

$$\delta_{k,q}(x) = \prod_{\alpha \in R} \frac{(e^{i\langle \alpha, x\rangle}; q)_\infty}{(q^{k_\alpha} e^{i\langle \alpha, x\rangle}; q)_\infty} = \prod_{\alpha \in R_+} \left| \frac{(e^{i\langle \alpha, x\rangle}; q)_\infty}{(q^{k_\alpha} e^{i\langle \alpha, x\rangle}; q)_\infty} \right|^2.$$

Hence $\delta_{k,q}$ is nonnegative, W-invariant, and invariant under translations by elements of $Q^\vee$. Then the *Macdonald polynomials* $P_\lambda^{(k,q)}$ are defined just as the Jacobi polynomials $P_\lambda^{(k)}$, but with the inner product in (4.2) replaced by

$$\langle f, g\rangle_{k,q} := \int_T f(x)\, \overline{g(x)}\, \delta_{k,q}(x)\, d\dot{x} \quad (f, g \in \mathcal{A}). \tag{5.1}$$

Thus the Macdonald polynomial $P_\lambda^{(k,q)}$ ($\lambda \in P_+$) is an element of $\mathcal{A}^W$ of the form

$$P_\lambda^{(k,q)} = \sum_{\substack{\mu \in P_+ \\ \mu \prec \lambda}} c_{\lambda,\mu}\, m_\mu$$

such that $c_{\lambda,\lambda} = 1$ and

$$\langle P_\lambda^{(k,q)}, m_\mu\rangle_{k,q} = 0 \quad \text{if } \mu \in P_+ \text{ and } \mu \nsucceq \lambda.$$

Note that $\lim_{q\uparrow 1} \delta_{k,q}(x) = \delta_k(x)$ by (2.6). Accordingly we have $\lim_{q\uparrow 1} P_\lambda^{k,q}(x) = P_\lambda^k(x)$, see [29, §11].

Just as in the case $q = 1$ we immediately see from the definition of Macdonald polynomials that the orthogonality

$$\langle P_\lambda^{(k)}, P_\mu^{(k)}\rangle_{k,q} = 0 \tag{5.2}$$

holds if $\mu \nsucceq \lambda$ or $\lambda \nsucceq \mu$.

In the case of root system A_1 we get, for $k \in [0, \infty)$ and $n = 0, 1, 2, \ldots$ that $P_n^{k,q}$ is of the form as given in the right-hand side of (4.5) and satisfying

$$\frac{1}{2\pi} \int_0^{2\pi} P_n^{(k)}(x) \cos((n - 2m)x) \left| \frac{(e^{2ix}; q)_\infty}{(q^k e^{2ix}; q)_\infty} \right|^2 dx = 0 \quad (m = 1, 2, \ldots, [\tfrac{1}{2}n]).$$

Hence $P_n^{k,q}(x) = \text{const.}\, C_n^{k,q}(\cos x)$, where $C_n^{k,q}$ is the q-ultraspherical polynomial of §2.3.

5.2. Minuscule and quasi-minuscule elements. Relative to the partial ordering on R (see (3.3)) there is a unique maximal element γ, which is of course a positive root (the so-called *maximal root*). Its decomposition in simple roots

$$\gamma = \sum_{i=1}^d k_i \alpha_i \tag{5.3}$$

has the property that $k_i > 0$ for all i (see [18, §10.4, Lemma A]).

Let the *dual weight lattice* $P^\vee$ be the weight lattice corresponding to the dual root system $R^\vee$. It has fundamental dominant weights $2\lambda_i/\langle\alpha_i,\alpha_i\rangle$ $(i=1,\ldots,d)$, where $\lambda_1,\ldots,\lambda_d$ are the fundamental dominant weights for P.

An element $\sigma \in V\backslash\{0\}$ is called *minuscule* (more precisely a minuscule fundamental weight for the root system $R^\vee$) if $\langle\sigma,\alpha\rangle = 0$ or 1 for all $\alpha \in R_+$. It can be shown (see [3, Chap. 6, Exercises §1, no. 24])) that the minuscule elements are the fundamental dominant weights $2\lambda_i/\langle\alpha_i,\alpha_i\rangle$ for $P^\vee$ precisely for those $i \in \{1,\ldots,d\}$ for which $k_i = 1$ in (5.3). The coefficients k_i of (5.3) are explicitly given in [18, §12.2, Table 2]. From this Table it can be concluded that the number of minuscule elements is n for root system A_n, 2 for root systems D_n, E_6, 1 for root systems B_n, C_n, E_7, and 0 for root systems E_8, F_4, G_2.

In these last three cases there still exists a *quasi-minuscule element*, i.e.. $\sigma \in V$ such that $\langle\sigma,\alpha\rangle = 0$, 1 or 2 for all $\alpha \in R_+$, with the values 1 and 2 both being attained. A very special choice of such a quasi-minuscule element can then be made such that the statements below will be true.

5.3. Macdonald's q-difference operators.

Macdonald gives some explicit q-difference operators of which the $P_\lambda^{(k,q)}$ are eigenfunctions. In order to state his results we need a few preparations.

Let $v \in V$. For f a holomorphic function on a suitable neighbourhood of V in $V+iV$ define a function $T_v f$ by

$$(T_v f)(x) := f(x - i(\log q)v) \qquad (x \in V).$$

Hence

$$T_v\, e^\lambda = q^{\langle v,\lambda\rangle}\, e^\lambda \qquad (\lambda \in P).$$

Let σ be a (quasi-)minuscule element as above and let W_σ be the stabilizer of σ in W. Put

$$E_\sigma^{(k,q)} f := |W_\sigma|^{-1} \sum_{w \in W} w\left(\frac{T_\sigma \delta_{k,q}^+}{\delta_{k,q}^+}\, T_\sigma f\right).$$

Then it can be shown in the minuscule case that $E_s i$ maps A^W into itself and that the operator E_σ is self-adjoint with respect to the inner product (5.1). In order to handle the quasi-minuscule case as well put

$$D_\sigma^{(k,q)} f := |W_\sigma|^{-1} \sum_{w \in W} w\left(\frac{T_\sigma \delta_{k,q}^+}{\delta_{k,q}^+}\,(T_\sigma f - f)\right).$$

If σ is minuscule this is a trivial change, since then

$$D_\sigma^{(k,q)} = E_\sigma^{(k,q)} - E_\sigma^{(k,q)}(1).$$

This will not be true in the quasi-minuscule case, but then it can still be proved that $D_s i$ maps A^W into itself and that the operator D_σ is self-adjoint with respect to the inner product (5.1).

Although these operators, except for root system A_n (where they were independently found by Ruijsenaars [38]) do not yet give a full commutative algebra of operators having the $P_\lambda^{(k,q)}$ as joint eigenfunctions, the additional parameter q gives enough freedom such that already the eigenvalue of one such operator separates the elements of P_+ for generic q, by which a positive answer to question 1 in §4.2 can be given for the case of Macdonald polynomials. Taking limits for $q \uparrow 1$ then yields the same positive answer for the case of Jacobi polynomials. This is an alternative to Heckman's approach via Problem 2. Macdonald also gives conjectured explicit expressions for the squared norms $\langle P_\lambda^{(k,q)}, P_\lambda^{(k,q)} \rangle_{k,q}$.

5.4. Askey-Wilson polynomials for root system BC_n. The author [23] introduced for the non-reduced root system BC_n a class of polynomials having two more parameters than Macdonald's class for BC_n. This extended class reduces for $n = 1$ to the Askey-Wilson polynomials [2]. In [23] only one explicit q-difference operator was given having the BC_n-polynomials as eigenfunctions, but this was sufficient for establishing orthogonality. Later, van Diejen [9] gave explicit expressions for the generators af a full commutative algebra of operators having the BC_n polynomials as joint eigenfunctions.

5.5. Cherednik's approach to Macdonald polynomials. Cherednik [5], [6], [7] succeeded to give positive answers to questions 2 and 3 in §4.2. In the context of certain representations of affine Hecke algebras he could realize a commutative algebra of operators which have the Macdonald polynomials as joint eigenfunctions. In the same context he could realize q-analogues of Opdam's shift operators and next, by the same technique as in Opdam, prove Macdonald's conjectures in the q-case.

It is beyond the scope of this short survey to explain Cherednik's approach in any detail. See the expository papers by I. G. Macdonald [32] and by Kirillov [20]. Let me here only give a few indications. Just as a Hecke algebra is a deformation of the group algebra of a Weyl group, an affine Hecke algebra (cf. [26]) deforms the group algebra of an affine Weyl group. If R is an irreducible root system in V with Weyl group W then the (extended) *affine Weyl group* is the semidirect product $\widetilde{W} := W \ltimes P^\vee$, where the *dual weight lattice* $P^\vee$ is defined as $P^\vee := \{\lambda \in V \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$, an abelian group under addition. Then $\widetilde{W}$ acts as a group of motions on V, with $P^\vee$ acting as a group of translations. The group $\widetilde{W}$ also acts on $\mathcal{A}$, with W acting as before and with the action of $P^\vee$ still depending on a parameter q.

The *affine Hecke algebra* H can be defined in terms of generators and relations which still depend on the values of a W-invariant function $\alpha \mapsto t_\alpha$ on R. Corresponding to a choice of R_+ we can define $P_+^\vee$. The embedding of $P_+^\vee$ in H then generates a commutative subalgebra $\mathcal{Y}$ of H.

For given q we can use the action of $\widetilde{W}$ on $\mathcal{A}$ in order to define an action of H on $\mathcal{A}$, by specifying the action for a set of generators of H (Demazure operators). This action depends on q and the t_α. Put $t_\alpha = q^{-k_\alpha/2}$, where $k: R \to [0, \infty)$ is a multiplicity function. Then Cherednik proves that the Macdonald polynomials $P_\lambda^{(k,q)}$ are the joint eigenfunctions of the W-invariant elements in the commutative algebra $\mathcal{Y}$. This answers question 2 in §4.2.

A next problem concerns the extension of Cherednik's approach to the BC_n polynomials. As van Diejen [9] already answered question 2 in §4.2 in a positive way for this case, it would be nice to complement van Diejen's constructive approach with the deep conceptual approach via affine Hecke algebras. In fact, Macdonald [32], at the end of his Bourbaki lecture, claims that he is able to do this.

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