Comment on the book "Chinese mathematics. A concise history" by Y. Li and S. R. Du, Oxford University Press, 1987

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On p.157, in the last line r should be replaced by n, so the whole formula due to Zhū Shìjié on the last two lines of p.157 should read:

$$\sum_{r=1}^{n} \frac{1}{p!}\, r(r+1)(r+2)\ldots(r+p-1) = \frac{1}{(p+1)!}\, n(n+1)(n+2)\ldots(n+p-1)(n+p).$$

This can be rewritten as

$${}_2F_1\left(\begin{matrix}-n+1, p+1\\ -n+1\end{matrix}; 1\right) = \frac{(p+2)_{n-1}}{(n-1)!}\,,$$

which is the special indefinite summation case of the (Chu-)Vandermonde formula (i.e., the summand is independent of the upper limit of the sum).

On l.−8 of p.158 the sum will run from $r = 1$ to n, so the whole formula due to Zhū Shìjié would read:

$$\sum_{r=1}^{n} r\, \frac{1}{p!}\, r(r+1)(r+2)\ldots(r+p-1) = \frac{1}{(p+2)!}\, n(n+1)(n+2)\ldots(n+p)\,\big((p+1)n+1\big).$$

This can be rewritten as

$$\begin{aligned}{}_2F_1\left(\begin{matrix}-n+1, p+1\\ -n+1\end{matrix}; 1\right) + (p+1)\; {}_2F_1\left(\begin{matrix}-n+2, p+2\\ -n+2\end{matrix}; 1\right) &= \frac{(p+2)_{n-1}}{(n-1)!} + (p+1)\,\frac{(p+3)_{n-2}}{(n-2)!}\\ &= \frac{(p+3)_{n-2}}{(n-1)!}\big((p+1)n+1\big).\end{aligned}$$

Li & Du refer for further details to the book

J. Hoe, *Les systèmes d'équations polynômes dans le Siyuan yujian (1303) par Chu Shih-chieh*, Mémoires de l'Institut des Hautes Études Chinoises, Vol. VI,
Institut des Hautes Études Chinoises, Collège de France, Paris, 1977.

There see in particular pp. 300–321 for the section on "Séries et interpolation". Not much more than already summarized in Li & Du can be found there. Two very simple instances of the (Chu-)Vandermonde sum beyond the case of indefinite summation occur there on p.314 and p.315:

$$\sum_{r=1}^{n} r(a+(n-r)b) = \frac{n(n+1)(bn+(3a-b))}{3!} \quad \text{(Hoe erroneously has } 3a+b \text{ instead of } 3a-b\text{)},$$

$$\sum_{r=1}^{n} \frac{r(r+1)}{2}\,(a+(n-r)b) = \frac{n(n+1)(n+2)(bn+(4a-b))}{4!}\,.$$

On the other hand, in the book

J. Needham, *Science and civilisation in China*, Vol. 3, Cambridge University Pres, 1959

on p.139 the following formula is ascribed to Chu:

$$\sum_{r=1}^{n} \frac{(r)_p}{p!} \frac{(n+1-r)_q}{q!} = \sum_{r=1}^{n} \frac{(r)_{p+q}}{(p+q)!}\,, \tag{1}$$

while we already saw (and is also mentioned by Needham) that Chu could evaluate the sum on the right. Thus, as observed on pp. 59–60 in the book

R. Askey, *Orthogonal polynomials and special functions, SIAM*, 1975,

(1) essentially yields the (Chu-)Vandermonde formula. It remains puzzling that (1) is missing in Li & Du and in Hoe. Can someone point out where this is given by Chu? How would he have obtained it? By repeated summation by parts? Anyhow, he could work with finite differences, since he had some form of Newton series, see Li & Du, p.160 and Hoe, p.308.