Charting the Askey and q-Askey schemes

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1. The classical orthogonal polynomials scheme

Limit from Jacobi to Laguerre

Jacobi polynomials:

$$\frac{P_n^{(\alpha,\beta)}(1-2x)}{P_n^{(\alpha,\beta)}(1)} = {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; x\right),$$

weight function on $[0,1]$: $x^\alpha(1-x)^\beta$.

Laguerre polynomials:

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} = {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}; x\right),$$

weight function on $[0,\infty)$: $x^\alpha e^{-x}$.

Limit from Jacobi to Laguerre

$$\frac{P_n^{(\alpha,\beta)}(1-2\beta^{-1}x)}{P_n^{(\alpha,\beta)}(1)} = {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \beta^{-1}x\right)$$

$$\Big\downarrow \beta \to \infty$$

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} = {}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}; x\right).$$

Or the limit for the weight functions:

$$x^\alpha(1-\beta^{-1}x)^\beta \xrightarrow{\beta\to\infty} x^\alpha e^{-x}.$$

Limits between Jacobi, Laguerre and Hermite

Orthogonal Polynomials (OPs) $p_n(x)$, monic:
$p_n(x) = x^n +$ terms of lower degree.

Classical OPs:

- Jacobi: $p_n^{(\alpha,\beta)}(x)$, $\quad w(x) = (1-x)^\alpha\,(1+x)^\beta$ on $[-1,1]$;
- Laguerre: $\ell_n^{(\alpha)}(x)$, $\quad w(x) = e^{-x}\,x^\alpha$ on $[0,\infty)$;
- Hermite: $h_n(x)$, $\quad w(x) = e^{-x^2}$ on $(-\infty,\infty)$.

$$\alpha^{n/2} p_n^{(\alpha,\alpha)}(x/\alpha^{1/2}) \to h_n(x), \quad (1-x^2/\alpha)^\alpha \to e^{-x^2},\ \alpha\to\infty\,;$$

$$(-\beta/2)^n\, p_n^{(\alpha,\beta)}(1-2x/\beta) \to \ell_n^{(\alpha)}(x), \quad x^\alpha(1-x/\beta)^\beta \to x^\alpha e^{-x},\ \beta\to\infty\,;$$

$$(2\alpha)^{-n/2}\,\ell_n^{(\alpha)}((2\alpha)^{1/2}x+\alpha) \to h_n(x), \quad (1+(2/\alpha)^{1/2}x)^\alpha e^{-(2\alpha)^{1/2}x} \to e^{-x^2},\quad \alpha\to\infty\,.$$

Uniform limits between classical OPs

$p_n(x) = \rho^n \, p_n^{(\alpha,\beta)}(\rho^{-1}x - \sigma)$ (rescaled monic Jacobi) with

$$\rho = \frac{(\alpha+\beta)^{3/2}}{\alpha^{1/2}\,\beta^{1/2}}\,,\quad \sigma = \frac{\alpha-\beta}{\alpha+\beta}\,.$$

satisfies three-term recurrence relation

$$x\,p_n(x) = p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x)$$

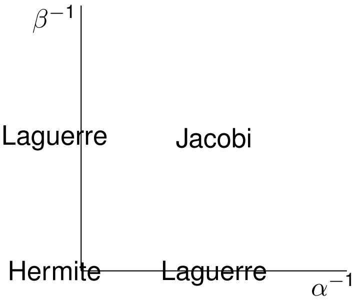
with $C_n > 0$ and $B_n \in \mathbb{R}$ given by

$$C_n = \frac{4n\,(1+n/\alpha)\,(1+n/\beta)\,(1+n/(\alpha+\beta))}{(1+(2n-1)/(\alpha+\beta))\,(1+2n/(\alpha+\beta))^2\,(1+(2n+1)/(\alpha+\beta))}\,,$$

$$B_n = (\beta^{-1/2}-\alpha^{-1/2})(\beta^{-1/2}+\alpha^{-1/2})^{1/2} \times \frac{4n+2+4n\,(n+1)/(\alpha+\beta)}{(1+2n/(\alpha+\beta))\,(1+(2n+2)/(\alpha+\beta))}\,.$$

B_n and C_n are continuous in α^{-1} and $\beta^{-1} \geq 0$.
For $\alpha^{-1} = 0$ or $\beta^{-1} = 0$ Laguerre. For $\alpha^{-1} = \beta^{-1} = 0$ Hermite.

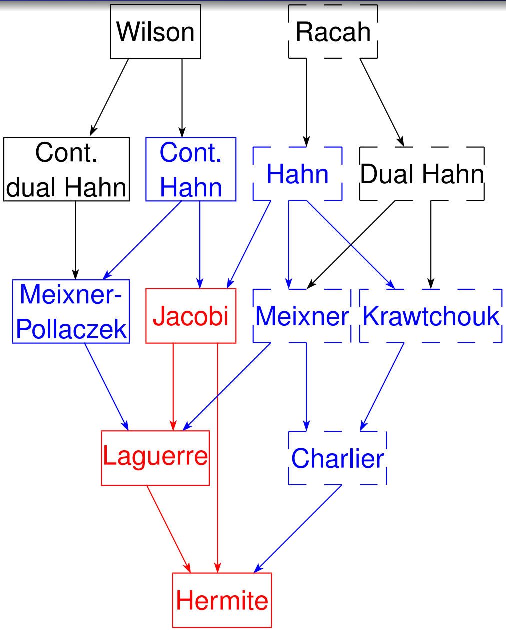
The $(\alpha^{-1}, \beta^{-1})$-parameter quarter plane

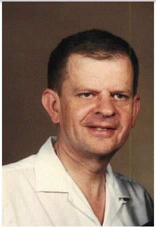


2. Uniform parameters for the Askey scheme

T. H. Koornwinder, The Askey scheme as a four-manifold with corners, *Ramanujan J.* **20** (2009), 409–439; arXiv:0909.2822.

Askey scheme





Dick Askey
1933–2019

discrete OP

quadratic lattice

Hahn class

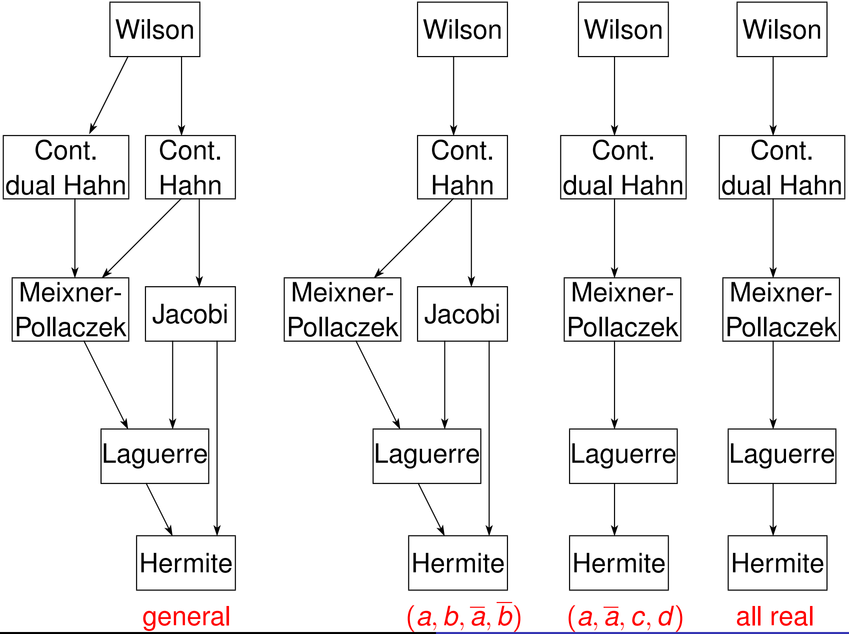
classical OP

Monic Wilson polynomials

$$w_n(x^2; a, b, c, d) := \frac{(-1)^n (a+b)_n (a+c)_n (a+d)_n}{(n+a+b+c+d-1)_n} \times {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1\right)$$

Either two pairs of complex conjugate parameters or one pair of complex conjugate parameters and two real parameters or four real parameters.

Wilson scheme and its subschemes



first Wilson subscheme

Rescaled monic Wilson polynomial in terms of new parameters a_1, a_2, a_3, a_4:

$$p_n(x; a_1, a_2, a_3, a_4) = \rho^n \, w_n(\rho^{-1} x - \sigma; a, b, \overline{a}, \overline{b}),$$

where

$$a = a_1^{-1} - \frac{1 - a_1^{1/2}\, a_2\, a_4}{2\, a_1^{3/2}\, a_2^2\, a_3\, a_4}\, \mathrm{i}, \quad b = a_1^{-1}\, a_2^{-1} - \frac{1 + a_1^{1/2}\, a_2\, a_4}{2\, a_1^{3/2}\, a_2^2\, a_3\, a_4}\, \mathrm{i},$$

$$\rho = 2^{3/2}\, a_1^2\, a_2^2\, a_3^2\, a_4,$$

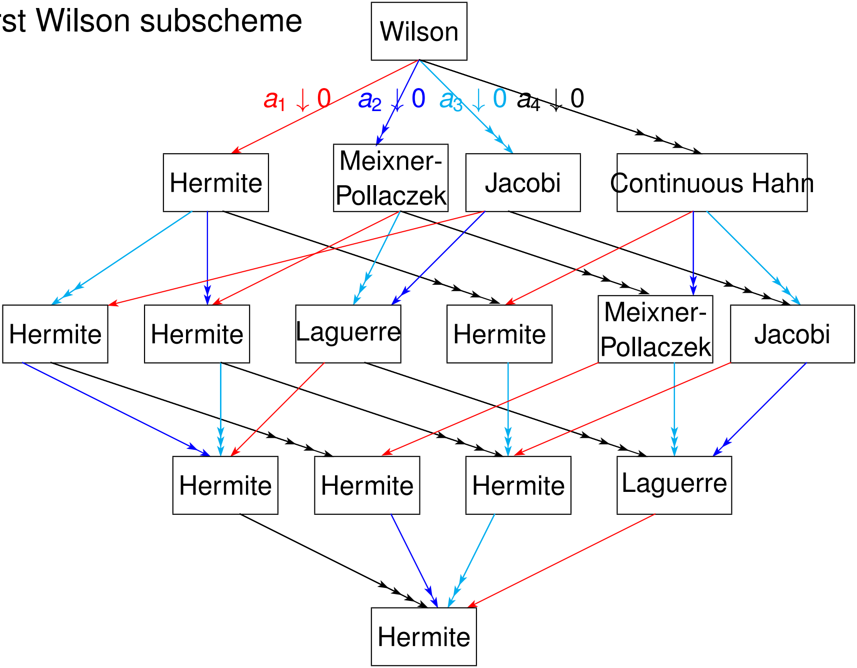
$$\sigma = -\frac{1}{4\, a_1^3\, a_2^4\, a_3^2\, a_4^2} + \frac{1 - a_2}{2\, a_1^{5/2}\, a_2^3\, (1 + a_2 - a_1\, a_2)\, a_3^2\, a_4}.$$

$p_n(x; a_1, a_2, a_3, a_4)$ satifies three-term recurrence relation

$$x\, p_n(x) = p_{n+1}(x) + B_n\, p_n(x) + C_n\, p_{n-1}(x)$$

with $C_n > 0$ and $B_n \in \mathbb{R}$ depending continuously on $a_1, a_2, a_3, a_4 \geq 0$.

first Wilson subscheme



3. Askey–Wilson polynomials

Askey–Wilson polynomials

***q*-Pochhammer symbol**:

$$(a;q)_k = (1-a)(1-aq)\dots(1-aq^{k-1}),$$

$$(a_1,\dots,a_r;q)_k = (a_1;q)_k \dots (a_r;q)_k\,.$$

Askey–Wilson polynomial, monic in $z+z^{-1}$:

$$\frac{a^n\,(q^{n-1}abcd;q)_n}{(ab,ac,ad;q)_n}\,p_n^{\mathrm{monic}}(z+z^{-1};a,b,c,d;q)$$
$$= {}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right)$$
$$=\sum_{k=0}^{n}\frac{(q^{-n},q^{n-1}abcd,az,az^{-1};q)_k}{(ab,ac,ad;q)_k\,(q;q)_k}\,q^k$$
$$=\sum_{k=0}^{n}\frac{q^k}{(ab,ac,ad;q)_k\,(q;q)_k}\,(q^{-n},q^{n-1}abcd;q)_k\,(az,az^{-1};q)_k\,.$$

Askey–Wilson polynomials

$$p_n^{\mathrm{monic}}(z+z^{-1};a,b,c,d;q)=\frac{(ab,ac,ad;q)_n}{a^n\,(q^{n-1}abcd;q)_n}$$

$$\times\sum_{k=0}^{n}\frac{q^k}{(ab,ac,ad;q)_k\,(q;q)_k}\,(q^{-n},q^{n-1}abcd;q)_k\,(az,az^{-1};q)_k$$

$$=\left(\prod_{j=0}^{n-1}\frac{g_{j+1}}{h_n-h_j}\right)\sum_{k=0}^{n}\prod_{j=0}^{k-1}\frac{(h_n-h_j)\,(z+z^{-1}-x_j)}{g_{j+1}},$$

where

$$x_k=aq^k+a^{-1}q^{-k},\qquad h_k=abcdq^{k-1}+q^{-k},$$
$$g_k=a^{-1}q^{-2k+1}(1-abq^{k-1})(1-acq^{k-1})(1-adq^{k-1})(1-q^k)$$

Verde-Star's idea

$$p_n^{\text{monic}}(x;a,b,c,d;q) = u_n(x) = \sum_{k=0}^{n} c_{n,k}\, v_k(x),$$

where $v_k(x) = (x-x_0)(x-x_1)\ldots(x-x_{k-1})$

and $c_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{h_n - h_j}$.

Then $Lu_n = h_n\, u_n$, where the operator L is determined by $Lv_n = h_n v_n + g_n v_{n-1}$.

Also the operator X of multiplication by x is determined by $Xv_n = x_n v_n + v_{n+1}$.

$$h_k = a_0 + a_1 q^k + a_2 q^{-k}, \quad x_k = b_0 + b_1 q^k + b_2 q^{-k},$$
$$g_k = d_0 + d_1 q^k + d_2 q^{-k} + d_3 q^{2k} + d_4 q^{-2k}, \quad \sum_{i=0}^{4} d_i = 0,$$
$$d_3 = q^{-1} a_1 b_1, \quad d_4 = q a_2 b_2.$$

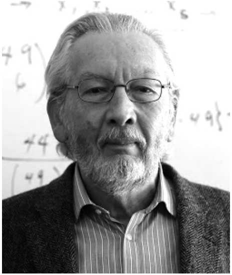
Luis Verde-Star, arXiv:2002.07932

The a_i, b_i, d_i parametrize the q-Askey scheme.

4. The (q-)Verde-Star scheme

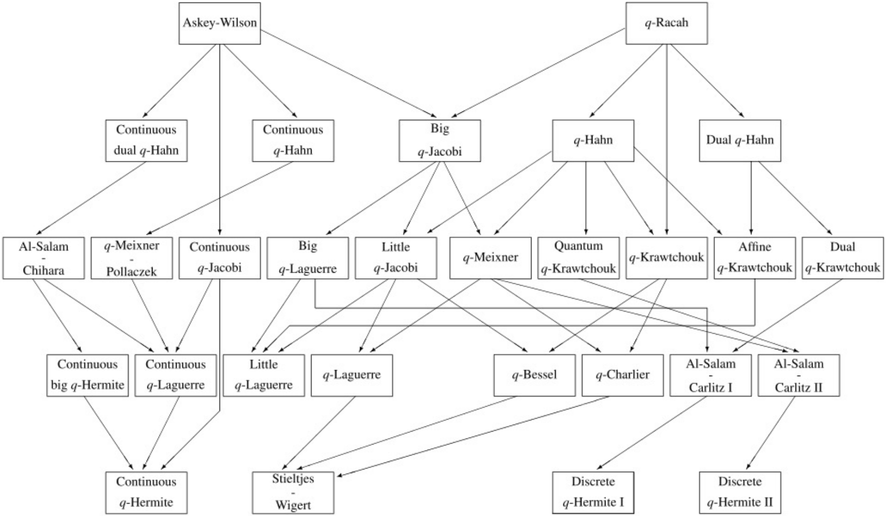
Inspired by the paper

L. Verde-Star, *A unified construction of all the hypergeometric and basic hypergeometric families of orthogonal polynomial sequences*, arXiv:2002.07932.



Luis Verde-Star

q-Askey scheme



q-Verde-Star scheme

$$u_n(x)=\sum_{k=0}^{n} c_{n,k}\,v_k(x),\quad v_k(x)=\prod_{j=0}^{k-1}(x-x_j),\ \ c_{n,k}=\prod_{j=k}^{n-1}\frac{g_{j+1}}{h_n-h_j}\,,$$

$$x_k=b_2q^{-k}+b_0+b_1q^k,\quad h_k=a_2q^{-k}+a_0+a_1q^k,$$
$$g_k=d_4q^{-2k}+d_2q^{-k}+d_0+d_1q^k+d_3q^{2k},\quad \textstyle\sum_{i=0}^4 d_i=0,$$
$$d_3=q^{-1}a_1b_1,\quad d_4=qa_2b_2.$$

Translation invariances: $x\to x+\sigma$, $x_k\to x_k+\sigma$, $h_k\to h_k+\tau$.
Dilation invariance: $u_n(x)\to\rho^{-n}u_n(\rho x)$, $v_k(x)\to\rho^{-k}u_k(\rho x)$, $x_k\to\rho x_k$, $g_k\to\rho g_k$.
Homogeneous of degree zero in h_k,g_k: $h_k\to\mu h_k$, $g_k\to\mu g_k$.

$3+3+5=11$ parameters, 3 constraints, 4 invariances:
Four essential parameters.

$q\leftrightarrow q^{-1}$ *exchange*: $a_1\leftrightarrow a_2,\ b_1\leftrightarrow b_2,\ d_1\leftrightarrow d_2,\ d_3\leftrightarrow d_4$.

q-Verde-Star scheme

Represent

$$\begin{array}{lll} x_k = & b_2 q^{-k} + b_0 + b_1 q^k & \\ g_k = d_4 q^{-2k} + & d_2 q^{-k} + d_0 + d_1 q^k + d_3 q^{2k} & \text{by} \quad \begin{array}{ccccc} & b_2 & b_0 & b_1 & \\ d_4 & d_2 & d_0 & d_1 & d_3 \\ & a_2 & a_0 & a_1 & \end{array} \\ h_k = & a_2 q^{-k} + a_0 + a_1 q^k & \end{array}$$

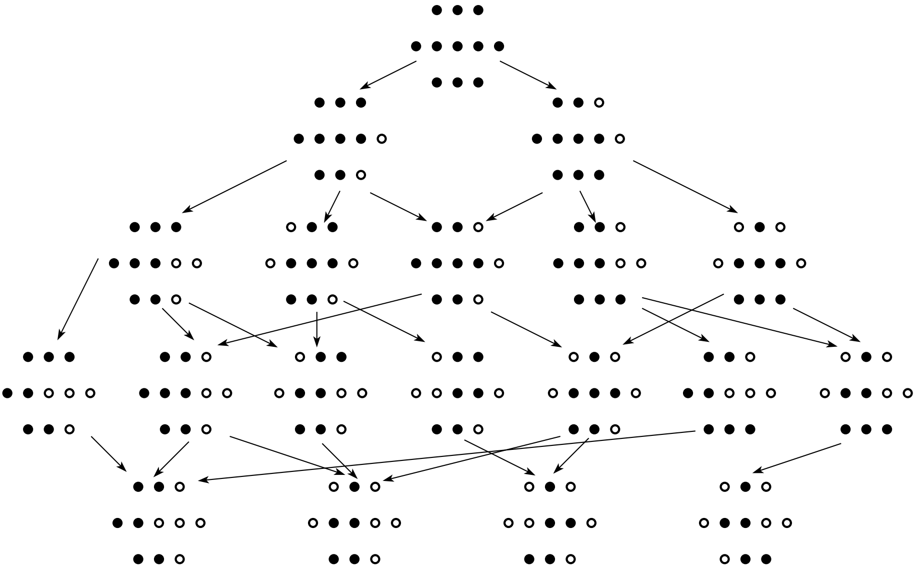
$\bullet$ denotes any parameter value and $\circ$ a zero parameter value.

So Askey–Wilson corresponds to the symbol $\begin{array}{ccccc} & \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet & \bullet \\ & \bullet & \bullet & \bullet & \end{array}$.

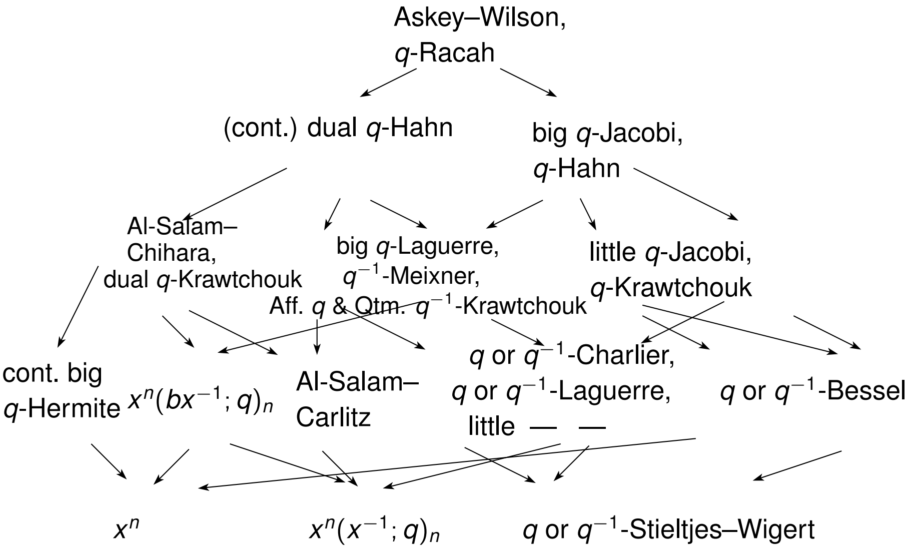
Rules:

1. If b_1 or a_1 is $\circ$ then d_3 is $\circ$; if b_2 or a_2 is $\circ$ then d_4 is $\circ$.
2. b_0 and a_0 are always $\bullet$.
3. In the second row no $\circ$ between two $\bullet$ ones.
4. In the second and third row at least two $\bullet$ ones.
5. Flipping a $\bullet$ into a $\circ$ causes an arrow between the symbols.
6. Reflection w.r.t. the central column means $q \leftrightarrow q^{-1}$.
7. Reflection w.r.t. the middle row means $x \leftrightarrow \lambda_n$ (duality).

q-Verde-Star scheme



q-Verde-Star scheme, the families



q-Verde-Star scheme. Remarks

- Not just classification of families of OPs, but in combination with families of generalized monomials in which the OPs are expanded.
- Therefore same family of OPs may occur twice in the scheme. See big q-Laguerre, little q-Jacobi, q-Bessel.
- The scheme does not consider orthogonality w.r.t. a positive measure, but it classifies families of q-hypergeometric polynomials which are eigenfunctions of an operator in the x-variable and which can be seen to satify a three-term recurrence relation.
- One position in the scheme may contain both a continuous and a discrete family.
- The scheme also contains a few degenerate cases.
- The continuous q-Hermite polynomials are missing in the scheme, because they have only an expansion in terms of $x^n, x^{n-2}, \ldots$. The discrete q-Hermite I, II polynomials are subfamilies of the Al-Salam–Carlitz I, II polynomials.

Verde-Star scheme for $q = 1$

$$u_n(x) = \sum_{k=0}^{n} c_{n,k}\, v_k(x), \quad v_k(x) = \prod_{j=0}^{k-1}(x - x_j), \ \ c_{n,k} = \prod_{j=k}^{n-1} \frac{g_{j+1}}{h_n - h_j}\,,$$

$$x_k = b_0 + b_1 k + b_2 k^2, \quad h_k = a_0 + a_1 k + a_2 k^2,$$

$$g_k = d_0 + d_1 k + d_2 k^2 + d_3 k^3 + d_4 k^4, \quad d_0 = 0,$$

$$d_4 = a_2 b_1, \quad d_3 = a_1 b_2 + a_2 b_1 - 4 a_2 b_2.$$

Obtained from rescaled x_k, h_k, g_k in the q-case and then $q \to 1$.

$$x_k = \widetilde{b}_2 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^2 + \widetilde{b}_1\, \tfrac{1-q^k}{1-q} + \widetilde{b}_0\,,$$

$$h_k = \widetilde{a}_2 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^2 + \widetilde{a}_1\, \tfrac{1-q^k}{1-q} + \widetilde{a}_0\,,$$

$$g_k = \widetilde{d}_4 q^{-2k} \left(\tfrac{1-q^k}{1-q}\right)^4 + \widetilde{d}_3 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^3 + \widetilde{d}_2 q^{-k} \left(\tfrac{1-q^k}{1-q}\right)^2 + \widetilde{d}_1\, \tfrac{1-q^k}{1-q}\,.$$

For $q = 1$ a similar but simpler scheme as in the q-case.
Hermite polynomials cannot be handled.

5. The (q-)Zhedanov scheme

Inspired by the paper

Ya. I. Granovskii, I. M. Lutzenko and A. S. Zhedanov, *Mutual integrability, quadratic algebras, and dynamical symmetry*, Ann. Physics 217 (1992), 1–20.

Alexei Zhedanov

q-Zhedanov scheme

Let K_1 and K_2 be operators acting on sequences $\{f_n\}_{n=0}^{\infty}$:
$(K_1 f)_n = h_n f_n + g_{n+1} f_{n+1}, \quad (K_2 f)_n = x_n f_n + f_{n-1}$, where
$x_k = b_2 q^{-k} + b_0 + b_1 q^k, \quad h_k = a_2 q^{-k} + a_0 + a_1 q^k,$
$g_k = d_4 q^{-2k} + d_2 q^{-k} + d_0 + d_1 q^k + d_3 q^{2k}, \quad \sum_{i=0}^4 d_i = 0,$
$d_3 = q^{-1} a_1 b_1, \quad d_4 = q a_2 b_2.$

Then

$$(q+q^{-1})K_2 K_1 K_2 - K_2^2 K_1 - K_1 K_2^2 = A_1(K_1 K_2 + K_2 K_1) + A_2 K_2^2 + C_1 K_1 + D K_2 + G_1,$$

$$(q+q^{-1})K_1 K_2 K_1 - K_1^2 K_2 - K_2 K_1^2 = A_2(K_1 K_2 + K_2 K_1) + A_1 K_1^2 + C_2 K_2 + D K_1 + G_2,$$

and the coefficients $A_1, A_2, C_1, C_2, D, G_1, G_2$ can be expressed in terms of the a_i, b_i, d_i. Scheme can be given depending on vanishing of some coefficients. Quite similar to the q-Verde-Star scheme, but not completely.

Thanks for listening.

Full moon above New Delhi, 5 November 2017