Orthogonal polynomials

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Contents

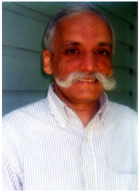
Some books on orthogonal polynomials

• G. Szegő, *Orthogonal polynomials*, Amer. Math. Soc., Fourth ed., 1975.

• T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach, 1978; reprinted, Dover, 2011.

• G. E. Andrews, R. Askey and R. Roy, *Special Functions*, Cambridge University Press, 1999.





• R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric orthogonal polynomials and their q-analogues*, Springer-Verlag, 2010; in particular Chapters 9, 14, based on the Koekoek-Swarttouw report

`http://aw.twi.tudelft.nl/~koekoek/askey/`



• *NIST Handbook of Mathematical Functions*,
Cambridge University Press, 2010;
`http://dlmf.nist.gov`;
in particular Ch. 18 on Orthogonal polynomials.

Definition of orthogonal polynomials

$\mathcal{P}$ is the space of all polynomials in one variable with real coefficients. This is a real vector space.

Assume a (positive definite) inner product $\langle f, g\rangle$ ($f, g \in \mathcal{P}$) on $\mathcal{P}$.

Orthogonalize the sequence $1, x, x^2, \ldots$ with respect to the inner product (Gram-Schmidt), resulting into $p_0, p_1, p_2, \ldots$.

So $p_0(x) = 1$ and, if $p_0(x), p_1(x), \ldots, p_{n-1}(x)$ are already produced and mutually orthogonal, then

$$p_n(x) := x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, p_k\rangle}{\langle p_k, p_k\rangle}\, p_k(x).$$

Indeed, $p_n(x)$ is a linear combination of $1, x, x^2, \ldots, x^n$, and

$$\begin{aligned}\langle p_n, p_j\rangle &= \langle x^n, p_j\rangle - \sum_{k=0}^{n-1} \frac{\langle x^n, p_k\rangle}{\langle p_k, p_k\rangle}\, \langle p_k, p_j\rangle \\ &= \langle x^n, p_j\rangle - \frac{\langle x^n, p_j\rangle}{\langle p_j, p_j\rangle}\, \langle p_j, p_j\rangle = 0 \quad (j = 0, 1, \ldots, n-1).\end{aligned}$$

Definition of orthogonal polynomials (cntd.)

Constants h_n and k_n:

$$\langle p_n, p_n \rangle = h_n, \qquad p_n(x) = k_n x^n + \text{polynomial of lower degree} \; .$$

The p_n are unique up to a nonzero constant real factor. We may take them, for instance, orthonormal ($h_n = 1$, if also $k_n > 0$ then unique) or monic ($k_n = 1$).

In general we want

$$\langle x\, f, g \rangle = \langle f, x\, g \rangle.$$

This is true, for instance, if for a weight function $w(x) \geq 0$:

$$\langle f, g \rangle := \int_a^b f(x)\, g(x)\, w(x)\, dx,$$

or if for weights $w_j > 0$:

$$\langle f, g \rangle := \sum_{j=0}^{\infty} f(x_j)\, g(x_j)\, w_j.$$

Intermezzo about measures

The cases with the weight function and with the weights are special cases of a (positive) measure μ on $\mathbb{R}$:

$d\mu(x) = w(x)\,dx$ on (a,b) and $= 0$ outside (a,b);

resp. $\mu = \sum_{j=1}^{\infty} w_j\,\delta_{x_j}$, where δ_{x_j} is a unit mass at x_j.

A measure μ on $\mathbb{R}$ can also be thought as a non-decreasing function μ on $\mathbb{R}$. Then $\displaystyle\int_{\mathbb{R}} f(x)\,d\mu(x) = \lim_{M\to\infty} \int_{-M}^{M} f(x)\,d\mu(x)$

can be considered as a Riemann-Stieltjes integral.

μ has in x a mass point of mass $c>0$ if μ has a jump c at x, i.e., if $\displaystyle\lim_{\delta\downarrow 0} \big(\mu(x+\delta)-\mu(x-\delta)\big) = c > 0$.

The number of mass points is countable.

More generally, the support of μ consists of all $x\in\mathbb{R}$ such that $\mu(x+\delta)-\mu(x-\delta)>0$ for all $\delta>0$.
This set $\mathrm{supp}(\mu)$ is always closed.

Definition of orthogonal polynomials (cntd.)

In the most general case let μ be a (positive) measure on $\mathbb{R}$ (of infinite support, i.e., not $\mu = \sum_{j=1}^{N} w_j\, \delta_{x_j}$) such that for all $n = 0, 1, 2, \ldots$

$$\int_{\mathbb{R}} |x^n|\, d\mu(x) < \infty,$$

and take

$$\langle f, g \rangle := \int_{\mathbb{R}} f(x)\, g(x)\, d\mu(x).$$

A system $\{p_0, p_1, p_2, \ldots\}$ obtained by orthogonalization of $\{1, x, x^2, \ldots\}$ with respect to such an inner product is called a system of orthogonal polynomials (OP's) with respect to the orthogonality measure μ. Typical cases are:

- (weight function) $d\mu(x) = w(x)\, dx$ on an interval I.
- (weights) $\mu = \sum_{j=1}^{\infty} w_j\, \delta_{x_j}$.

First examples of orthogonal polynomials

1. Legendre polynomials $P_n(x)$, orthogonal on $[-1,1]$ with respect to the weight function 1. Normalized by $P_n(1)=1$.
2. Hermite polynomials $H_n(x)$, orthogonal on $(-\infty,\infty)$ with respect to the weight function e^{-x^2}. Normalized by $k_n=2^n$.
3. Charlier polynomials $c_n(x,a)$, orthogonal on the points $x=0,1,2,\ldots$ with respect to the weights $a^x/x!$ $(a>0)$. Normalized by $c_n(0;a)=1$.

The h_n can be computed:

$$\tfrac{1}{2}\int_{-1}^{1} P_m(x)\,P_n(x)\,dx=\frac{1}{2n+1}\,\delta_{m,n}\,,$$

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty} H_m(x)\,H_n(x)\,e^{-x^2}\,dx=2^n n!\,\delta_{m,n}\,,$$

$$e^{-a}\sum_{x=0}^{\infty} c_m(x,a)\,c_n(x,a)\,\frac{a^x}{x!}=a^{-n}n!\,\delta_{m,n}\,.$$

Three-term recurrence relation

Theorem

Orthogonal polynomials $p_n(x)$ satisfy

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n > 0),$$
$$x p_0(x) = a_0 p_1(x) + b_0 p_0(x)$$

with a_n, b_n, c_n real constants and $a_n c_{n+1} > 0$. Also

$$a_n = \frac{k_n}{k_{n+1}}, \qquad \frac{c_{n+1}}{h_{n+1}} = \frac{a_n}{h_n}.$$

Indeed, $x p_n(x) = \sum_{k=0}^{n+1} \alpha_k p_k(x)$, and if $k \le n-2$ then

$$\langle x p_n, p_k \rangle = \langle p_n, x p_k \rangle = 0, \quad \text{hence } \alpha_k = 0.$$

Furthermore,

$$c_{n+1} = \frac{\langle x p_{n+1}, p_n \rangle}{\langle p_n, p_n \rangle} = \frac{\langle x p_n, p_{n+1} \rangle}{h_n} = \frac{\langle x p_n, p_{n+1} \rangle}{\langle p_{n+1}, p_{n+1} \rangle} \, \frac{h_{n+1}}{h_n} = a_n \, \frac{h_{n+1}}{h_n}.$$

Hence $a_n c_{n+1} = a_n^2 \, h_{n+1}/h_n > 0$. Hence $c_{n+1}/h_{n+1} = a_n/h_n$.

Three-term recurrence relation (cntd.)

Theorem (Favard)

If polynomials $p_n(x)$ of degree n ($n = 0, 1, 2, \ldots$) satisfy

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n > 0),$$
$$xp_0(x) = a_0 p_1(x) + b_0 p_0(x)$$

with a_n, b_n, c_n real constants and $a_n c_{n+1} > 0$ then there exists a (positive) measure μ on $\mathbb{R}$ such that the polynomials $p_n(x)$ are orthogonal with respect to μ.

Remarks

1. The measure μ may not be unique (up to constant factor).
2. If μ unique then the polynomials are dense in $L^2(\mu)$.
3. If there is a μ with bounded support then μ is unique.

Favard

http://fr.wikipedia.org/wiki/Jean_Favard

Il a depuis longtemps une belle notoriété dans le monde mathématique lorsqu'il est mobilisé en septembre 1939 comme officier d'artillerie. Fait prisonnier en juin 1940, il est envoyé à l'oflag XVIII, à Lienz (Autriche), où il crée une Faculté des Sciences dont il est le doyen. Des mathématiciens autrichiens veulent le faire libérer s'il consent à enseigner à Vienne; il refuse. Dès 1941, il a été nommé professeur à la faculté des Sciences de Paris, mais il ne prend ses fonctions à la Sorbonne qu'à sa libération en 1945.

Three-term recurrence relation (cntd.)

For orthonormal polynomials:

$$x p_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \quad (n > 0),$$
$$x p_0(x) = a_0 p_1(x) + b_0 p_0(x).$$

For monic orthogonal polynomials:

$$x p_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n > 0),$$
$$x p_0(x) = p_1(x) + b_0 p_0(x)$$

with $c_n = h_n / h_{n-1} > 0$.
If the orthogonality measure is even ($\mu(E) = \mu(-E)$) then

$$p_n(-x) = (-1)^n p_n(x),$$

hence $b_n = 0$, so $x p_n(x) = a_n p_{n+1}(x) + c_n p_{n-1}(x)$.
Examples: Legendre and Hermite polynomials.

Moment functional

The recurrence relation (with $a_n c_{n+1} > 0$) determines the orthogonal polynomials $p_n(x)$ (up to constant factor because of the choice of the constant p_0).

The p_n determine (up to constant factor) the moment functional $\pi \mapsto \langle \pi, 1 \rangle$ on $\mathcal{P}$ by the rule $\langle p_n, 1 \rangle = 0$ for $n > 0$. Thus the inner product $\langle f, g \rangle = \langle fg, 1 \rangle$ on $\mathcal{P}$ is determined by the recurrence relation, independent of the choice of the orthogonality measure μ.

The moment functional $\pi \mapsto \langle \pi, 1 \rangle$ on $\mathcal{P}$ is determined by the moments $\mu_n := \langle x^n, 1 \rangle$. The condition $a_n c_{n+1} > 0$ is equivalent to *positive definiteness* of the moments, stated as

$$\Delta_n := \det(\mu_{i+j})_{i,j=0}^{n} > 0 \quad (n = 0, 1, 2, \ldots).$$

Christoffel-Darboux kernel

$\mathcal{P}$: space of all polynomials;
$\mathcal{P}_n$: space of polynomials of degree $\le n$;
$p_n(x)$: orthogonal polynomials with respect to measure μ.
Christoffel-Darboux kernel:

$$K_n(x,y) := \sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{h_j}$$

Then $$(\Pi_n f)(x) := \int_{\mathbb{R}} K_n(x,y)\, f(y)\, d\mu(y)$$

defines an orthogonal projection $\Pi_n \colon \mathcal{P} \to \mathcal{P}_n$.

Indeed, if $f(y) = \sum_{k=0}^{\infty} \alpha_k p_k(y)$ (finite sum) then

$$(\Pi_n f)(x) = \sum_{j=0}^{n} p_j(x) \sum_{k=0}^{\infty} \frac{\alpha_k}{h_j} \int_{\mathbb{R}} p_j(y)\, p_k(y)\, d\mu(y) = \sum_{j=0}^{n} \alpha_j p_j(x).$$

Christoffel-Darboux formula

$$\sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{h_j} = \frac{k_n}{h_n k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \quad (x \neq y),$$
$$= \frac{k_n}{h_n k_{n+1}} \left(p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x)\right) \; (x = y).$$

Indeed,
$$xp_j(x) = a_j p_{j+1}(x) + b_j p_j(x) + c_j p_{j-1}(x),$$
$$yp_j(y) = a_j p_{j+1}(y) + b_j p_j(y) + c_j p_{j-1}(y).$$

Hence
$$\frac{(x-y)p_j(x)p_j(y)}{h_j} = \frac{a_j}{h_j}(p_{j+1}(x)p_j(y) - p_j(x)p_{j+1}(y)) - \frac{c_j}{h_j}(p_j(x)p_{j-1}(y) - p_{j-1}(x)p_j(y)).$$

Use $c_j/h_j = a_{j-1}/h_{j-1}$. Sum from $j = 0$ to n.

Use that $a_n = k_n/k_{n+1}$. We have the C-D formula for $x \neq y$.

Zeros of orthogonal polynomials

Theorem

Let $p_n(x)$ be an orthogonal polynomial of degree n.
Let μ have support within the closure of the interval (a, b).
Then p_n has n distinct zeros on (a, b).

Proof (for $(a, b) = (-\infty, \infty)$)

Assume $k_n > 0$ (no loss of generality).
Suppose p_n has $k < n$ sign changes on $\mathbb{R}$ at $x_1, x_2, \ldots, x_k$.
Hence $p_n(x)(x - x_1)\ldots(x - x_k) \geq 0$ on $\mathbb{R}$.
Hence $\int_{\mathbb{R}} p_n(x)(x - x_1)\ldots(x - x_k)\, d\mu(x) > 0$.
But by orthogonality $\int_{\mathbb{R}} p_n(x)(x - x_1)\ldots(x - x_k)\, d\mu(x) = 0$.
Contradiction. □

Zeros of orthogonal polynomials (cntd.)

By the Christoffel-Darboux formula: If $k_n, k_{n+1} > 0$ then

$$p'_{n+1}(x)p_n(x) - p'_n(x)p_{n+1}(x) = \frac{h_n k_{n+1}}{k_n} \sum_{j=0}^{n} \frac{p_j(x)^2}{h_j} > 0.$$

Hence, if y, z are two successive zeros of p_{n+1} then

$$p'_{n+1}(y)p_n(y) > 0, \quad p'_{n+1}(z)p_n(z) > 0.$$

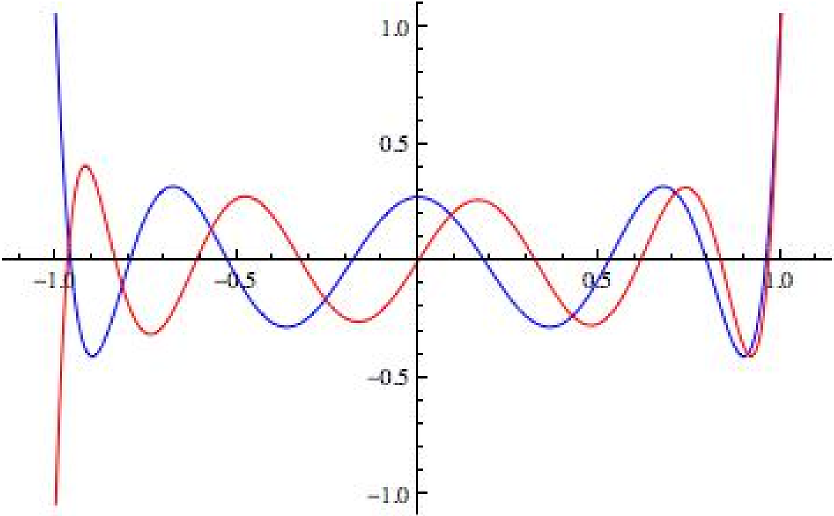
Since $p'_{n+1}(y)$ and $p'_{n+1}(z)$ will have opposite signs, $p_n(y)$ and $p_n(z)$ will have opposite signs.
Hence p_n must have a zero in (y, z).

Theorem

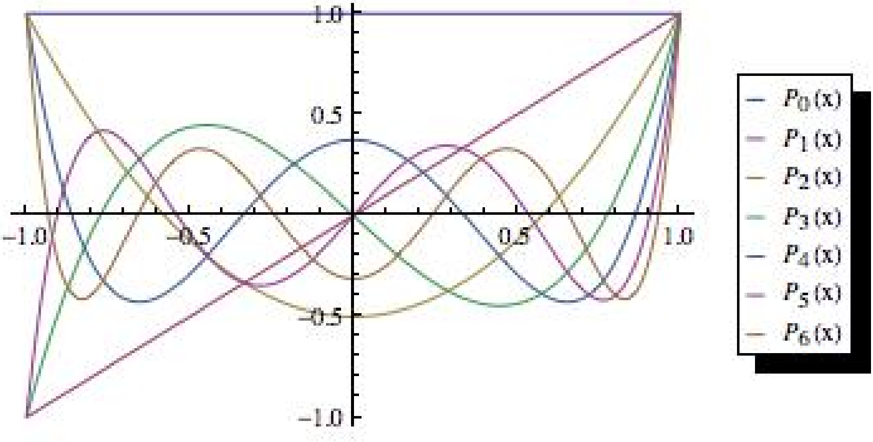
The zeros of p_n and p_{n+1} alternate.

Graphs of Legendre polynomials

Alternating zeros of Legendre polynomials $P_8(x)$ (blue graph) and $P_9(x)$ (red graph):



Graphs of Legendre polynomials (cntd.)



Definition of Jacobi polynomials

$$p_n(x) = P_n^{(\alpha,\beta)}(x),$$

$$d\mu(x) = w(x)\,dx \text{ on } [-1,1],$$

$$w(x) = (1-x)^\alpha(1+x)^\beta \quad (\alpha,\beta > -1),$$

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}\,.$$

Explicit expression (see Paule for hypergeometric functions)

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\;{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix};\tfrac12(1-x)\right).$$

Symmetry $P_n^{(\alpha,\beta)}(-x) = (-1)^n\,P_n^{(\beta,\alpha)}(x)$. Hence

$${}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix};z\right) = \frac{(-1)^n(\beta+1)_n}{(\alpha+1)_n}\;{}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \beta+1\end{matrix};1-z\right).$$

Jacobi polynomials (cntd.)

Second order differential equation for $p_n(x) = P_n^{(\alpha,\beta)}(x)$

$$(1-x^2)p_n''(x) + \big(\beta - \alpha - (\alpha+\beta+2)x\big)p_n'(x) \\ = -n(n+\alpha+\beta+1)\,p_n(x).$$

Shift operator relations

$$\frac{d}{dx}\,P_n^{(\alpha,\beta)}(x) = \tfrac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x),$$

$$(1-x^2)\,\frac{d}{dx}\,P_{n-1}^{(\alpha+1,\beta+1)}(x) + \big(\beta-\alpha-(\alpha+\beta+2)x\big)\,P_{n-1}^{(\alpha+1,\beta+1)}(x) \\ = (1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d}{dx}\Big((1-x)^{\alpha+1}(1+x)^{\beta+1}\,P_{n-1}^{(\alpha+1,\beta+1)}(x)\Big) \\ = -2n\,P_n^{(\alpha,\beta)}(x).$$

Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}\,(1-x)^{-\alpha}(1+x)^{-\beta} \\ \times \left(\frac{d}{dx}\right)^n \Big((1-x)^{\alpha+n}(1+x)^{\beta+n}\Big).$$

Jacobi polynomials (special cases)

Gegenbauer or ultraspherical polynomials $(\alpha = \beta = \lambda - \frac{1}{2})$

$$C_n^\lambda(x) := \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda - \frac{1}{2})}(x).$$

Legendre polynomials $(\alpha = \beta = 0)$

$$P_n(x) := P_n^{(0,0)}(x).$$

Chebyshev polynomials $(\alpha = \beta = \pm\frac{1}{2})$

$$T_n(\cos\theta) := \cos(n\theta) = \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2}, -\frac{1}{2})}(\cos\theta),$$

$$U_n(\cos\theta) := \frac{\sin(n+1)\theta}{\sin\theta} = \frac{(2)_n}{(\frac{3}{2})_n} P_n^{(\frac{1}{2}, \frac{1}{2})}(\cos\theta).$$

Laguerre polynomials

Definition of Laguerre polynomials

$$p_n(x) = L_n^\alpha(x),$$

$$d\mu(x) = w(x)\,dx \text{ on } [0,\infty),$$

$$w(x) = x^\alpha e^{-x} \quad (\alpha > -1),$$

$$L_n^\alpha(0) = \frac{(\alpha+1)_n}{n!}\,.$$

Explicit expression

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\;{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right).$$

Second order differential equation for $p_n(x) = L_n^\alpha(x)$

$$x\,p_n''(x) + (\alpha + 1 - x)\,p_n'(x) = -n\,p_n(x).$$

Shift operator relations

$$\frac{d}{dx}\,L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x),$$

$$x\,\frac{d}{dx}\,L_{n-1}^{\alpha+1}(x) + (\alpha + 1 - x)L_{n-1}^{\alpha+1}(x) = x^{-\alpha}e^x\,\frac{d}{dx}\left(x^{\alpha+1}\,e^{-x}\,L_{n-1}^{\alpha+1}(x)\right) = n\,L_n^\alpha(x).$$

Rodrigues formula

$$L_n^\alpha(x) = \frac{x^{-\alpha}e^x}{n!}\,\left(\frac{d}{dx}\right)^n \left(x^{n+\alpha}\,e^{-x}\right).$$

Hermite polynomials

Definition of Hermite polynomials

$$p_n(x) = H_n(x), \qquad d\mu(x) = e^{-x^2}\,dx, \qquad k_n = 2^n.$$

Explicit expression

$$H_n(x) = n! \sum_{j=0}^{[n/2]} \frac{(-1)^j (2x)^{n-2j}}{j!\,(n-2j)!}\,.$$

Second order differential equation

$$H_n''(x) - 2xH_n'(x) = -2nH_n(x).$$

Shift operator relations

$$H_n'(x) = 2n\,H_{n-1}(x), \quad H_{n-1}'(x) - 2xH_{n-1}(x) = -H_n(x)$$

Rodrigues formula

$$H_n(x) = (-1)^n\,e^{x^2}\left(\frac{d}{dx}\right)^n\left(e^{-x^2}\right).$$

Derivation of previous formulas

(a,b) open interval; $w, w_1 > 0$ on (a,b) and C^1.

On (a,b) monic OP's $p_n(x), q_m(x)$ with respect to w resp. w_1.

Then under suitable boundary assumptions for w and w_1:

$$\int_a^b p_n'(x)\, q_{m-1}(x)\, w_1(x)\, dx$$
$$= -\int_a^b p_n(x)\, w(x)^{-1} \frac{d}{dx}\big(w_1(x)\, q_{m-1}(x)\big)\, w(x)\, dx.$$

Suppose that for certain $a_n \neq 0$:

$$w(x)^{-1} \frac{d}{dx}\left(w_1(x)\, x^{n-1}\right) = -a_n\, x^n + \text{polynomial of degree} < n.$$

Then $\quad p_n'(x) = n\, q_{n-1}(x), \quad w(x)^{-1} \dfrac{d}{dx}\big(w_1(x)\, q_{n-1}(x)\big) = -a_n\, p_n(x),$

$$w(x)^{-1} \frac{d}{dx}\big(w_1(x)\, p_n'(x)\big) = -n a_n\, p_n(x),$$

$$n \int_a^b q_{n-1}(x)^2\, w_1(x)\, dx = a_n \int_a^b p_n(x)^2\, w(x)\, dx.$$

Derivation of previous formulas (cntd.)

Work with monic Jacobi polynomials $p_n^{(\alpha,\beta)}(x)$. Then
$(a,b)=(-1,1)$, $w(x)=(1-x)^\alpha(1+x)^\beta$, $p_n(x)=p_n^{(\alpha,\beta)}(x)$,
$w_1(x)=(1-x)^{\alpha+1}(1+x)^{\beta+1}$, $q_m(x)=p_m^{(\alpha+1,\beta+1)}(x)$.
Then $a_n=(n+\alpha+\beta+1)$,

$$\Big((1-x^2)\,\frac{d}{dx}+\big(\beta-\alpha-(\alpha+\beta+2)x\big)\Big)p_{n-1}^{(\alpha+1,\beta+1)}(x) = -(n+\alpha+\beta+1)\,p_n^{(\alpha,\beta)}(x).$$

For $x=1$: $\quad p_n^{(\alpha,\beta)}(1)=\dfrac{2(\alpha+1)}{n+\alpha+\beta+1}\,p_{n-1}^{(\alpha+1,\beta+1)}(1).$

Then iterate: $\quad p_n^{(\alpha,\beta)}(1)=\dfrac{2^n(\alpha+1)_n}{(n+\alpha+\beta+1)_n}\,.$

So we know $p_n(1)/k_n$, which is independent of the normalization.

Derivation of previous formulas (cntd.)

Hypergeometric series representation of Jacobi polynomials obtained by Taylor expansion:

$$p_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(x-1)^k}{k!} \left(\frac{d}{dx}\right)^k p_n^{(\alpha,\beta)}(x)\Big|_{x=1}$$
$$= \sum_{k=0}^{n} \frac{(x-1)^k}{k!} \frac{n!}{(n-k)!} p_{n-k}^{(\alpha+k,\beta+k)}(1).$$

Quadratic norm h_n obtained by iteration:

$$\int_{-1}^{1} p_n^{(\alpha,\beta)}(x)^2 (1-x)^\alpha (1+x)^\beta \, dx$$
$$= \frac{n}{n+\alpha+\beta+1} \int_{-1}^{1} p_{n-1}^{(\alpha+1,\beta+1)}(x)^2 (1-x)^{\alpha+1} (1+x)^{\beta+1} \, dx.$$

So we know h_n/k_n^2, which is independent of the normalization.

Very classical orthogonal polynomials

Jacobi, Laguerre and Hermite polynomials together, for the given parameter ranges, are called very classical orthogonal polynomials. Up to constant factors and up to transformations $x \to ax + b$ of the argument they are uniquely determined as OP's $p_n(x)$ satisfying any of the following three criteria:

• (Bochner's theorem) The p_n are eigenfunctions of a second order differential operator.

• The polynomlals $p'_{n+1}(x)$ are again orthogonal polynomials.

• The polynomials are orthogonal with respect to a positive C^∞ weight function $w(x)$ on an open interval I and there is a polynomial $X(x)$ such that the Rodrigues formula holds on I:

$$p_n(x) = \text{const.}\, w(x)^{-1} \left(\frac{d}{dx}\right)^n \big(X(x)^n w(x)\big).$$

Benjamin Olinde Rodrigues (1795–1851) lived in Paris. He had in his thesis the Rodrigues formula for the Legendre polynomials. Afterwards he became a banker and became a relatively wealthy man as he supported the development of the French railway system.

Rodrigues was an early socialist. He argued that working men were kept poor by lending at interest and by inheritance. He also argued in favour of mutual aid societies and profit-sharing for workers.

Rodrigues joined the Paris Ethnological Society. He argued strongly that *all races had equal aptitude for civilization in suitable circumstances* and that *women will one day conquer equality without any restriction*. These views were much criticised by other members: “Rodrigues was sentimental and science proved that he was wrong”.

Limits for very classical OP's

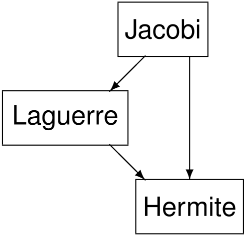
Monic versions:

- Jacobi: $p_n^{(\alpha,\beta)}(x)$, $\quad w(x) = (1-x)^\alpha\,(1+x)^\beta$ on $(-1,1)$
- Laguerre: $\ell_n^\alpha(x)$, $\quad w(x) = e^{-x}\,x^\alpha$ on $(0,\infty)$
- Hermite: $h_n(x)$, $\quad w(x) = e^{-x^2}$ on $(-\infty,\infty)$

$$\alpha^{n/2} p_n^{(\alpha,\alpha)}(x/\alpha^{1/2}) \to h_n(x), \qquad (1-x^2/\alpha)^\alpha \to e^{-x^2},\ \alpha\to\infty$$

$$(-\beta/2)^n\, p_n^{(\alpha,\beta)}(1-2x/\beta) \to \ell_n^\alpha(x), \qquad x^\alpha(1-x/\beta)^\beta \to x^\alpha e^{-x},\ \beta\to\infty$$

$$(2\alpha)^{-n/2}\,\ell_n^\alpha((2\alpha)^{1/2}x+\alpha) \to h_n(x), \quad (1+(2/\alpha)^{1/2}x)^\alpha e^{-(2\alpha)^{1/2}x} \to e^{-x^2},\ \alpha\to\infty$$



Electrostatic interpretation of zeros

Let $p_n(x) = P_n^{(2p-1,2q-1)}(x)/k_n = (x-x_1)(x-x_2)\ldots(x-x_n)$ be monic Jacobi polynomials ($p,q>0$). We know that

$$(1-x^2)p_n''(x) + 2(q-p-(p+q)x)p_n'(x) \\ = -n(n+2p+2q-1)p_n(x).$$

Hence $\quad (1-x_k^2)p_n''(x_k) + 2(q-p-(p+q)x_k)p_n'(x_k) = 0,$

i.e., $$\tfrac{1}{2}\frac{p_n''(x_k)}{p_n'(x_k)} + \frac{p}{x_k-1} + \frac{q}{x_k+1} = 0,$$

i.e., $$\sum_{j,\,j\neq k} \frac{1}{x_k-x_j} + \frac{p}{x_k-1} + \frac{q}{x_k+1} = 0,$$

i.e., $(\nabla V)(x_1,\ldots,x_n) = 0$, where $V(y_1,\ldots,y_n)$

$$= -\sum_{i<j}\log(y_j-y_i) - p\sum_j \log(1-y_j) - q\sum_j \log(1+y_j).$$

Logarithmic potential from charges $q,1,\ldots,1,p$ at $-1<y_1<\ldots<y_n<1$ achieves minimum at the zeros of $P_n^{(2p-1,2q-1)}(x)$.

Stieltjes

Thomas Stieltjes, 1856–1894.
1877 assistant at Leiden astronomical observatory.
Was corresponding with Hermite.
1884 honorary doctorate of Leiden University.
1885 professor in Toulouse.

Quadratic transformations

$P_{2n}^{(\alpha,\alpha)}(x)$ is polynomial $p_n(2x^2-1)$ of degree n in x^2. For $m \neq n$

$$0 = \int_0^1 p_m(2y^2-1)p_n(2y^2-1)\,(1-y^2)^\alpha\,dy$$

$$= \text{const.} \int_{-1}^1 p_m(x)p_n(x)\,(1-x)^\alpha(1+x)^{-\frac{1}{2}}\,dx.$$

Hence $\dfrac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \dfrac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)}$.

Similarly $\dfrac{P_{2n+1}^{(\alpha,\alpha)}(x)}{P_{2n+1}^{(\alpha,\alpha)}(1)} = \dfrac{xP_n^{(\alpha,\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,\frac{1}{2})}(1)}$.

Theorem

Let $p_n(x)$ be monic orthogonal polynomial with respect to even weight function $w(x)$ on $\mathbb{R}$. Then $p_{2n}(x) = q_n(x^2)$ and $p_{2n+1}(x) = x\,r_n(x^2)$ with $q_n(x)$ and $r_n(x)$ OP's on $[0,\infty)$ with respect to weight functions $x^{-\frac{1}{2}}w(x^{\frac{1}{2}})$ resp. $x^{\frac{1}{2}}w(x^{\frac{1}{2}})$.

Kernel polynomials

Christoffel-Darboux formula:

$$\sum_{j=0}^{n} \frac{p_j(x)p_j(y)}{h_j} = \frac{k_n}{h_n k_{n+1}} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y} \quad (x \neq y).$$

Suppose the orthogonality measure μ has support within $(-\infty, b]$ and fix $y \geq b$. Then for $k \leq n-1$:

$$\int_{-\infty}^{b} K_n(x,y)\, x^k\, (y-x)\, d\mu(x) = y^k(y-y) = 0.$$

Hence $x \mapsto q_n(x) = K_n(x,y)$ is an OP of degree n on $(-\infty, b]$ with respect to the measure $(y-x)\, d\mu(x)$. Hence

$$q_n(x) - q_{n-1}(x) = \frac{p_n(y)}{h_n}\, p_n(x),$$

$$p_n(y)p_{n+1}(x) - p_{n+1}(y)p_n(x) = \frac{h_n k_{n+1}}{k_n}\,(x-y)q_n(x).$$

True interval of orthogonality

Orthogonal polynomials $p_n(x)$.

Let $p_n(x)$ have zeros $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$.

Then $x_{i,i} > x_{i+1,i} > \ldots > x_{n,i} \downarrow \xi_i \geq -\infty$,

and $x_{j,1} < x_{j+1,2} < \ldots < x_{n,n-j+1} \uparrow \eta_j \leq \infty$.

Definition

The closure of the interval (ξ_1, η_1) is called the true interval of orthogonality of the OP's $p_n(x)$.

Remarks

The true interval of orthogonality I has the following properties.

1. I is the smallest closed interval containing all zeros $x_{n,i}$.
2. There is an orthogonality measure μ for the $p_n(x)$ such that I is the smallest closed interval containing the support of μ.
3. If μ is any orthogonality measure for the $p_n(x)$ and J is a closed interval containing the support of μ then $I \subset J$.

Criteria for bounded support of orthogonality measure

$$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (c_n > 0).$$

Theorem

1. $\{b_n\}$ *bounded,* $\{c_n\}$ *unbounded* $\Longrightarrow (\xi_1, \eta_1) = (-\infty, \infty)$.
2. $\{b_n\}$, $\{c_n\}$ *bounded* $\Longleftrightarrow [\xi_1, \eta_1]$ *bounded.*
3. $b_n \to b$, $c_n \to c$ *(b, c finite)* $\Longrightarrow \mathrm{supp}(\mu)$ *bounded with at most countably many points outside* $[b - 2\sqrt{c}, b + 2\sqrt{c}]$ *and* $b \pm 2\sqrt{c}$ *limit points of* $\mathrm{supp}(\mu)$.

Example

Monic Jacobi polynomials $k_n^{-1} P_n^{(\alpha,\beta)}(x)$:

$$b_n = \frac{\beta^2 - \alpha^2}{(2n + \alpha + \beta)(2n + \alpha + \beta + 2)} \to 0.$$

$$c_n = \frac{4n(n + \alpha)(n + \beta)(n + \alpha + \beta)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta)^2(2n + \alpha + \beta + 1)} \to \frac{1}{4}.$$

Hence $[b - 2\sqrt{c}, b + 2\sqrt{c}] = [-1, 1]$.

Criteria for uniqueness of orthogonality measure

(See Shohat & Tamarkin, *The problem of moments*, AMS, 1943.)
Let $p_n(x)$ be orthonormal polynomials, i.e., solutions of

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \quad (a_n > 0,\ b_n \in \mathbb{R}).$$

Put $$\rho(z) := \Big(\sum_{n=0}^{\infty} |p_n(z)|^2 \Big)^{-1} \quad (z \in \mathbb{C}).$$

Theorem

The orthogonality measure is not unique iff $\rho(z) > 0$ for all $z \in \mathbb{C}$. Hence it is unique iff $\rho(z) = 0$ for some $z \in \mathbb{C}$.

In fact, if there is a unique orthogonality measure μ then $\rho(x) = \mu(\{x\})$ if μ has a mass point at x, and $\rho(z) = 0$ for $z \in \mathbb{C}$ outside the mass points of μ.

In case of non-uniqueness, for each $x \in \mathbb{R}$ the largest possible jump of a measure μ at x is $\rho(x)$ and there is a measure realizing this jump.

Criteria for uniqueness of orthog. measure (cntd.)

Orthonormal polynomials $p_n(x)$.

$$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \quad (a_n > 0,\ b_n \in \mathbb{R}),$$

$$x\,\frac{p_n(x)}{k_n} = \frac{p_{n+1}(x)}{k_{n+1}} + b_n\,\frac{p_n(x)}{k_n} + a_{n-1}^2\,\frac{p_{n-1}(x)}{k_{n-1}} \quad \text{(monic version)},$$

$$\mu_n := \langle x^n, 1\rangle \quad \text{(moments)}.$$

Theorem (Carleman)

There is a unique orthogonality measure for the p_n if one of the following two conditions is satisfied.

1. $\displaystyle\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)} = \infty.$
2. $\displaystyle\sum_{n=1}^{\infty} a_n^{-1} = \infty.$

Hermite: $\mu_{2n} = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx = \Gamma(n + \frac{1}{2})$.

$\log \Gamma(n + \frac{1}{2}) = n \log(n + \frac{1}{2}) + O(n)$ as $n \to \infty$,

so $\mu_{2n}^{-1/(2n)} \sim (n + \frac{1}{2})^{-\frac{1}{2}}$. Hence $\sum_{n=1}^{\infty} \mu_{2n}^{-1/(2n)} = \infty$:

unique orthogonality measure.

Monic Laguerre $p_n(x) = k_n^{-1} L_n^{\alpha}(x)$:

$xp_n(x) = p_{n+1}(x) + (2n + \alpha + 1)p_n(x) + n(n + \alpha)p_{n-1}(x)$.

$\sum_{n=0}^{\infty} \frac{1}{(n(n + \alpha))^{1/2}} = \infty$: unique orthogonality measure.

Also $\frac{L_n^{\alpha}(0)^2}{h_n} = \frac{\Gamma(n + \alpha + 1)}{\Gamma(n + 1)\Gamma(\alpha + 1)} \sim n^{\alpha}$.

$\sum_{n=1}^{\infty} n^{\alpha} = \infty$ $(\alpha > -1)$: unique orthogonality measure.

Example of non-unique orthogonality measure

$$\int_{-\infty}^{\infty} e^{-u^2}(1 + C\sin(2\pi u))\,du = \pi^{1/2}.$$

Substitute $u = \log x - \frac{1}{2}(n+1)$ and take $-1 < C < 1$.

$$\pi^{-\frac{1}{2}} \int_0^{\infty} x^n(1 + C\sin(2\pi\log x))\,e^{-\log^2 x}\,dx = e^{(n+1)^2/4}.$$

The moments are independent of C. The corresponding orthogonal polynomials are the Stieltjes-Wigert polynomials.

Orthogonal polynomials and continued fractions

Let $p_n(x)$ be monic OP's given by $p_0(x) = 1$, $p_1(x) = x - b_0$,
$xp_n(x) = p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \quad (n \geq 1,\ c_n > 0)$.

The monic first associated OP's or numerator polynomials $p_n^{(1)}(x)$ are defined by $p_0^{(1)}(x) = 1, p_1^{(1)}(x) = x - b_1$,
$xp_n^{(1)}(x) = p_{n+1}^{(1)}(x) + b_{n+1} p_n^{(1)}(x) + c_{n+1} p_{n-1}^{(1)}(x) \quad (n \geq 1)$.

Recursively define $F_1(x) := \dfrac{1}{x - b_0}$, $F_2(x) := \dfrac{1}{x - b_0 - \frac{c_1}{x - b_1}}$,

$F_3(x) := \dfrac{1}{x - b_0 - \frac{c_1}{x - b_1 - \frac{c_2}{x - b_2}}}$, and $F_{n+1}(x)$ obtained from $F_n(x)$

by replacing b_{n-1} by $b_{n-1} + \dfrac{c_n}{x - b_n}$ (**continued fraction**).

Theorem (essentially Stieltjes)

$$F_n(x) = \frac{p_{n-1}^{(1)}(x)}{p_n(x)}, \quad p_{n-1}^{(1)}(y) = \frac{1}{\mu_0} \int_{\mathbb{R}} \frac{p_n(y) - p_n(x)}{y - x}\, d\mu(x).$$

OP's and continued fractions (cntd.)

$$F_n(z) := \frac{1}{z - b_0 -|} \; \frac{|c_1}{z - b_1 -|} \cdots \frac{|c_{n-2}}{z - b_{n-2} -|} \; \frac{|c_{n-1}}{z - b_{n-1}} = \frac{p^{(1)}_{n-1}(z)}{p_n(z)} \, .$$

Suppose that there is a (unique) orthogonality measure μ of bounded support for the p_n. Let $[\xi_1, \eta_1]$ be the true interval of orthogonality.

Theorem (Markov)

$$\lim_{n\to\infty} F_n(z) = \frac{1}{\mu_0} \int_{\xi_1}^{\eta_1} \frac{d\mu(x)}{z - x} \quad \textit{uniformly}$$

on compact subsets of $\mathbb{C} \backslash [\xi_1, \eta_1]$.

Measures in case of non-uniqueness

Take p_n and $p_n^{(1)}$ orthonormal: $p_0(x) = 1$, $p_1(x) = (x - b_0)/a_0$,
$xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + a_{n-1} p_{n-1}(x) \quad (n \geq 1,\, a_n > 0)$,
$p_0^{(1)}(x) = 1, p_1^{(1)}(x) = (x - b_1)/a_1$,
$xp_n^{(1)}(x) = a_{n+1} p_{n+1}^{(1)}(x) + b_{n+1} p_n^{(1)}(x) + a_n p_{n-1}^{(1)}(x). \quad (n \geq 1)$.
Let $\mu_0 = 1,\ \mu_1, \mu_2, \ldots$ be the moments for the p_n. Assume non-uniquness of μ satisfying $\int_{\mathbb{R}} x^n\, d\mu(x) = \mu_n$. The set of these μ is convex and weakly compact. Then the following functions are entire.

$$A(z) := z \sum_{n=0}^{\infty} p_n^{(1)}(0)\, p_n^{(1)}(z), \quad B(z) := -1 + z \sum_{n=1}^{\infty} p_{n-1}^{(1)}(0)\, p_n(z),$$

$$C(z) := 1 + z \sum_{n=1}^{\infty} p_n(0)\, p_{n-1}^{(1)}(z), \quad D(z) = z \sum_{n=0}^{\infty} p_n(0)\, p_n(z).$$

Measures in case of non-uniqueness (cntd.)

Theorem (Nevanlinna, M. Riesz)

The identity

$$\int_{\mathbb{R}} \frac{d\mu_\phi(t)}{t-z} = -\frac{A(z)\phi(z)-C(z)}{B(z)\phi(z)-D(z)} \quad (\operatorname{Im} z > 0)$$

gives a one-to-one correspondence $\phi \to \mu_\phi$ *between the set of functions* ϕ *being either identically* ∞ *or a holomorphic function mapping the open upper half plane into the closed upper half plane* (Pick function) *and the set of measures solving the moment problem.*

Furthermore the measures μ_t $(t \in \mathbb{R} \cup \{\infty\})$ *are precisely the extremal elements of the convex set, and also precisely the measures* μ *solving the moment problem for which the the polynomials are dense in* $L^2(\mu)$. *All measures* μ_t *are discrete.*

Gauss quadrature

Let be given n real points $x_1 < x_2 < \ldots < x_n$.

Put $p_n(x) := (x - x_1)\ldots(x - x_n)$.

Let $l_k(x)$ be the unique polynomial of degree $< n$ such that $l_k(x_j) = \delta_{k,j}$ $(j = 1, \ldots, n)$. Then (Lagrange interpolation polynomial)

$$l_k(x) = \frac{\prod_{j;\, j\neq k}(x - x_j)}{\prod_{j;\, j\neq k}(x_k - x_j)} = \frac{p_n(x)}{(x - x_k)\, p_n'(x_k)} \quad \text{and}$$

for all polynomials r of degree $< n$: $\quad r(x) = \sum_{k=1}^{n} r(x_k)\, l_k(x)$.

Theorem (Gauss quadrature)

Let p_n be an OP with respect to μ. Put $\lambda_k := \int_{\mathbb{R}} l_k(x)\, d\mu(x)$.
Then $\lambda_k = \int_{\mathbb{R}} l_k(x)^2\, d\mu(x) > 0$ and for all polynomials of degree $\leq 2n - 1$:
$\int_{\mathbb{R}} f(x)\, d\mu(x) = \sum_{k=1}^{n} \lambda_k\, f(x_k)$.

Gauss quadrature: Proof

Let $f(x)$ be polynomial of degree $\le 2n-1$. Then for certain polynomials $q(x)$ and $r(x)$ of degree $\le n-1$:

$f(x) = q(x)p_n(x) + r(x)$. Hence $f(x_k) = r(x_k)$ and

$$\int_{\mathbb{R}} f(x)\,d\mu(z) = \int_{\mathbb{R}} r(x)\,d\mu(x) = \sum_{k=1}^{n} r(x_k)\ \int_{\mathbb{R}} l_k(x)\,d\mu(x)$$

$$= \sum_{k=1}^{n} \lambda_k r(x_k) = \sum_{k=1}^{n} \lambda_k f(x_k).$$

Also $\lambda_k = \sum_{j=1}^{n} \lambda_j\, l_k(x_j)^2 = \int_{\mathbb{R}} l_k(x)^2\,d\mu(x) > 0.$ □

Finite systems of orthogonal polynomials

We saw: $\displaystyle\int_{\mathbb{R}} f(x)\,d\mu(x) = \sum_{k=1}^{n} \lambda_k\, f(x_k) \quad (f \in \mathcal{P}_{2n-1}).$

In particular, for $i, j \le n-1$,

$$h_j \delta_{i,j} = \int_{\mathbb{R}} p_i(x) p_j(x)\, d\mu(x) = \sum_{k=1}^{n} \lambda_k\, p_i(x_k)\, p_j(x_k).$$

Thus the finite system $p_0, p_1, \ldots, p_{n-1}$ forms a set of orthogonal polynomials on the finite set $\{x_1, \ldots, x_n\}$ of the n zeros of p_n with respect to the weights λ_k and with quadaratic norms h_j.

All information about this system is already contained in the finite system of recurrence relations

$$x p_j(x) = a_j p_{j+1}(x) + b_j p_j(x) + c_j p_{j-1}(x) \quad (j = 0, 1, \ldots, n-1)$$

with $a_j c_{j+1} > 0$ $(j = 0, 1, \ldots, n-2)$. In particular, the λ_k are obtained up to constant factor by solving the system

$$\sum_{k=1}^{n} \lambda_k p_j(x_k) = 0 \quad (j = 1, \ldots, n-1).$$

Finite systems of orthogonal polynomials (cntd.)

For exampe, consider orthogonal polynomials $p_0, p_1, \ldots, p_N$ on the zeros $0, 1, \ldots, N$ of the polynomial $p_{N+1}(x) := x(x-1)\ldots(x-N)$ with respect to nice explicit weights w_x $(x = 0, 1, \ldots, N)$ like:

1. $w_x := \binom{n}{x} p^x (1-p)^{N-x} \quad (0 < p < 1).$

   Then the $p_n(x)$ are the Krawtchouk polynomials

   $$K_n(x; p, N) := {}_2F_1\left(\begin{matrix}-n, -x \\ -N\end{matrix}; \frac{1}{p}\right) = \sum_{k=0}^{n} \frac{(-n)_k (-x)_k}{(-N)_k\, k!} \frac{1}{p^k}.$$

2. $w_x := \frac{(\alpha+1)_x}{x!} \frac{(\beta+1)_{N-x}}{(N-x)!} \quad (\alpha, \beta > -1).$

   Then the $p_n(x)$ are the Hahn polynomials

   $$Q_n(x; \alpha, \beta, N) := {}_3F_2\left(\begin{matrix}-n, n+\alpha+\beta+1, -x \\ \alpha+1, -N\end{matrix}; 1\right).$$

Hahn and Krawtchouk polynomials (cntd.)

Hahn polynomials are discrete versions of Jacobi polynomials:

$$Q_n(Nx;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix}-n,n+\alpha+\beta+1,-Nx\\ \alpha+1,-N\end{matrix};1\right) \to$$

$$ {}_2F_1\left(\begin{matrix}-n,n+\alpha+\beta+1\\ \alpha+1\end{matrix};x\right) = \text{const.}\, P_n^{(\alpha,\beta)}(1-2x)$$

and

$$N^{-1} \sum_{x\in\{0,\frac{1}{N},\frac{2}{N},\ldots,1\}} Q_m(Nx;\alpha,\beta,N)Q_n(Nx;\alpha,\beta,N)\, w_{Nx} \to$$

$$\text{const.} \int_0^1 P_m^{(\alpha,\beta)}(1-2x)P_n^{(\alpha,\beta)}(1-2x)\, x^\alpha(1-x)^\beta\, dx.$$

Jacobi and Krawtchouk polynomials are different ways of looking at the matrix elements of the irreps of $SU(2)$.

The $3j$ coefficients or Clebsch-Gordan coefficients for $SU(2)$ can be expressed as Hahn polynomials.

Classical orthogonal polynomials of Hahn class

Hahn and Krawtchouk polynomials are orthogonal polynomials $p_n(x)$ on $0, 1, \ldots, N$ which are eigenfunctions of a second order difference operator:

$$A(x)p_n(x-1) + B(x)p_n(x) + C(x)p_n(x+1) = \lambda_n\, p_n(x).$$

Moreover, the polynomials $q_n(x) := p_{n+1}(x+1) - p_{n+1}(x)$ are orthogonal polynomials on $0, 1, \ldots, N-1$.

If we also allow orthogonal polynomials on $0, 1, 2, \ldots$ then Meixner polynomials $M_n(x; \beta, c)$ and Charlier polynomials $C_n(x; a)$ also have these properties. Here

$$M_n(x; \beta, c) := {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}; 1 - \frac{1}{c}\right), \quad w_x := \frac{(\beta_x)}{x!}\, c^x,$$

$$C_n(x; a) := {}_2F_0(-n, -x; ; -a^{-1}), \quad w_x := a^x / x!.$$

Classical orthogonal polynomials

More generally we can ask for orthogonal polynomials which are eigenfunctions of a second order operator L of the form

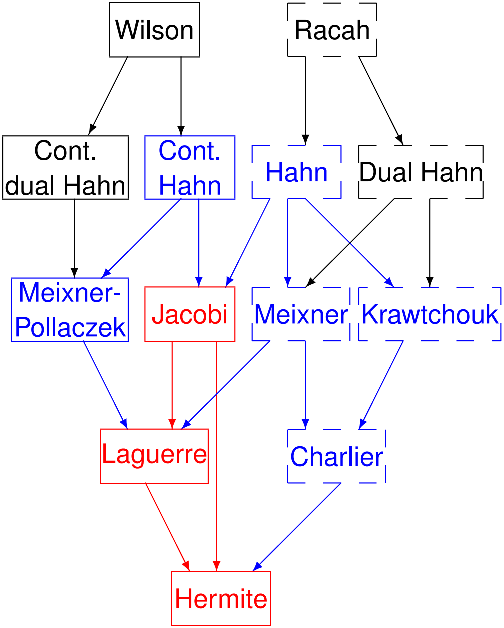
$$(Lf)(x) := A(x)f(x+i) + B(x)f(x) + C(x)f(x-i)$$

or (the so-called quadratic lattice)

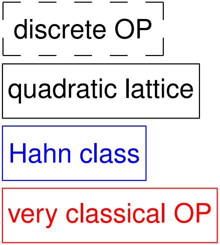
$$(Lf)(q(x)) := A(x)f(q(x+1)) + B(x)f(q(x)) + C(x)f(q(x-1)),$$

where $q(x)$ is a fixed polynomial of second degree.

All such orthogonal polynomials have been classified. There are only 13 families, all but the Hermite depending on parameters, at most four, and all expressible as hypergeometric functions, the most complicated as ${}_4F_3$. They can be arranged hierarchically according to limit transitions denoted by arrows.


Wilson
Racah
Cont. dual Hahn
Cont. Hahn
Hahn
Dual Hahn
Meixner-Pollaczek
Jacobi
Meixner
Krawtchouk
Laguerre
Charlier
Hermite


Dick Askey


discrete OP
quadratic lattice
Hahn class
very classical OP

The q-case

On top of the Askey-scheme is lying the q-Askey scheme, from which there are also arrows to the Askey scheme as $q \to 1$.

We take always $0 < q < 1$ and let $q \uparrow 1$ to the classical case. Some typical examples of q-analogues of classical concepts are (see Gasper & Rahman, *Basic hypergeometric series*):

- q-number: $[a]_q := \dfrac{1-q^a}{1-q} \to a$
- q-shifted factorial: $(a;q)_n := \displaystyle\prod_{k=0}^{n-1}(1-aq^k)$ (also for $n=\infty$).

  $\dfrac{(q^a;q)_k}{(1-q)^a} \to (a)_k$.
- q-hypergeometric series:

  $$ {}_{s+1}\phi_s\left(\begin{matrix} a_1,\ldots,a_{s+1} \\ b_1,\ldots,b_s \end{matrix}; q, z\right) := \sum_{k=0}^{\infty} \frac{(a_1;q)_k \ldots (a_{s+1};q)_k}{(b_1;q)_k \ldots (b_s;q)_k} \, \frac{z^k}{(q;q)_k}. $$

  $$ {}_{s+1}\phi_s\left(\begin{matrix} q^{a_1},\ldots,q^{a_{s+1}} \\ q^{b_1},\ldots,q^{b_s} \end{matrix}; q, z\right) \to {}_{s+1}F_s\left(\begin{matrix} a_1,\ldots,a_{s+1} \\ b_1,\ldots,b_s \end{matrix}; z\right). $$

The q-case (cntd.)

- q-derivative: $(D_q f)(x) := \dfrac{f(x) - f(qx)}{(1-q)x} \to f'(x)$.
- q-integral:
$$\int_0^1 f(x)\,d_qx := (1-q)\sum_{k=0}^{\infty} f(q^k)\,q^k \to \int_0^1 f(x)\,dx.$$

The q-case allows more symmetry which may be broken when taking limits for q to 1. In the elliptic case lying above the q-case there is even more symmetry.

Askey-Wilson polynomials (up to constant factor):

$$p_n(\cos\theta; a, b, c, d \mid q) := {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right).$$

Orthogonal with respect to a weight function on $(-1, 1)$.
A special case are the continuous q-ultraspherical polynomials ($a = -c = \beta^{\frac{1}{2}},\ b = -d = (q\beta)^{\frac{1}{2}}$).

Continuous q-ultraspherical polynomials

For $m \neq n$:

$$\int_0^\pi C_m(\cos\theta;\beta \mid q)\, C_n(\cos\theta;\beta \mid q) \left| \frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta};q)_\infty} \right|^2 d\theta = 0.$$

Generating function:

$$\left| \frac{(\beta e^{i\theta} t;q)_\infty}{(e^{i\theta} t;q)_\infty} \right|^2 = \sum_{n=0}^\infty C_n(x;\beta \mid q) t^n.$$

Limit formula to ultraspherical polynomials:
$C_n(x;q^\lambda \mid q) \to C_n^\lambda(x)$. These have generating function

$$(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^\infty C_n^\lambda(x) t^n.$$

SIAM Activity Group OPSF

The SIAM Activity Group on Orthogonal Polynomials and Special Functions

- Sends out a free bimonthly electronic newsletter;
- Organizes minisymposia on SIAM conferences;
- Awards the biennial Gábor Szegö Prize to an early-career researcher (at most 10 years after PhD) for outstanding research contributions in the area of orthogonal polynomials and special functions.
  Nominations before September 15, 2012.

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