Sklyanin algebra

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Evgeni Sklyanin

E. K. Sklyanin,
Some algebraic structures connected with the Yang-Baxter equation,
Functional Anal. Appl. 16 (1982), 263–270.

Idem, *Representations of a quantum algebra*,
Functional Anal. Appl. 17 (1983), 273–284.

Sklyanin algebra

Put $[A, B] := AB - BA, \quad \{A, B\} := AB + BA$.
α, β, γ means cyclic permutation of $1, 2, 3$.

Definition

Let J_{12}, J_{23}, J_{31} be complex constants such that

$$J_{12} + J_{23} + J_{31} + J_{12}\, J_{23}\, J_{31} = 0. \tag{1}$$

The Sklyanin algebra is the algebra $\mathcal{S}$ generated by S_0, S_1, S_2, S_3 with the six relations

$$\begin{aligned} [S_0, S_\alpha] &= i\, J_{\beta\gamma}\, \{S_\beta, S_\gamma\}, \\ [S_\alpha, S_\beta] &= i\, \{S_0, S_\gamma\}. \end{aligned} \tag{2}$$

If $J_1, J_2, J_3 \neq 0$ and $J_{\alpha\beta} = (J_\beta - J_\alpha)/J_\gamma$ then (1) holds.
If all $J_{\alpha\beta} = 0$ then $\mathcal{S}/\langle S_0 - 1\rangle \simeq \mathcal{U}(\mathrm{sl}(2, \mathbb{C}))$.

Structure constants

Let $J_{\alpha\beta}$ be complex constants. Finding J_1, J_2, J_3 such that

$$J_{\alpha\beta} = (J_\beta - J_\alpha)/J_\gamma \tag{3}$$

means solving the linear system

$$\begin{pmatrix} 1 & -1 & J_{12} \\ J_{23} & 1 & -1 \\ 1 & -J_{31} & -1 \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = 0.$$

The matrix has determinant $-(J_{12} + J_{23} + J_{31} + J_{12}\,J_{23}\,J_{31})$.
So (3) has nonzero solutions iff (1) holds.

If (1) holds then the matrix has rank 2, so the solutions of (3) are unique up to a constant factor.

There are degeneracies if $J_\gamma = 0$ for some γ or, equivalently, if $J_{\beta\gamma} = -1$, $J_{\gamma\alpha} = 1$ for some γ.

$J_{12} = J_{23} = J_{31} = 0$ iff $J_1 = J_2 = J_3$ (the sl(2) limit case).

Casimir operators

Theorem

The elements

$$K_0 := S_0^2 + S_1^2 + S_2^2 + S_3^2, \quad K_2 := J_1 S_1^2 + J_2 S_2^2 + J_3 S_3^2$$

are in the center of $\mathcal{S}$.

It is sufficient to prove that K_0 and K_2 commute wirh S_0 and S_1. Only the proof that $[K_0, S_1] = 0$ is straightforward.

For proving the other parts rewrite relations (2) such that an algorithmic proof may be possible:

$$S_\alpha S_0 = \frac{1 - J_{\beta\gamma}}{1 + J_{\beta\gamma}} S_0 S_\alpha - \frac{2iJ_{\beta\gamma}}{1 + J_{\beta\gamma}} S_\beta S_\gamma \,,$$
$$S_\gamma S_\beta = \frac{1 - J_{\beta\gamma}}{1 + J_{\beta\gamma}} S_\beta S_\gamma - \frac{2i}{1 + J_{\beta\gamma}} S_0 S_\alpha \,.$$

Then use the Mathematica package NCAlgebra 4.0.4, see `http://www.math.ucsd.edu/~ncalg/` .

Casimir operators (cntd.)

Then one can show by symbolic computation that K_0 and K_2 commute with the S_α. However, $[K_0, S_0]$ and $[K_2, S_0]$ reduce to cubic expressions which are not yet zero.

Alternatively, use reductions of $S_\alpha S_0, S_2 S_1, S_3 S_1, S_3 S_2$ to $S_0 S_\alpha, S_1 S_2, S_1 S_3, S_2 S_3$. (Note that this breaks the symmetry.) Then reduction of $[K_0, S_3], [K_2, S_3], [K_0, S_0], [K_2, S_0]$ runs forever. It turns out that this is already the case for reduction of $S_1 S_0 S_1$ and $S_1 S_0 S_2$ because, after a few reduction steps, one of the terms is a constant multiple of the starting expression. Since the other terms are of the form $S_i S_j S_k$ $(i \le j \le k)$, we can express $S_1 S_0 S_1$ and $S_1 S_0 S_2$ in terms of them and add these to the relations. Then, with the aid of these two added relations, every cubic expression can be reduced to a linear combination of $S_i S_j S_k$ $(i \le j \le k)$. Now indeed all commuting properties of K_0 and K_2 can be proved symbolically.

Casimir operators (cntd.)

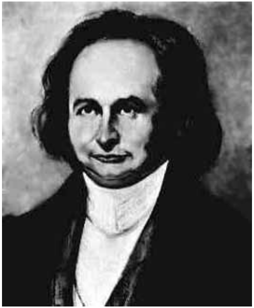
In the sl(2) limit case $J_1 = J_2 = J_3 = 1$ we have $K_0 = S_0^2 + S_1^2 + S_2^2 + S_3^2$, $K_2 = S_1^2 + S_2^2 + S_3^2$ while S_0 is in the center, so K_0 and K_2 are then essentially the same.

Sklyanin (1982) asks whether the space of homogeneous polynomials of degree p in the generators has the same dimension as the homogeneous polynomials of degree p in four commuting variables (clear already for $p = 2$ and $p = 3$). For $J_1, J_2, J_3 \neq 0$ this was answered positively by S. P. Smith & J. T. Stafford (Compositio Math. 83 (1992), 259–289).

Sklyanin (1982) also asks whether the center of $\mathcal{S}$ is generated by K_0, K_2. This turns out to be true in the generic case, see Levasseur & Smith (Bull. Soc. Math. France 121 (1993), 35–90, Proposition 6.12).

Jacobi theta functions

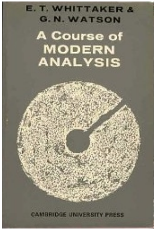
C. G. J. Jacobi (1829),
Fundamenta Nova Theoriae Functionum Ellipticarum



Jacobi

E. T. Whittaker

G. N. Watson



[WW]

Jacobi theta functions (cntd.)

Let $q = e^{i\pi\tau}$ $\quad (0 < |q| < 1, \operatorname{Im}\tau > 0)$.

Modified theta function (as in Gasper & Rahman):

$$\theta(w;q) := (w, q/w; q)_\infty = \frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{1}{2}k(k-1)} w^k.$$

Jacobi theta functions (θ_a as in Tannery & Molk, HTF and Sklyanin; ϑ_a as in Jacobi and in Whittaker & Watson):

$$\theta_a(z) = \theta_a(z,q) = \theta_a(z \mid \tau) = \vartheta_a(\pi z, q) \qquad (a = 1,2,3,4):$$

$$\theta_1(z) := i\, q^{1/4} (q^2;q^2)_\infty\, e^{-\pi i z}\, \theta(e^{2\pi i z}; q^2),$$

$$\theta_2(z) := q^{1/4} (q^2;q^2)_\infty\, e^{-\pi i z}\, \theta(-e^{2\pi i z}; q^2) = \theta_1(z + \tfrac{1}{2}),$$

$$\theta_3(z) := (q^2;q^2)_\infty\, \theta(-q\, e^{2\pi i z}; q^2) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2\pi i k z},$$

$$\theta_4(z) := (q^2;q^2)_\infty\, \theta(q\, e^{2\pi i z}; q^2) = \theta_3(z + \tfrac{1}{2}).$$

$\theta_1(z)$ is odd; $\theta_2(z), \theta_3(z), \theta_4(z)$ are even. $\qquad \theta_a := \theta_a(0)$.

Structure constants reparametrized

Fix η and τ. Put $J_\alpha = \dfrac{\theta_{\alpha+1}(2\eta)\,\theta_{\alpha+1}}{\theta^2_{\alpha+1}(\eta)}$ $(\alpha = 1,2,3)$. Then

$$J_{12} = \frac{\theta_4^2(\eta)\theta_1^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)}, \quad J_{23} = \frac{\theta_2^2(\eta)\theta_1^2(\eta)}{\theta_3^2(\eta)\theta_4^2(\eta)}, \quad J_{31} = -\frac{\theta_3^2(\eta)\theta_1^2(\eta)}{\theta_4^2(\eta)\theta_2^2(\eta)}.$$

For the proof compute $J_{\alpha\beta} = (J_\beta - J_\alpha)/J_\gamma$ by expressing $\theta_2(2\eta), \theta_3(2\eta), \theta_4(2\eta)$ in terms of the $\theta_a^2(\eta)$ [WW, p.488, Ex. 4]. Note that

$$\begin{aligned} &J_{12} + J_{23} + J_{31} + J_{12}\,J_{23}\,J_{31} \\ &= \frac{\theta_1^2(\eta)}{\theta_2^2(\eta)\theta_3^2(\eta)\theta_4^2(\eta)}\left(\theta_4^4(\eta)+\theta_2^4(\eta)-\theta_3^4(\eta)-\theta_1^4(\eta)\right) = 0 \ \text{(necessarily)}. \end{aligned}$$

This last identity is also in [WW, p.488, Ex. 4].

Representation on meromorphic functions

Fix ℓ. For $a = 1, 2, 3, 4$ put $i_a = i$ if $a = 3$ and $i_a = 1$ otherwise. For $f(z)$ meromorphic put

$$(S_{a-1}f)(z) := \frac{i_a\theta_a(\eta)}{\theta_1(2z)} \Big(\theta_a(2z - 2\ell\eta)f(z + \eta) \\ - \theta_a(-2z - 2\ell\eta)f(z - \eta)\Big) \quad (a = 1, 2, 3, 4). \quad (4)$$

Theorem

Formula (4) *defines a representation of the Sklyanin algebra* $\mathcal{S}$ *on the space of meromorphic functions.*

For the proof compute the terms of $f(z + 2\eta)$, $f(z)$, $f(z - 2\eta)$ in $(i_a i_b)^{-1}(S_{a-1}S_{b-1}f)(z)$.

Representation on meromorphic functions (Proof)

$$(i_a i_b)^{-1}(S_{a-1}S_{b-1}f)(z) = \frac{\theta_a(\eta)\theta_b(\eta)}{\theta_1(2z)}$$
$$\times \left(\frac{\theta_a(2z-2\ell\eta)}{\theta_1(2z+2\eta)}\,\theta_b(2z-2(\ell-1)\eta)\,f(z+2\eta) \right.$$
$$- \frac{\theta_a(2z-2\ell\eta)}{\theta_1(2z+2\eta)}\,\theta_b(-2z-2(\ell+1)\eta)\,f(z)$$
$$- \frac{\theta_a(-2z-2\ell\eta)}{\theta_1(2z-2\eta)}\,\theta_b(2z-2(\ell+1)\eta)\,f(z)$$
$$\left. + \frac{\theta_a(-2z-2\ell\eta)}{\theta_1(2z-2\eta)}\,\theta_b(-2z-2(\ell-1)\eta)\,f(z-2\eta) \right).$$

Now compute the coefficients of $f(z+2\eta)$, $f(z)$, $f(z-2\eta)$ in, for instance, $\big((S_0S_3+S_3S_0+i^{-1}S_2S_1-i^{-1}S_1S_2)f\big)(z)$. They turn out to vanish by combining formulas [WW, p.488, Ex.3].

Reps on meromorphic functions (Proof, cntd.)

For instance, derive from

$$\theta_1(y+z)\theta_4(y-z)\theta_2\theta_3 = \theta_1(y)\theta_4(y)\theta_2(z)\theta_3(z) + \theta_2(y)\theta_3(y)\theta_1(z)\theta_4(z), \quad (5)$$

$$\theta_2(y+z)\theta_3(y-z)\theta_2\theta_3 = \theta_2(y)\theta_3(y)\theta_2(z)\theta_3(z) - \theta_1(y)\theta_4(y)\theta_1(z)\theta_4(z) \quad (6)$$

that

$$\theta_1(z)\theta_4(z)\Big(\theta_1(y+z)\theta_4(y-z) + \theta_1(y-z)\theta_4(y+z)\Big) + \theta_2(z)\theta_3(z)\Big(\theta_2(y+z)\theta_3(y-z) - \theta_2(y-z)\theta_3(y+z)\Big) = 0,$$

$$\theta_2(z)\theta_3(z)\Big(\theta_1(y+z)\theta_4(y-z) - \theta_1(y-z)\theta_4(y+z)\Big) - \theta_1(z)\theta_4(z)\Big(\theta_2(y+z)\theta_3(y-z) + \theta_2(y-z)\theta_3(y+z)\Big) = 0.$$

Theta addition formulas

The master addition formula is the three-term identity

$$\theta(xy, x/y, uv, u/v; q^2) - \theta(xv, x/v, uy, u/y; q^2) = uy^{-1}\theta(yv, y/v, xu, x/u; q^2), \quad (7)$$

see Gasper & Rahman, (11.4.3). Curiously enough, [WW] only gives (7) in disguised form for the function $\sigma(z)$, see [WW, p.451, Ex.5; p.473, §21.43]. It is usually ascribed to Riemann, but [WW] ascribes it to Weierstrass.

Many addition formulas in [WW] are special cases of (7). For instance, (5) (on the previous page) can be rewritten as

$$\theta(yz, qy/z, -1, -q; q^2) = \theta(y, qy, -z, -qz; q^2) + \theta(-y, -qy, z, qz; q^2)$$

and then follows from (7) by the substitution $(x, u, v, y) \to (q^{\frac{1}{2}}y, q^{-\frac{1}{2}}z, q^{\frac{1}{2}}, -q^{\frac{1}{2}})$. Also use that $\theta(w; q) = -w\theta(qw; q)$.

Reps on meromorphic functions (Casimir operators)

Theorem

In the representation of $\mathcal{S}$ on the space of meromorphic functions the Casimir operators act as

$$K_0 = 4\theta_1^2\big((2\ell+1)\eta\big), \quad K_2 = 4\theta_1\big((2\ell+2)\eta\big)\,\theta_1(2\ell\eta).$$

For the proof use [WW, p.468, (iv)]:

$$\sum_{a=1}^{4} i_a^2\theta_a(z_1)\theta_a(z_2)\theta_a(z_3)\theta_a(z_4) = 2\theta_1(z_1')\theta_1(z_2')\theta_1(z_3')\theta_1(z_4'),$$

where $z_a' := \frac{1}{2}(z_1+z_2+z_3+z_4) - z_a$.

Finite dimensional subrepresentation

From $\theta(qw;q) = -w^{-1}\theta(w;q)$ we see

$$\theta_a(z+2\tau) = e^{-4\pi i(z+\tau)}\theta_a(z),$$
$$\theta_a(z+k\tau) = e^{-4\pi ikz}\theta_a(z-k\tau) \quad (k\in\mathbb{Z}).$$

Fix $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let V_ℓ be the space of holomorphic functions f such that $f(z+1) = f(z) = f(-z)$ and

$$f(z+\tau) = e^{-4\pi i\ell(2z+\tau)}f(z), \quad \text{hence} \quad f(z+\tfrac{1}{2}k\tau) = e^{-8\pi i\ell kz}f(z-\tfrac{1}{2}k\tau)$$

for $k\in\mathbb{Z}$. If $f\in V_\ell$ then $S_{a-1}f\in V_\ell$. Indeed, from

$$(S_{a-1}f)(z) = \frac{i_a\theta_a(\eta)}{\theta_1(2z)}\Big(\theta_a(2z-2\ell\eta)f(z+\eta)-\theta_a(-2z-2\ell\eta)f(z-\eta)\Big)$$

verify the three symmetries and check that the expression in brackets vanishes at the zeros $\frac{1}{2}k\tau$ ($k\in\mathbb{Z}$) of $\theta_1(2z)$.

Finite dimensional subrepresentation (cntd.)

Theorem

The space V_ℓ has dimension $2\ell+1$.

For the proof note that a holomorphic function f is in V_ℓ iff $f(z)=\sum_{j=-\infty}^{\infty} c_j e^{2\pi i j z}$ with $c_j=c_{-j}$ and $c_{j+4\ell}=e^{2\pi i(j+2\ell)\tau}c_j$. Hence V_ℓ has the $2\ell+1$ functions $f_j=g_j+g_{-j}$ $(j=0,\ldots,2\ell)$ as a basis, where

$$g_j(z):=\sum_{k=0}^{\infty} c_{j+4k\ell}e^{(2\pi i(j+4k\ell)z}.$$

Since $c_{j+4k\ell}=c_j e^{2\pi i(kj+2k^2\ell\tau)}$, we can take

$$g_j(z)=e^{2\pi ijz}\sum_{k=-\infty}^{\infty} e^{4\pi ik^2\ell\tau}e^{2\pi ik(4\ell z+j\tau)}=e^{2\pi ijz}\theta_3(4\ell z+j\tau\mid 4\ell\tau)$$
$$=(q^2;q^2)_\infty e^{2\pi ijz}\theta(-q^{4(\ell+j)}e^{8\pi i\ell z};q^{8\ell}).$$

Finite dimensional subrepresentation (cntd.)

As observed in Rosengren, Ramanujan J. 13 (2007), 131-166, Remark 5.2, any function of the form

$$f(z) = \prod_{j=1}^{2\ell} \theta(a_j e^{2\pi i z}, a_j e^{-2\pi i z}; q^2)$$

belongs to V_ℓ. Moreover, for generic a, b, p the functions

$$F_k(z) := \prod_{j=0}^{k-1} \theta(ap^j e^{2\pi i z}, ap^j e^{-2\pi i z}; q^2) \prod_{j=0}^{2\ell-k-1} \theta(bp^j e^{2\pi i z}, bp^j e^{-2\pi i z}; q^2)$$

$(k = 0, 1, \ldots, 2\ell)$ form a basis of V_ℓ.