Okounkov's *BC*-type interpolation Macdonald polynomials and their $q = 1$ limit

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Notes of lectures given at the 30th International Colloquium on Group Theoretical Methods in Physics, Gent, Belgium, 14–18 July 2014, at the conference *Exact Solvability and Symmetry Avatars*, Montréal, Canada, 25–29 August 2014, and in the Representation Theory Seminar, University of Amsterdam, 30 October 2014.

Last modfied: 3 March 2015.

Reference

See details of this talk in my preprint

T. H. Koornwinder,

Okounkov's BC-type interpolation Macdonald polynomials and their $q = 1$ *limit*,

http://arxiv.org/abs/1408.5993

Prelude: one variable, $q = 1$

monomial	$P_m(x)$	x^m
factorial	$P_m^{\mathrm{ip}}(x)$	$(-1)^m(-x)_m$
quadr. factorial	$P_m^{\mathrm{ip}}(x;\alpha)$	$(-1)^m(\alpha-x)_m(\alpha+x)_m$
Jacobi	$P_m(x;\alpha,\beta)$	$\frac{(-1)^m(\alpha+1)_m}{(m+\alpha+\beta+1)_m}\,{}_2F_1\left({-m,m+\alpha+\beta+1 \atop \alpha+1};x\right)$

$$P_m^{\mathrm{ip}}(x) = x(x-1)\ldots(x-m+1),$$
$$P_m^{\mathrm{ip}}(x;\alpha) = \big(x^2-\alpha^2\big)\big(x^2-(\alpha+1)^2\big)\big(x^2-(\alpha+m-1)^2\big),$$

(even) monic polynomials determined by vanishing properties.

Limits:

$$\left.\begin{matrix} P_m(x;\alpha,\beta) \\ P_m^{\mathrm{ip}}(x) \end{matrix}\right\} = P_m(x) + \text{lower degree},$$
$$P_m^{\mathrm{ip}}(x;\alpha) \qquad = P_m(x^2) + \text{lower degree};$$

$$(2\alpha)^{-m}P_m^{\mathrm{ip}}(x+\alpha,\alpha) \xrightarrow{\alpha\to\infty} P_m^{\mathrm{ip}}(x),$$
$$P_m(x;\alpha,\beta) \xrightarrow{\alpha\to\infty} P_m(x-1).$$

Prelude: one variable, $q = 1$, binomial formulas

monomial	$P_m(x)$	x^m
factorial	$P_m^{\mathrm{ip}}(x)$	$(-1)^m(-x)_m$
quadr. factorial	$P_m^{\mathrm{ip}}(x;\alpha)$	$(-1)^m(\alpha-x)_m(\alpha+x)_m$
Jacobi	$P_m(x;\alpha,\beta)$	$\frac{(-1)^m(\alpha+1)_m}{(m+\alpha+\beta+1)_m}\,{}_2F_1\left(\begin{matrix}-m,m+\alpha+\beta+1\\ \alpha+1\end{matrix};x\right)$

$$\frac{P_m(x;\alpha,\beta)}{P_m(0;\alpha,\beta)} = \sum_{k=0}^{m} \frac{(-m)_k(m+\alpha+\beta+1)_k}{(\alpha+1)_k k!}\,x^k$$

$$\stackrel{\alpha':=\frac{1}{2}(\alpha+\beta+1)}{=} \sum_{k=0}^{m} \frac{P_k^{\mathrm{ip}}(m+\alpha';\alpha')}{P_k^{\mathrm{ip}}(k+\alpha';\alpha')}\,\frac{P_k(x)}{P_k(0;\alpha)} \qquad \xrightarrow{\alpha\to\infty}$$

$$(1-x)^m = \sum_{k=0}^{m} \binom{m}{k}(-x)^k, \quad \text{i.e.,}$$

$$\frac{P_m(1-x)}{P_m(1)} = \sum_{k=0}^{m} \frac{P_k^{\mathrm{ip}}(m)}{P_k^{\mathrm{ip}}(k)}\,\frac{P_k(-x)}{P_k(1)}\,.$$

Prelude: one variable, q-case

Throughout $0 < q < 1$. Four classes of polynomials:

monomial	$P_m(x;q)$	x^m
q-factorial	$P_m^{\rm ip}(x;q)$	$x^m(x^{-1};q)_m$
quadr. q-factorial	$P_m^{\rm ip}(x;q,a)$	$\dfrac{(ax,ax^{-1};q)_m}{(-1)^m q^{\frac{1}{2}m(m-1)}a^m}$
Askey-Wilson	$\dfrac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}$	${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4\end{matrix};q,q\right)$

Put $\langle x;y\rangle := x+x^{-1}-y-y^{-1}$.

$$P_m^{\rm ip}(x;q) = (x-1)(x-q)\ldots(x-q^{m-1}),$$

$$P_m^{\rm ip}(x;q,a) = \prod_{j=0}^{m-1}(x+x^{-1}-a\,q^j-a^{-1}q^{-j}) = \prod_{j=0}^{m-1}\langle x;a\,q^j\rangle,$$

monic polynomials resp. monic symmetric Laurent polynomials determined by their vanishing properties.

Prelude: one variable, q-case, limits

monomial	$P_m(x;q)$	x^m
q-factorial	$P^{\rm ip}_m(x;q)$	$x^m(x^{-1};q)_m$
quadr. q-factorial	$P^{\rm ip}_m(x;q,a)$	$\dfrac{(ax,ax^{-1};q)_m}{(-1)^m q^{\frac12 m(m-1)} a^m}$
Askey-Wilson	$\dfrac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}$	${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4\end{matrix};q,q\right)$

$$\left.\begin{matrix} P_m(x;q;a_1,a_2,a_3,a_4) \\ P^{\rm ip}_m(x;q,a) \\ P^{\rm ip}_m(x;q) \end{matrix}\right\} = P_m(x;q) + \text{lower degree};$$

$$a^{-m}P^{\rm ip}_m(ax;q,a) \stackrel{a\to\infty}{\longrightarrow} P^{\rm ip}_m(x;q),$$

$$a_1^{-m}\,P_m(a_1x;q;a_1,a_2,a_3,a_4) \stackrel{a_1\to\infty}{\longrightarrow} P_m(x;q).$$

Prelude: one variable, q-case, binomial formulas

$$a^{-m}P^{\mathrm{ip}}_m(ax;q,a)\overset{a\to\infty}{\longrightarrow}P^{\mathrm{ip}}_m(x;q),$$

$$a_1^{-m}\,P_m(a_1x;q,t;a_1,a_2,a_3,a_4)\overset{a_1\to\infty}{\longrightarrow}P_m(x;q).$$

$$\frac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}=\sum_{k=0}^m\frac{q^k}{(a_1a_2,a_1a_3,a_1a_4,q;q)_k}$$

$$\times\,(q^{-m},q^{m-1}a_1a_2a_3a_4;q)_k\,(a_1x,a_1x^{-1};q)_k$$

$$\overset{a_1':=(q^{-1}a_1a_2a_3a_4)^{\frac12}}{=}\sum_{k=0}^m\frac{P^{\mathrm{ip}}_k(q^ma_1';q,a_1')}{P^{\mathrm{ip}}_k(q^ka_1';q,a_1')}\,\frac{P^{\mathrm{ip}}_k(x;q,a_1)}{P_k(a_1;q;a_1,a_2,a_3,a_4)}$$

$$\overset{x:=a_1x,\ a_1\to\infty}{\longrightarrow}$$

$$x^m={}_2\phi_0\left(\begin{matrix}q^{-m},x^{-1}\\ -\end{matrix};q,q^mx\right)=\sum_{k=0}^m\frac{(q^{-m},x^{-1};q)_k}{(-1)^kq^{\frac12k(k-1)}(q;q)_k}\,(q^mx)^k,$$

i.e.,
$$\frac{P_m(x;q)}{P_m(1;q)}=\sum_{k=0}^m\frac{P^{\mathrm{ip}}_k(q^m;q)}{P^{\mathrm{ip}}_k(q^k;q)}\,\frac{P^{\mathrm{ip}}_k(x;q)}{P_k(1;q)}\,.$$

Prelude: one variable, limits for $q \uparrow 1$

monomial	$P_m(x;q)$	x^m
q-factorial	$P_m^{\mathrm{ip}}(x;q)$	$x^m(x^{-1};q)_m$
quadr. q-factorial	$P_m^{\mathrm{ip}}(x;q,a)$	$\dfrac{(ax,ax^{-1};q)_m}{(-1)^m q^{\frac{1}{2}m(m-1)} a^m}$
Askey-Wilson	$\dfrac{P_m(x;q;a_1,a_2,a_3,a_4)}{P_m(a_1;q;a_1,a_2,a_3,a_4)}$	${}_4\phi_3\left(\begin{matrix} q^{-n},q^{n-1}a_1a_2a_3a_4,a_1x,a_1x^{-1} \\ a_1a_2,a_1a_3,a_1a_4 \end{matrix};q,q\right)$

$$P_m(x;q) \to P_m(x),$$

$$(q-1)^{-m}\,P_m^{\mathrm{ip}}(q^x;q) \to P_m^{\mathrm{ip}}(x),$$

$$P_m^{\mathrm{ip}}(x;q) \to P_m(x-1),$$

$$(1-q)^{-2m}P_m^{\mathrm{ip}}(q^x;q,q^\alpha) \to P_m^{\mathrm{ip}}(x;\alpha),$$

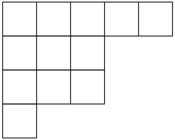
$$P_m^{\mathrm{ip}}(x;q,q^\alpha) \to (-4)^m\,P_m(\tfrac{1}{4}(2-x-x^{-1})),$$

$$P_m(x;q;q^{\alpha+1},-q^{\beta+1},1,-1) \to (-4)^m\,P_m(\tfrac{1}{4}(2-x-x^{-1});\alpha,\beta).$$

Partitions

Fixed n. $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n$ $(\lambda_1\ge\lambda_2\ge\ldots\ge\lambda_n\ge 0)$ is a *partition*. Set of all such λ is denoted by Λ_n. λ has *length* $\ell(\lambda):=|\{j\mid\lambda_j>0\}|$ and *weight* $|\lambda|:=\lambda_1+\cdots+\lambda_n$.

Example $\lambda=(5,3,3,1)$ has *Young diagram*



dominance partial ordering:

$\mu\le\lambda$ iff $\mu_1+\cdots+\mu_i\le\lambda_1+\cdots+\lambda_i$ $(i=1,\ldots,n)$.

inclusion partial ordering:

$\mu\subset\lambda$ iff $\mu_i\le\lambda_i$ $(i=1,\ldots,n)$.

$W_n:=S_n\ltimes(\mathbb{Z}_2)^n$.

$x=(x_1,\ldots,x_n)$, $x^\mu:=x_1^{\mu_1}\ldots x_n^{\mu_n}$, $\lambda=(\lambda_1,\ldots,\lambda_n)$ partition.

$m_\lambda(x):=\sum_{\mu\in S_n\lambda}x^\mu,\qquad \widetilde{m}_\lambda(x):=\sum_{\mu\in W_n\lambda}x^\mu.$

polynomials of A and BC type

Throughout $\quad n \geq 2, \quad 0 < q < 1, \quad 0 < t < 1, \quad \tau > 0.$

1	A	Macdonald	$P_\lambda(x;q,t)$
2	A	interpolation Macdonald	$P_\lambda^{\rm ip}(x;q,t)$
3	BC	BC_n interpolation Macdonald	$P_\lambda^{\rm ip}(x;q,t,a)$
4	BC	Koornwinder	$P_\lambda(x;q,t;a_1,a_2,a_3,a_4)$
5	A	Jack	$P_\lambda(x;\tau)$
6	A	interpolation Jack	$P_\lambda^{\rm ip}(x;\tau)$
7	BC	BC_n interpolation Jack	$P_\lambda^{\rm ip}(x;\tau,a)$ (new)
8	BC	Jacobi	$P_\lambda(x;\tau;\alpha,\beta)$

For 1,2,5,6,8 and for $P_\lambda^{\rm ip}(x^{\frac12};\tau,\alpha)$ in 7: $\quad = m_\lambda + \sum_{\mu<\lambda} u_{\lambda,\mu} m_\mu.$

For 3,4: $\quad = \widetilde{m}_\lambda + \sum_{\mu<\lambda} u_{\lambda,\mu} \widetilde{m}_\mu.$

Definition by orthogonality for 1,4,5,8.

Inventors of interpolation functions

interpolation Jack:
Sahi (1994), Knop & Sahi (1996); Okounkov & Olshanski (1997)

interpolation Macdonald:
Sahi (1996); Knop (1997); Okounkov (1998)

BC_n interpolation Macdonald:
Okounkov (1998)

Pictures

Macdonald

Knop

Sahi

Okounkov

Olshanskii

Orthogonality

$\Delta(z) := \Delta_+(z)\Delta_+(z^{-1})$, $\quad \oint \frac{dz}{z} := \prod_{j=1}^n \oint \frac{dz_j}{z_j}$.

Macdonald and Jack: $\displaystyle\oint P_\lambda(z)\, m_\mu(z^{-1})\, \Delta(z)\, \frac{dz}{z} = 0$ if $\mu < \lambda$.

$\Delta_+(z) = \prod_{1\le i<j\le n} \frac{(z_i z_j^{-1};q)_\infty}{(t z_i z_j^{-1};q)_\infty}$ resp. $\prod_{1\le i<j\le n}(1 - z_i z_j^{-1})^\tau$.
Homogeneous of degree $|\lambda|$.

Koornwinder: $\displaystyle\oint P_\lambda(z)\, \widetilde{m}_\mu(z)\, \Delta(z)\, \frac{dz}{z} = 0$ if $\mu < \lambda$.

$$\Delta_+(z) := \prod_{j=1}^n \frac{(z_j^2;q)_\infty}{(a_1 z_j, a_2 z_j, a_3 z_j, a_4 z_j;q)_\infty} \prod_{1\le i<j\le n} \frac{(z_i z_j, z_i z_j^{-1};q)_\infty}{(t z_i z_j, t z_i z_j^{-1};q)_\infty}.$$

Jacobi: $\int_{[0,1]^n} P_\lambda(x)\, m_\mu(x)\, \Delta(x)\, dx = 0$ if $\mu < \lambda$.

$$\Delta(x) := \prod_{j=1}^n x_j^\alpha (1 - x_j)^\beta \prod_{1\le i<j\le n} |x_i - x_j|^{2\tau}.$$

In all cases **full orthogonality** can be proved.

Let $\delta := (n-1, n-2, \ldots, 1, 0)$.

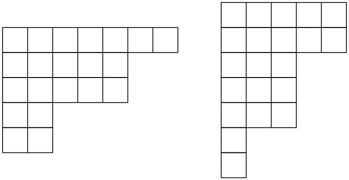
$$\left.\begin{array}{l} P_\lambda^{\rm ip}(\mu+\tau\delta;\tau) \\ P_\lambda^{\rm ip}(\mu+\tau\delta+\alpha;\tau,\alpha) \\ P_\lambda^{\rm ip}(q^\mu t^\delta;q,t) \\ P_\lambda^{\rm ip}(q^\mu t^\delta a;q,t,a) \end{array}\right\} = 0 \quad \text{if not } \lambda\subset\mu,$$

$$\text{in particular} \quad = 0 \quad \text{if } |\mu|\le|\lambda|,\ \mu\ne\lambda.$$

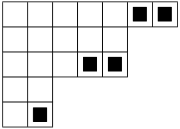
Diagrams

Young diagram of partition λ consists of boxes (i,j) with $i = 1,\ldots,\ell(\lambda)$ and $j = 1,\ldots,\lambda_i$. The *conjugate* partition λ' has transposed diagram. Then $\ell(\lambda') = \lambda_1$.

Example $\lambda = (7,5,5,2,2)$, $\lambda' = (5,5,3,3,3,1,1)$:



If $\mu \subset \lambda$ then $\lambda - \mu := \{s \in \lambda \mid s \notin \mu\}$ is a *skew diagram*.
$\lambda - \mu$ is a *horizontal strip* if $\lambda'_j - \mu'_j \le 1$ $(j = 1,\ldots,\lambda_1)$.
Then write $\mu \preccurlyeq \lambda$.

Example $\mu = (5,5,3,2,1)$, $\lambda - \mu =$ 

Tableaux

The diagram of $\lambda \in \Lambda_n$ becomes a *tableau* T if the boxes (i,j) of λ are filled by numbers $T(i,j)$.
Then T is called a *semistandard* tableau of *shape* λ with entries in $\{1,\ldots,n\}$ if $T(i,j) \in \{1,2,\ldots,n\}$ is weakly increasing in j and strongly increasing in i.

Similarly T is a *reverse semistandard* tableau if $T(i,j)$ is weakly decreasing in j and strongly decreasing in i. If T is semistandard then $S(i,j) := n+1-T(i,j)$ gives a reverse semistandard tableau S.

Example (semistandard tableau)
$\lambda = (7,5,5,2,2)$, $\ell(\lambda) = 5$, $n = 6$:

1	1	1	3	4	6	6
2	2	2	5	5		
3	3	5	6	6		
4	5					
5	6					

In general, let:

$\lambda^{(i)} := \{s \in \lambda \mid T(s) \le i\}$, this is a partition.

Then: $0^n = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(n-1)} \subset \lambda^{(n)} = \lambda$ and
$\lambda^{(i)} - \lambda^{(i-1)} = \{s \in \lambda \mid T(s) = i\}$.
T restricted to the diagram of $\lambda^{(i)}$ is a semistandard tableau of shape $\lambda^{(i)}$ with entries in $\{1, \ldots, i\}$. In the example:

$() \subset (3) \subset (3,3) \subset (4,3,2) \subset (5,3,2,1) \subset (5,5,3,2,1) \subset (7,5,5,2,2)$.

Combinatorial formula for Macdonald polynomials

$$P_\lambda(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\lambda} x_{T(s)}\,,$$

sum over all semistandard tableaux T of shape λ with entries in $\{1,\ldots,n\}$, so $T(s)=i$ for s in the horizontal strip $\lambda^{(i)}-\lambda^{(i-1)}$.

$$\psi_T(q,t)=\prod_{i=1}^n \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t).$$ Hence

$$P_\lambda(x;q,t)=\sum_T\prod_{i=1}^n\left(\psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t)\,x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|}\right)$$

$$=\sum_T\prod_{i=1}^n P_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i;q,t),\quad \text{where}$$

$P_{\lambda/\mu}(z;q,t):=\psi_{\lambda/\mu}(q,t)\,z^{|\lambda|-|\mu|}$ if $\lambda\in\Lambda_n$, $\mu\in\Lambda_{n-1}$, $\mu\preccurlyeq\lambda$,
and $:=0$ otherwise.

Hence we have the *branching formula*

$$P_\lambda(x_1,\ldots,x_{n-1},x_n;q,t))=\sum_\mu P_{\lambda/\mu}(x_n;q,t)\,P_\mu(x_1,\ldots,x_{n-1};q,t).$$

Combinatorial formula (cntd.)

$$P_\lambda(x;q,t) = \sum_T \prod_{i=1}^n \left(\psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q,t)\, x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|}\right).$$

Then $\psi_{\lambda/\mu}(q,t) = \psi'_{\lambda'/\mu'}(t,q)$, where

$$e_r(x)\, P_\lambda(x;q,t) = \sum_\mu \psi'_{\mu/\lambda}(q,t) P_\mu(x;q,t) \quad (\textit{Pieri formula}).$$

$$\psi_{\mu/\nu}(q,t) = \prod_{s\in(R\backslash C)_{\mu/\nu}} \frac{b_\nu(s;q,t)}{b_\mu(s;q,t)}\,.$$

Here $s \in (R\backslash C)_{\mu/\nu}$ iff $s \in \nu$ in a row of μ intersecting with $\mu - \nu$ but outside each column of μ intersecting with $\mu - \nu$.

$$b_\lambda(s;q,t) := \frac{1 - q^{a_\lambda(s)}\, t^{l_\lambda(s)+1}}{1 - q^{a_\lambda(s)+1}\, t^{l_\lambda(s)}}\,,$$

$$a_\lambda(i,j) := \lambda_i - j, \quad l_\lambda(i,j) := |\{k > i \mid \lambda_k \ge j\}|.$$

Combinatorial formula (cntd.)

For a reverse semistandard tableau T with entries in $\{1,\ldots,n\}$ put $\lambda^{(i)} := \{s \in T \mid T(s) > i\}$. Then $\lambda^{(i-1)} - \lambda^{(i)} = \{s \in T \mid T(s) = i\}$.

Now write the combinatorial formula for Macdonald polynomials as a sum over reverse semistandard tableaux:

$$P_\lambda(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\lambda} x_{T(s)}\,, \text{ where}$$

$$\psi_T(q,t) = \prod_{i=1}^n \psi_{\lambda^{(i-1)}/\lambda^{(i)}}(q,t).$$

Combinatorial formulas for interpolation polynomials

Macdonald polynomials:

$$P_\lambda(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\lambda} x_{T(s)}\,.$$

Interpolation Macdonald polynomials:

$$P_\lambda^{\mathrm{ip}}(x;q,t)=\sum_T \psi_T(q,t)\prod_{s\in\lambda} \big(x_{T(s)}-q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}\big)\,.$$

BC_n interpolation Macdonald polynomials:

$$P_\lambda^{\mathrm{ip}}(x;q,t,a)=\sum_T \psi_T(q,t)\prod_{s\in\lambda} \langle x_{T(s)};q^{a'_\lambda(s)}t^{n-T(s)-l'_\lambda(s)}a\rangle.$$

$$a'_\lambda(i,j):=j-1,\quad l'_\lambda(i,j):=i-1,\quad \langle x;y\rangle:=x+x^{-1}-y-y^{-1}.$$

The T-sums are over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$.

Combinatorial formulas ($q = 1$)

$$\psi_T(\tau) := \prod_{i=1}^{n} \prod_{s \in (R\backslash C)_{\lambda^{(i-1)}/\lambda^{(i)}}} \frac{b_{\lambda^{(i)}}(s;\tau)}{b_{\lambda^{(i-1)}}(s;\tau)}.$$

$$b_\lambda(s;\tau) := \frac{a_\lambda(s) + \tau(l_\lambda(s)+1)}{a_\lambda(s) + \tau l_\lambda(s) + 1}.$$

Jack polynomials: $\quad P_\lambda(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} x_{T(s)}.$

Interpolation Jack polynomials:

$$P_\lambda^{\mathrm{ip}}(x;\tau) = \sum_T \psi_T(\tau) \prod_{s\in\lambda} \left(x_{T(s)} - a'_\lambda(s) - \tau(n - T(s) - l'_\lambda(s))\right).$$

BC_n interpolation Jack polynomials: (new)

$$P_\lambda^{\mathrm{ip}}(x;\tau,\alpha) := \lim_{q\uparrow 1}(1-q)^{-2|\lambda|} P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau,q^\alpha) = \sum_T \psi_T(\tau) \\ \times \prod_{s\in\lambda} \left(x_{T(s)}^2 - \left(a'_\lambda(s) + \tau(n - T(s) - l'_\lambda(s)) + \alpha\right)^2\right).$$

The T-sums over all reverse tableaux T of shape λ with entries in $\{1,\ldots,n\}$.

Limits, $q \uparrow 1$

$$P_\lambda(x;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-x-x^{-1});\tau;\alpha,\beta),$$

$$(1-q)^{-2|\lambda|}P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau,q^\alpha) \to P_\lambda^{\mathrm{ip}}(x;\tau,\alpha),$$

$$P_\lambda^{\mathrm{ip}}(x;q,q^\tau,q^\alpha) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-x-x^{-1});\tau),$$

$$(q-1)^{-|\lambda|}\,P_\lambda^{\mathrm{ip}}(q^x;q,q^\tau) \to P_\lambda^{\mathrm{ip}}(x;\tau),$$

$$P_\lambda^{\mathrm{ip}}(x;q,q^\tau) \to P_\lambda(x-1^n,\tau),$$

$$P_\lambda(x;q,q^\tau) \to P_\lambda(x;\tau).$$

Limits, q fixed

Highest term:

$$\left.\begin{array}{l} P_\lambda(x;q,t;a_1,a_2,a_3,a_4) \\ P_\lambda^{\mathrm{ip}}(x;q,t,a) \\ P_\lambda^{\mathrm{ip}}(x;q,t) \end{array}\right\} = P_\lambda(x;q,t) + \text{degree lower than } |\lambda|;$$

$$\left.\begin{array}{l} P_\lambda(x;\tau;\alpha,\beta) \\ P_\lambda^{\mathrm{ip}}(x,\tau) \end{array}\right\} = P_\lambda(x;\tau) + \text{degree lower than } |\lambda|,$$

$$P_\lambda^{\mathrm{ip}}(x;\tau,\alpha) = P_\lambda(x^2;\tau) + \text{degree in } x^2 \text{ lower than } |\lambda|.$$

parameter to ∞:

$$a^{-|\lambda|} P_\lambda^{\mathrm{ip}}(ax;q,t,a) \overset{a\to\infty}{\longrightarrow} P_\lambda^{\mathrm{ip}}(x;q,t),$$

$$a_1^{-|\lambda|} P_\lambda(a_1x;q,t;a_1,a_2,a_3,a_4) \overset{a_1\to\infty}{\longrightarrow} P_\lambda(x;q,t) \quad \textcolor{red}{\text{(new)}}.$$

$$(2\alpha)^{-|\lambda|} P_\lambda^{\mathrm{ip}}(x+\alpha,\alpha,\tau) \overset{\alpha\to\infty}{\longrightarrow} P_\lambda^{\mathrm{ip}}(x,\tau),$$

$$P_\lambda(x;\tau;\alpha,\beta) \overset{\alpha\to\infty}{\longrightarrow} P_\lambda(x-1^n;\tau).$$

Binomial formula for Koornwinder polynomials

$$a_1' := (q^{-1}a_1a_2a_3a_4)^{\frac{1}{2}},$$

$$a_1'a_2' = a_1a_2,\quad a_1'a_3' = a_1a_3,\quad a_1'a_4' = a_1a_4.$$

Binomial formula (Okounkov, Transf. Groups, 1998):

$$\frac{P_\lambda(x;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\rm ip}(q^\lambda t^\delta a_1';q,t,a_1')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')}\,\frac{P_\mu^{\rm ip}(x;q,t,a_1)}{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}.$$

This implies the duality

$$\frac{P_\lambda(q^\nu t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)} = \frac{P_\nu(q^\lambda t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{P_\nu(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}$$

if $\displaystyle \frac{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}{P_\mu^{\rm ip}(q^\mu t^\delta a_1;q,t,a_1)} = \frac{P_\mu(t^\delta a_1';q,t;a_1',a_2',a_3',a_4')}{P_\mu^{\rm ip}(q^\mu t^\delta a_1';q,t,a_1')}.$

Clear if $a_1' = a_1$. In general clear from evaluation formulas.

Binomial formula for Macdonald polynomials

Binomial formula (Okounkov, Math. Research Letters, 1997):

$$\frac{P_\lambda(x;q,t)}{P_\lambda(t^\delta;q,t)}=\sum_{\mu\subset\lambda}\frac{P^{\mathrm{ip}}_\mu(q^\lambda t^\delta;q,t)}{P^{\mathrm{ip}}_\mu(q^\mu t^\delta;q,t}\,\frac{P^{\mathrm{ip}}_\mu(x;q,t)}{P_\mu(t^\delta;q,t)}\,.$$

This implies the duality $\displaystyle \frac{P_\lambda(q^\nu t^\delta;q,t)}{P_\lambda(t^\delta;q,t)}=\frac{P_\nu(q^\lambda t^\delta;q,t)}{P_\nu(t^\delta;q,t)}\,.$

Compare with binomial formula for Koornwinder polynomials.

$$\frac{P_\lambda(x;q,t;a_1,a_2,a_3,a_4)}{P_\lambda(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}$$

$$=\sum_{\mu\subset\lambda}\frac{P^{\mathrm{ip}}_\mu(q^\lambda t^\delta a_1';q,t,a_1')}{P^{\mathrm{ip}}_\mu(q^\mu t^\delta a_1';q,t,a_1')}\,\frac{P^{\mathrm{ip}}_\mu(x;q,t,a_1)}{P_\mu(t^\delta a_1;q,t;a_1,a_2,a_3,a_4)}\,.$$

Theorem

$$\lim_{a_1\to\infty} a_1^{-|\lambda|}\,P_\lambda(a_1x;q,t;a_1,a_2,a_3,a_4)=P_\lambda(x;q,t)$$

Proof Verify for $x=q^\delta$ in evaluation formulas.

Binomial formula for BC_n Jacobi polynomials

Let $\alpha' := \frac{1}{2}(\alpha+\beta+1)$.

Theorem

$$\frac{P_\lambda(x;\tau;\alpha,\beta)}{P_\lambda(0;\tau;\alpha,\beta)} = \sum_{\mu\subset\lambda} \frac{P^{\mathrm{ip}}_\mu(\lambda+\tau\delta+\alpha';\tau,\alpha')}{P^{\mathrm{ip}}_\mu(\mu+\tau\delta+\alpha';\tau,\alpha')} \, \frac{P_\mu(x;\tau)}{P_\mu(0;\tau;\alpha,\beta)}\,.$$

Proof Use $\displaystyle \frac{P_\lambda(z;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)}{P_\lambda(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)} =$

$$\sum_{\mu\subset\lambda} \frac{P^{\mathrm{ip}}_\mu(q^{\lambda+\tau\delta+\alpha'};q,q^\tau,q^{\alpha'})}{P^{\mathrm{ip}}_\mu(q^{\mu+\tau\delta+\alpha'};q,q^\tau,q^{\alpha'})} \, \frac{P^{\mathrm{ip}}_\mu(z;q,q^\tau,q^{\alpha+1})}{P_\mu(q^{\tau\delta+\alpha+1};q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1)}$$

together with (for $q\uparrow 1$) the limits

$$P_\lambda(z;q,q^\tau;q^{\alpha+1},-q^{\beta+1},1,-1) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-z-z^{-1});\tau;\alpha,\beta),$$

$$P^{\mathrm{ip}}_\lambda(z;q,q^\tau,q^\alpha) \to (-4)^{|\lambda|}\,P_\lambda(\tfrac14(2-z-z^{-1});\tau),$$

$$(1-q)^{-2|\lambda|}P^{\mathrm{ip}}_\lambda(q^x;q,q^\tau,q^\alpha) \to P^{\mathrm{ip}}_\lambda(x;\tau,\alpha).$$

Binomial formula for BC_n Jacobi polynomials (cntd.)

So $\quad P_\lambda(x;\tau;\alpha,\beta) = \sum_{\mu\subset\lambda} c_{\lambda,\mu} P_\mu(x;\tau)$

with $c_{\lambda,\mu}$, up to products of elementary factors, given by $P_\mu^{\mathrm{ip}}(\lambda+\tau\delta+\alpha';\tau,\alpha')$, which can be expressed by a certain sum over all reverse tableaux of shape μ with entries in $\{1,\ldots,n\}$.

On the other hand Macdonald (manuscript, 1987; arXiv:1309.4568) gives this as a certain sum over all standard tableaux of shape λ/μ and with entries in $1,\ldots,|\lambda-\mu|$.

The relationship between both expressions is not clear.

Binomial formula for Jack polynomials

$$\frac{P_\lambda(1+x,\tau)}{P_\lambda(1;\tau)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\mathrm{ip}}(\lambda+\tau\delta;\tau)}{P_\mu^{\mathrm{ip}}(\mu+\tau\delta;\tau)} \frac{P_\mu(x;\tau)}{P_\mu(1;\tau)}.$$

Hence $\dfrac{P_\mu^{\mathrm{ip}}(\lambda+\tau\delta;\tau)}{P_\mu^{\mathrm{ip}}(\mu+\tau\delta;\tau)} = \dbinom{\lambda}{\mu}_\tau$ (Lassalle, 1990).

Theorem (Beerends & K., unpublished; Rösler, K. & Voit, 2013)

$$\lim_{\alpha\to\infty} P_\lambda(x;\tau;\alpha,\beta) = P_\lambda(x-1;\tau).$$

Proof Let $\alpha\to\infty$ in

$$\frac{P_\lambda(x;\tau;\alpha,\beta)}{P_\lambda(0;\tau;\alpha,\beta)} = \sum_{\mu\subset\lambda} \frac{P_\mu^{\mathrm{ip}}(\lambda+\tau\delta+\alpha';\tau,\alpha')}{P_\mu^{\mathrm{ip}}(\mu+\tau\delta+\alpha';\tau,\alpha')} \frac{P_\mu(x;\tau)}{P_\mu(0;\tau;\alpha,\beta)}$$

and use that $(2\alpha)^{-|\lambda|} P_\lambda^{\mathrm{ip}}(x+\alpha;\tau,\alpha) \to P_\lambda^{\mathrm{ip}}(x;\tau), \quad \alpha\to\infty.$

It is now sufficient to verify the Theorem for $x=0$ by using the evaluation formulas.

Reduction formula for BC_n interpolation Macdonald

$$P^{\mathrm{ip}}_{\mu}(z;q,t,a) = (-a)^{n\mu_n} q^{-\frac{1}{2}n\mu_n(\mu_n-1)} \prod_{j=1}^{n} \left((z_j a;q)_{\mu_n}(z_j^{-1}a;q)_{\mu_n}\right)$$
$$\times P^{\mathrm{ip}}_{\mu-\mu_n 1^n}(z;q,t,q^{\mu_n}a).$$

In particular for $\mu = (\mu_1, \mu_2)$.

Also use that a sum over all reverse tableaux T of shape $(m,0)$ with entries in $\{1,2\}$ is a single sum.

Thus the combinatorial formula for $P^{\mathrm{ip}}_{m,0}(z_1,z_2;q,t,a)$ gives an explicit expression for $P^{\mathrm{ip}}_{m_1,m_2}(z_1,z_2;q,t,a)$.

Explicit formula for BC_2 interpolation Macdonald

$$P^{\mathrm{ip}}_{m_1,m_2}(z_1,z_2;q,t,a)=\frac{(-1)^{m_1-m_2}}{t^{m_1-m_2}a^{m_1+m_2}q^{\frac12 m_1(m_1-1)}q^{\frac12 m_2(m_2-1)}}$$
$$\times\,(z_1a,z_1^{-1}a,z_2a,z_2^{-1}a;q)_{m_2}\,(q^{m_2}z_1ta,q^{m_2}z_1^{-1}ta;q)_{m_1-m_2}$$
$$\times\ {}_4\phi_3\left(\begin{matrix}q^{-m_1+m_2},t,q^{m_2}z_2a,q^{m_2}z_2^{-1}a\\ q^{1-m_1+m_2}t^{-1},q^{m_2}z_1ta,q^{m_2}z_1^{-1}ta\end{matrix};q,q\right).$$

Thus, by Okounkov's binomial formula, this gives an explicit formula for Macdonald-Koornwinder polynomials $P_{n_1,n_2}(z_1,z_2;q,t;a_1,a_2,a_3,a_4)$.

Explicit formula for BC_2 interpolation Jack

$$P^{\mathrm{ip}}_{m_1,m_2}(x_1,x_2;\tau,\alpha)=(\alpha+x_1,\alpha-x_1,\alpha+x_2,\alpha-x_2)_{m_2}$$
$$\times(\tau+\alpha+x_1+m_2,\tau+\alpha-x_1+m_2)_{m_1-m_2}$$
$$\times\,{}_4F_3\left(\begin{matrix}-m_1+m_2,\tau,m_2+\alpha+x_2,m_2+\alpha-x_2\\1-m_1+m_2-\tau,\tau+\alpha+x_1+m_2,\tau+\alpha-x_1+m_2\end{matrix};1\right).$$

Together with ($C^{\tau}_{m_1-m_2}$ is Gegenbauer polynomial)

$$P_{m_1,m_2}(x_1,x_2;\tau)=\frac{(m_1-m_2)!}{(\tau)_{m_1-m_2}}(x_1x_2)^{\frac12(m_1+m_2)}C^{\tau}_{m_1-m_2}\left(\frac{x_1+x_2}{2(x_1x_2)^{\frac12}}\right)$$

we have by the binomial formula for BC_2 Jacobi polynomials an explicit expression for these polynomials.

Much earlier (1978) in a very different way obtained by K. & Sprinkhuizen.

Branching formula for Koornwinder polynomials

Van Diejen & Emsiz, arXiv:1408.2280, obtained a branching formula and hence a combinatorial formula for Koornwinder polynomials.

$$P_\lambda(x_1,\ldots,x_n;q,t;a_1,a_2,a_3,a_4)=\sum_\mu P_{\lambda/\mu}(x_n;q,t;a_1,a_2,a_3,a_4)\\ \times P_\mu(x_1,\ldots,x_{n-1};q,t;a_1,a_2,a_3,a_4).$$

The sum is over all μ such that $\ell(\mu)\le n-1$ and $\exists\nu$ with $\mu\preccurlyeq\nu\preccurlyeq\lambda$.

$$P_{\lambda/\mu}(x;q,t;a_1,a_2,a_3,a_4)=\sum_{k=0}^{d} B^k_{\lambda/\mu}(q,t;a_1,a_2,a_3,a_4)\,\langle x;a_1\rangle_{q,k}\,,$$

$$d=|\{j\mid \lambda'_j=\mu'_j+1\}|,\quad \langle x;a_1\rangle_{q,k}:=\prod_{j=0}^{k-1}\langle x;q^ja_1\rangle.$$

Branching formula for Koornwinder polynomials (cntd.)

For $\mu \subset m^n$ let $(m^n - \mu)_j := m - \mu_{n+1-j}$ $(j = 1, \ldots, n)$.

$$B^k_{\lambda/\mu}(q, t; a_1, a_2, a_3, a_4)$$
$$= (-1)^{k+|\lambda|-|\mu|} \, C^{n^{\lambda_1}-\lambda', \lambda_1}_{(n-1)^{\lambda_1}-\mu', \lambda_1-k}(t; q; a_1, a_2, a_3, a_4),$$

where $C^{\mu,n}_{\lambda,r}$ are Pieri coefficients:

$$E_r(x; t, a) := \sum_{1 \le j_1 < \ldots < j_r \le n} \prod_{i=1}^{r} \langle x_{j_i}; t^{j_i-1} a \rangle,$$

$$E_r(x; t, a) \, P_\lambda(x; q, t; a_1, a_2, a_3, a_4)$$
$$= \sum_{\mu} C^{\mu,n}_{\lambda,r}(q, t; a_1, a_2, a_3, a_4) P_\mu(x; q, t; a_1, a_2, a_3, a_4).$$

Here $\lambda \in \Lambda_n$ and the sum runs over all $\mu \in \Lambda_n$ such that there exists $\nu \subset \lambda \cap \mu$ with $\lambda - \nu$ and $\mu - \nu$ vertical strips and $|\lambda - \nu| + |\mu - \nu| \le r$.

Branching formula for Koornwinder polynomials (cntd.)

The branching formula for Koornwinder polynomials is much more complicated than the branching formula for Macdonald polynomials, but the latter one is obtained from the first one by taking the highest degree part at both sides of the identity.

There is no obvious relation between the Van Diejen-Emsiz branching formula for Koornwinder polynomials and Okounkov's branching formula for *BC* interpolation Macdonald polynomials.

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