NIEUW ARCHIEF VOOR WISKUNDE (3), XXVI (1978), 54-64

SOME DETAILS OF THE HISTORY OF THE KORTEWEG—DE VRIES EQUATION

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> *"In such excellent treatises on hydrodynamics as those of Lamb and Basset, we find that even when friction is neglected long waves in a rectangular canal must necessarily change their form as they advance, becoming steeper in front and less steep behind. Yet since the investigations of De Boussinesq, Lord Rayleigh and St. Venant on the solitary wave, there has been some cause to doubt the truth of this assertion. Indeed, if the reasons adduced were really decisive, it is difficult to see why the solitary wave should make an exception; but even Lord Rayleigh and McCowan, who have successfully and thoroughly treated the theory of this wave, do not directly contradict the statement in question. They are, as it seems to us, inclined to the opinion that the solitary wave is only stationary to a certain approximation.*
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> *It is the desire to settle this question definitively which has led us into the somewhat tedious calculations which are to be found at the end of our paper. We believe, indeed, that from them the conclusion may be drawn, that in a frictionless liquid there may exist absolutely stationary waves and that the form of their surface and the motion of the liquid below it may be expressed by means of rapidly convergent series."*

These are the first lines of an article by Dr. D.J. KORTEWEG, professor of Mathematics at the University of Amsterdam, and Dr. G. DE VRIES. The paper is entitled: *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*. It was published in the Philosophical Magazine, 1895, 5th series, Volume XXXIX, p.422-443.

Two Dutchmen in an English journal about a real typical Dutch subject, canals and waves? Alas, there seems to be no connection with the Dutch landscape; in the years around 1895 there was a great interest in the theory of waves and many papers on the subject were published in the Philosophical Magazine. On page 425 the authors write that their investigations were greatly influenced by a paper of Lord RAYLEIGH, Phil. Mag. 1876, Vol.I, page 257. KORTEWEG himself wrote a paper: *On a general theorem of the stability of the motion of a viscous fluid,* in Vol.XVI, 1883, page 112 of the Philosophical Magazine.

In the paper of KORTEWEG & DE VRIES a certain nonlinear differential equation is introduced:

$$\frac{\partial\eta}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{\ell}}\cdot\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right), \tag{1}$$

where η is the surface elevation above the equilibrium level, x the coordinate along the canal, g the acceleration of gravity, ℓ the wave length and α and σ physical constants. This equation is now called the *KdV* or *Korteweg-de Vries equation.*

> *"In spite of this early derivation of the KdV equation, not until 1960 was a new application of the model equation found in the study of collision-free hydromagnetic waves by Gardner and Morikawa. This is surprising because, in general, the KdV equations describes the unidirectional propagation of small-but-finite amplitude waves in a nonlinear dispersive medium. Since 1960, numerous other applications of the equation have been found. For a derivation of the KdV equation in different contexts, see the various original papers cited and the articles by Jeffrey and Kakutani, and Leibovich and Seebass.*
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> *Kruskal and Zabusky showed the KdV equation governs longitudinal waves propagating in a one-dimensional lattice of equal masses coupled by nonlinear springs, the Fermi-Pasta-Ulam problem. Other applications to plasma physics were given by Berezin and Karpman and by Washimi and Taniuti in their study of ion-accoustic waves in a cold plasma. Wijngaarden found it described pressure waves in a liquid-gas bubble mixture. Naraboli showed it governed waves in elastic rods. Shen derived the KdV equation in the study of 3-dimensional water waves and Leibovich showed it described the axial component of velocity in a rotating fluid flow down a tube. Su and Gardner, and Taniuti and Wei showed it arises from several general classes of equations.*
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> *Obviously, this fecundity of applications which cuts across various areas of science associated with solids, liquids, gases, and plasmas amply justifies extensive study of the Korteweg-de Vries equation. However, it is*

noteworthy that much of the recent intensive research on the equation has been motivated and stimulated primarily by interesting features of the solutions and properties of the equation.

This paper is devoted to a survey of some of the results of these studies carried out over the past 10 years for the Korteweg-de Vries equation which we rewrite as

$$(2) \qquad u_t - 6uu_x + u_{xxx} = 0$$

where subscripts denote partial differentiations. To get the specific form above, which will be convenient later, we have rescaled and translated the dependent and independent variables in the various applications to eliminate the physical constants. Any desired coefficients can be inserted into the equation by such transformations. From the original form of the KdV equation (1) the transformations

$$t' \equiv \frac{1}{2}\sqrt{\frac{g}{\ell\sigma}}\, t, \quad x' \equiv -\frac{x}{\sqrt{\sigma}}, \quad u' \equiv -\frac{1}{2}\eta - \frac{1}{3}\alpha.$$

give us (2), where we have dropped the primes.

In addition to obtaining (2) from other forms of the equation, there are transformations of the variables which leave it invariant. Clearly (2) is invariant to arbitrary translations in x *and* t *since they appear only in the differentiations. Also, because all derivates are of odd order, reversing the signs of both* x *and* t *does not alter the equation. Moreover, the KdV equation is Galilean invariant, i.e. (2) is unchanged by the transformation*

$$t' \equiv t, \quad x' \equiv x - ct, \quad u'(x',t') \equiv u(x,t) + \frac{1}{6}c,$$

where c *is some constant. This corresponds to going to a steady moving reference frame with velocity* c."

This second quotation is from the introduction of a paper by ROBERT M. MIURA: *The Korteweg-de Vries equation: A Survey of Results*, S.I.A.M. Reviews, Vol.18, 1976, p.412-459. The list of references of this paper contains 109 items, most of them published after 1960. I now take one of them, 72 : A.C. SCOTT, F.Y.F. CHU & D.W. McLAUGHLIN: *The soliton: A new concept in applied science*, Proc. I.E.E.E. 61, 1973, p.1443-1483 as a new starting point. It looks like a dangerous procedure, since this paper has a list of 267 references, most of them connected with the Korteweg-de Vries equation. The introduction of ALWYN SCOTT et al.'s paper brings us back in history, as far as 1844, starting with a paper of J. SCOTT-RUSSELL: *Report on waves*, Proc. Roy. Soc. Edinburgh, 1844; Brit. Assoc. Rep.

> *"The concept of a solitary wave was introduced to the budding science of hydrodynamics well over a century ago by SCOTT-RUSSELL with the following delightful description:*
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> *I was ovserving the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon ....*
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> *In 1895 KORTEWEG & DE VRIES provided a simple analytic foundation for the study of solitary waves by developing an equation for shallow water waves, ..."*

It was already remarked by MIURA that the Korteweg-de Vries equation has not only a large number of quite different applications in mathematical physics, but that it has also a great interest from the side of pure mathematics. I only mention a few examples: the work of H.P. McKEAN & P. VAN MOERBEKE (see e.g. *Sur le spectre de quelques opérateurs et les variétés de Jacobi*, Séminaire Bourbaki, 28 (1975/76, p.474), the papers of P.D. LAX on periodic solutions of the KdV equation and a paper of I.M. GEL'FAND & L.A. DIKI: *Asymptotic behaviour of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equation*, Uspekhi Mat. Nauk 30: 5 (1975), p.67-100, Russian Math. Surveys 30: 5 (1975), p.77-113.

Now we come to the following minor problem. KORTEWEG & DE VRIES wrote about water waves in 1895. The new stream of publications about different applications of the KdV equation started in 1960. And what can be said of the time in between? But let us start at the beginning.

A biography of DIEDERIK JOHANNES KORTEWEG (31.3.1848 - 5.10.1941) is published in the Yearbook of the Royal Netherlands Academy of Arts and Sciences; 1945-1946. This biography is written by H.J.E. BETH & W. VAN DER WOUDE.

KORTEWEG studied for his doctors thesis with J.D. VAN DER WAALS, he was awarded the degree cum laude on July 12, 1878. He was the first doctor of the University of Amsterdam. His thesis was a study concerning the propagation speed of waves in elastic tubes. The problem to be solved in his thesis was about the propagation speed of the heart-beat in the blood of the arteries. In the introduction of his thesis KORTEWEG deals with studies of B. AIRLY and of WERTHEIM (*Annales de Chimie,* 3 série, Vol.23) concerning the velocity of sound in water. WERTHEIM has immersed organ pipes in water and forcing the water through it in the same manner as air; a sort of musical tone was produced. The velocities for water of the Seine varied from 1173 M. to 1480 M. per second; all much inferior to those found by direct experiment. Among possible causes of the difference, WERTHEIM suggested the yielding of the sides of the tube. KORTEWEG remarks that water waves influenced by wind have in deep water a propagation speed proportional to the square root of the wave length. KORTEWEG refers to G. KIRCHHOF: *Vorlesungen über mathematische Physik,* Leipzig, 1877, pag. 358.

The interest of D.J. KORTEWEG in the phenomenon of waves is obvious. But KORTEWEG has published about quite a lot of other mathematical subjects. We only remind a paper concerning voting procedures, one on a subject from classical mechanics: *Über eine ziemlich verbreitete unrichtige Behandlungsweise der rollenden Bewegung und ins Besondere über kleine rollende Schwingungen um eine Gleichgewichtslage.* In his paper *Sur les points de plissement* he studies singularities of surgaces as they occur in the theory of the Ψ-surface of VAN DER WAALS. KORTEWEG was also chairman of the committee for the publication of the *Oeuvres complètes de Christiaan Huygens.*

He was the promotor of Gustav de Vries, who wrote a thesis *Bijdrage tot de kennis der lange golven* (Contribution to the theory of long waves). GUSTAV DE VRIES defended his thesis on Saturday, December 1st in 1894. In the introduction of GUSTAV DE VRIES we find the problem of the solitary wave in a channel. There is of course the story of SCOTT-RUSSELL on horseback. DE VRIES tells somewhat more. SCOTT-RUSSELL built a long wooden tank, filled it with water to a rather small height and introduced a wave. G. DE VRIES gives a mathematical theory concerning long waves, that can run through the channel without changing its form.

He refers to former authors concerning the solitary wave: McCOWAN, LORD RAYLEIGH, ST. VENANT. I think it is important to mention that he also refers to J. DE BOUSSINESQ, 1871: *Théorie de l'intumescense liquide appelée onde solitaire ou de translation se propageant dans un canal rectangulaire,* Comptes Rendus 72, pag.755-759. We also mention another paper of DE BOUSSINESQ: *Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant un liquide continue dans ce canal des vitesses sensiblement pareilles de la surface au fond,* J. Math. Pures et Appl. (2) 17 (1872), p.55-108.

GUSTAV DE VRIES gives on page 9 of his thesis an equation, which he proposes to use for studying the change in the form of a solitary wave. This equation is:

$$\frac{\partial\mu}{\partial t} = \frac{3q_0}{2\ell} \cdot \frac{\partial}{\partial x}\left\{\frac{1}{2}\mu^2 + \frac{2}{3}\alpha\mu + \frac{1}{3}\sigma\frac{\partial^2\mu}{\partial x^2}\right\} .$$

And this is the now well-known *Korteweg-de Vries equation*. Since the thesis was written in Dutch an excerpt and translation of it was published by DE VRIES, with cooperation of his promotor KORTEWEG in the Philosophical Magazine dated January 1895. In the Netherlands it was, and still is, prescribed that with the written theses a number of short "thesis" on scientific subjects are proposed. One of these theses of GUSTAV DE VRIES was: *The solitary wave is no special phenomenon, but it can be considered as a special case of long waves.*

One of the theses of KORTEWEG was: "It is a pity that Stas meant to explain the approximate similarity of the great majority of smaller atomweights with integers as pure change. On the contrary this fact may lead to the key for important conclusions concerning the nature of atoms.")

Perhaps it is good to remark that GUSTAV DE VRIES was not one of the Dutch professors in mathematics of that surname. The name "De Vries" is a rather common surname in the Netherlands and it is difficult to find biographical facts concerning G. DE VRIES. He was a member of the Wiskundig Genootschap (Dutch Mathematical Society) since 1892. He was a teacher at the Gymnasium in Alkmaar and Haarlem. After his thesis and the joint paper with Korteweg he published two papers in the *Verhandelingen* of the Royal Netherlands Academy of Arts and Sciences. In Volume 5 (1896): *Les équations du mouvement des Cyclones* and in 6 (1897): *Le tourbillon cyclonal.*

There is a reference to the Korteweg-de Vries paper by professor GIBSON (Glasgow) in the Jahrbuch über die Fortschr. der Math. 26 (1895) p.881-884. In the third printing of H. LAMB's classical book *Hydrodynamics* (1906[3]) the work of KORTEWEG-DE VRIES is mentioned. In the encyclopädie der Mathematischen Wissenschaften IV, 3 on p.140, there is in the paper of A.E.H. LOVE: *Hydrodynamic II,* a remark about a special type of wave, *cnoidal waves,* introduced in the KORTEWEG-DE VRIES paper. The KdV equation is not mentioned by LOVE.

In April 1924 the first international Congress for applied Mechanics was held in Delft. One of the main lectures was given by T. LEVI-CIVITA. In his lecture LEVE-CIVITA refers to a paper, published in the Mathematische Annalen 93 (1925): *Détermination rigoureuse des ondes permanentes d'ampleur finie.* On page 266 there is a reference to KORTEWEG-DE VRIES. The problem dealt with is not the real KdV equation, but the solution of the "time independent" solitary waves. The paper of LEVI-CIVITA was followed by a paper of D.J. STRUIK: *Détermination rigoureuse des ondes irrotationelles périodiques dans un canal à profondeur finie.* Math. Ann. 95 (1926), p.595-634; and of M.L. DUBREIL-JACOTIN *Sur la détermination rigoureuse des ondes permanentes d'ampleur finie.* Journal de Mathem. 13 (1924), p.217-291.

We also mention that in the proceedings of the Delft congress there is a remark on a study of A.I. NEKRASSOV: *On steady waves,* and that the same author also published in 1951 on the exact theory of steady waves on the surface of a heavy fluid.

Another paper concerning water waves was published in 1948 in Comm. Pure and Applied Math. 1, p.323-339 by J.B. KELLER, in which we find a reference to the KORTEWEG-DE VRIES paper of 1895.

In the literature on waves many more times reference is made to the paper of KORTEWEG-DE VRIES; we only mention the paper of TH. VON KÁRMÁN: *The engineer grapples with nonlinear problems,* Bull. Amer. Math. Soc. 46 (1940), p.616-683. We also refer to the nice historical story on p. 342 in J.J. STOKER: *Water waves* (New York, 1957).

In the Handbuch der Physik 1960, Vol. IX there is a long description of the KORTEWEG-DE VRIES paper and their results concerning solitary waves. JOHN V. WELHAUSEN & EDMUND V. LAITONE write on p.702-712 about KORTEWEG-DE VRIES' introduction of the cnoidal wave and so on.

Up to this point we have only dealt with KORTEWEG-DE VRIES theory of water waves. But in 1960 we suddenly find a new field of research referring to the Korteweg-de Vries equation. There seem to be two different lines. One refers to a study of R.Z. SAGDEEV: *Voprosy magnituoi gidrodinamiki,* Riga; 1960 (see YU.A. BEREZIN & V.I. KARPMAN, J.E.T.P. 46 (1964), p.1880-1890 = J.E.P.T. 19 (1964), p.1265-1271; they refer as well to SAGDEEV in connection with

$$\frac{\partial f}{\partial \zeta} - f\,\frac{\partial f}{\partial \mu} + \frac{\partial^3 f}{\partial \mu^3} = 0$$

and water waves in canals of finite depth as to the work of C.S. GARDNER mentioned below).

I shall more explicitly deal with a second one. One of the oldest "new time" references to this nonlinear description of waves seems to be: GARDNER et al: *Proceedings of the conference on peaceful uses of atomic energy,* Geneva, 1958, vol.31, pag.230 and GARDNER, C.S. & MORIKAWA, G.K.: *Courant institute of math. sciences reports,* N.Y.O. 9082, 1960. Professor MORIKAWA was so kind as to inform me about his knowledge of the history of the connection between the classical water waves and the recent applications of the KvD equation. I quote two of his remarks:

"It is appropriate to take one step backward in time to the Boussinesq equation:

$$u_{tt} - g(u+3u^2/2 + u_{xx}/3)_{xx} = 0.$$

This equation is an asymptotic description of wave motion with double characteristics from which the one-sided (or single characteristic) KdV equation can be derived also. Reference: J. Math. pures et appl. (2), 17, 55-108 (1872). A more recent paper concerning the Boussinesq equation is: F. Ursell, Proc. Cambridge Phil. Soc. 49, 685-694 (1953)".

"Gardner and I rediscovered the KdV equation in/or about 1958 when I wrote up a manuscript but did not submit it to the Geneva Conference on peaceful uses of atomic energy. In retrospect, I find it remarkable that neither of us were made aware of the work of Korteweg and De Vries for more than one year, since the Courant Institute at that time was considered one of the centers for work in mathematical fluid mechanics. Fortunately, I finally got a reproduction of the KdV paper from the New York Public Library when I was in the process of finishing the CIMS Report (1960) which you have referenced. As you have noted, since the middle 60's a flood of publications have ensued concerning the KdV equation and related subjects."

We still mention some of the first papers after 1960 on the subject, which refer explicitly to the KdV equation: In 1965 in Phys. Rev. Lett. 15 (1965) p.240 N.J. ZABUSKY & M.D. KRUSKAL: *Interaction of solitons in a collisionless plasma and the recurrence of initial states.* In 1966 H. WASHIMI & T. TANIUTI, *Propagation of ion-acoustic solitary waves of smaal amplitude* was published in Phys. Rev. Lett. 17 p.996-998. As a further application we remind of the Fermi-Pasta-Ulam problem. As a reference we give M.D. KRUSKAL & N.J. ZABUSKY: *Stroboscopic perturbation procedure for treating a class of nonlinear wave equations,* J. of Math. Phys. 5 (1964), p.231-244. On this Fermi-Pasta-Ulam problem there has been done a lot of computation on the KdV equation, which started in 1955. More details can be found in the paper of A. SCOTT et al., Proc.I.E.E.E. 61, 1973, p.1443-1483. We remark that this paper deals with solitons, that is special solitary waves, shich have the special property that they do not scatter upon collision. So the solitary waves that are solitons, emerge from a collision having the same shapes and velocities with which they entered it. This phenomenon is also studied for other equations as the KdV equation.

Perhaps it is even worthwhile to deal with several generalisations of the KdV equation, for instance such as

$$\frac{\partial\phi}{\partial t} + \alpha\phi^p \frac{\partial\phi}{\partial x} + \frac{\partial^{2r+1}\phi}{\partial x^{2r+1}} = 0 \quad . \tag{3}$$

The case p = 2, r = 1 has been used to describe acoustic waves in certain unharmonic lattices. As well of this generalised KdV equations as of the KdV equation itself films were made by ZABUSKY, DEEM & KRUSKAL and by TAPPERT.

It is quite remarkable that in the original paper of KORTEWEG and De VRIES the equation is written in the form of a conservation law:

$$\frac{\partial}{\partial t}\eta = \frac{\partial}{\partial x}\left\{\frac{3}{2}\sqrt{\frac{g}{\ell}}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2\eta}{\partial x^2}\right)\right\} , \tag{4}$$

or using formula (2):

$$u_t = (3u^2 - u_{xx})_x \quad . \tag{5}$$

There are other conservation laws to be deduced from the KdV equation:

$$(u^2)_t = (4u^3 - 2uu_{xx} + u_x^2)_x \tag{6}$$

and

$$(u^3 + \tfrac{1}{2}u_x^3)_t = (\tfrac{9}{2}u^4 - 3u^2u_{xx} + 6uu_x^2 - u_xu_{xxx} + \tfrac{1}{2}u_{xx}^2)_x \quad . \tag{7}$$

This gives the possibility to use variational principles of Lagrangian and Hamiltonian type. MIURA tells in his paper (S.I.A.M., Reviews, vol.18, 1976, p.421) that during a week's vacation at a beautiful lake near Peterboro, Ontario, he found a tenth conservation law for the KdV equation. Later on it was proved that there are infinitely many conservation laws. It is quite remarkable that for the generalised KdV with $p \geq 3$, $r = 1$, there are only three conservation laws.

In the book *Linear and nonlinear waves,* by G.B. WHITHAM, New York, 1974, we find many recent results concerning the KORTEWEG-DE VRIES equation. In several necrologies of D.J. KORTEWEG published after his death in the age of 93 in 1941, the work of KORTEWEG, concerning the *Van der Waals surface,* concerning the *edition of Huygens' Oevres,* concerning *a simple connection between the perturbations of a plane periodic movement in three successive periods,* known as the theorem of KORTEWEG, was mentioned. But we do not find any reference to the paper with G. DE VRIES in the Phil. Mag. 1895. Nobody could predict in 1941, nearly 50 years after its publication in the doctors thesis of G. DE VRIES, that 80 years after its publication in 1894, it would become a hot topic in pure and applied mathematics.