Invertible Homogeneous Versors are Blades

T.A. Bouma
University of Amsterdam
G. Memowich

April 23rd, 2001 CE

Abstract

This paper proves the theorem "Invertible Homogeneous Versors are Blades" for finite dimensional Geometric Algebras. This implies for Euclidean and Anti-Euclidean spaces that all homogeneous versors are blades. Furthermore, since an element is a blade regardless of the metric and since there are easy tests to determine if an element is homogeneous or a versor in a Euclidean space this translates directly into an easy test to determine if an arbitrary element is a blade.

1 Introduction

In this paper we show that all invertible homogeneous versors are blades.

The following formulas are adapted from reference[3]:

$$xY = x\rfloor Y + x \wedge Y \tag{1}$$

$$X\rfloor(Y\rfloor Z) = (X \wedge Y)\rfloor Z \tag{2}$$

$$x \wedge (YZ) = (x\rfloor Y)Z + \overline{Y}(x \wedge Z). \tag{3}$$

Let $\mathcal{B}$ be the set of blades. Let $\mathcal{V}$ be the set of versors. Let $\mathcal{I}$ be the set of invertible multivectors. Let $\mathcal{H}_r$ be the set of r-vectors and let $\mathcal{H}$ be the set of homogeneous multivectors.

1.1 Versor Facts

A versor is defined to be the product of vectors in a Geometric Algebra[3]. A blade is a k-vector that can be factored into k mutually anticommuting vectors, hence it is a special kind of versor.

$$X \in \mathcal{H}_r \text{ and } A \in \mathcal{V} \cap \mathcal{I} \Rightarrow AXA^{-1} \in \mathcal{H}_r \tag{4}$$

As described in reference[2], for each $A \in \mathcal{I} \cap \mathcal{V}$ there is a projection operator, P_A, "onto" A, namely:

$$P_A(X) = \frac{1}{2}(X - \overline{A}XA^{-1}). \tag{5}$$

2 The Theoreom

Theorem:

$$\mathcal{I} \cap \mathcal{H} \cap \mathcal{V} \subset \mathcal{B} \tag{6}$$

Proof:

Let $B \in \mathcal{I} \cap \mathcal{H}_r \cap \mathcal{V}$. As described in the appendix, let $f_\wedge$ be a LIFT that isomorphically embeds B into a nondegenerate Geometric Algebra over an n dimensional vector space. Then $B \in \mathcal{B} \iff A = f_\wedge(B) \in \mathcal{B}$, and $A \in \mathcal{I} \cap \mathcal{H}_r \cap \mathcal{V}$. If $\mathcal{A}$ is defined as $\{x \in \mathcal{H}_1 : x \wedge A = 0\}$, then it is clear that $\mathcal{A}$ is a vector space. Since $A \in \mathcal{V}$ it follows from equation (4) that

$$x \in \mathcal{H}_1 \Rightarrow P_A(x) \in \mathcal{H}_1. \tag{7}$$

In fact, $P_A(x)A = \frac{1}{2}(x - \overline{A}xA^{-1})A = \frac{1}{2}(xA - \overline{A}x) = x \rfloor A \in \mathcal{H}_{r-1}$, so $P_A(x) \wedge A = 0$, which implies the stronger statement

$$x \in \mathcal{H}_1 \Rightarrow P_A(x) \in \mathcal{A}. \tag{8}$$

If $\mathcal{A} = \{0\}$ then $\forall x \in \mathcal{H}_1 \; x \rfloor A = P_A(x)A = 0$ therefore by lemma (12), $A \in \mathcal{H}_1$ (and hence $A \in \mathcal{V}$). Therefore assume $\mathcal{A} \neq \{0\}$ and let $e_1, e_2, ..., e_k$ be an orthogonal basis for $\mathcal{A}$.

Choose a set of (n-k) vectors $p_1, p_2, ..., p_{n-k}$ such that $e_1, e_2, ..., e_k, p_1, p_2, ..., p_{n-k}$ is a basis for $\mathcal{H}_1$. Then let $e_{k+i} = p_i - P_A(p_i)$. The set $e_1, e_2, ..., e_n$ is a basis because $P_A(p_i) \in \mathcal{A}$. This basis is constructed so that $i > k \Rightarrow e_i \rfloor A = 0$. Now since $i > k \Rightarrow e_i A = p_i A - p_i \rfloor A = p_i \wedge A \in \mathcal{H}_{r+1}$, it follows that $i > k \Rightarrow e_i \rfloor A = e_i A - e_i \wedge A = 0$. Because

$$\begin{aligned} e_i \rfloor ((e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A) &= (e_i \wedge e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A & (9) \\ &= (-1)^k (e_1 \wedge e_2 \wedge ... \wedge e_k \wedge e_i) \rfloor A & (10) \\ &= (-1)^k e_1 \rfloor (e_2 \rfloor (...(e_k \rfloor (e_i \rfloor A))...)) & (11) \end{aligned}$$

it follows that line (9) is zero if $i \leq k$ and line (11) is zero if $i > k$. Since $e_i \rfloor ((e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A)$ is zero $\forall i$ it follows that $x \rfloor ((e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A) = 0$ $\forall x \in \mathcal{H}_1$. Therefore lemma (12) applies and $\alpha = (e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A \in \mathcal{H}_0$.

Let $E_i = e_{i+1} e_{i+2} ... e_k \; \forall i \in \{1, ..., k-1\}$ and let $E_k = 1$. Then since $e_1, e_2, ..., e_k$ is an orthogonal basis, it follows $\forall i \in \{1, ..., k\}$ that $e_i \rfloor E_i = 0$. Since $e_i \wedge A = 0$ it then follows from equation (1) and (3) that $\forall i \in \{1, ..., k\}$ $e_i E_i A = e_i(E_i A) = e_i \rfloor (E_i A) + e_i \wedge (E_i A) = e_i \rfloor (E_i A) + (e_i \rfloor E_i) A + \overline{E_i}(e_i \wedge A) = e_i \rfloor (E_i A)$. Using that identity k times and equation (2) it follows that

$$\begin{aligned} e_1 e_2 ... e_k A &= e_1 E_1 A = e_1 \rfloor (E_1 A) \\ &= e_1 \rfloor (e_2 E_2 A) = e_1 \rfloor (e_2 \rfloor (E_2 A)) \\ &= ... = e_1 \rfloor (e_2 \rfloor (...(e_k \rfloor A)...)) \\ &= (e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A \end{aligned}$$

so $(e_1 \wedge e_2 \wedge ... \wedge e_k) \rfloor A = e_1 e_2 ... e_k A$.

Therefore $A = \frac{A^\dagger A}{\alpha} e_1 \wedge e_2 \wedge ... \wedge e_k$, and hence A is a blade.

3 Corollaries

This theoreom immediatly translates into a test to determine if an arbitrary multivector is blade. Given an element B, check to see if it is homogeneous, if not then it is not a blade, if the it is then fix a LIFT, $f_\wedge$, to a Euclidean metric. Then $A = f_\wedge(B) \in \mathcal{B} \iff B \in \mathcal{B}$. Every nonzero blade in a Euclidean space has an inverse, so test to see if B has an inverse. If it does not have an inverse then it is not a blade (unless it is zero in which case it is a blade). If it does have an inverse then let $e1, e2, ..., e_n$ be a basis for $\mathcal{H}_1$ in the Euclidean space. Then compute $a_i = \overline{A} e_i A^{-1}$. If each a_i is a vector then A is a versor (for a proof see reference[2] or a more classical text). Now if A is not a versor, then it is not a blade and if it is a versor, then since $B \in \mathcal{H}$ and a LIFT sends $\mathcal{H} \mapsto \mathcal{H}$ then $A \in \mathcal{H} \cap \mathcal{V} \cap \mathcal{I}$ so $A \in \mathcal{B}$.

4 Conclusion

Invertible homogeneous versors are blades. In a Euclidean or Anti-Euclidean space all the nonzero versors are inveritble, so in a Euclidean or Anti-Euclidean space all homogeneous versors are blades (and all blades are homogeneous versors). This is useful because tests exist to determine if an invertible multivector is a versor, so now tests exists to determine if an invertible multivector in a Euclidean or Anti-Euclidean space is a blade. In fact since a LIFT[1] is a blade-preserving map, tests now exist to determine if any multivector, in any space, is a blade.

Appendix

The appendix contains two results used in the earlier proofs.

LIFT

As taken from[1], a LIFT ('linear injective function' transformation) from one Geometric Algebra to another Geometric Algebra is defined as a linear injective map that preserves the outer product and the scalars. In more detail, given two geometric algebras, $\mathcal{R}_{p,q,r}$ and $\mathcal{R}_{a,b,c}$, and a linear injective function, f, from the vectors of $\mathcal{R}_{p,q,r}$ to the vectors of $\mathcal{R}_{a,b,c}$ then $f_\wedge$ is a LIFT between the two algebras, where $f_\wedge$ is the outermorphism of f.

This paper uses a LIFT to isomorphically embed a degenerate Geometric Algebra into a nondegenerate algebra as described in reference[1]. Let $\{p_1, ..., p_p, q_1, ..., q_q, r_1, ..., r_r\}$ be an orthogonal basis for the vectors in $\mathcal{R}_{p,q,r}$ such that $p_i^2 = 1$, $q_j^2 = -1$, and $r_k^2 = 0$ and let $\{e_1, ..., e_{p+r}, f_1, ..., f_{q+r}\}$ be an orthogonal basis for the vectors in $\mathcal{R}_{p+r,q+r,0}$ such that $e_i^2 = 1$ and $f_j^2 = -1$. Then let f be a linear function such that $f(p_i) = e_i$, $f(q_j) = f_j$, and $f(r_k) = e_{p+k} + f_{q+k}$. Then $f_\wedge$ is an isomorphism embedding $\mathcal{R}_{p,q,r}$ into $\mathcal{R}_{p+r,q+r,0}$.

Contraction Lemma

Let $\mathcal{G}$ be a Geometric Algebra over a nondegenerate finite dimensional vector space $\mathcal{R}^{p,q,0}$ then

$$x \rfloor A = 0 \quad \forall x \in \mathcal{R}^{p,q,0} \Rightarrow A \in \mathcal{R} \tag{12}$$

Proof: Since $\mathcal{R}^{p,q,0}$ is nondegenerate it has an orthogonal basis of invertible vectors $e_1, ..., e_n$. The set $\{e_I : I \subset \{1, 2, ..., n\}\}$ is a basis for $\mathcal{G}$ where $e_\emptyset = 1$ and $e_I = \prod_{i_k \in I} e_{i_k}$. If $A = \sum \alpha^I e_I$ then $e_i \rfloor A = 0$ implies that $\alpha^I = 0$ when $i \in I$. Since $e_i \rfloor A = 0$ for each e_i it is clear that $\alpha^I = 0$ for all $I \neq \emptyset$, therefore A is a scalar.

References

[1] Bouma, T.A., Dorst, L. and Pijls, H.G.J.: Geometric Algebra for Subspace Operations, (*Submitted for Publication*), available on-line at **http://xxx.lanl.gov/abs/math.LA/0104102** .

[2] Bouma, T.A.: Generalized Projection Operators in Geometric Algebra, (*Submitted for Publication*), available on-line at **http://xxx.lanl.gov/abs/math.LA/0104159** .

[3] Hestenes, D. and Sobczyk, G.: *Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht, 1984.