Objects in contact: the common 'wave propagation' geometry of collision detection, object growing and milling

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July 15, 1999

Invited presentation at 5th International Conference on Clifford Algebras and their Applications, Ixtapa, Mexico, June 1999.

Abstract

We provide representations in geometric algebra for m-dimensional boundaries, to describe and analyze the geometry of wave propagation. This operation also is the essence of collision avoidance in robotics, object growing in graphics, and milling. The most promising result is to represent boundaries as versors in an $(m+1,1)$-dimensional Minkowski space; wave propagation then becomes a geometric product of versors.

1 Collision detection, object growing, milling: forms of wave propagation

When we perform a Huyghens-style wave propagation, we place copies of a 'propagator' $\mathcal{A}$ at each position on a wave front $\mathcal{B}$. The caustic of these secondary wave fronts then forms the resulting propagated front, which we denote $\mathcal{A}\check{\oplus}\mathcal{B}$ (see figure 1b). Each point P of $\mathcal{B}$ in this way 'causes' a point in the result (at least when $\mathcal{A}$ is convex and $\mathcal{B}$ is differentiable), see figure 1a. By performing a linear approximation to $\mathcal{B}$ at P, it is easy to see that at every point of the resulting caustic, the tangent is equal to that of the point that caused it, and equal to the tangent at the corresponding point of the 'propagator' $\mathcal{A}$. Also (in a Euclidean space) the outwardly directed unit normal vectors at the corresponding points of the original front, the propagator and the resulting front are identical.

*Work performed while on a sabbatical with David Hestenes' group at ASU, Tempe, AZ, USA.

Now consider figure 1c, which depicts the collision of a movable object $\mathcal{A}'$ with a fixed obstacle $\mathcal{B}$. The reference point of $\mathcal{A}$ is prevented from moving freely due to the collision at P. Computing such local contacts for all translational motions of $\mathcal{A}'$ generates the boundary of the 'free space' for the reference point of $\mathcal{A}'$. A linear approximation shows that the tangents at P of $\mathcal{B}$, $\mathcal{A}'$ and the reference point on the result are proportional (with the outwardly directed normal vector at $\mathcal{A}'$ having an opposite sense).

In fact, the two operations of wave propagation (or 'object growing', or 'morphological dilation') are mathematically closely related. Indeed, the boundary of free space in the collision problem is precisely $(-\mathcal{A}')\check{\oplus}\mathcal{B}$, i.e. the propagation of $\mathcal{B}$ with the propagator $-\mathcal{A}' \equiv \{-a \mid a \in \mathcal{A}'\}$. Thus an analysis of either is applicable to the other. Another related problem is 'milling', in which one desires to move a milling tool $\mathcal{A}'$ over some boundary $\mathcal{C}$ such that a desired object $\mathcal{B}$ is produced. This is in this sense a destructive form of collision, obviously restrained by $\mathcal{C} = (-\mathcal{A}')\check{\oplus}\mathcal{B}$ in the class of boundaries $\mathcal{B}$ that can be produced with a given tool $\mathcal{A}'$. In this paper, we will use the wave propagation application to provide the basic terminology, when required, since it has the simplest relationship between tangents (no inverse orientation of the normal of $\mathcal{A}'$).

The computation and detection of collisions is an important problem in robotics and computer graphics, but there has been very little quantitative analysis of the mathematical structure of the interaction. Instead the focus has been on algorithms for some object representations, or on topological properties (see e.g. [7] for an interesting qualitative analysis of the milling problem). An exception is [1] which computes the differential geometry of such interactions of boundaries for the 2-dimensional case, using Legendre transformations and their coordinate free generalizations. There is an obvious need for extension of those methods to higher dimensional spaces, but dealing with the boundaries as *oriented hypersurfaces* (with a definite inside and outside) is not easily achieved using conventional geometric techniques. In this paper, we use geometric algebra [2, 3] to analyze the operations. It is crucial to first develop a proper representation of m-dimensional boundaries, extending classical differential geometry with a *parametrization-free notion of 'inside'*. Geometric calculus [2] can be used to do this very compactly, as we show in section 2, enabling consistent treatment of the 'signs' involved in orienting boundaries. But for a good implementational and analytical framework, we need a better representation of boundaries as geometric objects; this is developed in section 3, where we show that geometric algebra permits a powerful description of the boundaries of objects from Euclidean m-dimensional space, using an embedding in a $(m+1,1)$-dimensional Minkowski space. We show in section 4 that in this space, wave propagation assumes a simple additive form, and analyze some of the differential properties (such as: Gaussian curvatures are additive under propagation). Then in section 5 we go one step further: we show that this representation can be cast in terms of *versors*, specifying the boundary as a direction-dependent translation on a point at the origin. In this representation many boundary operations (such as rotation, translation and propagation) are represented as geometric products of versors.

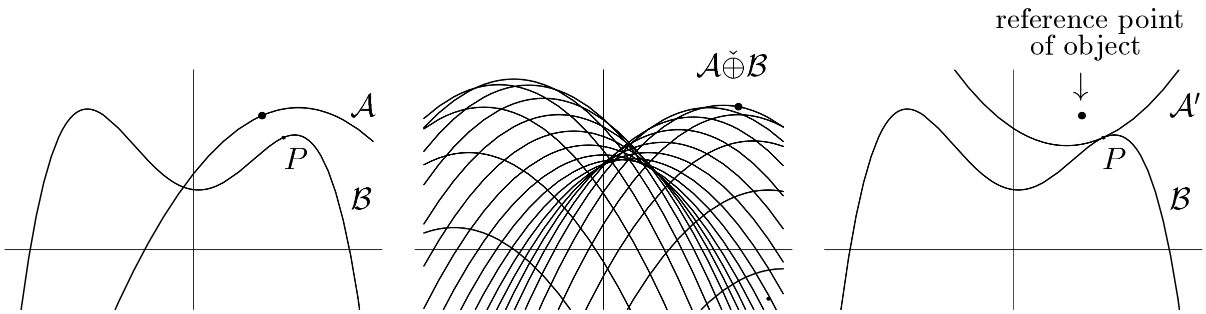


Figure 1: *Wave propagation and collision are mathematically similar (see text).*

2 Boundary geometry

In classical differential geometry, one speaks of curves and surfaces without necessarily viewing them as *boundaries*, i.e. as borders in the appropriate space with an inside and an outside. An *ad hoc* introduction afterwards of a 'positive side' to such a curve or surface using a 'left-hand rule' bases the indication of inside on a parametrization, and is therefore not geometrically invariant. This creates problems when studying outward wave propagation as an operation on boundaries, for the non-convex parts of an original wave front will eventually generate secondary wave fronts in which this rule does not provide the correct outward propagation direction, notably in the pieces where swallowtail catastrophes have occurred during propagation (see figure 2). We thus need to have a consistent geometrically invariant way of specifying the 'sides' of a boundary. In [1] this was done for the 2-dimensional case by embedding the boundaries into a projective space, in classical terminology. In this paper, we treat the m-dimensional case, using geometric algebra and geometric calculus [2, 3] as our language of expression for oriented (sub)spaces.

2.1 The oriented tangent space

In an m-dimensional Euclidean space $\mathcal{G}^1(\mathbf{I}_m)$, with pseudoscalar $\mathbf{I}_m$, we consider an object, noting specifically its boundary. This boundary is an $(m-1)$-dimensional hypersurface, with locally two 'sides': an inside and an outside. Assume the boundary to be smooth (we will not treat edges in this paper); then at every point $\mathbf{p}$ of the boundary, the boundary surface has a local tangent space with pseudoscalar $\mathbf{I}[\mathbf{p}]$ of grade $m-1$ (which we will mostly denote by $\mathbf{I}$, with $\mathbf{p}$ understood). Dually, we may represent this by its normal vector $\mathbf{n}$ (with again $\mathbf{p}$ understood as parameter), defined by the geometric product with the inverse pseudoscalar:

$$\mathbf{n} = -\mathbf{I}\mathbf{I}_m^{-1}. \tag{1}$$

The '$-$' is introduced here to avoid awkward signs later on.

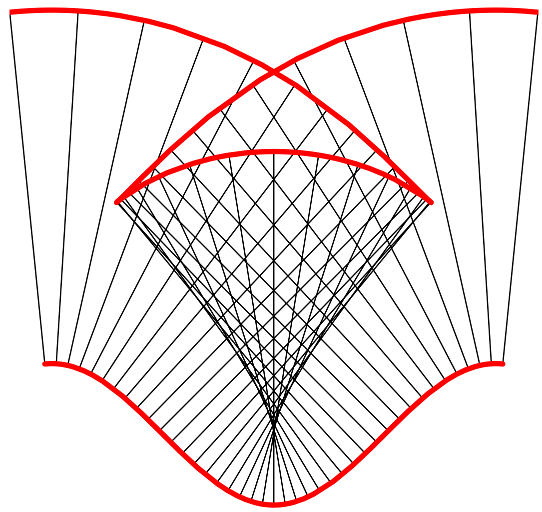


Figure 2: *Close-up of a swallowtail catastrophe in wave propagation. Note the apparent reversion of order of points between the cusps, which may seem to be a sign change in the first derivative of the normal vector (it is* not*, as we will see).*

We wish to denote the notion of 'inside' geometrically in the boundary representation. This involves orienting $\mathbf{n}$ (and hence $\mathbf{I}$). The usual convention for a circular blob in 2D (with the usual right-handed pseudoscalar $\mathbf{I}_2$) is 'when following the contour, the object is at the left-hand side'. So with a tangent $\mathbf{I}$ in the direction of motion, we have $\mathbf{I}\mathbf{I}_2$ as *inward* pointing direction. Therefore $\mathbf{n} = -\mathbf{I}\mathbf{I}_2^{-1} = \mathbf{I}\mathbf{I}_2$ is the *inward pointing normal.* We generalize this to m dimensional space, deriving the sign of $\mathbf{I}$ using eq.(1) from the desire to have $\mathbf{n}$ be the locally *inward* pointing normal.

2.2 Differential geometry of the boundary

The boundary surface at position $\mathbf{p}$ in $\mathcal{G}^1(\mathbf{I}_m)$ has tangent $\mathbf{I}[\mathbf{p}]$; when we move along the boundary surface, the tangent will change. The description of these changes can be found in [2](chapter 4 & 5), and we repeat the elements relevant to our analysis.

A surface is described by the points on it; this is the *identity function* of the surface, denoted by $\mathbf{id}(\mathbf{x})$ for a point $\mathbf{x}$. (Example: an origin-centered sphere with radius ρ is defined by $\mathbf{id}(\mathbf{x}) = \rho\mathbf{x}/|\mathbf{x}|$.) This identity function determines the formula for the projection onto the tangent space $\mathbf{I}$ at any point $\mathbf{p}$ of the surface: *the projection is the differential of the identity.* In formula, the projection of a vector $\mathbf{a}$ to the tangent space $\mathbf{I}$ at $\mathbf{p}$ is:

$$\mathbf{P}(\mathbf{a}) \equiv (\mathbf{a} \cdot \boldsymbol{\partial}_\mathbf{x})\mathbf{id}(\mathbf{x}) \quad \text{evaluated at} \quad \mathbf{p} \tag{2}$$

(We follow Hestenes' convenient notation of suppressing the argument $\mathbf{p}$, and we will write $\boldsymbol{\partial}$ for derivatives to $\mathbf{x}$. We used a bold $\mathbf{P}$ and $\boldsymbol{\partial}$ to denote that these operators are in the space $\mathcal{G}^1(\mathbf{I}_m)$. If we need to denote the point $\mathbf{p}$ at which a quantity is to be evaluated, we will do so using square brackets.) (Example of eq.(2): for the sphere, we obtain the familiar $\mathbf{P}(\mathbf{a}) = \mathbf{p}(\mathbf{p}^{-1} \wedge \mathbf{a}) = \mathbf{a} - \mathbf{p}(\mathbf{p}^{-1} \cdot \mathbf{a})$.) This same projection operator to the tangent space with pseudoscalar $\mathbf{I}$ can be written in terms of the normal $\mathbf{n}$ at $\mathbf{p}$ as:

$$\mathbf{P}(\mathbf{a}) = (\mathbf{a} \cdot \mathbf{I})\mathbf{I}^{-1} = (\mathbf{a} \cdot (\mathbf{n}\mathbf{I}_m))\mathbf{I}_m^{-1}\mathbf{n}^{-1} = (\mathbf{a} \wedge \mathbf{n})\mathbf{n}^{-1} = \mathbf{n}^{-1}(\mathbf{n} \wedge \mathbf{a}) = \mathbf{a} - (\mathbf{a} \cdot \mathbf{n})\mathbf{n}^{-1}. \quad (3)$$

Therefore a unit normal to the surface is:

$$\mathbf{n} = \pm \frac{\mathbf{a} - \mathbf{P}(\mathbf{a})}{|\mathbf{a} - \mathbf{P}(\mathbf{a})|}. \quad (4)$$

The sign needs to be chosen to represent the desired inside of the boundary, since the definition of the boundary surface in terms of $\mathbf{id}(\mathbf{x})$ does not yet include this specification – it could bound a 'blob' or a 'hole'.

If the positions on the boundary surface are implicitly defined by a scalar-valued function equation $\phi(\mathbf{p}) = 0$, then there is no need to solve for $\mathbf{p}$ to derive $\mathbf{id}(\mathbf{x})$ and perform the procedure above. For taking the differential of $0 = \phi(\mathbf{p})$ to $\mathbf{a}$ we obtain

$$0 = (\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{x}})\phi(\mathbf{p}) = (\mathbf{a} \cdot \boldsymbol{\partial}_{\mathbf{x}})\phi(\mathbf{id}(\mathbf{x})) = (\mathbf{P}(\mathbf{a}) \cdot \boldsymbol{\partial}_{\mathbf{p}})\phi(\mathbf{p}). \quad (5)$$

But $(\mathbf{P}(\mathbf{a}) \cdot \boldsymbol{\partial}_{\mathbf{p}})\phi(\mathbf{p})$ is the $\mathbf{P}(\mathbf{a})$ component of the vector derivative $\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})$. Since this is zero, $\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})$, which is a vector since ϕ is scalar valued, is proportional to $\mathbf{n}$. It still needs to be normalized and endowed with the appropriate sign, about which (as before) the equation specifying the position of the boundary is silent. We thus obtain:

For a boundary implicitly given as $\phi(\mathbf{p}) = 0$, the inward pointing normal is:

$$\mathbf{n} = \pm \frac{\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})}{|\boldsymbol{\partial}_{\mathbf{p}}\phi(\mathbf{p})|} \quad (6)$$

with the sign determined by the desired 'inside' at $\mathbf{p}$.

We will refer to this sign as $\boldsymbol{\sigma}[\mathbf{p}]$, or just $\boldsymbol{\sigma}$ with $\mathbf{p}$ understood.

The second order differential structure of the boundary is obtained by differentiating such a properly oriented $\mathbf{n}$ using a vector derivative in some direction $\mathbf{b}$. We denote the resulting position-dependent linear function of $\mathbf{b}$ by $\underline{\mathbf{n}}(\mathbf{b})$ (leaving the dependence on $\mathbf{p}$ understood implicitly):

$$\underline{\mathbf{n}}(\mathbf{b}) \equiv (\mathbf{b} \cdot \boldsymbol{\partial})\mathbf{n}. \quad (7)$$

Since $\underline{\mathbf{n}}(\mathbf{b})$ is a linear function of the vector argument $\mathbf{b}$, we may extend it to arbitrary multivector arguments as an outermorphism (i.e. a $\wedge$-preserving linear operator). Specifically, we can form $\underline{\mathbf{n}}(\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \mathbf{b}_{m-1})$, with the $\mathbf{b}_i$ forming a basis for the tangent space $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$ at $\mathbf{p}$. The quantity $\underline{\mathbf{n}}(\mathbf{I})$ thus denotes the total change of $\mathbf{n}$ as we move over a local tangent volume the size of the tangent pseudoscalar $\mathbf{I}$; the ratio with $\mathbf{I}$ (which is the determinant of the mapping $\underline{\mathbf{n}}$) is related to the *Gaussian curvature* of the boundary at $\mathbf{p}$ by:

$$\boldsymbol{\kappa} \equiv (-1)^{m-1}\underline{\mathbf{n}}(\mathbf{I})\mathbf{I}^{-1}, \tag{8}$$

where we obtain an extra sign relative to the standard convention for the definition of $\boldsymbol{\kappa}$ which uses the *outward pointing normal*, due to the fact that $\mathbf{n} \to -\mathbf{n}$ gives $\underline{\mathbf{n}}(\mathbf{I}) \to (-1)^{m-1}\underline{\mathbf{n}}(\mathbf{I})$). (Although this is not totally standard; [9] uses both conventions – switching between them verbally due to a lack of convenient mathematics to denote orientation.)

The vector $\underline{\mathbf{n}}(\mathbf{a})$ gives the change in unit normal when moving in the $\mathbf{a}$ direction; this value is unique for the regular surfaces we treat. It is one of the properties of differentials that $\underline{\mathbf{n}}(\mathbf{P}(\mathbf{a})) = \underline{\mathbf{n}}(\mathbf{a})$ for all $\mathbf{a}$ – and so a unique inverse to $\underline{\mathbf{n}}(\cdot)$ does not exist. But even when we limit the inverse to have values in the tangent space $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$, there may not be a unique solution. (An example is a cylinder with axis $\mathbf{z}$, where only the component of $\mathbf{P}(\mathbf{a})$ perpendicular to $\mathbf{z}$ determines the value of $\underline{\mathbf{n}}(\mathbf{a})$.) Therefore the inverse of $\underline{\mathbf{n}}(\cdot)$ usually produces a *set* of vectors in the space with pseudoscalar $\mathbf{I}_m$ based at $\mathbf{p}$ (we will denote this space by $\mathcal{G}^1(\mathbf{I}_m[\mathbf{p}])$). We prefer to limit the values to the local tangent space $\mathcal{G}^1(\mathbf{I}[\mathbf{p}])$, and so define:

$$\underline{\mathbf{n}}^{-1} : \mathcal{G}^1(\mathbf{I}[\mathbf{p}]) \to \mathcal{G}^1(\mathbf{I}[\mathbf{p}]) : \quad \underline{\mathbf{n}}^{-1}(\mathbf{m}) \equiv \{\mathbf{a} \mid \underline{\mathbf{n}}(\mathbf{a}) = \mathbf{m}\}. \tag{9}$$

Such set-valued functions can be added, using the *Minkowski sum* (denoted by $\oplus$) as set addition:

$$A \oplus B = \{a + b \mid a \in A, b \in B\} \tag{10}$$

(Note that if one of the arguments is $\emptyset$, then so is the result!).[1]

3 The boundary as a geometric object

In the representation of the boundary so far, we required a description of the position $\mathbf{p}$ (of which the differential structure gives us the local tangent space, characterizable by $\mathbf{I}$ or $\mathbf{n}$) and an orientation $\boldsymbol{\sigma}[\mathbf{p}]$ to specify 'inside' (which then gives the proper sign to $\mathbf{I}$ or $\mathbf{n}$). Thus the boundary is not yet a *single geometric object*. It is actually possible to combine this into a single geometrical representation, but we should look towards higher-dimensional spaces to construct it. A classical construction is an embedding into a projective space (this technique is called 'homogeneous coordinates'), where we have to be careful to use *oriented* projective geometry to maintain the proper signs (see [1]). We start with that, but we will see how it naturally leads to the use of 'generalized homogeneous points', a recent development in geometric algebra [6].

[1] The 'inverse' should be interpreted in the sense of sets; there is no identity but a containment relationship for $\underline{\mathbf{n}}^{-1} \circ \underline{\mathbf{n}}$: $\mathbf{a} \in \underline{\mathbf{n}}^{-1}(\underline{\mathbf{n}}(\mathbf{a}))$, although for the regular surfaces we study here, we do have: $\underline{\mathbf{n}}(\underline{\mathbf{n}}^{-1}(\mathbf{m})) = \{\mathbf{m}\}$ (where we naturally extended $\underline{\mathbf{n}}$ to work on sets by $\underline{\mathbf{n}}(A) \equiv \{\underline{\mathbf{n}}(\mathbf{a}) \mid \mathbf{a} \in A\}$). In what follows, we will not be pedantic about overloading our functions to operate on sets of operands, sometimes omitting the set brackets in our expressions.

3.1 Embedding in oriented projective space

We embed the boundaries of the Euclidean space $\mathcal{G}^1(\mathbf{I}_m)$ in a higher dimensional space $\mathcal{G}^1(I_{m+1})$. Choose a basis vector e_0 in this space, perpendicular to $\mathbf{I}_m$, and construct the pseudoscalar I_{m+1} such that $I_{m+1} = e_0 \mathbf{I}_m$. Note that $e_0 \cdot \mathbf{x} = 0$ for any $\mathbf{x} \in \mathbf{I}_m$.

Let the point $\mathbf{p}$ be embedded as

$$p = e_0 + \mathbf{p} \tag{11}$$

(we will consistently use **bold** symbols for elements of $\mathcal{G}(\mathbf{I}_m)$, and *italic* for elements of $\mathcal{G}(I_{m+1})$ not also in $\mathcal{G}(\mathbf{I}_m)$). As we have seen, a boundary element at $\mathbf{p}$ is characterized by a pseudoscalar $\mathbf{I}[\mathbf{p}]$, which we will often write as $\mathbf{I}$. In this paper, we take the boundary to be regular, so that $\mathbf{I}$ is an $(m-1)$-dimensional blade. In the embedding space $\mathcal{G}^1(I_{m+1})$, this tangent is represented by the homogeneous blade (see [4]):

$$(e_0 + \mathbf{p}) \wedge \mathbf{I} = p \wedge \mathbf{I}. \tag{12}$$

Its dual is a vector n in $\mathcal{G}^1(I_{m+1})$, and it is going to be our representation (as $\mathbf{p}$ varies over the boundary). We compute the dual as:

$$n \equiv (p \wedge \mathbf{I}) I_{m+1}^{-1} = p \cdot (\mathbf{I}\mathbf{I}_m^{-1} e_0^{-1}) = p \cdot (e_0^{-1}\mathbf{n}) = \mathbf{n} - e_0^{-1}(\mathbf{p} \cdot \mathbf{n}). \tag{13}$$

We will write e^0 for e_0^{-1}, noting that so far the only demand on this inverse is that $e_0 \cdot e^0 = 1$, i.e. e^0 is reciprocal to e_0. We will also need the obvious $e_0 \wedge e_0 = 0$ and $e^0 \wedge e^0 = 0$ and of course $e^0 \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in \mathcal{G}^1(\mathbf{I}_m)$.

The n-representation is a function of $\mathbf{n}$ and $\mathbf{p}$, non-linear in $\mathbf{p}$ but linear in $\mathbf{n}$. We denote it by $\mathcal{R}(\mathbf{n})$, with $\mathbf{p}$ understood as usual. Differentiating this to $\mathbf{a}$ we obtain:

$$(\mathbf{a}\cdot\boldsymbol{\partial})\mathcal{R}(\mathbf{n}) = \underline{\mathbf{n}}(\mathbf{a}) - e^0(\mathbf{P}(\mathbf{a})\cdot\mathbf{n}) - e^0(\mathbf{p}\cdot\underline{\mathbf{n}}(\mathbf{a})) = p\cdot(e^0\underline{\mathbf{n}}(\mathbf{a})) = \mathcal{R}(\underline{\mathbf{n}}(\mathbf{a})) = \mathcal{R}((\mathbf{a}\cdot\boldsymbol{\partial})\mathbf{n}). \tag{14}$$

so the representation commutes with differentiation to an element of $\mathcal{G}^1(\mathbf{I}_m[\mathbf{p}])$. This differential vector $\underline{\mathbf{n}}(\mathbf{a})$ is in $\mathbf{I}[\mathbf{p}]$, and so we find by linearity that *any vector* $\mathbf{a} \in \mathcal{G}^1(\mathbf{I}_m[\mathbf{p}])$ (note: $\mathbf{I}_m$ rather than merely $\mathbf{I}$!) can be represented by

$$\mathcal{R}(\mathbf{a}) = \mathbf{a} - e^0(\mathbf{p}\cdot\mathbf{a}) = p\cdot(e^0\mathbf{a}). \tag{15}$$

Since this is linear in $\mathbf{a}$, we may extend it as an outermorphism to *any blade* in $\mathcal{G}(\mathbf{I}_m[\mathbf{p}])$:

$$\mathcal{R}(\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \cdots \wedge \mathbf{a}_k) = \mathcal{R}(\mathbf{a}_1) \wedge \mathcal{R}(\mathbf{a}_2) \wedge \cdots \wedge \mathcal{R}(\mathbf{a}_k), \tag{16}$$

and then by linearity to all of $\mathcal{G}(\mathbf{I}_m[\mathbf{p}])$. Thus we obtain as our representation:

$$\mathcal{R}(\mathbf{A}) = \mathbf{A} - e^0(\mathbf{p}\cdot\mathbf{A}) = p\cdot(e^0\mathbf{A}), \quad \text{for any } \mathbf{A} \in \mathcal{G}(\mathbf{I}_m[\mathbf{p}]). \tag{17}$$

(Note that for a scalar α this yields $\mathcal{R}(\alpha) = \alpha$, as it should. Note also that the representation of $\mathbf{p}$ itself: $p = e_0 + \mathbf{p}$, does *not* satisfy this formula – but of course, it is not an element of its own affine space $\mathcal{G}^1(\mathbf{I}_m[\mathbf{p}])$.)

Thus the representation of eq.(17) maps the whole geometric algebra at $\mathbf{p}$ to $\mathcal{G}(I_{m+1})$. However, it does *not* necessarily preserve the geometric product, for the outermorphism property gives:

$$\mathcal{R}(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} \tag{18}$$

whereas

$$\mathcal{R}(\mathbf{a}) \cdot \mathcal{R}(\mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + (e^0 \cdot e^0)(\mathbf{p} \cdot \mathbf{a})(\mathbf{p} \cdot \mathbf{b}), \tag{19}$$

and the two are unequal if $e^0 \cdot e^0 \neq 0$. Therefore the *inner* product is not preserved, and so neither is the *geometric* product. We will mend this in the next section.

The representation $\mathcal{R}(\mathbf{A})$ satisfies:

$$p \cdot \mathcal{R}(\mathbf{A}) = p \cdot \Big(p \cdot (e^0 \mathbf{A})\Big) = (p \wedge p) \cdot (e^0 \mathbf{A}) = 0, \tag{20}$$

so that p is perpendicular to the representation of *any* of the elements of the algebra $\mathcal{G}(\mathbf{I}_m[\mathbf{p}])$ based at $\mathbf{p}$. Since $\mathcal{R}(\mathbf{n})$ spans an m-dimensional hypersurface within an $(m+1)$-dimensional space, we can use this property to obtain $\mathbf{p}$ from the differential structure of the representation, as follows:

If we assume $e^0 e_0 = 1$, then:

$$\begin{aligned} p &= (-1)^m \mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) e_0 (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}))^{-1} & (21) \\ &= \mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1} e_0 / \boldsymbol{\kappa}. & (22) \end{aligned}$$

with $\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})$ computable from $\mathcal{R}(\mathbf{n})$ by:

$$\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}) = e_0 \cdot \Big(e^0 \wedge \mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}))\Big) \tag{23}$$

Proof:

$$\begin{aligned} \mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) e_0 \mathbf{I}_m^{-1} &= \Big(p \cdot \Big(e^0 (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}))\Big)\Big) e_0 \mathbf{I}_m^{-1} \\ &= p \wedge \Big(e^0 (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) e_0 \mathbf{I}_m^{-1}\Big) \\ &= (-1)^m p \wedge \Big(e^0 (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1} e_0\Big) \\ &= (-1)^m (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1}\ p \wedge (e^0 e_0) \\ &= (-1)^m (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1}\ p, \end{aligned}$$

where we use the fact that $(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1}$ is scalar. In fact its value is:

$$\begin{aligned} (-1)^m (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1} &= (-1)^{m-1} (\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}^{-1} \mathbf{n} \\ &= (-1)^{m-1} \mathbf{n} \cdot (\underline{\mathbf{n}}(\mathbf{I}) \mathbf{I}^{-1} \mathbf{n}) = \mathbf{n} \cdot (\boldsymbol{\kappa} \mathbf{n}) = \boldsymbol{\kappa} \end{aligned}$$

so that we obtain

$$\mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) e_0 \mathbf{I}_m^{-1} = \boldsymbol{\kappa} p \tag{24}$$

GP	e_0	e^0	E
e_0	0	$1-E$	e_0
e^0	$1+E$	0	$-e^0$
E	$-e_0$	e^0	1

$\cdot$	e_0	e^0	E
e_0	0	1	e_0
e^0	1	0	$-e^0$
E	$-e_0$	e^0	1

$\wedge$	e_0	e^0	E
e_0	0	$-E$	0
e^0	E	0	0
E	0	0	0

Figure 3: *The multiplication tables for e_0, e^0 and $E = e_0 \wedge e^0$ for the geometric product (GP), the inner product $\cdot$ and the outer product $\wedge$, respectively. First factor is the column index, second factor the row index.*

(Note that dually, it is quite clear that $\mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}))$ should be a multiple of p, since each of the factors in the wedge product is of the form $p \cdot (e^0 \mathbf{A})$; therefore the dual of the total wedge product is a meet with factors of the form $p \wedge (\mathbf{A}\mathbf{I}_m)$. The meet of m such blades with p as common term *must* be proportional to p.)

It is easily verified that the representation of an element $\mathbf{A}$ from the tangent algebra $\mathcal{G}(\mathbf{I}_m[\mathbf{p}])$ is invertible by the standard projective split method:

$$\mathcal{R}^{-1}(\mathcal{R}(\mathbf{A})) = e_0 \cdot (e^0 \wedge \mathcal{R}(\mathbf{A})) = e_0 \cdot \Big(e^0 \wedge (\mathbf{A} - e^0(\mathbf{p} \cdot \mathbf{A})\Big) = e_0 \cdot (e^0 \mathbf{A}) = \mathbf{A} \tag{25}$$

and this relates $\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})$ to $\mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I}))$. $\square$

In practice, one only needs to compute $\mathcal{R}(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) e_0 \mathbf{I}_m^{-1}$, and then perform scalar rescaling to bring it in the form $e_0 + \mathbf{p}$. This provides $\mathbf{p}$ – and, incidentally, also $\boldsymbol{\kappa}$.

3.2 Generalized homogeneous representation of boundaries

We have seen in eq.(19) that the inner product is not preserved under the mapping $\mathcal{R}(\cdot)$ if $e^0 e^0 \neq 0$. So it is now reasonable to turn e^0 into a *null vector*, i.e. a vector with norm zero. This is not possible if we take $\mathcal{G}^1(I_{m+1})$ to be Euclidean – we must extend it to a Minkowski space. Also, if we still want to preserve the relationship $e_0 \cdot e^0 = 0$, then e_0 must be a null vector as well. We thus find ourselves with *two* extra dimensions required beyond m for a product-preserving representation. This is precisely what was done in the 'generalized homogenous point' embedding for Euclidean spaces used in [6], where a Euclidean $\mathcal{G}(\mathbf{I}_m)$ was mapped into a Minkowskian $\mathcal{G}(I_{m+1,1})$, into which basis vectors (e_0 and $e = -e^0$) were then chosen on the null cone. We denote the pseudoscalar formed by e_0 and e^0 by E:

$$E = e_0 \wedge e^0 \tag{26}$$

and find the multiplication tables of figure 3. We will follow the same convention as in the previous section, indicating elements of the Euclidean space in **bold**, and those of the embedding space by *italic* (unless they are contained in $\mathbf{I}_m$ as well). The total pseudoscalar for the Minkowski algebra we define as:

$$I_{m+1,1} \equiv E\mathbf{I}_m. \tag{27}$$

In [6] it is shown that we can now homogeneously embed the point $\mathbf{p}$ by the null vector:

$$p' = e_0 + \mathbf{p} - \tfrac{1}{2}\mathbf{p}^2 e^0 \tag{28}$$

(Check that indeed $(p')^2 = 0$.) We will use the notation p' to distinguish this embedding from the projective embedding $p = e_0 + \mathbf{p}$ in the previous section. Note that, under identification of e_0 in both cases, the two are related by:

$$e^0 \wedge p' = e^0 \wedge p \quad (= -E + e^0\mathbf{p}), \tag{29}$$

i.e. if we take the wedge product with e^0, the algebraic structures should behave similarly. In [6] it is shown that $e^0 \wedge p' \wedge q' \wedge \cdots$ is the representation of a hyperplane through the points $\mathbf{p}$, $\mathbf{q}$, $\cdots$, which we may call the *flat* through $\mathbf{p}$, $\mathbf{q}$, $\cdots$ following [8]. In this view, by eq.(29) the projective representation of $\mathbf{p}$ is similar to considering $\mathbf{p}$ as a flat, i.e. as a 0-dimensional linear subspace. (For your intuition, it may be useful to know that $-e^0$ represents the point at infinity, and e_0 obviously represents the point at the origin.)

The tangent space is a flat with pseudoscalar $\mathbf{I}$ at $\mathbf{p}$, and this is represented by:

$$(e^0 \wedge p') \wedge \mathbf{I}, \tag{30}$$

which is, dually in $\mathcal{G}^1(I_{m+1,1})$:

$$\begin{aligned} \mathcal{R}'(\mathbf{n}) &= (e^0 \wedge p' \wedge \mathbf{I})\mathbf{I}_m^{-1} E = (e^0 \wedge p) \cdot (\mathbf{I}\mathbf{I}_m^{-1} E) \\ &= (p \wedge e^0) \cdot (E\mathbf{n}) = p \cdot (e^0 \mathbf{n}) = \mathcal{R}(\mathbf{n}) \end{aligned}$$

So the dual representation of a flat is the same as the dual projective representation of a hyperplane (although the interpretation of e^0 differs slightly in both cases).

As before, we can establish by differentiation and linearity that a general element $\mathbf{A}$ from the algebra $\mathcal{G}(\mathbf{I}_m[\mathbf{p}])$ can be represented as:

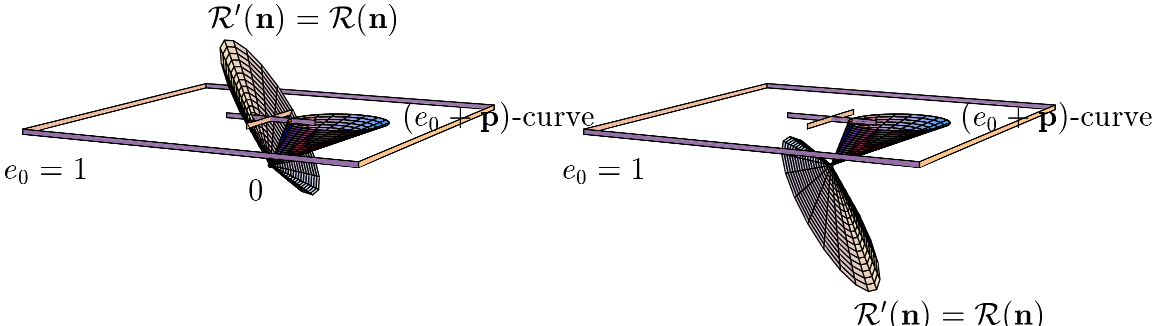
$$\mathcal{R}'(\mathbf{A}) = (p' \wedge e^0) \cdot (E\mathbf{A}), \tag{31}$$

and this is numerically the same as $\mathcal{R}(\mathbf{A})$, so that we could now drop the distinction in notation. But there is an important difference in interpretation, since the Minkowski representation *preserves the geometric product*:

$$\begin{aligned} \mathcal{R}'(\mathbf{a})\mathcal{R}'(\mathbf{b}) &= \Big((p' \wedge e^0) \cdot (E\mathbf{a})\Big)\Big((p' \wedge e^0) \cdot (E\mathbf{b})\Big) \\ &= \Big(\mathbf{a} - e^0(\mathbf{p} \cdot \mathbf{a})\Big)\Big(\mathbf{b} - e^0(\mathbf{p} \cdot \mathbf{b})\Big) \\ &= \mathbf{ab} - e^0\left((\mathbf{p} \cdot \mathbf{a})\mathbf{b} - \mathbf{a}(\mathbf{p} \cdot \mathbf{a})\right) + 0 \\ &= (\mathbf{ab}) - e^0(\mathbf{p} \cdot (\mathbf{ab})) = p \cdot \Big(e^0(\mathbf{ab})\Big) \\ &= (p' \wedge e^0) \cdot (E\mathbf{ab}) = \mathcal{R}'(\mathbf{ab}) \end{aligned}$$

Therefore this representation of the tangent algebra is *isometric*. As a consequence,

$$(\mathcal{R}'(\mathbf{n}))^2 = \mathbf{n}^2 = 1 \tag{32}$$



(a) *a circular hole* (b) *a circular blob*

Figure 4: *The representation of a circular hole and a blob in* $\mathbf{I}_2$. *For the* $(e_0 + \mathbf{p})$*-curve (we factor out the '*$e^0\wedge$*' from the full representation, for simplicity) the vertical axis denotes* e_0*; this curve resides in the plane* $e_0 = 1$ *which is indicated. For the* $\mathcal{R}'(\mathbf{n})$ *curves, the vertical axis is* e^0*, and the curve resides on the sphere* $\mathcal{R}'(\mathbf{n})^2 = 1$*, which due to the Minkowski geometry looks like a Euclidean cylinder. The curves have been made into cones to better indicate their spatial nature, and to help see that* $(e_0 + \mathbf{p})$ *is everywhere perpendicular to* $\mathcal{R}'(\mathbf{n} \wedge \mathbf{n}(\mathbf{I}))$ *Alternatively, the figure can be viewed as a depiction of the embedding* $\mathcal{R}(\mathbf{n})$ *into projective space endowed with a Euclidean metric; then the vertical axis denotes* $e_0 = e^0$*. (See also [1].)*

so the $(m-1)$-dimensional $\mathcal{R}'(\mathbf{n})$-surface resides on the intersection of the $(m+1)$-dimensional unit sphere in $I_{m+1,1}$ with the plane $e^0 \cdot \mathcal{R}'(\mathbf{n}) = 0$. It is essentially the Gaussian sphere of directions augmented with a perpendicular fiber in the e^0 direction. (Due to $(e_0)^2 = (e^0)^2 = 0$, when drawn this section of the unit sphere looks like an $(m+1)$-dimensional cylinder to our Euclidean eyes, see figure 4.) Similar to eq.(21), we can now derive an expression for $\mathbf{p}$ in terms of the representation $\mathcal{R}'(\mathbf{n})$:

With a boundary represented by $\mathcal{R}'(\mathbf{n})$*, one can recover the positional aspects as:*

$$
\begin{aligned}
(p' \wedge e^0) &= (-1)^m \mathcal{R}'(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) E \left(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})\right)^{-1} \\
&= (-1)^m \mathcal{R}'(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) E \mathbf{I}_m^{-1} / \boldsymbol{\kappa},
\end{aligned}
\tag{33}
$$

with $\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})$ *recoverable from* $\mathcal{R}'(\mathbf{n})$ *as* $e_0 \cdot (e^0 \wedge \mathcal{R}'(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})))$.

Proof: This proof follows that of eq.(21):

$$
\begin{aligned}
\mathcal{R}'(\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) E \mathbf{I}_m^{-1} &= \Big((p' \wedge e^0) \cdot (E\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \Big) E \mathbf{I}_m^{-1} \\
&= (p' \wedge e^0) \wedge \Big((\mathbf{n} \wedge \underline{\mathbf{n}}(\mathbf{I})) \mathbf{I}_m^{-1} \Big) = (-1)^m \boldsymbol{\kappa}\, (p' \wedge e^0).
\end{aligned}
$$

□

Thus as $\mathcal{R}(\mathbf{n})$, the representation $\mathcal{R}'(\mathbf{n})$ of the boundary contains sufficient information to reconstruct both its position representation and its inward pointing normal – it is a complete boundary representation.

4 Propagation of boundaries

We are interested in studying the properties of boundaries under the operation of *propagation* – a term borrowed from the similarity of this operation to Huyghens wave propagation. As we showed in the introduction, this operation is essentially equivalent to translational collision detection in robotics, object growing in graphics, and an inversion of the milling process.

4.1 Definition

Propagation combines two boundaries $\mathcal{A}$ and $\mathcal{B}$ to produce a boundary $\mathcal{A}\check{\oplus}\mathcal{B}$ according to the following rules (which can be taken as the definition of propagation, or alternatively derived from a formulaic definition as in [1]):

Propagation definition:

- The resulting position vector after combining a point $\mathbf{p}_\mathcal{A}$ on $\mathcal{A}$ and a point $\mathbf{p}_\mathcal{B}$ on $\mathcal{B}$ is the position $\mathbf{p}_\mathcal{A} + \mathbf{p}_\mathcal{B}$:

$$\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}} = \mathbf{p}_\mathcal{A} + \mathbf{p}_\mathcal{B} \tag{34}$$

- The points $\mathbf{p}_\mathcal{A}$ and $\mathbf{p}_\mathcal{B}$ *must* have the same inward pointing normal (to $\mathcal{A}$ and $\mathcal{B}$, respectively), and this is also the inward pointing normal at the resulting position in the resulting boundary. Symbolically:

$$\mathbf{n}_{\mathcal{A}\check{\oplus}\mathcal{B}}[\mathbf{p}_{\mathcal{A}\check{\oplus}\mathcal{B}}] = \mathbf{n}_\mathcal{A}[\mathbf{p}_\mathcal{A}] = \mathbf{n}_\mathcal{B}[\mathbf{p}_\mathcal{B}]. \tag{35}$$

See figure 2 for an illustration of these conditions in 2-dimensional space.

These conditions together fully determine the propagation result and the dependence of its geometry on the geometry of $\mathcal{A}$ and $\mathcal{B}$.

4.2 Propagation in the embedded representations

We denote the embedded boundary representation by n (since numerically, $\mathcal{R}(\mathbf{n})$ and $\mathcal{R}'(\mathbf{n})$ are identical, see eq.(31), there is no need to distinguish between them).

Since n is a full description of a boundary, if we can construct the n-representation of the wave propagation result then we know what the resulting boundary is. But this is extremely simple, since the n-representation lends itself to direct implementation of the definition of propagation of eq.(34) and eq.(35):

Let $\mathbf{p}_\mathcal{A}[\mathbf{n}]$ *be defined as:*

$$\mathbf{p}_\mathcal{A}[\mathbf{n}] = \{\mathbf{x} \in \mathcal{A} \mid \mathbf{n}_\mathcal{A}[\mathbf{x}] = \mathbf{n}\}, \tag{36}$$

and similarly for $\mathbf{p}_\mathcal{B}[\cdot]$. *Then the propagation result of two boundaries*

$$n_\mathcal{A} = \mathbf{n} - e^0(\mathbf{p}_\mathcal{A}[\mathbf{n}] \cdot \mathbf{n}) \tag{37}$$

and

$$n_{\mathcal{B}} = \mathbf{n} - e^0(\mathbf{p}_{\mathcal{B}}[\mathbf{n}] \cdot \mathbf{n}) \tag{38}$$

is

$$\begin{aligned} n_{\mathcal{C}} &= \mathbf{n} - e^0\left((\mathbf{p}_{\mathcal{A}}[\mathbf{n}] \cdot \mathbf{n}) \oplus (\mathbf{p}_{\mathcal{B}}[\mathbf{n}] \cdot \mathbf{n})\right) \\ &= \mathbf{n} - e^0\left((\mathbf{p}_{\mathcal{A}}[\mathbf{n}] \oplus \mathbf{p}_{\mathcal{B}}[\mathbf{n}]) \cdot \mathbf{n}\right), \end{aligned} \tag{39}$$

so:

$$\mathbf{p}_{\mathcal{C}}[\mathbf{n}] = \mathbf{p}_{\mathcal{A}}[\mathbf{n}] \oplus \mathbf{p}_{\mathcal{B}}[\mathbf{n}] \tag{40}$$

So basically, the e^0 components add up (we use the $\oplus$ since there may be several values of $\mathbf{p}[\mathbf{n}]$ for a given $\mathbf{n}$ in each boundary, if the boundaries are non-convex).

This is a simple result – in fact, deceptively simple since the fact that both $n_{\mathcal{A}}$ and $n_{\mathcal{B}}$ are written in terms of $\mathbf{n}$ may require an inversion and a reparametrization to obtain $\mathbf{p}$ as a function of $\mathbf{n}$ valid over a finite domain (if the boundaries were originally given in terms of parametrized position). However, this can be done, if necessary numerically; and then the result is useful to construct the resulting boundary (in its n-representation form), and to derive its properties. Collision detection and wave propagation (and the other, equivalent operations of the introduction) can be done fully in this representation. It is only when one desires the result to be drawn as a positional surface again that the inversion formula eq.(21) or eq.(33) needs be invoked (so really only for rendering purposes).

Figure 5 was generated in this way: adding the representations of a cardioid and a circular blob, and then 'inverting' the result to a position curve with a local notion of 'inside'. Note the occurrence of swallowtail catastrophes in some of the wave fronts (see also figure 2). Classically, these have been considered hard to treat, and even non-differentiable; however they are fully differentiable in our *directed* representation of the tangent space. In fact, these *spatial cusps correspond to inflection points* in the n-representation, and are thus nothing more unusual than a vanishing second derivative of $\mathbf{n}$ (please convince yourself in figure 2 that $\underline{\mathbf{n}}(\mathbf{a})$ retains its sign – it is the *second* derivative of $\mathbf{n}$ which undergoes a sign change). The locations where the boundary surface self-intersect (important for the analysis of the 'millability' of surfaces) correspond to non-local properties of the n-representation; the intersection point corresponds to a straight part of the convex hull of n (for some more details see [1] – but the computational consequences are a subject of ongoing research).

4.3 Analysis of propagation

We can derive a differential property of the propagation operation:

The propagated boundary $\mathcal{C} = \mathcal{A}\check{\oplus}\mathcal{B}$ obeys the 'velocity law':

$$\underline{\mathbf{n}}_{\mathcal{C}}^{-1}(\mathbf{m}) = \underline{\mathbf{n}}_{\mathcal{A}}^{-1}(\mathbf{m}) \oplus \underline{\mathbf{n}}_{\mathcal{B}}^{-1}(\mathbf{m}), \tag{41}$$

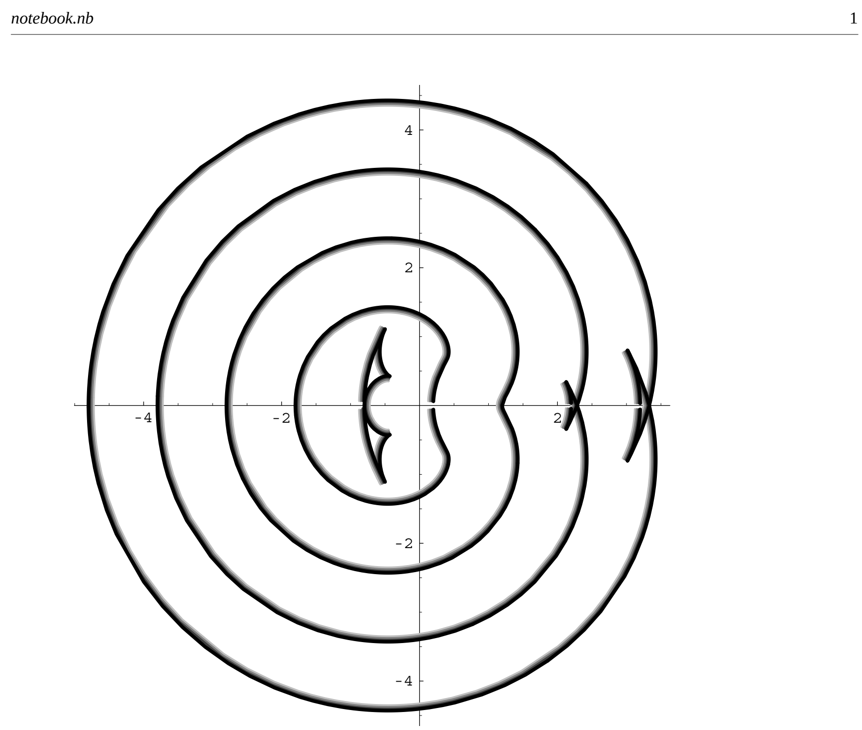


Figure 5: *The propagation of waves from a cardioid shape, both two units inward and outward, using a circular propagator. Computed using Mathematica via the embedded representation.*

where the quantities are to be evaluated at the related points of eq.(34). (So a fuller description is:

$$\underline{\mathbf{n}}^{-1}_{\mathcal{A}\check{\oplus}\mathcal{B}}(\mathbf{m})[\mathbf{p}_\mathcal{A} \oplus \mathbf{p}_\mathcal{B}] = \underline{\mathbf{n}}^{-1}_\mathcal{A}(\mathbf{m})[\mathbf{p}_\mathcal{A}] \oplus \underline{\mathbf{n}}^{-1}_\mathcal{B}(\mathbf{m})[\mathbf{p}_\mathcal{B}]. \tag{42}$$

but it is convenient to omit the point references.)
The result is $\emptyset$ *for* $\mathbf{m}$ not *in the common range of* $\underline{\mathbf{n}}_\mathcal{A}[\mathbf{p}_\mathcal{A}](\cdot)$ *and* $\underline{\mathbf{n}}_\mathcal{B}[\mathbf{p}_\mathcal{B}](\cdot)$.

Proof: Introduce three tangent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, to measure the derivative on each of the surfaces, and use the chain rule of [2] to rewrite them in terms of derivatives of $\mathbf{p}[\mathbf{n}]$:

$$\mathbf{a} = \mathbf{P}_\mathcal{A}(\mathbf{a}) = (\mathbf{a} \cdot \boldsymbol{\partial})\mathbf{p}_\mathcal{A} = (\underline{\mathbf{n}}_\mathcal{A}(\mathbf{a}) \cdot \boldsymbol{\partial}_\mathbf{p})\mathbf{p}_\mathcal{A}[\mathbf{n}]$$

$$\mathbf{b} = \mathbf{P}_\mathcal{B}(\mathbf{b}) = (\mathbf{b} \cdot \boldsymbol{\partial})\mathbf{p}_\mathcal{B} = (\underline{\mathbf{n}}_\mathcal{B}(\mathbf{b}) \cdot \boldsymbol{\partial}_\mathbf{p})\mathbf{p}_\mathcal{B}[\mathbf{n}]$$

$$\mathbf{c} = \mathbf{P}_\mathcal{C}(\mathbf{c}) = (\mathbf{c} \cdot \boldsymbol{\partial})\mathbf{p}_\mathcal{C} = (\underline{\mathbf{n}}_\mathcal{C}(\mathbf{c}) \cdot \boldsymbol{\partial}_\mathbf{p})\mathbf{p}_\mathcal{C}[\mathbf{n}]$$

Now select these such that $\underline{\mathbf{n}}_\mathcal{A}(\mathbf{a}) = \underline{\mathbf{n}}_\mathcal{B}(\mathbf{b}) = \underline{\mathbf{n}}_\mathcal{C}(\mathbf{c}) = \mathbf{m}$. This implies that

$$\mathbf{a} \in \underline{\mathbf{n}}^{-1}_\mathcal{A}(\mathbf{m}), \quad \mathbf{b} \in \underline{\mathbf{n}}^{-1}_\mathcal{B}(\mathbf{m}), \quad \mathbf{c} \in \underline{\mathbf{n}}^{-1}_\mathcal{C}(\mathbf{m}). \tag{43}$$

We then find from the above that these tangents add as position vectors:

$$\mathbf{c} = (\mathbf{m} \cdot \boldsymbol{\partial}_\mathbf{p})\mathbf{p}_\mathcal{C}[\mathbf{n}] = (\mathbf{m} \cdot \boldsymbol{\partial}_\mathbf{p})\,(\mathbf{p}_\mathcal{A}[\mathbf{n}] + \mathbf{p}_\mathcal{B}[\mathbf{n}]) = \mathbf{a} + \mathbf{b}, \tag{44}$$

and therefore, over all possibilities of choosing $\mathbf{a}$ and $\mathbf{b}$ given $\mathbf{c}$, we obtain:

$$\underline{\mathbf{n}}^{-1}_\mathcal{C}(\mathbf{m}) = \underline{\mathbf{n}}^{-1}_\mathcal{A}(\mathbf{m}) \oplus \underline{\mathbf{n}}^{-1}_\mathcal{B}(\mathbf{m}). \tag{45}$$

The right hand side produces $\emptyset$ for any element not common to both sets contributing to the Minkowski sum; hence only elements in both ranges contribute – which implies that $\mathbf{m}$ must be in the common range of $\underline{\mathbf{n}}_\mathcal{A}(\cdot)$ and $\underline{\mathbf{n}}_\mathcal{B}(\cdot)$ at $\mathbf{p}_\mathcal{A}$ and $\mathbf{p}_\mathcal{B}$, respectively. □

This interaction of the local differential geometries can produce involved results, especially for surfaces with torsion. However, there is an interestingly simple property when we 'lump' over all tangent directions at $\mathbf{p}$:

In a propagation operation, Gaussian curvatures add reciprocally:

$$\boldsymbol{\kappa}^{-1}_\mathcal{C} = \boldsymbol{\kappa}^{-1}_\mathcal{A} + \boldsymbol{\kappa}^{-1}_\mathcal{B} \tag{46}$$

(locally, at every triple of corresponding points).

Proof: We extend $\underline{\mathbf{n}}$ to an outermorphism over all of $\mathbf{I}$. Eq.(8) gives $\underline{\mathbf{n}}_\mathcal{A}(\mathbf{I}) = (-1)^{m-1}\boldsymbol{\kappa}_\mathcal{A}\mathbf{I}$. Then $\underline{\mathbf{n}}^{-1}_\mathcal{A}(\mathbf{I}) = \overline{\mathbf{n}}_\mathcal{A}(\mathbf{I}^2)\mathbf{I}^{-1}/\det(\underline{\mathbf{n}}) = (-1)^{m-1}\mathbf{I}/\boldsymbol{\kappa}_\mathcal{A}$ (where $\overline{\mathbf{n}}_\mathcal{A}$ is the adjoint of $\mathbf{n}_\mathcal{A}$, see [2]), and similarly for $\mathcal{B}$ and $\mathcal{C}$, and the result follows from eq.(41) since the Minkowski sum translates into a sum for each triple of corresponding points. □

5 Object boundaries as versors

XXX observe somewhere: in to out is -1

The equation for the representation $\mathcal{R}'(\mathbf{n})$ can be written in an interesting alternative form using *versors* in $\mathcal{G}(I_{m+1,1})$ (versors are multivectors that can be factored into geometric products of vectors, see [2] pg.103)

$$\mathcal{R}'(\mathbf{n})[\mathbf{p}] = \mathbf{n} - e^0(\mathbf{p}\cdot\mathbf{n}) = (1 - e^0\mathbf{p}/2)\mathbf{n}(1 + e^0\mathbf{p}/2) = e^{-e^0\mathbf{p}/2}\mathbf{n}e^{e^0\mathbf{p}/2}. \tag{47}$$

Thus the $\mathbf{n}$-representation can be constructed from a vector $\mathbf{n}$ via the general versor equation

$$\underline{U}(\mathbf{x}) = U\mathbf{x}\widehat{U}^{-1} \tag{48}$$

(we follow Hestenes' convenient convention of indicating the action of a versor U by its underlined symbol; $\widehat{\cdot}$ denotes the grade involution) using the versor

$$U = T_\mathbf{p} \equiv e^{-e^0\mathbf{p}/2} = 1 - e^0\mathbf{p}/2. \tag{49}$$

(This happens to be the versor $T_\mathbf{p}$ of a translation over $\mathbf{p}$ in the standard homogeneous model of a Euclidean space $\mathcal{G}^1(\mathbf{I}_m)$ in the Minkowski space $\mathcal{G}^1(E\mathbf{I}_m)$, see [6]. However, those versors work on vectors represented as $e_0 + \mathbf{x} - e^0\mathbf{x}^2/2$, not on $\mathbf{n}$. Note that this equals $\underline{T}_\mathbf{x}(e_0)$, a translation of the 'point at the origin' e_0 over the vector $\mathbf{x}$.)

We thus have, in versor notation:

$$\mathcal{R}'(\mathbf{n})[\mathbf{p}] = \underline{T}_{\mathbf{p}[\mathbf{n}]}(\mathbf{n}). \tag{50}$$

So in this view, we can see an object boundary (as represented by $\mathcal{R}'(\mathbf{n})$) as an $\mathbf{n}$-dependent translation $\mathbf{p}[\mathbf{n}]$ applied to the unit normal vector $\mathbf{n}$. Since the latter is the n-representation of a point object at the origin as a (trivial) function of its orientation, this provides the view:

> *Any object boundary can be represented as a deformation by orientation-dependent translation of a point object at the origin.*

Non-convex objects may have a particular inward pointing normal $\mathbf{n}$ at different points $\mathbf{p}$, so for those the function $\mathbf{p}[\mathbf{n}]$ should be considered set-valued. We will overload our notation to include this.

As before, the representation commutes with differentiation, and so we can extend this representation from $\mathbf{n}$ to tangent vectors, and then by linearity to the whole algebra $\mathcal{G}(\mathbf{I}_m[\mathbf{p}])$ of multivectors at $\mathbf{p}$. Interpretation for this representation, back to a positional representation of the boundaries, can be done by using its differential structure as in eq.(33).

boundary operation	action on boundary versor
identity	versor remains $T_{\mathbf{p}} = 1 - e^0\mathbf{p}/2$
translation over $\mathbf{t}$	left-multiply by $T_{\mathbf{t}} = 1 - e^0\mathbf{t}/2$
wave propagation by a boundary $T_{\mathbf{q}}$	left-multiply by $T_{\mathbf{q}[\mathbf{n}]}$ (same $\mathbf{n}$!)
rotation (center $\mathbf{c}$, spinor $\mathbf{R}$)	left-multiply by $R_{\mathbf{c},\mathbf{R}} = \mathbf{R} - e^0(\mathbf{c}\cdot\mathbf{R})$
mirroring in hyperplane (support $\mathbf{d} = \delta\mathbf{m}$)	left-multiply by $M_{\mathbf{d}} = \mathbf{m} - e^0\delta/2$
point reflection in $\mathbf{c}$	left-multiply by $P_{\mathbf{c}} = \mathbf{I}_m - e^0(\mathbf{c}\cdot\mathbf{I}_m)$
scaling by λ	replace by $(1 - e^0\mathbf{p}\lambda/2)$

Figure 6: *Changes of the boundary versor $T_{\mathbf{p}}$ used in the boundary representation $T_{\mathbf{p}}\mathbf{n}T_{\mathbf{p}}^{-1}$ under some common operations on the boundary.*

5.1 Versor representation of boundary operations

The advantage of the versor representation is that many important operations (including wave propagation!) have a simple representation as a versor pre-multiplier. We show this now; results are collected in figure 6.

- *Boundary translation*
  When the boundary translates over $\mathbf{t}$, the point $\mathbf{p}[\mathbf{n}]$ should become $\mathbf{p}[\mathbf{n}] + \mathbf{t}$; differentiation shows that $\mathbf{n}$ does not change. Thus the new versor for construction of the representation should act on the same $\mathbf{n}$, and be $(1 - e^0(\mathbf{p} + \mathbf{t})/2)$. This is achieved by multiplying the versor $T_{\mathbf{p}}$ by $T_{\mathbf{t}} = (1 - e^0\mathbf{t}/2)$, for:

$$\underline{T}_{\mathbf{t}+\mathbf{p}}(\mathbf{n}) = \underline{T}_{\mathbf{t}}(\underline{T}_{\mathbf{p}}(\mathbf{n})) \tag{51}$$

- *Wave propagation*
  Rewriting the combination law eq.(39) to produce the n-representation of the result from its constituents in terms of versors, we find:

  The versor of the wave propagation result $\mathcal{A}\check{\oplus}\mathcal{B}$ is the geometric product of the versors of $\mathcal{A}$ and $\mathcal{B}$ of wave front and propagator:

$$T_{\mathcal{A}\check{\oplus}\mathcal{B}} = T_{\mathcal{A}}\,T_{\mathcal{B}} = T_{\mathcal{B}}\,T_{\mathcal{A}} \tag{52}$$

  (where, for set-valued T, all combinations of products should be performed).

  Proof: For single -valued $\mathbf{p}[\mathbf{n}]$, this is essentially the same as the translation above:

$$T_{\mathcal{B}}\,T_{\mathcal{A}} = \left(1 - e^0\mathbf{p}_{\mathcal{B}}[\mathbf{n}]/2\right)\left(1 - e^0\mathbf{p}_{\mathcal{A}}[\mathbf{n}]/2\right) = \left(1 - e^0(\mathbf{p}_{\mathcal{B}}[\mathbf{n}] + \mathbf{p}_{\mathcal{A}}[\mathbf{n}])/2\right).$$

For set-valued $\mathbf{p}[\cdot]$s, we should combine all possibilities in the product, to produce the result agreeing with eq.(39):

$$1 - e^0(\mathbf{p}_{\mathcal{B}}[\mathbf{n}] \oplus \mathbf{p}_{\mathcal{A}}[\mathbf{n}])/2. \tag{53}$$

We assume this to be included in an overload of the geometric product notation. □

The somewhat strange 'addition law' eq.(39) for the n-representations ('add only the e^0-components, for the same $\mathbf{n}$') is thus just a disguised form of the geometric product of translational versors.

- *Boundary rotation*
  When the boundary rotates around $\mathbf{c}$ over a bivector angle characterized by a spinor $\mathbf{R}$ (we use boldface since this spinor is in $\mathcal{G}(\mathbf{I}_m)$), then $\mathbf{p}$ becomes $(\mathbf{R}(\mathbf{p}-\mathbf{c})\mathbf{R}^{-1}+\mathbf{c})$. This is achieved on its versor $T_{\mathbf{p}}$ by: $T_{\mathbf{p}'} = T_{\mathbf{c}}(\mathbf{R}(T_{-\mathbf{c}}T_{\mathbf{p}})\mathbf{R}^{-1})$, as is easily verified. Differentiating yields that $\mathbf{n}$ is rotated as well, to $\mathbf{n}' = \mathbf{R}\mathbf{n}\mathbf{R}^{-1}$. Therefore the new n-representation is achieved by applying $T_{\mathbf{p}'}$ to $\mathbf{n}'$ as a versor:

$$\begin{aligned} \underline{T}_{\mathbf{p}'}(\mathbf{n}') &= T_{\mathbf{p}'}\mathbf{n}'T_{\mathbf{p}'}^{-1} \\ &= \left(T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}}T_{\mathbf{p}}\mathbf{R}^{-1}\right)\mathbf{R}\mathbf{n}\mathbf{R}^{-1}\left(\mathbf{T}_{\mathbf{c}}\mathbf{R}\mathbf{T}_{-\mathbf{c}}\mathbf{T}_{\mathbf{p}}\mathbf{R}^{-1}\right)^{-1} \\ &= (T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}})T_{\mathbf{p}}\mathbf{n}T_{\mathbf{p}}^{-1}(T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}})^{-1}, \end{aligned}$$

  so the total result is the application of a new versor to $\mathbf{n}$ which is $T_{\mathbf{p}}$ left-multiplied by:

$$T_{\mathbf{c}}\mathbf{R}T_{-\mathbf{c}} = (1-e^0\mathbf{c}/2)\mathbf{R}(1+e^0\mathbf{c}/2) = \mathbf{R} - e^0(\mathbf{c}\cdot\mathbf{R}) \equiv R_{\mathbf{c},\mathbf{R}}. \tag{54}$$

- *Mirroring*
  Under reflection in a hyperplane with unit normal $\mathbf{m}$, a point $\mathbf{p}$ becomes $-\mathbf{m}\mathbf{p}\mathbf{m}$; reflection in a plane with support vector $\mathbf{d}/2 = \delta\mathbf{m}/2$ (with $\delta \geq 0$) results in $-\mathbf{m}\mathbf{p}\mathbf{m}+\mathbf{d}$. Differentiating gives $\mathbf{n}' = -\mathbf{m}\mathbf{n}\mathbf{m}$.

  We observe that the transformation of the versor $T_{\mathbf{p}}$ of $\mathbf{p}$ can be achieved by sandwiching between two versors $M_{\mathbf{d}} \equiv \mathbf{m} - e^0\delta/2$ and $\mathbf{m}$:

$$M_{\mathbf{d}}T_{\mathbf{p}}\mathbf{m} = (\mathbf{m} - e^0\delta/2)(1-e^0\mathbf{p}/2)\mathbf{m} = 1 - e^0(-\mathbf{m}\mathbf{p}\mathbf{m}+\mathbf{d})/2, \tag{55}$$

  and so combining with the change in $\mathbf{n}$ we obtain for the new boundary versor:

$$-(M_{\mathbf{d}}T_{\mathbf{p}}\mathbf{m})\mathbf{m}\mathbf{n}\mathbf{m}(M_{\mathbf{d}}T_{\mathbf{p}}\mathbf{m})^{-1} = -M_{\mathbf{d}}T_{\mathbf{p}}\mathbf{n}(M_{\mathbf{d}}T_{\mathbf{p}})^{-1} = M_{\mathbf{d}}T_{\mathbf{p}}\mathbf{n}(\widehat{M_{\mathbf{d}}T_{\mathbf{p}}})^{-1}. \tag{56}$$

  Note that the required minus sign gets absorbed into the grade involution of the general versor equation eq.(48), since the versor is of odd grade. So we simply need to multiply the boundary versor by $M_{\mathbf{d}}$ (in its scaled form used here, or the unscaled form $(\mathbf{d} - e^0|\mathbf{d}|/2)$; the inversion in the general versor equation takes care of normalization [2]).

- *Point reflection*
  A reflection $P_\mathbf{c}$ in a point $\mathbf{c}$ can be constructed using m planar reflections. This yields for $\mathbf{p}$ (translating back to 0 and doing it there): $P_0(\mathbf{p} - \mathbf{c}) + \mathbf{c}$, and for $\mathbf{n}$ it gives $P_0(\mathbf{n})$. The m reflections in planes with perpendicular unit vectors $\mathbf{e}_i$ are represented by the versor $\Pi_{i=1}^m \mathbf{e}_i = \mathbf{I}_m$. For instance, $P_0(\mathbf{n}) = -\mathbf{n} = \mathbf{I}_m \mathbf{n} \widehat{\mathbf{I}}_m^{-1}$. We obtain:

$$\left(T_\mathbf{c} \mathbf{I}_m T_{-\mathbf{c}} T_\mathbf{p} \widehat{\mathbf{I}}_m^{-1}\right) \mathbf{I}_m \mathbf{n} \widehat{\mathbf{I}}_m^{-1} \left(T_\mathbf{c} \mathbf{I}_m T_{-\mathbf{c}} T_\mathbf{p} \widehat{\mathbf{I}}_m^{-1}\right)^{-1} = (T_\mathbf{c} \mathbf{I}_m T_{-\mathbf{c}} T_\mathbf{p})\, \mathbf{n} \left(T_\mathbf{c} \widehat{\mathbf{I}}_m T_{-\mathbf{c}} T_\mathbf{p}\right)^{-1}, \quad (57)$$

  so that the versor representing point reflection is:

$$P_\mathbf{c} = T_\mathbf{c} \mathbf{I}_m T_{-\mathbf{c}} = \mathbf{I}_m - e^0 (\mathbf{c} \cdot \mathbf{I}_m). \quad (58)$$

  (Note that in even dimensions $\mathbf{I}_m$ is even and can be written as a product of rotors, which itself is a rotor. Thus in those dimensions, point reflection can be implemented as a rotation, a well-known fact.)

- *Scaling*
  Scaling relative to 0 replaces $\mathbf{p}$ by $\lambda \mathbf{p}$ and does not change $\mathbf{n}$. It changes the versor of $\mathbf{p}$ to $(1 - e^0 \lambda \mathbf{p}/2)$, which cannot be represented by a left-multiplication.

For our application in wave propagation and collision avoidance, the Euclidean operations of *translation* and *rotation* are especially relevant, plus the *point reflection* (since it is the operation that connects 'collision avoidance' to 'wave propagation', see section 1), and of course the *wave propagation* operation itself. Fortunately, table 6 shows that all of those can be represented by a simple left-multiply by an appropriate versor. We can thus study them in combination. In particular, this waives the restriction on the use of purely translational collisions only, still present in [1]: we can now analyze the effects of rotations on the propagation operation.

5.2 Rotations and wave propagation

In almost all earlier analyses of the wave propagation (or morphological dilation), results were limited to purely translational propagation, in which the propagator was used in a fixed orientation. The main reason may have been that the parameter space for this problem (translation vectors) is isomorphic to the space in which the objects are defined – propagation of a boundary in the Euclidean m-dimensional space produces a boundary in Euclidean m-space. However, for the use in collision avoidance, this is an unnatural restriction: the possible collisions form a boundary in the full parameter space of the robot, and this includes rotational motion parameters (unless the robot is a sphere).

[2]And even that is not required since we can derive the final scaling factor from the fact that the representation should have the form $\mathbf{n} - e^0(\mathbf{p}' \cdot \mathbf{n})$), with $\mathbf{n}$ a unit vector.

It is thus necessary to understand the collisions as a function of orientation. The versor formulation enables us to treat those in the same framework as the translational collisions. Many results need to be established; here we study how a small translation affects the collision boundary. As always, we use the wave propagation terminology, so as not to be bothered by point reflections upsetting the symmetry of the problem.

The versor representation of the central rotation of a boundary $\mathcal{A}$ operating on a normal vector $\mathbf{m}$ is:

$$\mathbf{R} - e^0\mathbf{R}\mathbf{p}[\mathbf{m}]/2,$$

which leads by eq.(48) to the n-representation:

$$\mathbf{R}\mathbf{m}\mathbf{R}^{-\mathbf{1}} - \mathbf{e^0}\left(\mathbf{p}[\mathbf{m}]\cdot\mathbf{m}\right). \tag{59}$$

For combination in a versor product with another boundary $\mathcal{B}$ of the form $\mathbf{n}-e^0\mathbf{p}_{\mathcal{B}}[\mathbf{n}]\cdot\mathbf{n}$, we need to make sure that the versors operate on the same $\mathbf{n}$, or it will not be a representation of the propagation result. We 'reparametrize' the boundary $\mathcal{A}$ to be a function of $\mathbf{n}$ as well. Obviously we need to set $\mathbf{R}\mathbf{m}\mathbf{R}^{-\mathbf{1}} = \mathbf{n}$, so $\mathbf{m} = \mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}$, yielding the versor product:

$$\begin{aligned}\left(\mathbf{R} - e^0\mathbf{R}\mathbf{p}[\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}]/\mathbf{2}\right)\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}\left(\mathbf{R} - \mathbf{e^0}\mathbf{R}\mathbf{p}[\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}]/\mathbf{2}\right)^{-\mathbf{1}} &=\\ = \left(1 - e^0\mathbf{R}\mathbf{p}[\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}]\mathbf{R}^{-\mathbf{1}}/\mathbf{2}\right)\mathbf{n}\left(1 + e^0\mathbf{R}\mathbf{p}[\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}]\mathbf{R}^{-\mathbf{1}}/\mathbf{2}\right)^{-1}&.\end{aligned}$$

Thus the rotation of $\mathcal{A}$ is represented, on the original direction $\mathbf{n}$, by the versor:

$$1 - e^0\mathbf{R}\mathbf{p}[\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}]\mathbf{R}^{-\mathbf{1}}/\mathbf{2}. \tag{60}$$

Since $\mathbf{p}$ is an arbitrary function (it is capable of characterizing arbitrary boundaries $\mathcal{A}$), this reparametrization can not be simplified in general and related to the original $\mathbf{p}$ in any simple manner. However, if we consider small rotations, we can linearize $\mathbf{p}$ and study the local effects of rotation. We set $\mathbf{R} = e^{-\mathbf{i}\phi/2} = 1 - \mathbf{i}\phi/2$ in first order in ϕ, and obtain:

$$\begin{aligned}\mathbf{R}\mathbf{p}[\mathbf{R}^{-\mathbf{1}}\mathbf{n}\mathbf{R}]\mathbf{R}^{-\mathbf{1}} &= (1 - \mathbf{i}\phi/2)\mathbf{p}\left[(1 + \mathbf{i}\phi/2)\mathbf{n}(1 - \mathbf{i}\phi/2)\right](1 + \mathbf{i}\phi/2)\\ &= (1 - \mathbf{i}\phi/2)\mathbf{p}\left[\mathbf{n} - \mathbf{n}\cdot\mathbf{i}\phi\right](1 + \mathbf{i}\phi/2)\\ &= (1 - \mathbf{i}\phi/2)\left(\mathbf{p}[\mathbf{n}] - \underline{\mathbf{n}}^{-1}(\mathbf{n}\cdot\mathbf{i}\phi)\right)(1 + \mathbf{i}\phi/2)\\ &= \mathbf{p}[\mathbf{n}] + \mathbf{p}[\mathbf{n}]\cdot\mathbf{i}\phi - \underline{\mathbf{n}}^{-1}(\mathbf{n}\cdot\mathbf{i}\phi) \quad \text{(to first order in } \phi)\end{aligned}$$

Therefore the versor for the rotated boundary is, to first order in ϕ:

$$1 - e^0\left(\mathbf{p}[\mathbf{n}] + \mathbf{p}[\mathbf{n}]\cdot\mathbf{i}\phi - \underline{\mathbf{n}}(\mathbf{n}\cdot\mathbf{i}\phi)\right)/2, \tag{61}$$

and this produces the new boundary:

$$\mathbf{n} - e^0\mathbf{p}\cdot\mathbf{n} - e^0(\mathbf{n}\wedge\mathbf{p})\cdot\mathbf{i}\phi = \mathcal{R}'(\mathbf{n}) - e^0(\mathbf{n}\wedge\mathbf{p})\cdot(\mathbf{i}\phi). \tag{62}$$

The tangent $\mathbf{I}$ thus shifts over a perpendicular distance $(\mathbf{n}\wedge\mathbf{p})\cdot(\mathbf{i}\phi)$. (Note that this is orientation-dependent, and changes sign when $\mathbf{n}$ and $\mathbf{p}$ go beyond being parallel, as it should. Note also that it does not depend on the second order differential structure of $\mathbf{p}[\mathbf{n}]$, for $\underline{\mathbf{n}}$ does not occur in the result.)

6 Conclusions

This paper demonstrates that the rather involved operation of wave front propagation in m-dimensional space (which also represents collision detection, object growing and milling) can be represented as a geometric product of versors. These versors represent boundaries in Euclidean m-space, within a Minkowski space of dimension $(m+1, 1)$, as *direction-dependent translations* of the point 0.

This representation combines well with Euclidean operations on the boundaries. We can use it to analyze differential properties of the propagation operation by Hestenes' geometric calculus. We are currently working on the practical implications of this simple and possibly powerful representation of the propagation operation on boundaries, for robotic collision avoidance.

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