Geometric Algebra for Subspace Operations

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Abstract. The set theory relations $\in$, $\backslash$, Δ, $\cap$, and $\cup$ have corollaries in subspace relations. Geometric Algebra is introduced as a useful framework to explore these subspace operations. The relations $\in$, $\backslash$, and Δ are easily subsumed by Geometric Algebra for Euclidean metrics. A short computation shows that the meet ($\cap$) and join ($\cup$) are resolved in a projection operator representation with the aid of one additional product beyond the standard Geometric Algebra products. The result is that the join can be computed even when the subspaces have a common factor, and the meet can be computed without knowing the join. All of the operations can be defined and computed in any signature (including degenerate signatures) by transforming the problem to an analogous problem in a different algebra through a transformation induced by a linear invertible function (a LIFT to a different algebra). The new results, as well as the techniques by which we reach them, add to the tools available for subspace computations.

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1. Introduction

Operations on subspaces are useful in applications everywhere. Geometric Algebra, the intriguing algebra promoted by David Hestenes to unify and simplify many areas of mathematics[2, 3, 4], is introduced as the ideal framework to explore subspace operations. A large repertoire of operations to compute subspace operations are made available by Geometric Algebra, but a few holes remain, notably with respect to the meet and join of subspaces. This paper should resolve the outstanding issues. A section on preliminaries makes this treatment reasonably self-contained. Four subspace operations are introduced (some more basic than others), motivated by the set theory relations ($\backslash$, Δ, $\cap$, and $\cup$), common usage, and applied needs. Four operations in Geometric Algebra are defined in this paper, one to mirror each of the subspace operations, no prior knowledge of Geometric is assumed. Experts will note that Geometric Algebra deals with oriented subspaces and that in general there is no oriented solution to the meet and join problem[7].

This problem is avoided by representing unoriented subspaces by orthogonal projection operators. This allows us to extend the meet and join defined in the previous literature[1, 3] to give meaningful (nonzero) results for any subspaces. The price we pay for this extension is that the meet and join presented in this paper are not linear. This is not a disadvantage, because our meet and join agree with the previous literature except when the previous literature results are zero. Many operations in this paper have an arbitrary scale and orientation. This is addressed in the penultimate section, where a geometrical significance can be given to the linear result of zero from the previous literature.

2. Preliminaries

This section includes the definitions and motivation for four subspace operations. Following the new definitions is a review of Geometric Algebra.

2.1. The Subspace Operations

The set theory operations of $\backslash$, Δ, $\cap$, and $\cup$ can be applied to subspaces of $\mathcal{R}^n$. However, in general the outcome will not be a subspace, since the set theory operations do not respect the linear structure of the subspaces. Four subspaces operations are defined that are motivated by the set theory operations, but that respect the linear structure and thus always produce subspaces. Let $\mathcal{A}$ and $\mathcal{B}$ be two subspaces of $\mathcal{R}^n$.

1. *The Meet Operation*
   The meet of $\mathcal{A}$ and $\mathcal{B}$ is the set $\{\mathbf{x} \in \mathcal{R}^n : \mathbf{x} \in \mathcal{A} \text{ and } \mathbf{x} \in \mathcal{B}\}$. In words it is the largest common subspace. It shall be denoted by $\mathcal{A} \cap \mathcal{B}$.

2. *The Join Operation*
   The join of $\mathcal{A}$ and $\mathcal{B}$ is the set $\{\mathbf{x} \in \mathcal{R}^n : \exists \mathbf{a} \in \mathcal{A} \text{ and } \exists \mathbf{b} \in \mathcal{B} \text{ such that } \mathbf{x} = \mathbf{a} + \mathbf{b}\}$. In words it is the span of the two subspaces (i.e. the smallest common superspace). It shall be denoted by $\mathcal{A} \cup \mathcal{B}$.

3. *The Difference Operation*
   The difference of $\mathcal{A}$ and $\mathcal{B}$ is the set $\{\mathbf{a} \in \mathcal{A} : \forall \mathbf{b} \in \mathcal{B} \quad \mathbf{a} \cdot \mathbf{b} = 0\}$, provided $\mathcal{B}$ is a subspace of $\mathcal{A}$. In words it is the orthogonal complement of $\mathcal{B}$ in $\mathcal{A}$. It shall be denoted by $\mathcal{A} \backslash \mathcal{B}$.

4. *The Symmetric Difference Operation*
The symmetric difference of $\mathcal{A}$ and $\mathcal{B}$ is the set $\{\mathbf{x} \in \mathcal{R}^n : \exists \mathbf{a} \in \mathcal{A}$ and $\exists \mathbf{b} \in \mathcal{B}$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\forall \mathbf{c} \in \mathcal{A} \cap \mathcal{B} \quad \mathbf{x} \cdot \mathbf{c} = 0\}$. In words it is the orthogonal complement of the meet in the join. It shall be denoted by $\mathcal{A} \Delta \mathcal{B}$. Clearly, $\mathcal{A} \Delta \mathcal{B} = (\mathcal{A} \cup \mathcal{B}) \backslash (\mathcal{A} \cap \mathcal{B})$.

This paper always uses the symbols $\backslash$, Δ, $\cap$, and $\cup$ to denote subspace operations.

2.2. A Review of Geometric Algebra

Geometric Algebra is based on Clifford Algebra over a real vector space. A Clifford Algebra is an algebra generated by the scalars and the elements of a vector space with a metric. The algebra has a linear, associative, distributive product, and the square of a vector is the squared length determined by the metric (for more details on Clifford Algebras see[5]). Let $\mathcal{R}^{p,q,r}$ be a $(p+q+r)$-dimensional real vector space with a set of linearly independent, mutually orthogonal vectors, $\{\mathbf{p}_1, ..., \mathbf{p}_p, \mathbf{q}_1, ..., \mathbf{q}_q, \mathbf{r}_1, ..., \mathbf{r}_r\}$, such that $\mathbf{p}_i^2 = 1$, $\mathbf{q}_j^2 = -1$, and $\mathbf{r}_k^2 = 0$. The Clifford Algebra and the Geometric Algebra generated by $\mathcal{R}^{p,q,r}$ shall be denoted $\mathcal{R}_{p,q,r}$.

The essential difference between a Geometric Algebra and a Clifford Algebra is that the elements of a Geometric Algebra are given a geometric interpretation. This leads to a focus on operations that are defined on geometrically meaningful subsets of the algebra and therefore to the introduction of additional structure (and more products between elements), so that a consistent geometric interpretation can be maintained on the results of computations. To emphasize this difference, we call the elements of the Geometric Algebra *multivectors* and we call the standard (Clifford) product between elements in the Algebra the *geometric product*. The geometric product is denoted by juxtaposition of operands, as in AB.

A summary of the extra features and terminology of Geometric Algebra follows.

1. *Blades*
If a nonzero multivector can be written as the geometric product of r mutually anticommuting vectors, then it is called an r-blade. The word blade refers to an r-blade with the value of r unspecified. Real numbers are considered 0-blades and are often called scalars. Vectors are considered 1-blades. The square of a blade is a scalar. For a vector the square is the squared length determined by the

metric. Zero is considered an r-blade for any value of r, but any nonzero blade is an r-blade for only one value of r[2].

2. *Steps (or Grades)*
   A linear combination of r-blades will be called an r-vector, and will be said to have *step* (or grade) r. The space of r-vectors is a linear subspace of the entire Clifford Algebra. An arbitrary multivector, A, can be uniquely written as $\sum \langle A \rangle_r$ where $\langle A \rangle_r$ is the r-vector part of A if r is a nonnegative integer and $\langle A \rangle_r$ is zero if r is not a nonnegative integer.

3. *Outer Product*
   The outer product of an r-vector A and an s-vector B is defined to be $\langle AB \rangle_{s+r}$. It is denoted by $A \wedge B$ and is extended by linearity to arbitrary multivectors. The outer product is associative between all multivectors and anti-symmetric between vectors. The outer product of two blades is a blade (see the appendix).

4. *(Contraction) Inner Product*
   The (contraction) inner product of an r-vector A and an s-vector B is defined to be $\langle AB \rangle_{s-r}$. It is denoted by $A \rfloor B$ and is extended by linearity to arbitrary multivectors. This inner product differs slightly from the inner product of Hestenes[2]. It has the useful properties that $(A \wedge B) \rfloor C = A \rfloor (B \rfloor C)$ and $\mathbf{a}A = \mathbf{a} \rfloor A + \mathbf{a} \wedge A$ for any multivectors A, B, C and any vector $\mathbf{a}$. These details are obvious from considering this definition and looking at the proofs in Hestenes[2]. The inner product is explicitly expanded for the vectors $\mathbf{a}$, $\mathbf{c}_1$, $\mathbf{c}_2$, ... , and $\mathbf{c}_k$ as

$$\mathbf{a} \rfloor (\mathbf{c}_1 \wedge \mathbf{c}_2 \wedge ... \wedge \mathbf{c}_k) = \sum (-1)^{r+1} (\mathbf{a} \rfloor \mathbf{c}_i) \mathbf{c}_1 \wedge ... \wedge \check{\mathbf{c}}_r \wedge ... \wedge \mathbf{c}_k \quad (1)$$

   where the inverted circumflex indicates that the rth vector was omitted from the product. The proof in reference[2] carries over to the contraction inner product with no modification. The inner product of two blades is a blade (see the appendix).

5. *Subspaces*
   One of the geometric interpretations common in geometric algebra is to use blades to represent subspaces. This works because blades are closely related to subspaces. Given a nonzero blade, $\mathbf{A}$, the set $\{\mathbf{x} \in \mathcal{R}^n : \mathbf{x} \wedge \mathbf{A} = 0\}$ is a subspace of $\mathcal{R}^n$. Similarly given an oriented basis $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}$ for a k-dimensional subspace $\mathcal{A}$, there is a nonzero blade that corresponds to that oriented basis. If $k = 0$

then that blade is 1. If $k \neq 0$ then that blade is $\mathbf{x}_1 \wedge \mathbf{x}_2 \wedge ... \wedge \mathbf{x}_k$. The identity $\mathbf{x} \in \mathcal{A} \iff \mathbf{x} \wedge \mathbf{A} = 0$ is the means by which Geometric Algebra subsumes the operation $\in$ from set theory. Since the square of a blade is a scalar, the inverse of a blade (if it has one), is equal to the blade divided by its square, so the inverse is just a scalar multiple of the original blade, and hence a blade and its inverse represent the same (unoriented) subspace.

6. *Pseudoscalars*
   Given an algebra, $\mathcal{R}_{p,q,r}$, and a set of $(p+q+r)$ vectors $\{\mathbf{v}_1, ..., \mathbf{v}_{p+q+r}\}$, the outer product $\mathbf{v}_1 \wedge ... \wedge \mathbf{v}_{p+q+r}$ is called a pseudoscalar. Often a particular nonzero pseudoscalar is singled out. This preferred pseudoscalar serves to determine the reference orientation for the vector space and sometimes is used to perform duality operations. Usually the preferred pseudoscalar is called *the* pseudoscalar and denoted $\mathbf{I}$. This can be seen as merely a special case of the previous section on subspaces, by noticing that every nonzero k-blade is a pseudoscalar for the k dimensional subspace it represents.

7. *Orthogonal Projection Operators*
   For a Euclidean metric, blades are closely related to orthogonal projection operators. Given a nonzero blade, $\mathbf{A}$, representing the subspace $\mathcal{A}$, the vector $P_{\mathbf{A}}(\mathbf{x}) = (\mathbf{x} \rfloor \mathbf{A})\mathbf{A}^{-1}$ is the orthogonal projection of the vector $\mathbf{x}$ onto the subspace $\mathcal{A}$. As taken from[2], the identities

$$\mathbf{AB} = \mathbf{A} \wedge \mathbf{B} \Rightarrow P_{\mathbf{AB}} = P_{\mathbf{B}} + P_{\mathbf{A}} \tag{2}$$

   and

$$\mathbf{AB} = \mathbf{A} \rfloor \mathbf{B} \Rightarrow P_{\mathbf{AB}} = P_{\mathbf{B}} - P_{\mathbf{A}} \tag{3}$$

   hold for any nonzero blades $\mathbf{A}$ and $\mathbf{B}$. Equation (2) implies that an orthogonal projection operator can be decomposed analogously to the way its corresponding blades can be factored.

8. *Outermorphism*
   Given a linear function, $f : \mathcal{R}^{p,q,r} \rightarrow \mathcal{R}^{p,q,r}$, there is an extension of the function to arbitrary multivectors called the outermorphism of f. It is denoted by $\underline{f}$ and is defined to be the identity when restricted to the scalars. The condition $\underline{f}(A \wedge (B+C)) = \underline{f}(A) \wedge \underline{f}(B)$ + $\underline{f}(A) \wedge \underline{f}(C)$ is then sufficient to define the outermorphism on arbitrary multivectors. This extension is well-established[4].

A notation convention is adopted to aid the reader in easily making meaningful distinctions between different multivectors. Lowercase Greek letters are reserved for scalars. Lowercase Latin letters are reserved for integers or functions when not in bold face, while lowercase Latin letters are reserved for vectors when in bold face. Bold face is reserved for blades. Lastly, uppercase Latin letters are used when it is impossible or unnecessary to be more specific about the nature of a multivector. This notation convention simplifies the reading of equations and emphasizes that different geometric interpretations are applied to different elements of the algebra.

3. The Euclidean Metrics

Since every blade represents an oriented subspace and blades are easy to compute with, they are a natural candidate for subspace computations. The extra scalar degree of freedom allows the future potential for more precise calculations with subspaces that attach meaning to the magnitude of a blade. Therefore in this paper four blade operations are introduced to correspond to the four subspace operations. For example, the delta product is introduced to correspond to the symmetric difference operation on subspaces. For a Euclidean space the delta product of two nonzero blades is another nonzero blade and it corresponds to the symmetric difference of the two subspaces. Idealy all four operations would have a corresponding blade product. However, for the meet and join there is no consistent way to assign an orientation to the result[7], for geometric algebra that corresponds to a sign. Therefore we know that if we stayed in geometric algebra we wouldn't be able to consistently produce oriented (signed) outcomes. Two approaches have been used in the past. One is to abandon the insistence on oriented outcome, and therefore sometimes get the zero blade instead of the meet or join[1, 3]. The other is to abandon Geometric Algebra and go to Linear Algebra to solve the meet and join and pull to result back to Geometric Algebra without a scale or orientation[6]. We have already noted that orthogonal projection operators sit nicely inside Geometric Algebra. We show in sections (3.4) and (3.5) that going all the way to Linear Algebra is not necessary because the new Delta product already does all the hard parts and the rest can be done with projection operators. Since computing the meet and join directly is impossible (no orientation/no sign), we compute the orthogonal projection onto the meet or join. The we simply use the results of section (3.1) to make an operation from blades to blades, with limited consistency of scale and orientation. The new Delta product and the relationship between projection operators and blades is strongest in a Euclidean space, so first we restrict attention to Euclidean spaces.

In this section we introduce the correspondence from projection operators to blades and four blade operations. Each blade operation is shown to faithfully mirror its corresponding subspace operation.

3.1. The Blade Correspondence

Here we give a straight forward algorithm to construct a blade from its corresponding orthogonal projection operator. The algorithm has an arbitrary scale and orientation inherited from an arbitrarily chosen basis, which is the best that can be expected. Let $P_{\mathcal{A}}$ be an idempotent linear operator on $\mathcal{R}^n$ such that the image of $\mathcal{R}^n$ is an unknown t-dimensional subspace, $\mathcal{A}$. The algorithm constructs a blade that characterizes $\mathcal{A}$ as follows. First we construct a set of candidate blades, and then show that all the nonzero candidate blades represent the subspace $\mathcal{A}$. Finally we show that at least one of the candidate blades is, in fact, nonzero. Let $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k$ be k t-blades such that $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k\}$ is a basis for the space of t-vectors. Let $\mathbf{T}_i = \underline{P_{\mathcal{A}}}(\mathbf{V}_i)$. The set $\{\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_k\}$ is our set of candidate blades. By the properties of the outermorphism, each $\mathbf{T}_i$ is a t-blade. Each $\mathbf{T}_i$ is clearly the outer product of t vectors and each of these t vectors is in $\mathcal{A}$. If the t vectors are linearly dependent then $\mathbf{T}_i = 0$, if not then they form a basis for $\mathcal{A}$ so $\mathbf{T}_i \neq 0$ and $\mathbf{T}_i$ is exactly the kind of blade to characterize the subspace $\mathcal{A}$. All that remains is to show that at least one of the candidate blades is nonzero. Since a blade that characterizes $\mathcal{A}$ exists we know that it is a linear combination of $\{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k\}$, so by the linearity of the outermorphism there must be a $\mathbf{V}_i$ such that $\mathbf{T}_i \neq 0$.

The existence of such a correspondence allows us to translate operations between orthogonal projection operators into operations between blades, except for a loss of the scale and orientation.

3.2. The Inner Division Operation

Consider two nonzero blades, $\mathbf{A}$ and $\mathbf{B}$, that characterize the subspaces $\mathcal{A}$ and $\mathcal{B}$ respectively. When $\mathcal{B}$ is a subspace of $\mathcal{A}$ we use the expression $\mathbf{A}\backslash\mathbf{B}$ to denote the quantity $\mathbf{B}^{-1}\mathbf{A}$ and the call the operation *inner division*. The inner division operation is motivated by the difference operation for subspaces. The justification requires showing two points, first that under such conditions, $\mathbf{B}^{-1}\mathbf{A}$ is a blade, and second that $\mathbf{x} \wedge (\mathbf{B}^{-1}\mathbf{A}) = 0 \iff \mathbf{x} \in \mathcal{A}\backslash\mathcal{B}$.

Let $\{\mathbf{b}_1, ..., \mathbf{b}_s\}$ be an orthogonal basis for $\mathcal{B}$. Let $\{\mathbf{a}_1, ..., \mathbf{a}_r\}$ be an orthogonal basis for $\mathcal{A}\backslash\mathcal{B}$. Clearly $\{\mathbf{b}_1, ..., \mathbf{b}_s, \mathbf{a}_1, ..., \mathbf{a}_r\}$ is an orthogonal basis for $\mathcal{A}$. Since $\{\mathbf{b}_1, ..., \mathbf{b}_s\}$ is an orthogonal basis for $\mathcal{B}$ and since $\mathbf{B}^{-1}$ characterizes the subspace $\mathcal{B}$ it follows that $\mathbf{B}^{-1}$ is a nonzero scalar multiple of $\mathbf{b}_1\mathbf{b}_2...\mathbf{b}_s$. Similarly $\mathbf{A}$ is a nonzero scalar multiple of $\mathbf{b}_1\mathbf{b}_2...\mathbf{b}_s\mathbf{a}_1\mathbf{a}_2...\mathbf{a}_r$. It then follows that there exists two nonzero scalars α and β such that $\mathbf{B}^{-1}\mathbf{A} = \alpha(\mathbf{b}_1\mathbf{b}_2...\mathbf{b}_s)(\mathbf{b}_1\mathbf{b}_2...\mathbf{b}_s\mathbf{a}_1\mathbf{a}_2...\mathbf{a}_r) = \beta\mathbf{a}_1\mathbf{a}_2...\mathbf{a}_r$. Thus $\mathbf{B}^{-1}\mathbf{A}$ is a blade and it characterizes the subspace $\mathcal{A}\backslash\mathcal{B}$.

A quick look at the step of the output reveals that when $\mathcal{B}$ is a subspace of $\mathcal{A}$ then $\mathbf{B}^{-1}\mathbf{A} = \mathbf{B}^{-1}\rfloor\mathbf{A}$. Therefore, while it appears that the inner division operation is based on the geometric product it is also just as easily based on the inner product. It is useful whenever a product between two blades can be written as either of two products, because either definition can be used from line to line of a computation, depending on which product gives simplifications at that particular moment. An example is the identity,

$$\mathbf{A} = \mathbf{B}(\mathbf{A}\backslash\mathbf{B}) = \mathbf{B} \wedge (\mathbf{A}\backslash\mathbf{B}) \tag{4}$$

which is proved by decomposing the inner division first into the geometric product and then into the inner product and looking at the step of the outcome.

3.3. The Delta Product

Consider two nonzero blades, $\mathbf{A}$ and $\mathbf{B}$, that characterize the subspaces $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and let $\mathbf{C}$ be any blade characterizing that subspace. When $\mathcal{C} = \{0\}$, $\mathbf{A} \wedge \mathbf{B} \neq 0$. When $\mathcal{C} \neq \{0\}$, $\mathbf{A} \wedge \mathbf{B} = 0$. In the latter case we can define $\mathbf{A}_\perp = \mathbf{A}\mathbf{C}$ and $\mathbf{B}_\perp = \mathbf{C}^{-1}\mathbf{B}$. Since $\mathbf{C}$ and $\mathbf{C}^{-1}$ both represent $\mathcal{C}$, which is a subspace of both $\mathcal{A}$ and $\mathcal{B}$, the previous section makes it clear that $\mathbf{A}_\perp$ and $\mathbf{B}_\perp$ are blades. The intersection of the subspaces characterized by $\mathbf{A}_\perp$ and $\mathbf{B}_\perp$ contains only the element zero, so $\mathbf{A}_\perp \wedge \mathbf{B}_\perp \neq 0$.

By construction $\mathbf{A}\mathbf{B} = \mathbf{A}_\perp\mathbf{B}_\perp$, so the highest step portion of $\mathbf{A}\mathbf{B}$ is $\mathbf{A}_\perp \wedge \mathbf{B}_\perp$, and therefore a blade. This motivates a new product for blades which we call the *delta product*. The delta product of two blades, $\mathbf{A}$ and $\mathbf{B}$ is denoted $\mathbf{A}\Delta\mathbf{B}$ and defined to be the highest step portion of $\mathbf{A}\mathbf{B}$. The delta product is motivated by the symmetric difference operation for subspaces. The justification requires showing that $\mathbf{x} \wedge (\mathbf{A}\Delta\mathbf{B}) = 0 \iff \mathbf{x} \in \mathcal{A}\Delta\mathcal{B}$.

Let $\{\mathbf{a}_1, ..., \mathbf{a}_r\}$ be an orthogonal basis for $\mathcal{A}\backslash\mathcal{C}$. Let $\{\mathbf{b}_1, ..., \mathbf{b}_s\}$ be an orthogonal basis for $\mathcal{B}\backslash\mathcal{C}$. Similar arguments as in the previous section demonstrate that there exists a nonzero scalar α such that $\mathbf{AB} = \alpha\mathbf{a}_1\mathbf{a}_2...\mathbf{a}_r\mathbf{b}_1\mathbf{b}_2...\mathbf{b}_s$, so $\mathbf{A}\Delta\mathbf{B} = \alpha\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge ... \wedge \mathbf{a}_r \wedge \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge ... \wedge \mathbf{b}_s$. Clearly $\mathbf{x} \wedge (\mathbf{A}\Delta\mathbf{B}) = 0 \Rightarrow \mathbf{x} \in \mathcal{A}\Delta\mathcal{B}$. Therefore assume $\mathbf{x} \in \mathcal{A}\Delta\mathcal{B}$ and we will show that $\mathbf{x} \wedge (\mathbf{A}\Delta\mathbf{B}) = 0$. By definition $\exists\mathbf{a} \in \mathcal{A}$ and $\exists\mathbf{b} \in \mathcal{B}$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\forall\mathbf{c} \in \mathcal{C} \quad \mathbf{x} \cdot \mathbf{c} = 0$. Let $\mathbf{a}_0 = \mathbf{a} + P_{\mathcal{C}}(\mathbf{b})$ and $\mathbf{b}_0 = \mathbf{b} + P_{\mathcal{C}}(\mathbf{a})$. Clearly $\mathbf{a}_0 + \mathbf{b}_0 = \mathbf{x} + P_{\mathcal{C}}(\mathbf{x}) = \mathbf{x}$. Clearly $\mathbf{a}_0 \in \mathcal{A}$ and $\mathbf{a}_0 \rfloor \mathbf{C} = \mathbf{a} \rfloor \mathbf{C} - \mathbf{b} \rfloor \mathbf{C} = 0$, so $\forall\mathbf{c} \in \mathcal{C}$, $\mathbf{c} \rfloor \mathbf{a}_0 = 0$ therefore $\mathbf{a}_0 \in \mathcal{A}\backslash\mathcal{C}$. This means $\mathbf{a}_0 \wedge \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge ... \wedge \mathbf{a}_r = 0$ so $\mathbf{a}_0 \wedge (\mathbf{A}\Delta\mathbf{B}) = 0$. Similarly for $\mathbf{b}_0$. Therefore $\mathbf{x} \wedge (\mathbf{A}\Delta\mathbf{B}) = 0$.

The delta product is very different than the inner or outer product in its algebraic properties. The major difference is that $\mathbf{A}\Delta(\mathbf{B} + \mathbf{C}) \neq \mathbf{A}\Delta\mathbf{B} + \mathbf{A}\Delta\mathbf{C}$, so the delta product cannot be extended by linearity to arbitrary multivectors. The delta product can only be used on blades. More care must be taken with implementations of the delta product because even a small change in either $\mathbf{A}$ or $\mathbf{B}$ can cause a change in the step of $\mathbf{A}\Delta\mathbf{B}$.

3.4. The Meet Operation

Consider two nonzero blades, $\mathbf{A}$ and $\mathbf{B}$, that characterize the subspaces $\mathcal{A}$ and $\mathcal{B}$ respectively. The blade correspondence from section (3.1) can be used to define a new product for blades called the *meet* and denoted $\mathbf{A} \cap \mathbf{B}$. $\mathbf{A} \cap \mathbf{B}$ is defined modulo a scale and an orientation as the blade corresponding to the projection operator $P_{\mathbf{A}\cap\mathbf{B}}$, where $P_{\mathbf{A}\cap\mathbf{B}}$ is defined as:

$$P_{\mathbf{A}\cap\mathbf{B}}(\mathbf{x}) = \frac{P_{\mathbf{B}}(\mathbf{x}) - P_{\mathbf{A}\Delta\mathbf{B}}(\mathbf{x}) + P_{(\mathbf{A}\Delta\mathbf{B})\mathbf{B}^{-1}}(\mathbf{x})}{2} \qquad (5)$$

The justification requires that $P_{\mathbf{A}\cap\mathbf{B}}$ be the orthogonal projection onto $\mathcal{A} \cap \mathcal{B}$. Let $\mathbf{C}$ be a blade characterizing the subspace $\mathcal{A} \cap \mathcal{B}$, then define $\mathbf{A}_\perp = \mathbf{AC}$ and $\mathbf{B}_\perp = \mathbf{C}^{-1}\mathbf{B}$ as above. Now the result follows from a simple application of equation (2). First note that $\mathbf{B}_\perp = \mathbf{B}\backslash\mathbf{C}$, so equation (4) implies that $\mathbf{B} = \mathbf{C}\mathbf{B}_\perp = \mathbf{C} \wedge \mathbf{B}_\perp$, therefore we have the following identity:

$$P_{\mathbf{B}} = P_{\mathbf{B}_\perp} + P_{\mathbf{C}} \qquad (6)$$

Similarly $\mathbf{A}_\perp \wedge \mathbf{B}_\perp = \mathbf{B}_\perp((\mathbf{A}_\perp \wedge \mathbf{B}_\perp)\backslash \mathbf{B}_\perp) = \mathbf{B}_\perp \wedge ((\mathbf{A}_\perp \wedge \mathbf{B}_\perp)\backslash \mathbf{B}_\perp)$, therefore we have the following identity:

$$P_{\mathbf{A}\Delta\mathbf{B}} = P_{\mathbf{B}_\perp^{-1}(\mathbf{A}\Delta\mathbf{B})} + P_{\mathbf{B}_\perp} \tag{7}$$

Finally $(\mathbf{A}\Delta\mathbf{B})\mathbf{C}^{-1} = (\mathbf{A}\Delta\mathbf{B}) \wedge \mathbf{C}^{-1}$, because every vector in $\mathbf{C}^{-1}$ is orthogonal to every vector in $\mathbf{A}\Delta\mathbf{B}$. Since $(\mathbf{A}\Delta\mathbf{B})\mathbf{B}_\perp^{-1}$ represents a subspace of $\mathbf{A}\Delta\mathbf{B}$ it is just as true that $((\mathbf{A}\Delta\mathbf{B})\mathbf{B}_\perp^{-1})\mathbf{C}^{-1} = ((\mathbf{A}\Delta\mathbf{B})\mathbf{B}_\perp^{-1}) \wedge \mathbf{C}^{-1}$, therefore we have the following identity:

$$P_{(\mathbf{A}\Delta\mathbf{B})\mathbf{B}^{-1}} = P_{(\mathbf{A}\Delta\mathbf{B})\mathbf{B}_\perp^{-1}} + P_{\mathbf{C}^{-1}} \tag{8}$$

The linear combination of the three projection operators has now been *reduced* to the linear combination of four projection operators. A quick appeal to equation (3) implies that for two blades, if their geometric product is a scalar then their orthogonal projection operators are equal. Now $\mathbf{C}\mathbf{C}^{-1}$ is a scalar, and so is $\mathbf{B}_\perp^{-1}(\mathbf{A}\Delta\mathbf{B})(\mathbf{A}\Delta\mathbf{B})\mathbf{B}_\perp^{-1}$. Therefore the terms $P_{(\mathbf{A}\Delta\mathbf{B})\mathbf{B}_\perp^{-1}}$ and $P_{\mathbf{B}_\perp^{-1}(\mathbf{A}\Delta\mathbf{B})}$ cancel and the terms $P_{\mathbf{C}^{-1}}$ and $P_\mathbf{C}$ combine. The result, (equation (5)), then follows from equations (6), (7), and (8).

A small commentary is in order. The first comment is that the calculation of the blade correspondence will be simplified by the fact that if the steps of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}\Delta\mathbf{B}$ are r, s, and q respectively, then the step of the meet is $\frac{r+s-q}{2}$. The second comment is that the blade correspondence is not precise about the scale and orientation of the blade because the blade correspondence inherits an arbitrary scale and orientation from an arbitrary basis of blades. Since the meet for blades is defined by the blade correspondence, this lack of precision is then passed on to the meet for blades, except for the *disjoint* case. The disjoint case occurs when $\mathcal{A} \cap \mathcal{B} = \{0\}$, and in this case one can choose a basis *a priori*. This is possible because in this case the meet for blades is a scalar. Therefore one can choose the scalar '1' for the basis, and then since the blade correspondence uses the outermorphism of the projection operator and an outermorphism is the identity when restricted to the scalars, the blade correspondence gives a determinate answer for the meet, namely it gives '1' back again. The third comment is that the meet for blades given here is different from previous literature[1, 3], which only relates nontrivially to our definition when $\mathcal{A} \cup \mathcal{B} = \mathcal{R}^n$. When $\mathcal{A} \cup \mathcal{B} \neq \mathcal{R}^n$ the previous literature gives the zero blade as the result, while we can also treat that case. The price we pay is linearity. Like the delta product, the meet is not linear.

3.5. The Join Operation

Previously we noted that $(\mathbf{A}\Delta\mathbf{B})\mathbf{C} = (\mathbf{A}\Delta\mathbf{B}) \wedge \mathbf{C}$, in fact $(\mathbf{A}\Delta\mathbf{B}) \wedge \mathbf{C}$ characterizes $\mathcal{A} \cup \mathcal{B}$, therefore using equation (2), we find that $P_{\mathbf{A}\cup\mathbf{B}} = P_{\mathbf{A}\Delta\mathbf{B}} + P_{\mathbf{A}\cap\mathbf{B}}$. We have two alternatives. The first alternative is to define the projection operator for the join directly as

$$P_{\mathbf{A}\cup\mathbf{B}}(\mathbf{x}) = \frac{P_{(\mathbf{A}\Delta\mathbf{B})\mathbf{B}^{-1}}(\mathbf{x}) + P_{\mathbf{A}\Delta\mathbf{B}}(\mathbf{x}) + P_{\mathbf{B}}(\mathbf{x})}{2} \tag{9}$$

and the blade correspondence from section (3.1) is speeded up by knowing the step of the join in advance. If the steps of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{A}\Delta\mathbf{B}$ are r, s, and q respectively, then the step of the join is $\frac{r+s+q}{2}$. The second alternative is to define the join for blades directly in terms of the meet for blades and the inner division operation through the equation $\mathbf{A} \cup \mathbf{B} = \mathbf{A} \wedge (\mathbf{B} \backslash (\mathbf{A} \cap \mathbf{B}))$. Just as the meet had a definite scale and orientation only in the disjoint case, this definition allows the join to inherit the definite scale and orientation from the meet, since in that case $\mathbf{A} \cup \mathbf{B} = \mathbf{A} \wedge \mathbf{B}$. It bears mentioning that this join for blades only agrees with the previous literature[1, 3] in the disjoint case, but again definitions in the previous literature are merely zero when $\mathcal{A} \cap \mathcal{B} \neq \{0\}$, so this definition is an extension. As with the meet the price we pay is linearity.

4. The Non-Euclidean Metrics

In a Non-Euclidean space, a nonzero blade, might not have an inverse, in which case, it is obvious that the formulas above will break down. Also, some of the operations, such as the Delta product, are sensitive to error. A single new tool, the LIFT, can deal with both problems. Working around the lack inverses and taming of error can be accomplished with judicious use of the LIFT. This new tool is introduced to allow the subspace operations to be performed in any metric and then each of the four operations is investigated in turn.

4.1. A LIFT Between Clifford Algebras

Given two algebras, $\mathcal{R}_{p,q,r}$ and $\mathcal{R}_{a,b,c}$, such that $p+q+r= a+b+c=n$, and a linear invertible function, f, from the vectors of $\mathcal{R}_{p,q,r}$ to $\mathcal{R}_{a,b,c}$

then $\underline{f}$ is a linear invertible map between the two algebras. Call the extended function $\underline{f}$ a LIFT ('linear invertible function' transformation) of $\mathcal{R}_{p,q,r}$ to $\mathcal{R}_{a,b,c}$.

A LIFT can be used to transfer a problem with subspaces to another algebra, preserving incidence relations but allowing the metric to change. Often a LIFT is taken to a Euclidean space. After solving the problem in that space, subspace results can be pulled back to the original space. Examples that extend the previous results on the subspace operations follow.

A minor variation of the LIFT is for f to go into an n-dimensional subspace of a Clifford Algebra over a larger vector space, then the outermorphism is a linear invertible map onto a subalgebra. This is especially nice when the outermorphism is an isomorphism between the original algebra and the subalgebra. Such a LIFT is called an embedding LIFT (or e-LIFT). This is especially common for degenerate algebras $\mathcal{R}_{p,q,r}$, for which an e-LIFT to $\mathcal{R}_{p+r,q+r,0}$ always exists. To see the existence of the e-LIFT, let $\{\mathbf{p}_1, ..., \mathbf{p}_p, \mathbf{q}_1, ..., \mathbf{q}_q, \mathbf{r}_1, ..., \mathbf{r}_r\}$ be an orthogonal basis for the vectors in $\mathcal{R}_{p,q,r}$ such that $\mathbf{p}_i^2 = 1$, $\mathbf{q}_j^2 = -1$, and $\mathbf{r}_k^2 = 0$ and let $\{\mathbf{e}_1, ..., \mathbf{e}_{p+r}, \mathbf{f}_1, ..., \mathbf{f}_{q+r}\}$ be an orthogonal basis for the vectors in $\mathcal{R}_{p+r,q+r,0}$ such that $\mathbf{e}_i^2 = 1$ and $\mathbf{f}_j^2 = -1$. Then let f be a linear function such that $f(\mathbf{p}_i) = \mathbf{e}_i$, $f(\mathbf{q}_j) = \mathbf{f}_j$, and $f(\mathbf{r}_k) = \mathbf{e}_{p+k} + \mathbf{f}_{q+k}$. Then $\underline{f}$ is the promised isomorphism.

4.2. The Meet Operation

If one fixes an arbitrary LIFT, $\underline{f}$, from $\mathcal{R}_{p,q,r}$ to $\mathcal{R}_{p+q+r,0,0}$ then the meet, $\mathbf{A} \cap \mathbf{B}$, between two blades $\mathbf{A}$ and $\mathbf{B}$ can be defined by:

$$\mathbf{A} \cap \mathbf{B} = \underline{f}^{-1}(\underline{f}(\mathbf{A}) \cap \underline{f}(\mathbf{B})) \tag{10}$$

The scale and orientation of $\underline{f}(\mathbf{A}) \cap \underline{f}(\mathbf{B})$ are indeterminate except when $\underline{f}(\mathbf{A})$ and $\underline{f}(\mathbf{B})$ are disjoint, which only happens when $\mathbf{A}$ and $\mathbf{B}$ are disjoint. The LIFT is an outermorphism, so it is the identity on the scalars, so the meet has a definite scale and orientation in the disjoint case, and they are the same scale and orientation as in the Euclidean space. More importantly, in the disjoint case, the scale and orientation are independent of which LIFT, $\underline{f}$, was chosen. The preservation of the outer product and the scalars makes it clear that this meet corresponds to $\mathcal{A} \cap \mathcal{B}$. This means that this definition has a well-defined scale and orientation in exactly the cases where the Euclidean definition did, and it has an arbitrary scale and orientation in exactly the cases where the Euclidean definition did.

4.3. The Join Operation

If one fixes fixes an arbitrary LIFT, $\underline{f}$, from $\mathcal{R}_{p,q,r}$ to $\mathcal{R}_{p+q+r,0,0}$ then the join, $\mathbf{A} \cup \mathbf{B}$, between two blades $\mathbf{A}$ and $\mathbf{B}$ can be defined by:

$$\mathbf{A} \cup \mathbf{B} = \underline{f}^{-1}(\underline{f}(\mathbf{A}) \cup \underline{f}(\mathbf{B})) \tag{11}$$

The scale and orientation of $\underline{f}(\mathbf{A}) \cup \underline{f}(\mathbf{B})$ are indeterminate except when $\underline{f}(\mathbf{A})$ and $\underline{f}(\mathbf{B})$ are disjoint, which only happens when $\mathbf{A}$ and $\mathbf{B}$ are disjoint, in which case the join should reduce to the outer product. The LIFT is an outermorphism, so it preserves the outer product, so clearly $\underline{f}^{-1}(\underline{f}(\mathbf{A}) \wedge \underline{f}(\mathbf{B})) = \underline{f}^{-1}(\underline{f}(\mathbf{A} \wedge \mathbf{B})) = \mathbf{A} \wedge \mathbf{B}$. Therefore the join has a definite scale and orientation in the disjoint case, and the scale and orientation are independent of which LIFT, $\underline{f}$, was chosen. The preservation of the outer product and the scalars makes it clear that this join corresponds to $\mathcal{A} \cup \mathcal{B}$. This means that this definition has a well-defined scale and orientation in exactly the cases where the Euclidean definition did, and it has an arbitrary scale and orientation in exactly the cases where the Euclidean definition did.

4.4. The Inner Division Operation

Consider two nonzero blades, $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{R}_{p,q,r}$, that characterize the subspaces $\mathcal{A}$ and $\mathcal{B}$ respectively such that $\mathcal{B} \subseteq \mathcal{A}$. When $\mathbf{B}^{-1}$ exists we can calculate $\mathbf{A}\backslash\mathbf{B} = \mathbf{B}^{-1}\mathbf{A}$ as usual. Otherwise, we need an e-LIFT, f, from $\mathcal{R}_{p,q,r}$ to $\mathcal{R}_{p+r,q+r,0}$. Let $\mathbf{I}$ be the pseudoscalar of $\mathcal{R}_{p+r,q+r,0}$. Then define $\mathbf{A}\backslash\mathbf{B} = \underline{f}^{-1}(\underline{f}(\mathbf{A}) \cap (\underline{f}(\mathbf{B})I))$. This meets the definition for the subspace operation, but now the scale and orientation has an arbitrary dependence on f.

4.5. The Delta Product

Consider two nonzero blades, $\mathbf{A}$ and $\mathbf{B}$ in $\mathcal{R}_{p,q,r}$, that characterize the subspaces $\mathcal{A}$ and $\mathcal{B}$ respectively. If $\mathbf{A} \cap \mathbf{B}$ has an inverse then the geometric product, $\mathbf{AB}$, is nonzero and the highest step portion represents the symmetric difference, $\mathbf{A}\Delta\mathbf{B}$ as usual. Otherwise, we need an e-LIFT, f, from $\mathcal{R}_{p,q,r}$ to $\mathcal{R}_{p+r,q+r,0}$. Let $\mathbf{I}$ be the pseudoscalar of $\mathcal{R}_{p+r,q+r,0}$. Then define $\mathbf{A}\Delta\mathbf{B} = \underline{f}^{-1}((\underline{f}(\mathbf{A}) \cup \underline{f}(\mathbf{B})) \cap ((\underline{f}(\mathbf{A}) \cap \underline{f}(\mathbf{B}))I))$. This meets the definition for the subspace operation, but now the scale and orientation has an arbitrary dependence on f. Note that the symmetric difference was used to compute the meet and join for Euclidean signatures, but the meet and join are used to define the symmetric difference for Non-Euclidean signatures.

5. Linearity and the Meaning of Zero

The meet and join for blades presented in this paper are not linear, (e.g. in general $(\mathbf{A}+\mathbf{B})\cap\mathbf{C}\neq\mathbf{A}\cap\mathbf{C}+\mathbf{B}\cap\mathbf{C}$, even when $\mathbf{A}+\mathbf{B}$ is a nonzero blade). The previous literature[1, 3] have linear results for the meet and join. In our notation the meet and join of the previous literature are $(A\mathbf{I})\rfloor B$ and $A\wedge B$ respectively.

The linear versions can operate on any multivector (by linear extension), but the geometric interpretation of the computation becomes confusing. Also, even when the linear versions operate on nonzero blades, they can disagree with the subspace operations by giving a result of zero. It is not surprising that the subspace operations disagree with the linear operations, because it was exactly the preponderance of the answer 0 for many meaningful computations that motivated the creation of the new subspace operations of this paper. However the opposing versions can be reconciled by pursuing a geometric interpretation of the zero blade.

An interpretation of the zero blade that is consistent with the general enterprise of representing oriented subspaces by blades is to have the zero blade represent an *indeterminate* oriented subspace. At first glance, it appears that the zero blade represents the whole space (i.e. $\{\mathbf{x}\in\mathcal{R}^n : \mathbf{x}\wedge 0 = 0\} = \mathcal{R}^n$), but this interpretation would imply that the zero blade represents a different subspace depending on a particular enveloping pseudoscalar (a property that destroys natural subalgebra and enveloping algebra relationships). But since this is implicit in stating that the zero blade can represent *any* subspace this is actually support for the interpretation proposed here. Furthermore, since the hope of representing subspaces by blades is to eventually be able to deal with uncertainty in geometrical computations, the scale factor would naturally be used to represent how well-determined the blade is. This implies that the zero blade represents a completely undetermined subspace. Lastly the linear versions give the zero blade as the result if either of the input blades is the zero blade, this adsorbing property is consistent with the indeterminacy the zero blade represents.

When two oriented subspaces do not span the pseudoscalar, the linear meet gives the result of the zero blade because the orientation of the meet cannot be determined. The linear meet is a fully functional quantitative operation, which can give a quantitative meet, but only *if* given a quantitative join first (in the role of the pseudoscalar). Similarly when two oriented subspaces have a nontrivial intersection, the linear join gives the result of the zero blade because the join has an undetermined orientation.

With this interpretation for the zero blade, the linear meet and join can be compared to the versions presented in this paper. The linearity can be an advantage for implementation for some applications, and if that advantage outweighs the costs of getting the zero blade as a result, then an educated decision to implement the linear versions can be made for that application. The interpretation for the zero blade can also be used to extend the subspace operations for nonzero blades to the zero blade by declaring the inner division, the delta product, the meet, and the join to be zero if either input of the two input blades is the zero blade.

6. Conclusion

Nonzero blades can represent subspaces, and in applications we need to perform operations on subspaces. Therefore we would naturally want operations on blades that mirror the results of computations that we would have liked to perform on subspaces. The four blade operations (inner division, delta product, meet, and join) are the first four operations in Geometric Algebra from nonzero blades to nonzero blades. Geometric Algebra is a useful extension of Linear Algebra to consistently use scale and orientation. The hope is that these four blade operations can contribute to quantitative computations with oriented subspaces.

The four subspace operations are different because the meet and join can be defined independently of the other subspace operations and without the structure of a metric. If there is already a metric on the space, then the space has a Geometric Algebra. In this paper we show that the four corollary blade operation can be carried out entirely within Geometric Algebra. When defining the four blade operations, the meet and join no longer are special, in fact the delta product and the inner division are used to define the meet and join in Euclidean signatures, while the meet and join are used to define the delta product and inner division in Non-Euclidean signatures. This indicates that they are all fundamental (though not as fundamental as the geometric product) and that they are all interconnected.

Standard concepts in geometric algebra needed to be augmented because the meet and join for blades cannot be defined[7] with an orientation due to fundamental geometric problems. This fundamental problem was solved by using orthogonal projection operators to represent unoriented subspaces. Within this solution the delta product helps to compute the meet and join for blades.

The price to be paid for this augmentation is that the new blade operations are not linear and cannot be extended to arbitrary multivectors. The authors believe that their non-linearity might be the reason that the operations have not been used previously. It is only by sacrificing linearity, and thus losing applicability to arbitrary multivectors, that one can solve the meet and join for blades.

The four subspace operations are tools intended for general use in applications, however no applied examples are included in this paper. Readers looking for examples of the inner division and the delta product need look no farther than the proof of the meet for Euclidean metrics, and readers looking for examples of the meet and join need look no farther than the inner division and delta product for Non-Euclidean metrics. Beyond the four subspace operations, this paper utilizes another tool of general applicability. This tool is the LIFT ('linear invertible function' transformation). It is an invertible map between algebras of different signatures. This tool can be used to advantageously transform problems that are independent of signature to whichever signature is most helpful at any particular moment. This tool is demonstrated in the paper by extending the results of the meet and join from the Euclidean case to the Non-Euclidean case (even to degenerate signatures).

Lastly some loose ends are resolved. A geometric interpretation is given to the zero blade that explains the results of the previous literature and facilitates the educated choice between different versions of the subspace operations. The final loose end is resolved by the appendix, which includes proofs to demonstrate that the inner and outer products go from blades to blades (even in degenerate signatures).

7. Appendix

In this Appendix we prove that the outer product of two blades is a blade and that the inner product of two blades is a blade. To our surprise, this does not appear to have been proved before, but is less trivial than may have been assumed when degenerate algebras are considered.

7.1. The Outer Product

By associativity of the outer product it suffices to show that the outer product of a vector and an r-blade is a blade. The result is trivial because the $(r+1)$-vector determines an $(r+1)$-dimensional subspace and that subspace has an orthogonal basis. However a more constructive proof is desirable, to assist in the proof for the inner product and to see how such a factorization can be made. The proof proceeds by induction on r, the base case is $r = 1$, or the outer product of two vectors is a blade.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two vectors. Since $\frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2)(\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{v}_1 \wedge \mathbf{v}_2 + \frac{\mathbf{v}_1^2 - \mathbf{v}_2^2}{2}$, this gives a factorization of $\mathbf{v}_1 \wedge \mathbf{v}_2$ when $\mathbf{v}_1^2 = \mathbf{v}_2^2$. If $\mathbf{v}_1^2 \neq \mathbf{v}_2^2$ then either $\mathbf{v}_1^2$ or $\mathbf{v}_2^2$ is not equal to 0. Since $\mathbf{v}_1 \wedge \mathbf{v}_2 = -\mathbf{v}_2 \wedge \mathbf{v}_1$, we may assume without loss of generality that $\mathbf{v}_1^2 \neq 0$. Then we note that $\mathbf{v}_1^{-1} \rfloor (\mathbf{v}_1 \wedge \mathbf{v}_2)$ is a vector and that $\mathbf{v}_1(\mathbf{v}_1^{-1} \rfloor (\mathbf{v}_1 \wedge \mathbf{v}_2)) = \mathbf{v}_1 \wedge \mathbf{v}_2$, so this gives a factorization of $\mathbf{v}_1 \wedge \mathbf{v}_2$. Therefore in both cases the outer product of two vectors can be factored.

Assume that the outer product of r vectors is a blade. Let $\{\mathbf{a}, \mathbf{c}_1, ..., \mathbf{c}_r\}$ be $r+1$ vectors. The inductive step has three cases.

1. The first case is when $\mathbf{a} \rfloor \mathbf{c}_i = 0$ for each i . Then by equation (1), $\mathbf{a} \wedge \mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r = \mathbf{a}(\mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r)$. By the inductive hypothesis, $\mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r$ is a blade, so there exist r anticommuting vectors, $\{\mathbf{a}_1, ..., \mathbf{a}_r\}$, such that $\mathbf{a}_1\mathbf{a}_2...\mathbf{a}_r = \mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r$. Each $\mathbf{a}_i$ is a linear combination of the set $\{\mathbf{c}_1, ..., \mathbf{c}_r\}$, so each $\mathbf{a}_i$ anticommutes with $\mathbf{a}$, so $\mathbf{a}\mathbf{a}_1...\mathbf{a}_r$ is a factorization of $\mathbf{a} \wedge \mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r$, therefore $\mathbf{a} \wedge \mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r$ is an $(r+1)$-blade.

2. The next case is when $\mathbf{a}^2 \neq 0$. Let $\mathbf{b}_i = \mathbf{a}^{-1}(\mathbf{a} \wedge \mathbf{c}_i)$. Then $\mathbf{a} \wedge \mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r = \mathbf{a} \wedge \mathbf{b}_1 \wedge ... \wedge \mathbf{b}_r$. Now we have guaranteed that $\mathbf{a} \rfloor \mathbf{b}_i = 0$ for each i, so the previous case resolves the factorization.

3. The last case is when there exists a k such that $\mathbf{a} \rfloor \mathbf{c}_k \neq 0$ and $\mathbf{a}^2 = 0$. Without loss of generality we may assume $k = 1$ since the order of

the vectors only affects the sign of the outcome. By the base case, $\mathbf{a} \wedge \mathbf{c}_1$ is a 2-blade. Direct computation shows that the square of $\mathbf{a} \wedge \mathbf{c}_1$ is $(\mathbf{a} \rfloor \mathbf{c}_1)^2$, hence nonzero, therefore there exist two invertible vectors $\mathbf{c}$ and $\mathbf{d}_1$ such that $\mathbf{a} \wedge \mathbf{c}_1 = \mathbf{c}\mathbf{d}_1$. Let $\mathbf{d}_i = (\mathbf{c}\mathbf{d}_1)^{-1}((\mathbf{c}\mathbf{d}_1) \wedge \mathbf{c}_i)$ for $i \in \{2, ..., r\}$. Then $\mathbf{a} \wedge \mathbf{c}_1 \wedge ... \wedge \mathbf{c}_r = \mathbf{c} \wedge \mathbf{d}_1 \wedge \mathbf{d}_2 \wedge ... \wedge \mathbf{d}_r$. Now since $\mathbf{c}^2 \neq 0$ the previous case resolves the factorization.

7.2. The (Contraction) Inner Product

Let $\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge ... \wedge \mathbf{a}_m$ and $\mathbf{B}$ be two blades in $\mathcal{R}_{p,q,r}$. By the properties of the inner product, $\mathbf{A} \rfloor \mathbf{B} = \mathbf{a}_1 \rfloor (\mathbf{a}_2 \rfloor (... \rfloor (\mathbf{a}_m \rfloor \mathbf{B})...))$, so it suffices to show that $\mathbf{a} \rfloor \mathbf{C}$ is a blade for any vector $\mathbf{a}$ and any blade $\mathbf{C}$. If $\mathbf{C}$ is a scalar then $\mathbf{a} \rfloor \mathbf{C} = 0$ so it is a blade. Otherwise let $\mathbf{C} = \mathbf{c}_1 \wedge \mathbf{c}_2 \wedge ... \wedge \mathbf{c}_k$. The nonzero blade $\mathbf{C}$ characterizes a subspace with signature (s, t, u) and hence a subalgebra $\mathcal{R}_{s,t,u}$. Using equation (1), it is clear that $\mathbf{a} \rfloor \mathbf{C}$ resides in $\mathcal{R}_{s,t,u}$. Let $\underline{f}$ be a LIFT from $\mathcal{R}_{s,t,u}$ to $\mathcal{R}_{k,0,0}$. Since $\underline{f}(X)$ is a blade if and only if X is a blade then to show that $\mathbf{a} \rfloor \mathbf{C}$ is a blade it suffices to show that $\underline{f}(\mathbf{a} \rfloor \mathbf{C})$ is a blade. From equation (1), it follows that $\underline{f}(\mathbf{a} \rfloor \mathbf{C})$ is the sum of (k-1)-blades. Let $\mathbf{I}$ be a nonzero pseudoscalar for $\mathcal{R}_{k,0,0}$. Let $\{\mathbf{H}_1, ..., \mathbf{H}_m\}$ be m (k-1)-blades in $\mathcal{R}_{k,0,0}$. Then $\mathbf{H}_1 + ... + \mathbf{H}_m = (\mathbf{H}_1 + ... + \mathbf{H}_m)\mathbf{I}^{-1}\mathbf{I} = \mathbf{v}\mathbf{I}$, where $\mathbf{v} = \mathbf{H}_1\mathbf{I}^{-1} + ... + \mathbf{H}_m\mathbf{I}^{-1} = \mathbf{H}_1 \rfloor \mathbf{I}^{-1} + ... + \mathbf{H}_m \rfloor \mathbf{I}^{-1}$ is a vector. If $\mathbf{v} = 0$ then the sum, $\mathbf{H}_1 + ... + \mathbf{H}_m = \mathbf{v}\mathbf{I}$, is zero and trivially a blade. If not then $\mathbf{v} \neq 0$ and therefore $\mathbf{v}$ has an inverse and by the details of the proof for the outer product, it is clear that $\mathbf{I}$ can be factored under the geometric product with $\mathbf{v}^{-1}$ as a factor. Then $\mathbf{H}_1 + ... + \mathbf{H}_m = \mathbf{v}\mathbf{I}$ is clearly a blade.

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