The KdV equation all around us

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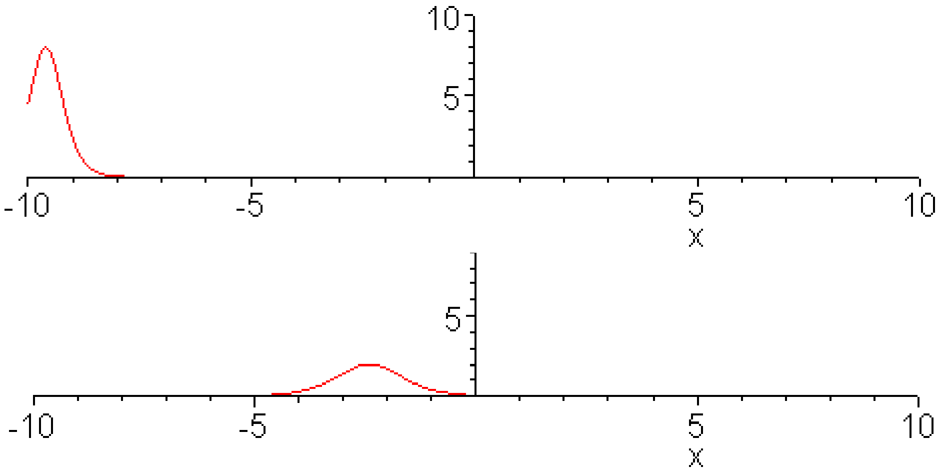
Plan of the presentation

1. Brief history of the KdV equation
2. KdV and KP
3. The key feature: Integrability
4. Identification of period lattices
5. Semi-infinite Grassmannian
6. Intersection theory of moduli spaces
7. Modern perspectives

Brief history of the KdV equation

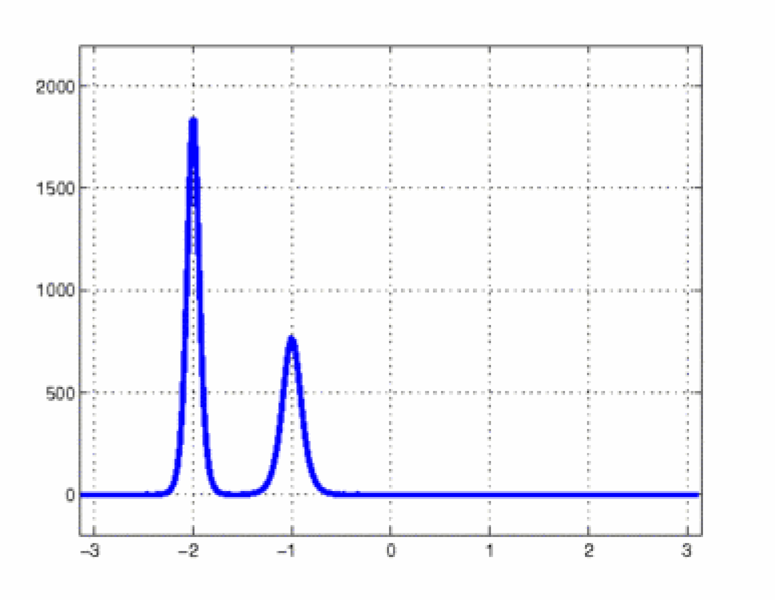
John Scott Russell
1808-1882
Scottish naval engineer

- In 1834 Russell observed solitary waves on the Union Canal. This observation had no theoretical explanation at that time.



Solitary waves or *solitons*

The Union Canal, runs from Edinburgh to Glasgow



Two-soliton system

- Russell went further in his experiments
- 1870-1875: important contributions by Lord Rayleigh and Joseph Boussinesq
- In 1895 Diederick Korteweg and his student Gustav de Vries derived a nonlinear PDE that is now called the KdV equation

$$u_t + u_{xxx} - 6\,u\,u_x = 0$$

Diederick Korteweg
1848 — 1941

Professor at UvA
1881 — 1918

Gustav de Vries
1866 — 1934

PhD 1984, Thesis:
«Bijdrage tot de Kennis der Lange Golven»

KdV and KP

Boris Kadomtsev
1928 — 1998
Soviet physicist

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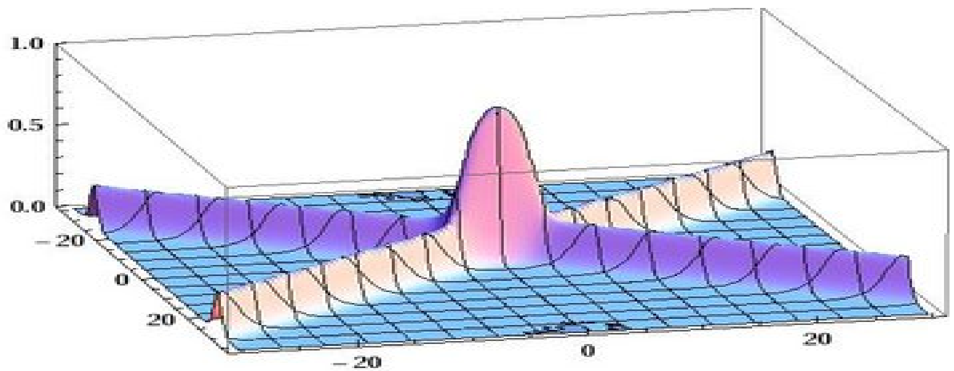
IN THE INTERNET

Vladimir Petviashvili
1936 — 1993
Soviet physicist

- In 1970 Boris Kadomtsev and Vladimir Petviashvili derived a generalization fo the KdV equation that takes into account transverse perturbations.

$$\frac{\partial}{\partial x}(u_t + u_{xxx} - 6\,u\,u_x) \pm u_{yy} = 0.$$

- The KdV (KP) equation is a universal integrable system in one (two) spatial dimension(s) since many other integrable systems can be obtained as reductions.



Integrability

$$H_0 = \int dx\, u$$

$$H_1 = \int dx\, \frac{1}{2}u^2$$

$$H_2 = \int dx\, \left(\frac{1}{3!}u^3 - \frac{1}{2}u_x^2\right)$$

$$H_3 = \int dx\, \left(\frac{1}{4}u^4 - 3uu_x + \frac{9}{5}u_{xx}^2\right)$$

$$H_4 = \int dx\, \left(\frac{1}{5}u^5 - 6u^2u_x^2 + \frac{36}{5}uu_{xx}^2 - \frac{108}{35}u_{xxx}^2\right)$$

$$\vdots$$

- The KdV equation has an infinite series of conserved quantities!

- The KdV equation has Lax representation!

$$\frac{\partial L}{\partial t} = [B, L]$$

$$L = \partial^2 + \frac{1}{6}u$$

$$B = 4\,\partial^3 + \frac{1}{2}(\partial u + u\partial)$$

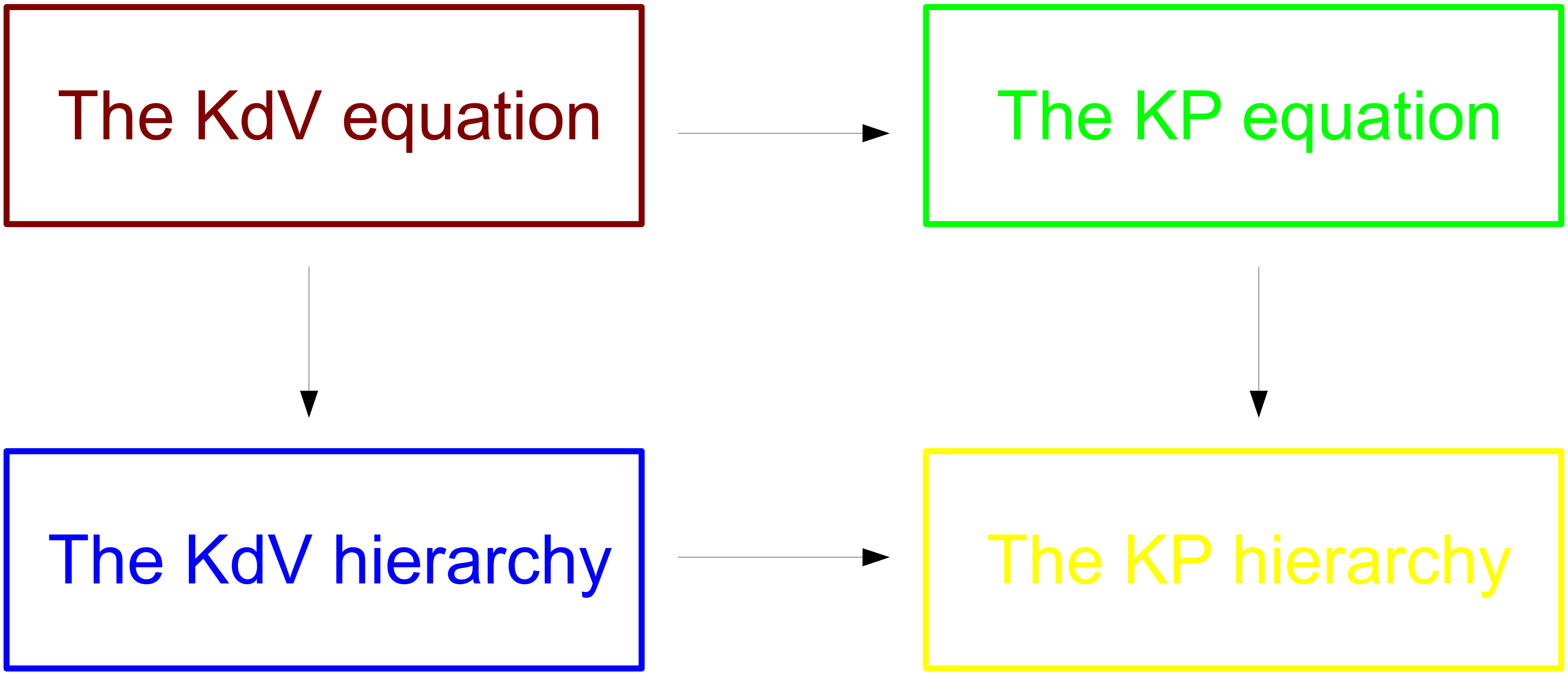
- The KdV equation has two Hamiltonian structures!

$$\{u(x), u(y)\}_1 = \mathcal{D}_1\delta(x-y)$$

$$\mathcal{D}_1 = \frac{\partial}{\partial x}$$

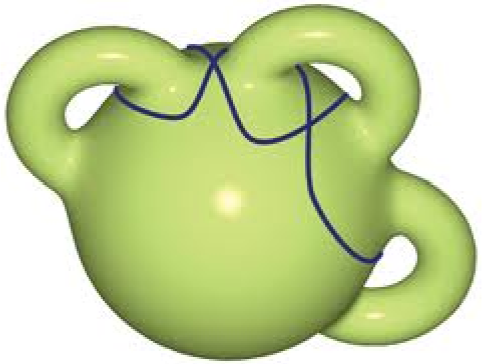
$$\{u(x), u(y)\}_2 = \mathcal{D}_2\delta(x-y)$$

$$\mathcal{D}_2 = \frac{\partial^3}{\partial x^3} + \frac{1}{3}\left(\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}\right)$$



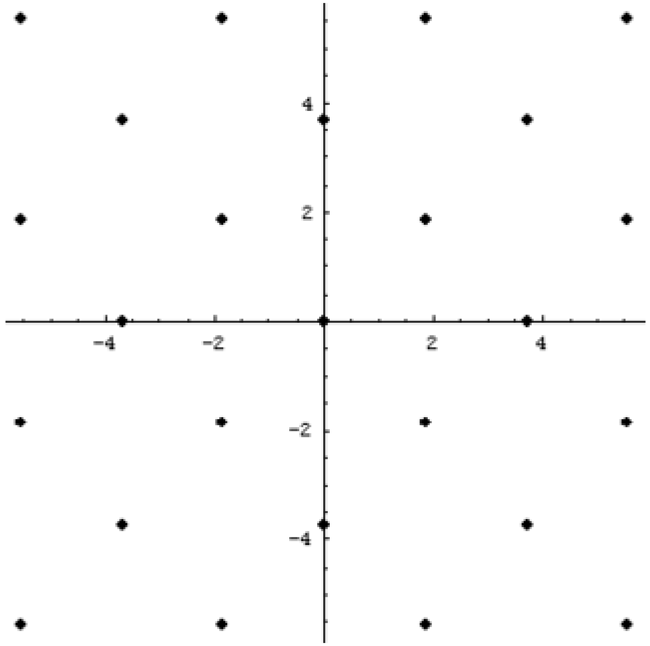
The study of these equations and hierarchies has led to the development of a very general and powerful **theory of soliton equations.**

Identification of period lattices



Riemann surface
= 1d complex manifold

The intrinsic integrals over closed contours (periods) lie on a ***lattice***



Q.: How to ***distinguish period lattices*** among all possible lattices?

A.: Associate to a lattice a distinguished function called ***theta-function***. Check, whether it ***solves the KP equation***!

Semi-infinite Grassmannian

Universal tools to study ***finite symmetry*** are intrinsic objects associated to the ***semi-infinite Grassmannian***

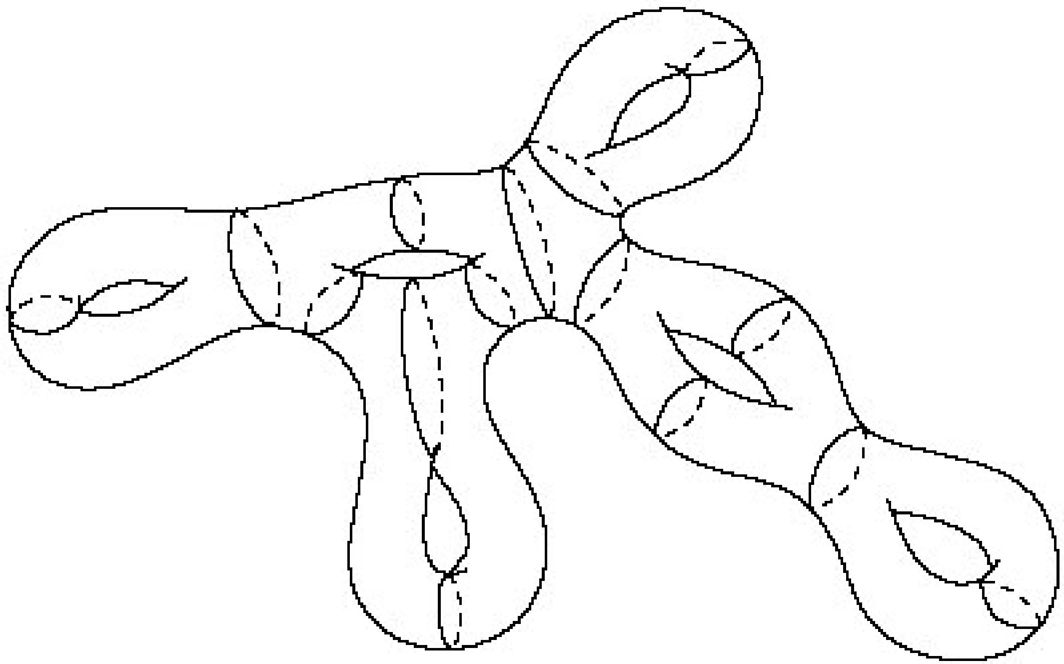
A point in the semi-infinite Grassmannian corresponds to a semi-infinite matrix →

a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	...
a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	...
a_{31}	a_{32}	a_{33}	a_{34}	a_{35}	a_{36}	...
a_{41}	a_{42}	a_{43}	a_{44}	a_{45}	a_{46}	...
a_{51}	a_{52}	a_{53}	a_{54}	a_{55}	a_{56}	...
a_{61}	a_{62}	a_{63}	a_{64}	a_{65}	a_{66}	...
...	...	...	...	...	...	...

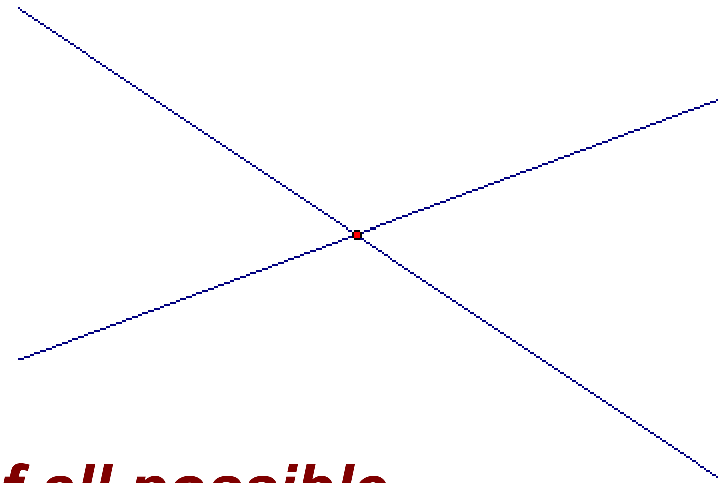
Q.: How to describe ***all possible universal relations*** among all finite volumes hidden in this matrix?

A.: This system of relations, properly re-arranged, is ***equivalent to the KP hierarchy***!

Intersection theory of moduli spaces



The space of all possible complex structures on a given 2d surface is called ***the moduli space of curves.***



Q.: How to describe ***all intersection numbers of all possible natural subvarieties*** of the moduli space of curve?

A.: The ***natural minimal set of data*** needed to reconstruct all intersection numbers form a special ***solution of the KdV hierarchy***!

Modern perspectives

Q.: Whether we can explain all these miracles
and/or make them not so exceptional?

A.: With collaborators we have developed very powerful deformation methods that allow us to extend special properties of KdV to a large class of integrale hierarchies.

Alexandr Buryak

Hessel Posthuma




Carel Faber

Dimitri Zvonkine

Thank you for your attention!