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Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes

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Laser light with a Laguerre-Gaussian amplitude distribution is found to have a well-defined orbital angular momentum. An astigmatic optical system may be used to transform a high-order Laguerre-Gaussian mode into a high-order Hermite-Gaussian mode reversibly. An experiment is proposed to measure the mechanical torque induced by the transfer of orbital angular momentum associated with such a transformation.

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I. INTRODUCTION

It is well known from Maxwell's theory that electromagnetic radiation carries both energy and momentum. The momentum may have both linear and angular contributions; angular momentum has a spin part associated with polarization [1] and an orbital part associated with spatial distribution [2]. Any interaction between radiation and matter is inevitably accompanied by an exchange of momentum. This often has mechanical consequences some of which are related to radiation pressure. Although an experimental demonstration of the mechanical torque created by the transfer of angular momentum of a circularly polarized light beam was performed over 50 years ago [1], the work associated with the mechanical influence of light beams on atoms and matter has been almost exclusively concerned with linear momentum [3–5].

Beth [1] made the first observation of the angular momentum of light following Poynting [6], who inferred from a mechanical analogy that circularly polarized light should exert a torque on a birefringent plate and that the ratio of angular to linear momentum is equal to $\lambda/2\pi$. In his experiment a half-wave plate was suspended by a fine quartz fiber [see Fig. 1(a)]. A beam of light, circularly polarized by a fixed quarter-wave plate, passed through the plate which transformed right-handed circularly polarized light into left-handed circularly polarized light and transferred $2\hbar$ of spin angular momentum for each photon to the birefringent plate. It was found that the measured torque agreed in sign and magnitude with that predicted by both wave and quantum theories of light. The ratio of the angular momentum of N photons in the beam, $J=\pm N\hbar$, to their energy, $W=N\hbar\omega$, is $\pm 1/\omega$ and Beth's measurement is sometimes referred to as the measurement of the spin angular momentum of the photon.

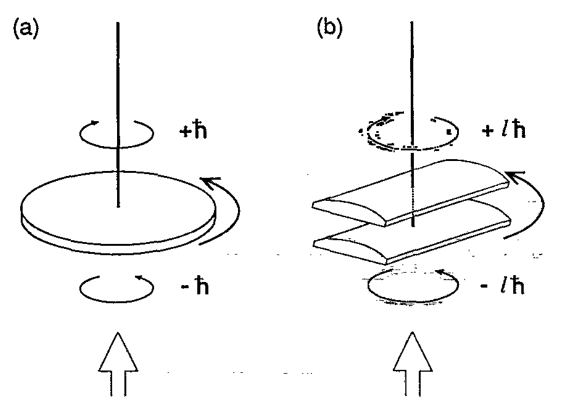

FIG. 1. (a) A suspended $\lambda/2$ birefringent plate undergoes torque in transforming right-handed into left-handed circularly polarized light. (b) Suspended cylindrical lenses undergo torque in transforming a Laguerre-Gaussian mode of orbital angular momentum $-l\hbar$ per photon, into one with $+l\hbar$ per photon.

The purpose of this paper is to investigate whether a Gaussian mode may be said to possess orbital angular momentum and to propose a study of the mechanical consequences which might then arise. The amplitude of a Laguerre-Gaussian mode has an azimuthal angular dependence of $\exp(-il\phi)$, where l is the azimuthal mode index. Analogy between quantum mechanics and paraxial optics [7] suggests that such modes are the eigenmodes of the angular momentum operator L_z and carry an orbital angular momentum of $l\hbar$ per photon. The transverse amplitude distribution of laser light is usually described in terms of a product of Hermite polynomials $H_n(x)H_m(y)$ and associated with TEM_{nmq} modes. Laguerre polynomial distributions of amplitude, TEM_{plq} modes, are also possible but occur less often in actual lasers. This allows the analogy with quantum mechanics to be stressed even more strongly. It is well known [8] that the one-dimensional quantum-mechanical harmonic oscillator has a solution in the form of a Hermite polynomial and that in two dimensions the solutions may be written as Laguerre polynomials with energy $(n+m+1)\hbar\omega$, while the eigenvalue of the two-dimensional angular momentum operator is $(n-m)\hbar$. It seems likely, therefore, that TEM_{plq} modes possess well-defined orbital angular momenta.

II. ORBITAL ANGULAR MOMENTUM OF A LAGUERRE-GAUSSIAN MODE

The angular momentum density associated with the transverse electromagnetic field may be shown [2] to be

$$\mathbf{M}=\epsilon_0\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) \tag{1}$$

while the total angular momentum of the field is

$$\mathbf{J}=\epsilon_0\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d\mathbf{r} \ . \tag{2}$$

In atomic physics it is normal to expect that

$$\mathbf{J}=\mathbf{L}+\mathbf{S} \ , \tag{3}$$

where the first term is identified with the orbital angular momentum **L**, and the second with the spin **S**. But there is doubt as to whether **L** and **S** are, in general, separately physically observable [9] for vector fields.

Clearly the linear momentum of a transverse plane wave, $\mathbf{E}\times\mathbf{B}$, is in the direction of propagation, z, and there cannot be a component of angular momentum $\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$ in the same direction [10]. However, the fields of the laser modes TEM_{nmq} or TEM_{plq}, unlike those in a coaxial metal waveguide of infinite length, are not strictly transverse [11]. They have small components in the direction of propagation, z. A convenient representation of a linearly polarized laser mode is achieved [12] in the Lorentz gauge using the vector potential

$$\mathbf{A}=\mathbf{x}u(x,y,z)e^{-ikz} \tag{4}$$

where $\mathbf{x}$ is the unit vector in the x direction. The expression $u(x,y,z)$, or $u(r,\phi,z)$, is the complex scalar function describing the distribution of the field amplitude which satisfies the wave equation in the paraxial approximation. In this approximation the second derivatives of E and B fields, and the products of first derivatives, are ignored and $\partial u/\partial z$ taken to be small compared with ku. The cylindrically symmetric solutions $u_{pl}(r,\phi,z)$ which describe Laguerre-Gaussian beams are given by

$$u_{pl}(r,\phi,z)=\frac{C}{(1+z^2/z_R^2)^{1/2}}\left[\frac{r\sqrt{2}}{w(z)}\right]^l L_p^l\left[\frac{2r^2}{w^2(z)}\right] \times\exp\left[\frac{-r^2}{w^2(z)}\right]\exp\frac{-ikr^2z}{2(z^2+z_R^2)}\exp(-il\phi) \times\exp\left[i(2p+l+1)\tan^{-1}\frac{z}{z_R}\right] , \tag{5}$$

where z_R is the Rayleigh range, $w(z)$ is the radius of the beam, L_p^l is the associated Laguerre polynomial, C is a constant, and the beam waist is at $z=0$. The Lorentz gauge has the advantage of being readily amenable in all coordinate systems and leads in this case to considerable symmetry in the x and y directions although the results are best expressed in cylindrical coordinates.

Within this description we have shown that the time average of the real part of $\epsilon_0\mathbf{E}\times\mathbf{B}$, which is the linear momentum density, is given by

$$\frac{\epsilon_0}{2}(\mathbf{E}^*\times\mathbf{B}+\mathbf{E}\times\mathbf{B}^*)=i\omega\frac{\epsilon_0}{2}(u^*\nabla u-u\nabla u^*) +\omega k\epsilon_0|u|^2\mathbf{z} \ , \tag{6}$$

for a beam of unit amplitude, where $\mathbf{z}$ is the unit vector in the z direction. We may recognize that $u^*\nabla u$ closely echoes the quantum-mechanical expression for the expectation value of linear momentum of a wave function. To achieve this appealing form we have retained the term $\partial u/\partial z$ in the expression for the magnetic field **B**, which could have been ignored.

When applied to the Laguerre-Gaussian distribution given by Eq. (5) for linearly polarized light, the momentum density per unit power is found to be

$$\mathcal{P}=\frac{1}{c}\left[\frac{rz}{(z^2+z_R^2)}|u|^2\mathbf{r}+\frac{l}{kr}|u|^2\boldsymbol{\phi}+|u|^2\mathbf{z}\right] , \tag{7}$$

where $\mathbf{r}$ and $\boldsymbol{\phi}$ are unit vectors and $|u|^2\equiv|u(r,\phi,z)|^2$. Here the $\partial u/\partial z$ term has now been neglected. It may be seen that the Poynting vector, given by $c^2\mathcal{P}$, spirals along the direction of propagation; see Fig. 2. The $\mathbf{r}$ component relates to the spread of the beam; the $\boldsymbol{\phi}$ component gives rise to orbital angular momentum in the z direction and the $\mathbf{z}$ component relates to the linear momentum P in the direction of propagation.

Calculation of the time averaged angular momentum density, $\epsilon_0\mathbf{r}\times\langle\mathbf{E}\times\mathbf{B}\rangle$, per unit power yields

$$\mathbf{M}=-\frac{l}{\omega}\frac{z}{r}|u|^2\mathbf{r}+\frac{r}{c}\left[\frac{z^2}{(z^2+z_R^2)}-1\right]|u|^2\boldsymbol{\phi}+\frac{l}{\omega}|u|^2\mathbf{z} \ . \tag{8}$$

The radial and azimuthal components are symmetric about the axis, so that integration over the beam profile leaves only the $\mathbf{z}$ component. The ratio of the flux of angular momentum to that of energy is $L/cP=l/\omega$, while the ratio of angular momentum to linear momentum is now $L/P=l(\lambda/2\pi)$. Our conviction that the Laguerre-Gaussian mode possesses a well-defined orbital angular momentum has thus been justified.

At position (r,ϕ,z) the magnitude of angular momentum density per unit power is given by $M=l/\omega(1+z^2/r^2)^{1/2}|u|^2$, oriented at an angle $\theta=\tan^{-1}z/r$ to the z axis. Locally we have $M_z/\mathcal{P}_z=l(\lambda/2\pi)$ where M_z is the $\mathbf{z}$ component of angular momentum density and $\mathcal{P}_z$ that of the linear momentum density. There is, however, also a local radial component.

We have so far considered linearly polarized light; when the vector potential is generalized to arbitrary polarization we find

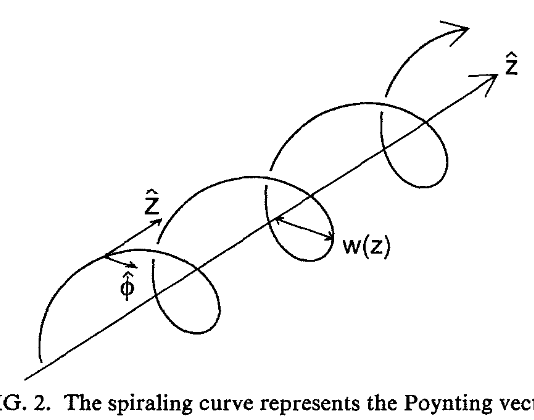

FIG. 2. The spiraling curve represents the Poynting vector of a linearly polarized Laguerre-Gaussian mode of radius $w(z)$.

$$\frac{\epsilon_0}{2}(\mathbf{E}^*\times\mathbf{B}+\mathbf{E}\times\mathbf{B}^*)=i\omega\frac{\epsilon_0}{2}(u^*\nabla u-u\nabla u^*)+\omega k\epsilon_0|u|^2\mathbf{z}+\omega\sigma_z\frac{\epsilon_0}{2}\frac{\partial|u|^2}{\partial r}\mathbf{\Phi}\ , \tag{9}$$

where the first two terms are polarization independent and relate to orbital angular momentum and the final term is a polarization, or spin, part. These lead to a $\mathbf{z}$ component of total angular momentum density per unit power

$$M_z=\frac{l}{\omega}|u|^2+\frac{\sigma_z r}{2\omega}\frac{\partial|u|^2}{\partial r}\ . \tag{10}$$

The ratio of the angular momentum flux to energy flux now becomes $J/cP=(l+\sigma_z)/\omega$ where $\sigma_z=\mp1$ for right-handed or left-handed circularly polarized light and $\sigma_z=0$ for linearly polarized light.

We may note that in the paraxial approximation the spin-dependent part of angular momentum density depends upon the gradient of the intensity. Thus at a particular local point the $\mathbf{z}$ component of angular momentum flux divided by energy flux does not yield a simple value. However, when the total angular momentum flux is calculated, the integration across the beam profile leads to the simple result of the preceding paragraph.

III. THE DETECTION OF ORBITAL ANGULAR MOMENTUM

In the Beth experiment right-handed circularly polarized light, $-\hbar$, was converted to left-handed circularly polarized light, $+\hbar$, so that $2\hbar$ of spin angular momentum per photon was imparted to the birefringent plate. In the same way the maximum transfer of orbital angular momentum would take place if a Laguerre-Gaussian beam possessing $l\hbar$ angular momentum per photon were converted into one with $-l\hbar$ per photon. The torque arising from the transfer of momentum must then be measured by an appropriate equivalent of the birefringent plate. It is, however, more convenient to discuss the conversion to a beam with zero orbital angular momentum. This is readily possible by transforming a Laguerre-Gaussian to a Hermite-Gaussian distribution, which can be done using a mode convertor with astigmatic optical components.

Abramochkin and Volostnikov [13] have considered the mathematical transformation of laser beams undergoing astigmatism and found an integral transformation of Hermite-Gaussian into Laguerre-Gaussian beams. They recognized that passing the beam through a cylindrical lens can perform the desired conversion. Tamm and Weiss [14] have similarly employed a "mode convertor" involving cylindrical lenses.

Some insight into the mechanism for transformation of modes of arbitrarily high order is readily possible. A Hermite-Gaussian laser beam with its axes along the axes of a cylindrical lens is not changed, apart from a difference in size along the two axes. Therefore the operation of a cylindrical lens on an arbitrary beam pattern is most easily discussed in terms of its Hermite-Gaussian components along the axes of the lens.

The decomposition of a Laguerre-Gaussian mode in terms of Hermite-Gaussian modes can be seen in Eq. (A3) in the Appendix of Abramochkin and Volostnikov's paper [13]. It connects a combination of products of Hermite polynomials to a Laguerre polynomial,

$$\sum_{k=0}^{n+m}(2i)^k P_k^{(n-k,m-k)}(0)H_{n+m-k}(x)H_k(y)=2^{n+m}\times\begin{cases}(-1)^m m!(x+iy)^{n-m}L_m^{n-m}(x^2+y^2) & \text{for } n\ge m\\ (-1)^n n!(x-iy)^{m-n}L_n^{m-n}(x^2+y^2) & \text{for } m>n\ ,\end{cases} \tag{11}$$

where

$$P_k^{(n-k,m-k)}(0)=\frac{(-1)^k}{2^k k!}\frac{d^k}{dt^k}[(1-t)^n(1+t)^m]\bigg|_{t=0}\ .$$

Also in the appendix of Ref. [13] there is an unnumbered equation which connects the same Hermite polynomials to their 45° transformed value,

$$\sum_{k=0}^{n+m}(-2)^k P_k^{(n-k,m-k)}(0)H_{n+m-k}(x)H_k(y)=(\sqrt{2})^{n+m}H_n\left[\frac{x-y}{\sqrt{2}}\right]H_m\left[\frac{x+y}{\sqrt{2}}\right]\ . \tag{12}$$

It may be seen that the summation in Eq. (11) is the same as that in Eq. (12), save for a factor of $(-i)^k$ where k is the integer associated with the Hermite polynomial in the y direction, $H_k(y)$. But $(-i)^k$, of course, corresponds to an additional change of phase $\pi/2$ for each integer increase in the value of k.

A simple example perhaps makes the result of this clear. From Eq. (12) we may show that

$$(\sqrt{2})^5 H_3\left[\frac{x-y}{\sqrt{2}}\right]H_2\left[\frac{x+y}{\sqrt{2}}\right]=H_5(x)H_0(y)-H_4(x)H_1(y)-2H_3(x)H_2(y)+2H_2(x)H_3(y)+H_1(x)H_4(y)-H_0(x)H_5(y) \tag{13}$$

while from Eq. (11)

$$32re^{i\phi}L_2^1(r^2)=H_5(x)H_0(y)+iH_4(x)H_1(y)+2H_3(x)H_2(y)+2iH_2(x)H_3(y)+H_1(x)H_4(y)+iH_0(x)H_5(y) . \quad (14)$$

We see that a Hermite-Gaussian mode with spatial dependence $H_n(x)H_m(y)$, may, provided the appropriate $\pi/2$ change of phase is achieved, become a single Laguerre-Gaussian mode well defined by L_p^l, or specifically $L_m^{n-m}(r^2)$ for $n \geq m$. We may identify $(n-m)$ as the orbital angular momentum of the photon in units of $\hbar$. Therefore we have converted a Hermite-Gaussian mode with zero orbital angular momentum into a Laguerre-Gaussian mode with $L=l\hbar$. If a change of phase other than $\pi/2$ is introduced, distributions between a Laguerre-Gaussian and a Hermite-Gaussian mode will result with ill-defined orbital angular momentum because such a mode is not an eigenstate of L_z.

As $H_k(y)=(-1)^k H_k(-y)$ we should note that if the left-hand side of Eq. (11) is multiplied inside the summation by $(-1)^k$, the right-hand side then relates to $(x-iy)$ for $n \geq m$ and $(x+iy)$ for $m \geq n$. Therefore the sign of the ϕ dependence of the Laguerre mode becomes changed. In other words, an additional π change of phase for all terms will transform a Laguerre-Gaussian mode with an angular momentum of $l\hbar$ to one with $-l\hbar$.

The equations allow us to recognize the nature of the decomposition of mode patterns into modes along the axes of a cylindrical lens. The presence of the focusing along only one of the axes and not the other, allows the components to propagate with different Gouy phase shifts [15]. In this way the necessary $\pi/2$ phase change to convert a Laguerre-Gaussian mode to a Hermite-Gaussian mode, or vice versa, may be achieved as Tamm and Weiss [14] realized for first-order modes.

Only for the TEM_{10} mode is the intuition that it must be combined with an out of phase version of itself to produce a radially continuous field or intensity distribution, actually true. Using Eqs. (11) and (12) it is easy to show

$$H_1\left[\frac{x-y}{\sqrt{2}}\right]H_0\left[\frac{x+y}{\sqrt{2}}\right]e^{-r^2/w^2} \rightarrow rL_0^1(r^2)e^{-r^2/w^2}e^{i\phi} . \quad (15)$$

It follows from Eq. (14) that to obtain higher-order Laguerre modes it is necessary to combine a number of Hermite-Gaussian distributions: an example is shown in Fig. 3. We have thus established a general recipe for the transformation of Laguerre-Gaussian modes of well-defined angular orbital momentum into Hermite-Gaussian modes, or into a Laguerre-Gaussian mode of opposite angular symmetry. In the experiments currently in progress in our laboratory Hermite-Gaussian modes have been converted into pure, nondegenerate, Laguerre-Gaussian modes by a simple astigmatic optical arrangement which confirms the analysis outlined in the preceding section. The full analysis of the effect of astigmatism on the Gouy phase of laser modes and the detailed design

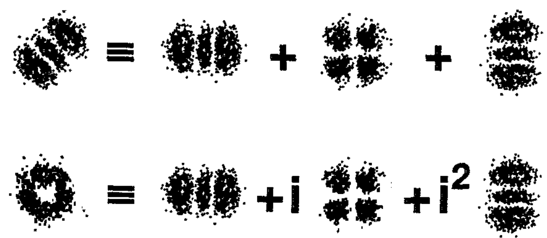

FIG. 3. The decomposition of a TEM_{02} Hermite-Gaussian mode at 45° into a set of Hermite-Gaussian modes and the decomposition of a Laguerre-Gaussian mode into the same, rephased, set.

of appropriate mode converters are planned to be published elsewhere [16].

IV. THE MEASUREMENT OF MECHANICAL TORQUE

Spin angular momentum in the form of circularly polarized light is both produced and detected by birefringence. In the Beth experiment [1] the torque per unit volume is given by $\mathbf{P}\times\mathbf{E}$. As $\mathbf{D}=\kappa\mathbf{E}=\epsilon_0\mathbf{E}+\mathbf{P}$ this may be written as

$$\frac{d\mathbf{J}}{dt}=(\mathbf{D}\times\mathbf{E}) . \quad (16)$$

The torque on the birefringent material arises because κ is a tensor; that is, $\mathbf{E}$ is not parallel to $\mathbf{P}$; in fact $\kappa_{xx}=n_x^2$ and $\kappa_{yy}=n_y^2$. The same "tensorial" attributes are required in the measurement of orbital angular momentum.

We have already indicated that a cylindrical lens can introduce phase differences between the Hermite-Gaussian components. Various combinations of cylindrical, or tilted spherical lenses, may be used to yield "retardation plates" of arbitrary thickness for orbital angular momentum. Just as appropriate phasing of the x and y components of the electric field in any light beam is capable of transforming linearly polarized light into circularly, or elliptically, polarized light so the rearrangement of phase introduced by an astigmatic optical element can destroy or create orbital angular momentum. Any measurement of mechanical torque must rely on such a change.

The measurement of a mechanical torque arising from orbital angular momentum should, thus, closely resemble that of spin. A suspended combination of two cylindrical lenses may be made to transform a Laguerre-Gaussian mode to the one with opposite orbital angular momentum, employing the anisotropy of the Gouy phase in the space in between the two lenses [see Fig. 1(b)]. The resultant torque which is predictable in terms of intensity and the value of l will then be compared with that measured from the rotation of the fiber suspension. An appropriate astigmatic birefringent element, or a combination of astigmatic and birefringent parts, would be capable of responding to both orbital and spin contributions simultaneously. This would allow the observation of total angular momentum J. However, a purely astigmatic or birefringent detector will respond only to the uncoupled values of L and S, respectively. It appears that there is no sense in which the spin and orbital contributions are intrinsically coupled except through the mechanical torque applied to such as astigmatic and birefringent device. It may be that perfect decoupling of **L** and **S** is limited to those cases where the paraxial approximation is valid; the decoupling of polarization state and scalar field amplitude is a feature of this approximation.

V. CONCLUSIONS

We have demonstrated that a Laguerre-Gaussian laser mode has a well-defined orbital angular momentum equal to $l\hbar$ per photon, with l the azimuthal mode index. We have outlined how such orbital angular momentum may be removed from the mode and converted into a mechanical torque. We have shown that such a transformation may be achieved by the use of astigmatic optical elements, which may also be used to produce Laguerre-Gaussian modes from the more commonly occurring Hermite-Gaussian modes.

It would appear that all light beams which possess field gradients, and which are not therefore plane waves, will possess a measure of orbital angular momentum. Indeed a badly phased transformation between transverse laser amplitude distributions will in general lead to ill-defined orbital angular momentum. For this reason it is important that stable, nondegenerate, propagating Laguerre-Gaussian polynomial modes are created and entirely transformed. A meaningful measurement of orbital angular momentum will not otherwise result. Although Laguerre-Gaussian modes may be created within the laser [14,19] they are usually degenerate, either simultaneously possessing $\exp(\pm il\phi)$ components or randomly fluctuating between them in time, and so have zero average orbital angular momentum. The reason why Hermite-Gaussian modes are the dominant ones in lasers is because of the presence of intracavity astigmatism [17].

We have been concerned in this paper with the orbital, or the total, angular momentum of the whole light beam. The ratio of the flux of total angular momentum to that of energy only gives $(l+\sigma_z)/\omega$ when we integrate over the whole beam profile. We have not investigated the consequences of the local ratio l/ω inside a linearly polarized Laguerre-Gaussian beam. It is in this regime that interactions with either atoms or small particles will take place; what the effect of orbital momentum of the light will be in this case remains to be investigated.

Other areas of study currently exist which invoke inhomogeneous, transverse, laser fields with field gradients. These include laser cooling, the manipulation of atoms and, expressly, mechanical effects in the interaction of neutral atoms and cylindrically symmetric modes: see [4,18,19]. It is interesting to speculate on whether such investigations might be changed by the existence of orbital angular momentum. It would also be interesting to know the extent to which the language of angular momentum might be a meaningful alternative to that already used in these areas.

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