Topological constraints on magnetostatic traps

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We theoretically investigate properties of magnetostatic traps for cold atoms that are subject to externally applied uniform fields. We show that Ioffe-Pritchard traps and other stationary points of B are confined to a two-dimensional curved surface, or manifold $\mathcal{M}$, defined by $\det(\partial B_i/\partial x_j)=0$. We describe how stationary points can be moved over the manifold by applying external uniform fields. The manifold also plays an important role in the behavior of points of zero field. Field zeroes occur in two distinct types, in separate regions of space divided by the manifold. Pairs of zeroes of opposite type can be created or annihilated on the manifold. Finally, we give examples of the manifold for cases of practical interest.

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I. INTRODUCTION

Magnetic trapping of neutral particles has first been achieved for cold neutrons [1] and has since become a widely used tool in cold-atom physics [2]. More recently, the flexibility to design complicated magnetic trapping potentials has been boosted tremendously by the development of atom chips [3–5]. The field sources are defined by microfabrication on a planar substrate, taking the form of either current-carrying wires or patterns in a permanent magnetic film [6–8].

Magnetostatic traps are defined by magnetic field minima. In this paper, we investigate the occurrence of field minima from a general perspective [9]. We introduce a conceptual tool, a curved surface, or manifold $\mathcal{M}$, to which all stationary points of B (nonzero minima and saddle points) are confined. We derive expressions for the movement of stationary points over this manifold in response to a change of an external uniform control field. We also show that the same manifold plays an important role in the creation and merging of field zeroes.

The typical application that we have in mind is a situation where a magnetic field configuration is fixed by, e.g., permanent magnets, and control of the field is limited to the application of uniform external fields. This situation occurs for instance in atom chip experiments [6–8], where field gradients can become very large. Control of the movement of Ioffe-Pritchard (IP) traps [10,11] is of importance in loading procedures and in experiments that require dynamical splitting or movement of atomic clouds. During loading, for instance, it is important to avoid regions of zero field because this will lead to losses due to Majorana spin flips to untrapped states. It is also important to avoid unwanted splitting of the trap during the transport and compression of the cloud to the final trap. Furthermore, quantum information processing applications on an atom chip may require the movement of qubits to regions where they can be "read out" or manipulated. In this case, it is also of importance to keep track of the individual phase evolutions of the atoms, i.e., to control the trapping parameters during transport.

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Although here we investigate magnetic traps, most of our conclusions also apply to traps based on electrostatic fields insofar as they rely on the field being rotation and divergence free. Electrostatic traps can be used to trap molecules with an electric dipole moment [12–14].

This paper is structured as follows. After introducing our notation in Sec. II, in Sec. III we derive the expression for the manifold to which stationary points must be confined. We also show how to create an IP trap in a given point on this manifold. In Sec. IV, we derive an expression for the movement of stationary points along the manifold, under the influence of an external uniform field. In Sec. V, we investigate the relationship between the manifold and points of zero field. We show how field zeroes can be moved and how pairs of zeroes can be created and annihilated on the manifold. Finally, in Sec. VI, we investigate the shape of the manifold for some cases of experimental interest.

II. NOTATION

Magnetic traps are usually operated in the regime where moving particles experience a magnetic field that varies slowly compared to the Larmor spin precession frequency. The spin component parallel to the field is then conserved due to adiabatic following of the local direction of the magnetic field. The effective potential is proportional to the modulus of the field: $B(\mathbf{r})=|\mathbf{B}(\mathbf{r})|$. We are interested in stationary points and trapping frequencies in this potential. For this purpose it is equivalent to use $B^2(\mathbf{r})$, since a minimum or saddle point of B is also a minimum or saddle point of B^2. We shall use $B^2(\mathbf{r})$ for convenience and define

$$U(\mathbf{r}) = B^2(\mathbf{r}). \tag{1}$$

Throughout this paper, we adopt the convention that summation over repeated indices is implied, e.g., $B_iB_i \equiv B_x^2+B_y^2+B_z^2=B^2$. Points where U is stationary are defined by $\partial_i U \equiv \partial U/\partial x_i=0$ for all $i=1,2,3$. In order to decide whether a stationary point is a local minimum or a saddle point, we will need also the second derivatives. Therefore, let us expand $\mathbf{B}(\mathbf{r})$ to second order in the relative coordinates x_i around some point of interest,

$$B_i = u_i + g_{ij}x_j + \tfrac{1}{2}c_{ijk}x_jx_k + \text{(higher-order terms)} \tag{2}$$

where $g_{ij}=\partial_j B_i$ is a tensor describing the gradient of the vector field $\mathbf{B}(\mathbf{r})$ and $c_{ijk}=c_{ikj}=\partial_j\partial_k B_i$ is a curvature tensor.

Using Maxwell's equations for stationary fields in vacuum, we can impose some restrictions on the tensor components g_{ij} and c_{ijk}. From the conditions div $\mathbf{B}=0$ and curl $\mathbf{B}=\mathbf{0}$ for stationary fields in empty space, we see that the gradient tensor must be both traceless and symmetric,

$$g_{ii} = 0 \tag{3}$$

$$g_{ij} = g_{ji}. \tag{4}$$

This leaves five independent parameters for g_{ij}, which can be interpreted as follows. In the coordinate frame where g_{ij} is diagonal, two independent gradients can be chosen. Three angles are needed to specify the orientation of the coordinate frame.

Similarly, the curvature tensor c_{ijk} must be fully symmetric under permutation of indices and all its partial traces must vanish

$$c_{iij} = c_{iji} = c_{jii} = 0 \tag{5}$$

$$c_{ijk} = c_{ikj} = c_{kji}. \tag{6}$$

This leaves seven independent parameters for c_{ijk}.

Throughout the paper, we adopt the convention that eigenvalues of a tensor are written in capital letters and eigenvectors are written in capital boldface letters. Thus $(\mathbf{G_1},\mathbf{G_2},\mathbf{G_3})$ are the eigenvectors of g_{ij} and (G_1,G_2,G_3) are the corresponding eigenvalues.

III. STATIONARY POINTS

A. The manifold det$(g_{ij})=0$

In order to find the stationary points of $U=B_iB_i$, we substitute the expansion of Eq. (2) and collect terms up to second order in the coordinates,

$$U = u_iu_i + 2u_ig_{ij}x_j + (u_ic_{ijk} + g_{ij}g_{ik})x_jx_k + \text{(higher-order terms)} \tag{7}$$

For a stationary point in $x_i=0$, we set $\partial_p U=2u_ig_{ip}=0$. A trivial solution is that the field is zero, $u_i=0$. Note, however, that this is generally only a stationary point of U, not of B. For stationary points at nonzero field, such as the minimum in a Ioffe-Pritchard trap, we must require that g_{ip} has an eigenvalue zero, i.e.,

$$\det(g_{ij}) = 0. \tag{8}$$

Furthermore, for a stationary point to occur, the field u_i must be parallel to the eigenvector of g_{ip} with eigenvalue zero. In the case of a IP trap, this direction is usually called the axial direction. The above condition, Eq. (8), expresses the fact that a IP trap requires a point where the magnetic field locally looks like a cylindrical quadrupole field, and that the axis of the quadrupole is the trap axis.

Since the gradient is a function of the spatial coordinates, $g_{ij}=g_{ij}(\mathbf{r})$, the condition of Eq. (8) defines a two-dimensional curved surface that we shall call the manifold $\mathcal{M}$. The points on the manifold are those points in space where a stationary point for B can be created by choosing u_i along the zero eigenvector of g_{ij}. Some examples of such manifolds in situations of practical interest are shown in Sec. VI.

B. Absence of field maxima in empty space

If the condition $u_ig_{ip}=0$ is fulfilled, Eq. (7) is simplified (up to second order) to

$$U = u_iu_i + t_{jk}x_jx_k, \tag{9}$$

where we defined

$$t_{jk} = u_ic_{ijk} + g_{ij}g_{ik}. \tag{10}$$

Note that the tensor t_{jk} is symmetric. Its trace is a sum of squares and, therefore, always non-negative

$$t_{kk} = g_{ik}g_{ik} \geq 0, \tag{11}$$

where we used the vanishing of partial traces, Eq. (5). We note that this gives the well-known result that U cannot have a local maximum in empty space, since a maximum would imply three negative eigenvalues of t and thus a negative trace. The absence of maxima in empty space is known as Wing's theorem [15] and is here retrieved by a different route. Note that the non-negative trace relies on $\mathbf{B}$ being irrotational and divergence free. Therefore, the same conclusion holds for electrostatic fields in vacuum. On the other hand, time-dependent $\mathbf{E}$ and $\mathbf{B}$ fields are not irrotational and in fact do allow for a maximum of field magnitude in empty space [16,17].

C. Trapping frequencies

The potential for an atom in a magnetic field is given by

$$V = m_Fg_F\mu_B\sqrt{U}, \tag{12}$$

with m_F the magnetic quantum number, g_F the Landé factor, and μ_B the Bohr magneton. If we make a harmonic approximation around the potential minimum, we find that the trap frequencies are given by

$$\omega_n = \sqrt{\frac{m_Fg_F\mu_BT_n}{mu}}, \tag{13}$$

with $u=\sqrt{u_iu_i}$, with T_n the eigenvalues of t_{jk} and m the mass of the atom. Here, we assume that the eigenvalues $T_n \geq 0$.

Combining this expression with Eq. (11), we find

$$\omega_1^2 + \omega_2^2 + \omega_3^2 \propto \frac{1}{u}g_{ik}g_{ik}. \tag{14}$$

Thus, remarkably, we find that this combination of trap frequencies is independent of the curvature c_{ijk} and depends only on the gradient g_{ij} and the uniform field u_i.

D. Where can a IP trap be created?

Having established that stationary points, including IP traps, can only be found on the manifold $\mathcal{M}$, we now address the question whether a IP trap can be created in any arbitrary point on the manifold $\mathcal{M}$. For convenience, we choose a coordinate frame that diagonalizes g_{ij}, such that the zero eigenvector lies along coordinate direction $\hat{e}_3$. The gradient tensor then takes a very simple form, with $g_{11}=-g_{22}=a$ as the only nonzero components. Since u_i must be chosen along the zero eigenvector, we write $\mathbf{u}=u_3\,\hat{e}_3$. We can then write the tensor t_{jk} as

$$(t_{jk}) = \begin{pmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + u_3 c_{3jk}. \tag{15}$$

Thus, t_{jk} depends on a single parameter u_3 that is a multiplier of the symmetric and traceless tensor c_{3jk}.

We can easily see the qualitative behavior of the eigenvalues of t_{jk} as a function of u_3. Obviously, for $u_3=0$ the eigenvalues are $(T_1,T_2,T_3)=(a^2,a^2,0)$. For small values of u_3, we can use perturbation theory to obtain the lowest eigenvalue to first order, yielding

$$T_3 \approx u_3 c_{333}. \tag{16}$$

This shows that T_3 can be made either positive or negative by choosing the sign of u_3. For small values of u_3, the other two eigenvalues will remain positive. This means that for small u_3 the stationary point will be either a IP trap or a saddle point with Morse index of 1, Morse index being the number of negative eigenvalues. Note that we can only make a IP trap if $c_{333}\neq 0$. In fact, for the case that $c_{333}=0$ a counterexample is easily found.

For large enough positive or negative values of u_3, the term c_{3jk} will become the dominant term. Since c_{3jk} is traceless, it has signature $(+,+,-)$ or $(+,-,-)$. Therefore, for large values of u_3 we will always have a saddle point. The sign of u_3 will determine whether the Morse index is 1 or 2.

Since for small u_3 the eigenvalues of t_{jk} are given by $(a^2+u_3c_{113}, a^2+u_3c_{223}, u_3c_{333})$, we can tune the two nonaxial trap frequencies with u_3 but the axial frequency is fixed by c_{333} [Eq. (13)].

IV. MOVING STATIONARY POINTS

We now investigate how stationary points can be moved over the manifold by changing the uniform field u_i. We consider the situation that the spatial dependence of the magnetic field is defined, e.g., by a configuration of permanent magnets. We can influence the magnetic field pattern by applying a uniform external field. In terms of the above quantities, g_{ij} and c_{ijk} are fixed, u_i is our control parameter.

A. Moving Ioffe-Pritchard traps

The movement of IP traps, which must clearly be constrained to the manifold, is important in applications that require trapped atoms to move, such as beam splitters and conveyor belts [18]. We now calculate a displacement tensor $d_{jq}\equiv \partial x_j/\partial u_q$ in a stationary point $x_j=0$ on the manifold. This tensor describes how the position of a stationary point moves when the uniform field is changed. To calculate it, we use the condition for a stationary point, $\partial_p U=0$, with U as in Eq. (7),

$$\partial_p U = 2u_i g_{ip} + 2(u_i c_{ijp} + g_{ij} g_{ip}) x_j = 0. \tag{17}$$

Taking the derivative with respect to u_q and setting $x_j=0$, we solve for d_{jq} and find

$$d_{jq} \equiv \frac{\partial x_j}{\partial u_q} = -(t^{-1})_{jp} g_{pq} \tag{18}$$

where $(t^{-1})_{jp}$ denotes the inverse tensor of t_{jp}. In the basis where g_{pq} is diagonal, we see that $d_{j3}=0$, i.e., a small field in the axial direction of a IP trap will not displace it. Since this means that the eigenvalue D_3 is zero, d_{jq} is singular. Therefore, d_{jq} is a mapping of a three-dimensional vector u_q onto a two-dimensional space, namely, the tangential plane to the manifold. It is spanned by the two eigenvectors $(\mathbf{D_1},\mathbf{D_2})$ corresponding to the nonzero eigenvalues. The vector $\mathbf{D_1}\times\mathbf{D_2}$ is normal to the manifold and is found to be proportional to $(c_{133},c_{233},c_{333})$.

Note that during the movement the radial trap frequencies can be controlled using a bias field in the axial direction of the IP trap [Eq. (13)].

V. THE MANIFOLD AND FIELD ZEROES

The manifold $\mathcal{M}$ is not only a powerful concept in the description of stationary points, it also has significance in the occurrence and movement of field zeroes. Such points of zero field are also minima of B and can thus serve as atom traps. However, these so-called quadrupole traps (QT) suffer from higher trap loss rates due to Majorana spin flips near the region of zero field. The movement of QTs is important in loading procedures, i.e., to transport atoms into the final IP trap. In this section, we investigate how field zeroes can be moved and what happens when they approach the manifold.

The manifold is the boundary between two regions of space V^+, V^-, where $\det(g_{ij})>0$ and <0, respectively. Since g_{ij} is traceless, and $\det(g_{ij})$ is the product of the eigenvalues, g_{ij} must have two negative and one positive eigenvalues in V^+. Similarly, in V^- it has one negative and two positive eigenvalues. It is impossible to move between the two regions V^+, V^- without one of the eigenvalues going through zero. This means that the manifold imposes restrictions on the movement of field minima that have a field zero.

A. Moving quadrupole traps

Since a quadrupole trap is a zero of the magnetic field, it is straightforward to give a prescription for how to move it by applying an external field. We call the stationary field $\mathbf{B}_{\rm stat}(\mathbf{r})$ and the desired trajectory $\mathbf{r}(t)$. In order to move the field zero along the trajectory, all we need to do is to use the external field $\mathbf{B}_{\rm ext}(t)$ to cancel the local magnetic field,

$$\mathbf{B}_{\rm ext}(t) = -\mathbf{B}_{\rm stat}(\mathbf{r}(t)). \tag{19}$$

As with the movement of stationary points, we can also express the movement of zeroes in terms of a displacement tensor $\partial x_j/\partial u_q$, Eq. (18). For field zeroes, this tensor takes a very simple form

$$d_{jq} = \frac{\partial x_j}{\partial u_q} = -(g^{-1})_{jq}. \tag{20}$$

Thus, generally speaking, field zeroes do not disappear when the uniform field u_j is changed; they simply move through three-dimensional space.

The situation is different when a field zero approaches the manifold between the regions V^+, V^-. On the manifold, $\det(g_{ij})=0$ so that d_{jq} does not exist. In fact, since one of the eigenvalues of g_{ij} vanishes, the magnetic confinement along one direction of the magnetic QT vanishes.

B. Pairs of zero field

Having established how field zeroes (QTs) and IP traps move, the natural question arises as to what happens when a field zero approaches the manifold. We have just noted that field zeroes can be broadly categorized into two types, according to whether the signature of the gradient is $(+,+,-)$ or $(+,-,-)$. It turns out that the approach of the manifold by a field zero is accompanied by the approach by another field zero, of the opposite type, from the other side of the manifold.

If a zero is sufficiently close to $\mathcal{M}$, we first assume that we can choose a point $x_i=0$ on $\mathcal{M}$ such that the position of the zero $x_i=\xi_i$ is in the direction $\mathbf{G_3}$ of the local quadrupole axis. Thus, we have $\xi_1=\xi_2=0$ and $\xi_3\neq 0$. Furthermore, $g_{i3}=0$ on $\mathcal{M}$ so that the requirement of zero field in ξ_i simplifies to

$$u_i + c_{i33}\xi_3^2 = 0. \tag{21}$$

This shows immediately that the replacement $\xi_3 \to -\xi_3$ yields another zero. The two zeroes are symmetrically placed around the point on $\mathcal{M}$, in the direction of the $\mathbf{G_3}$ axis. The zeroes must be of opposite type, since they are on opposite sides of $\mathcal{M}$. Finally, we note that the solution for ξ_3 of the above Eq. (21), namely, $\xi_3=\pm\sqrt{-u_i/c_{i33}}$, implies that the local field direction u_i must be proportional to c_{i33}. This is just the normal vector to the manifold as mentioned above, so the local field u_i is normal to the manifold $\mathcal{M}$.

The choice of a point $x_i=0$ on $\mathcal{M}$ such that $\xi_i \| \mathbf{G_3}$ is possible unless $\mathbf{G_3}$ is parallel to the manifold. In this special case, where $c_{333}=0$, a zero arriving on the manifold can transform into a line of zero field, which lies entirely on the manifold. Such lines of zero require a treatment that goes beyond the second-order field expansion. Evidence from numerical examples and some highly symmetric analytical examples suggests that such lines are closed loops on the manifold. Small perturbing fields can split this loop into a number of zero pairs, which can be macroscopically separated.

Finally, we can also address the question whether the merging of field zeroes must necessarily take place on $\mathcal{M}$. We assume that two field zeroes are sufficiently close so that it is sufficient to expand the field up to second order. We define local coordinates x_i around the first zero and x_i' around the other. For the field expansion, we can then write

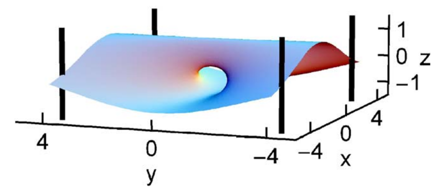

FIG. 1. (Color online) Manifold corresponding to the standard Ioffe-Pritchard trap, described by the field of Eq. (23) using $\epsilon=1$. The black bars indicate the orientation of the Ioffe bars.

$$g_{ij}x_j + \tfrac{1}{2}c_{ijk}x_jx_k = g'_{ij}x'_j + \tfrac{1}{2}c'_{ijk}x'_jx'_k. \tag{22}$$

Note that both u_i and u_i' are zero because we have two field zeros. We introduce the separation vector ξ_i and substitute $x_i=\xi_i+x_i'$. Equating terms of equal powers in x_i', we then obtain $(g_{ij}+g'_{ij})\xi_j=0$ and thus $\det(g_{ij}+g'_{ij})=0$. Thus, in the second-order field expansion, the midway point between the two zeroes x_j and x_j' lies on the manifold. Furthermore, the two field zeroes must be of opposite type. Apparently, in order for field zeroes of *similar* type to approach each other the leading term in the field expansion must be higher than second order.

VI. PRACTICAL IOFFE TRAPS

In this section, we consider the shape of the manifold $\det(g_{ij})=0$ in some cases of practical interest.

A. Standard Ioffe-Pritchard trap

The prototypical IP trap [10,11] consists of four long current-carrying wires ("Ioffe bars"), for creating a cylindrical quadrupole field, in combination with two pinch coils creating confinement in the axial direction. The field at the IP can be approximated by

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} + a\begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} + \frac{c}{2}\begin{pmatrix} -xz \\ -yz \\ z^2 - \frac{1}{2}(x^2+y^2) \end{pmatrix}, \tag{23}$$

where u, a, and c are the uniform field, radial gradient, and axial curvature, respectively. The manifold produced by this field is shown in Fig. 1. It is described by

$$\epsilon(x^2-y^2) + z(2z^2+y^2+x^2-2\epsilon^2) = 0, \tag{24}$$

where $\epsilon=2a/c$. Note that for $\epsilon=0$, this describes the flat surface $z=0$. The hole in the manifold has typical size ϵ. In the point $(0,0,0)$, the eigenvectors of d_{jq} point in the x, y, and z directions and, thus, the lowest order displacement of the IP under influence of an external field is in a direction prependicular to the axial direction. The shape shown is only realistic in the region where Eq. (23) is a good approximation of the field.

B. Z-wire Ioffe trap

A method routinely used in atom chip experiments for creating IP traps involves a current-carrying wire bent into a

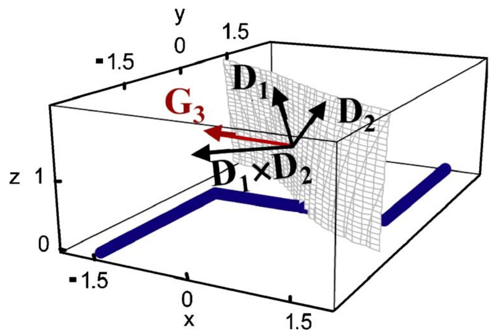

FIG. 2. (Color online) Manifold created by a Z-shaped wire. Since the manifold is given by the gradient, its shape does not depend on the bias field. The current is only a multiplier and is also of no importance for the shape. The vectors $\mathbf{D_1}$, $\mathbf{D_2}$, $\mathbf{G_3}$, and $\mathbf{D_1} \times \mathbf{D_2}$ have been drawn in the point (0,0,1). The vector $\mathbf{G_3}$ gives the axial direction of the IP, $\mathbf{D_1} \times \mathbf{D_2}$ is normal to the manifold.

Z-shape in combination with a uniform bias field [19]. In Fig. 2, the manifold for such a wire is shown together with the eigenvectors of d_{jq} and the IP axis $\mathbf{G_3}$. The vector $\mathbf{G_3}$, straight above the middle of the central wire, always lies in the xy plane. We find that the azimuthal angle φ between $\mathbf{G_3}$ and the x axis is given by

$$\varphi = -\arctan\left(\frac{\cos 2\theta \sec \theta}{2 + \cot^2 \theta}\right). \tag{25}$$

Here, $\tan \theta = 2z/s$, where s is the separation between the two wires in the y direction and z is the height above the central wire. This result is plotted in Fig. 3. For $z=0$, $\mathbf{G_3}$ points in the x direction; in the limit $z \to \infty$, it points in the y direction. For a given s, $\mathbf{G_3}$ points in the x direction if we choose $z = \frac{1}{2}s$.

C. Array of Ioffe traps

The exact shape of the manifold is of particular importance in arrays of IP traps that might be used as conveyor belts or shift registers. Such devices are promising for quantum information processing applications [20]. Atoms sitting at a lattice site that is connected to another via the manifold can be shifted there using a uniform bias field, while remaining a IP trap. Moreover, the trap frequencies along the way can be tuned using Eq. (13).

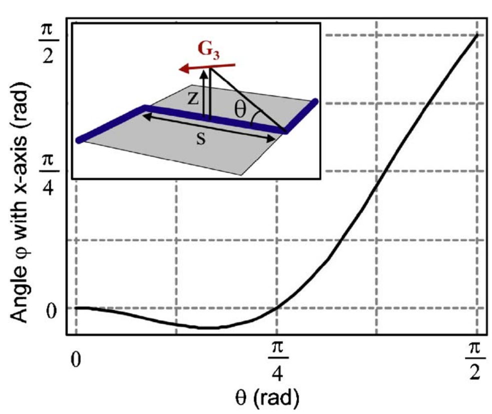

FIG. 3. (Color online) The azimuthal angle φ of $\mathbf{G_3}$ with respect to the x axis as a function of θ. Note that directly above the middle of the central wire $\mathbf{G_3}$ always lies in the xy plane. The inset shows the geometrical meaning of θ.

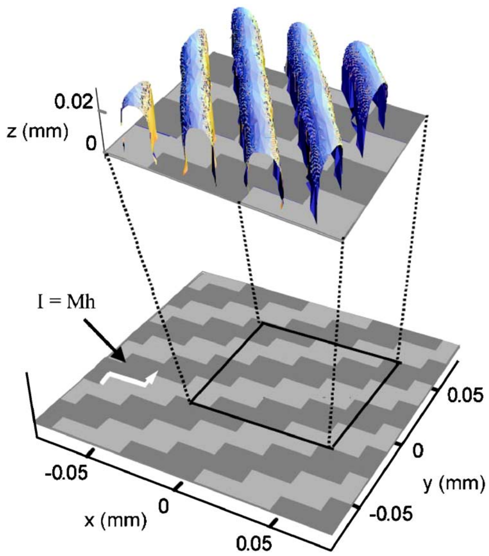

FIG. 4. (Color online) Array of magnetic material (dark regions). The magnetization is out of plane. The equivalent edge current forms Z-shapes at every lattice point. In combination with a bias field in the y direction, IP traps can be created. The shape of the manifold is shown in the inset.

To make these ideas more explicit, we discuss an array of IP traps created by permanent magnetic material in combination with a uniform bias field. In Fig. 4, the array of magnetic material is shown. Since its magnetization is out of plane, we can think of the material as having an equivalent current of magnitude Mh running around its edges, where M is the magnetization and h is the height of the material. This equivalent current forms a Z-shape at every lattice site in the array. We apply a uniform bias field in the y direction to create Ioffe traps at all lattice sites.

In this particular design, the magnetization is $M = 800$ kA/m and the height of the material is $h = 250$ nm. A bias field of 17.9 G in the y direction then produces IP traps at 10 μm from the surface with trap frequencies (21, 20, 54) kHz and a residual field of 4.5 G at the trap bottom.

We are interested in where the individual IP traps can be moved. Therefore, we draw the manifold over a region of the array containing several unit cells. As can be seen in Fig. 4, IP traps are only connected in a diagonal direction from lower right to upper left. This means that this array can be used as a shift register only in this direction. The manifold clearly does not allow shifting the IP traps in the perpendicular direction. One could try to move the traps in the perpendicular direction as field zeroes (QT). However, we find that this leads to a sequence of splitting and recombination of zeroes, every time the manifold is crossed. Thus, in spite of its appearance, this is a one-dimensional shift register.

Finally, for the sake of completeness, we show in Fig. 5 how the axial direction of the Ioffe traps varies over the manifold.

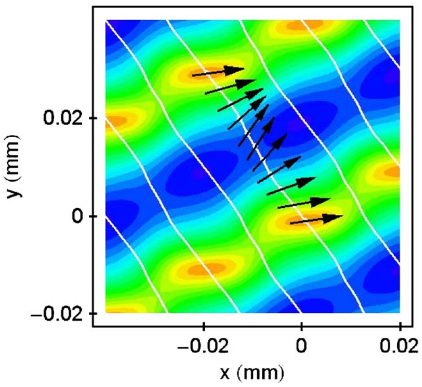

FIG. 5. (Color online) Two-dimensional cross section showing the magnetic potential and the manifold at $z=10$ μm. The IP traps can be transferred over the manifold. For one IP, the route to the neighboring lattice site is shown. The arrows indicate how the trap axis varies along the way.

VII. CONCLUSION

In conclusion, we have shown that Ioffe Pritchard traps as well as other stationary points of $B(\mathbf{r})$ are confined to a two-dimensional curved manifold $\mathcal{M}$ defined by $\det(g_{ij}) \equiv \det(\partial B_i/\partial x_j)=0$. Furthermore, in any point of $\mathcal{M}$ where the local quadrupole axis is not parallel to $\mathcal{M}$ a IP trap or other stationary point can be created by choosing the magnetic field parallel to this axis, i.e., the eigenvector of g_{ij} corresponding to eigenvalue zero. We have given an expression for the movement of stationary points over the manifold, in response to a change of an external uniform control field.

We have shown that the same manifold plays an important role in the behavior of field zeroes and separates regions of space where field zeroes are of an opposite type. These field zeros can be moved anywhere in their respective region of space but can only disappear on the manifold. In this annihilation, two field zeros of opposite type merge and form a Ioffe Pritchard trap. Similarly, an IP on the manifold can be made to split into two field zeroes of opposite type, on opposite sides of the manifold.

The manifold is a conceptual tool in designing magnetostatic or electrostatic potentials. Understanding how IP traps can be moved over this manifold is of great use in experiments that require some level of dynamics, such as shift registers for trapped atoms or double well experiments. The shape of the manifold is important in loading protocols for permanent magnetic atom chips, defining possible routes for transfering Ioffe-Pritchard traps.

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