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A Classical Analogy of Entanglement

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A classical analogy of quantum mechanical entanglement is presented, using classical light beams. The analogy can be pushed a long way, only to reach its limits when we try to represent multiparticle, or nonlocal, entanglement. This demonstrates that the latter is of exclusive quantum nature. On the other hand, the entanglement of different degrees of freedom of the same particle might be considered classical. The classical analog cannot replace Einstein–Podolsky–Rosen type experiments, nor can it be used to build a quantum computer. Nevertheless, it does provide a reliable guide to the intuition and a tool for visualizing abstract concepts in low-dimensional Hilbert spaces.

1. INTRODUCTION

Entanglement is one of the central concepts in quantum mechanics. It plays a role in many of those effects where quantum physics is spectacularly different from classical physics. Famous examples are Schrödinger's cat *Gedankenexperiment*,[1–3] the Einstein–Podolsky–Rosen paradox,[4] Bell's inequality,[5, 6] and more recently quantum cryptography[7] and quantum computation.[8, 9] Not surprisingly, entanglement has come to be regarded as something typically quantum mechanical.

However, we will argue in this paper that some examples of entanglement found in the literature are in fact not of quantum mechanical nature. In order to demonstrate this point, a classical analog of entanglement is constructed using classical light beams. This classical analog displays most of the features that one would typically associate with entanglement. We explore how far the analogy can be pushed before it breaks down.

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It turns out that the analogy stops whenever nonlocality comes into play. This is a consequence of a profound difference that exists between two types of entanglement: (i) entanglement between separate particles and (ii) entanglement between different properties of a single particle.

A prototypical example of type (i) is the well-known singlet state of two spins-$\frac{1}{2}$, $|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2$. An example of type (ii) is provided by the entangled internal (electronic, $|\uparrow\rangle$, $|\downarrow\rangle$) and external (motional, $|x_1\rangle$, $|x_2\rangle$) states of a single ion, $|x_1\rangle |\uparrow\rangle + |x_2\rangle |\downarrow\rangle$.[2] These two prototypes have in common that the state vector cannot be written as the product of two kets. In this paper we take this nonfactorizability as synonymous with entanglement. It has been argued that only type (i) should properly be called "entanglement."[10] However, this would not appreciate the remarkable similarities between the two types (the minus sign is not significant).

The classical analogy discussed in this paper suggests that single-particle entanglement, type (ii), might also be called "classical entanglement." The other, type (i) or multi-particle entanglement, could then be called "nonlocal entanglement." The latter appears to be exclusively quantum mechanical in nature and it seems unlikely that a classical analog can be given. A proof of this suspicion would be a worthwhile extension of this paper.

The paper is organized as follows In Sec. 2 a classical Hilbert space is introduced mimicking that of a quantum mechanical two-state system, or spin-$\frac{1}{2}$. In Sec. 3 these concepts are extended to construct the product of two such spaces. This will then allow the identification of entangled states. The limits of this analogy are explored in Sec. 4. Conclusions are drawn in Sec. 5.

2. CLASSICAL HILBERT SPACES AND "CEBITS"

2.1. The Hilbert Space of Jones' Polarization Vectors

It is well known that a formal equivalence exists between the quantum mechanical state vector of a spin-$\frac{1}{2}$ and the polarization vector of a classical electromagnetic (light) wave.[11] This becomes obvious when both are written as a two-component complex vector or spinor. Alternatively, one can represent this spinor as a real three-dimensional (unit) vector, commonly known as "Bloch vector" in the case of the spin-$\frac{1}{2}$. For polarization vectors the same procedure leads to real vectors forming the so-called "Poincaré sphere." In this section we elaborate on this formal equivalence and introduce a convenient notation, reminiscent of Dirac's famous bra-ket notation for quantum mechanics.

Let us consider a classical, monochromatic light beam propagating in the $+z$-direction, with electric field $\mathbf{E}(z, t) = \mathbf{E}_0 \exp[i(kz - \omega t)] + \text{c.c.}$, where c.c. stands for complex conjugate. The complex polarization vector $\mathbf{E}_0$ has two transverse components, which we label by h (horizontal) and v (vertical): $\mathbf{E}_0 = \binom{\varepsilon_h}{\varepsilon_v}$. This vector is sometimes called the Jones vector.[12] In this paper we will also call it a "c-spin" (where "c" stands for "classical") or "cebit," the latter in analogy with the "qubit" in discussions on quantum computation.[13] We will use the terms cebit or c-spin in a more general meaning, not restricted to the special case of optical polarization. In the next section we will combine also other pairs of electric field amplitudes into a cebit. This will be necessary once we need to represent more than one cebit. Where necessary, we will then use a prefix to specify which cebit we mean, e.g., "polarization-cebit." The (polarization-) cebits build a Hilbert space, with a Hermite product that has the meaning of a light intensity.

Let us now introduce a convenient notation for the cebits, similar to the bracket notation. (Note that all we are doing is introducing notation, the light beam is still assumed classical.) We write the above column vector as $|\chi)$ and call it a "thesis" vector, and write its Hermitian conjugate as $(\chi|$, called "parent" vector,

$$(\chi| = (\varepsilon_h^*, \varepsilon_v^*), \qquad |\chi) = \begin{pmatrix} \varepsilon_h \\ \varepsilon_v \end{pmatrix} \tag{1}$$

The Hermite product, the "parentheses" $(\chi\,|\,\chi) = |\varepsilon_h|^2 + |\varepsilon_v|^2$, is proportional to the intensity measured by a polarization-insensitive photodetector. We are not interested in the overall intensity of the light beam and will assume from here on that the theses are normalized, $(\chi\,|\,\chi) = 1$.

Measurements on the state of polarization can be made by sending the light through a polarizing beam splitter and placing photodetectors at its output ports; see Fig. 1. Let us assume that the ε_h component is transmitted, the ε_v component is reflected, and label the corresponding output ports by "+" and "−," respectively. The intensities $I_\pm$ measured at the beam splitter outputs can be written in a form reminiscent of an expectation value, $I_+ = (\chi|\,P_+\,|\chi) \equiv (P_+) = |\varepsilon_h|^2$, where $P_\pm$ are projection operators,

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

The projectors $P_\pm$ can also be expressed in the Pauli spin matrices, $P_\pm = \frac{1}{2}(I_2 \pm \sigma_z)$, where I_2 is the 2×2 unit matrix and σ_z is one of the Pauli spin matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

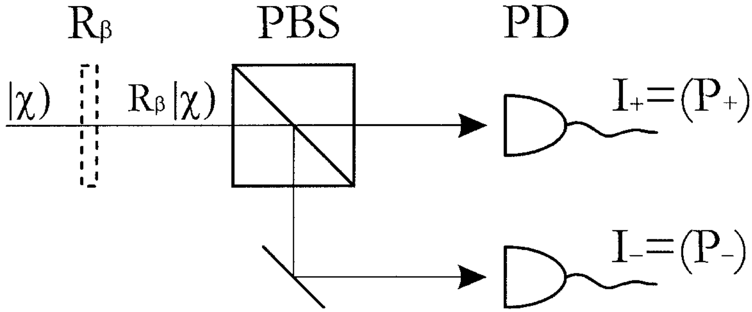

Fig. 1. Measurement of a polarization cebit $|\chi)$ using a polarizing beam splitter PBS and photodetectors PD. The photocurrents $I_\pm$ are "expectation values" of projection operators $(P_\pm)$. Measurements in a different basis can be performed using a polarization rotator R_β.

The measured intensities can thus also be written as

$$I_\pm = \tfrac{1}{2}[1 \pm (\sigma_z)] \tag{4}$$

Note that the coordinates in c-spin space should not be confused with the coordinates in real space, $\{x, y, z\}$.

The polarizing beam splitter is thus the analog of a Stern–Gerlach magnet separating the spin z-components. For example, horizontal polarization yields $I_+ = 1$, $I_- = 0$, corresponding with c-spin "up;" vertical polarization is c-spin "down." For circular polarization the h and v components have equal amplitudes, $\varepsilon_h = \pm i\varepsilon_v$, so that $I_\pm = \frac{1}{2}$ and the expectation value of c-spin component z vanishes, $(\sigma_z) = I_+ - I_- = 0$.

Other spin components can be measured by preceding the beam splitter by an optical component that changes the polarization, described by a unitary operation, $|\chi') = U|\chi)$. For example, a rotation of the polarization by an angle β is described by a rotation matrix, $U = R_\beta$,

$$R_\beta = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \tag{5}$$

For the measured intensities we now get the expectation value (σ_z) in a transformed basis:

$$I_\pm = \tfrac{1}{2}[1 \pm (\chi'|\sigma_z|\chi')] = \tfrac{1}{2}[1 \pm (\chi|\sigma_\beta|\chi)] \tag{6}$$

where

$$\sigma_\beta = R_\beta^\dagger \sigma_z R_\beta = [\cos 2\beta]\,\sigma_z + [\sin 2\beta]\,\sigma_x \tag{7}$$

The c-spin is thus measured in a rotated basis. It can be measured in any other desired basis by performing the corresponding unitary operation U, for example using a combination of quarter- and half-wave plates.

Having established the "parentheses" notation for a classical Hilbert space of cebits, we now proceed toward a description of entanglement of *two* such cebits. At this point it is a tempting mistake to simply add a second light beam and identify its polarization as cebit No. 2. This would yield a Hilbert space which is the direct *sum* of the individual spaces. However, in quantum mechanics the Hilbert space of two spins is the *product* of the individual spaces. Before constructing this product space with classical light, let us first introduce an alternative to the Hilbert space of Jones vectors.

2.2. The Hilbert Space of Position Cebits

As an alternative to the two-dimensional Hilbert space of Jones vectors or polarization cebits, let us consider two spatially separated light beams, both of which can be described by a single complex amplitude. This is possible, for example, if we give both beams the same polarization and use no optical components that change the polarization. We draw the two beams above each other (see Fig. 2) and write the complex amplitudes of the upper and lower beam as ε_u, ε_l. The two amplitudes are combined into a thesis $|\theta)$, which we will again take to be normalized. Since the components of $|\theta)$ are electric field amplitudes of beams that propagate in different positions, we will call $|\theta)$ a *position cebit*.

The definitions introduced for polarization cebits need only minor modifications. A measurement of c-spin component z is now performed by placing photodetectors in each beam path, the upper detector measuring

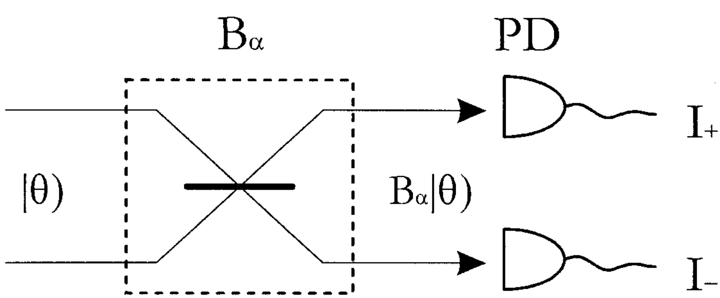

Fig. 2. Measurement of a position cebit $|\theta)$—a pair of light beams—by placing photodetectors PD in each beam path. Measurements in a different basis are performed by first interfering the two amplitudes on a beam splitter B_α.

(P_+), the lower one (P_-). So a bright upper beam together with a dark lower beam encodes for c-spin "up" (for measurement axes along z). As before, in order to measure the c-spin in a different basis, a unitary operation precedes the photodetectors, which can now be, e.g., a beam splitter. A 50/50 beam splitter that combines the two amplitudes with adjustable phase α is described by $|\theta')= B_\alpha|\theta)$, where B_α is the unitary matrix

$$B_\alpha = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\alpha} & e^{-i\alpha} \\ -e^{i\alpha} & e^{-i\alpha} \end{pmatrix} \tag{8}$$

The measured intensities are $I_\pm = \frac{1}{2}[1 \pm (\sigma_\alpha)]$, where

$$\sigma_\alpha = B_\alpha^\dagger \sigma_z B_\alpha = [\cos 2\alpha]\,\sigma_x + [\sin 2\alpha]\,\sigma_y \tag{9}$$

3. A HILBERT SPACE FOR TWO CEBITS

The construction of the product space of the polarization-cebit space and position-cebit space is now straightforward. The product space is spanned by the basis $\{|u,h),|u,v),|l,h),|l,v)\}$. The elements of the product space are complex four-vectors

$$|\Psi) = \begin{pmatrix} \varepsilon_{uh} \\ \varepsilon_{uv} \\ \varepsilon_{lh} \\ \varepsilon_{lv} \end{pmatrix} = \varepsilon_{uh}|u,h) + \varepsilon_{uv}|u,v) + \varepsilon_{lh}|l,h) + \varepsilon_{lv}|l,v) \tag{10}$$

where ε_{uv} is the vertical polarization component of the upper beam, etc.

One may think of the four-vector $|\Psi)$ as consisting of two Jones vectors placed above each other, describing the polarization of the upper and lower beam, respectively. *However, the two Jones vectors may* not *be identified, one by one, with the two cebits.* Such an identification would yield the sum space rather than the product space. The basis theses $|u,h)$, etc., should be compared to the basis kets, $|\uparrow_1,\uparrow_2\rangle$, etc. of a Hilbert space of two spins $\frac{1}{2}$ in quantum mechanics. Incidentally, the sum and product spaces both have dimension four ($2+2=2\times 2$). This is no longer true once more spins are added. The product space of N spins has dimension 2^N, whereas the sum space has dimension $2N$. Consequently, for every extra spin we have to *double* the number of classical light beams to obtain a classical analog in terms of cebits.

In accordance with the above, it is also instructive to note that the two Jones vectors are not individually normalized. Only $|\Psi)$ is normalized, $(\Psi|\Psi)=1$. Where previously the $+z$ component of the position cebit was described by a single amplitude ε_u, we now have two amplitudes $\varepsilon_{uh}, \varepsilon_{uv}$, which together carry the information. Information about either of the two cebits is contained in *the state of the entire beam pair*. We always need all four components of $|\Psi)$, even if we are only interested in one of the cebits. The correct way to read out the two cebits from the four-vector $|\Psi)$ will be clarified by a few examples.

3.1. Product States

The simplest examples are those where the two cebits are in a product state, i.e., the two cebits can be assigned independent values. Suppose, e.g., that both cebits are in a state of "*c*-spin up." Position "up" means that the lower beam must be dark, and polarization "up" means that the light is horizontally polarized. So the only nonzero amplitude in $|\Psi)$ is $\varepsilon_{uh}=1$.

More generally, if the position cebit is described by $|\theta)=\binom{\varepsilon_u}{\varepsilon_l}$ and the polarization cebit by $|\chi)=\binom{\varepsilon_h}{\varepsilon_v}$, the product state is given by

$$|\Psi)=|\theta)\otimes|\chi)=\begin{pmatrix}\varepsilon_u\\ \varepsilon_l\end{pmatrix}\otimes\begin{pmatrix}\varepsilon_h\\ \varepsilon_v\end{pmatrix}=\begin{pmatrix}\varepsilon_u\begin{pmatrix}\varepsilon_h\\ \varepsilon_v\end{pmatrix}\\ \varepsilon_l\begin{pmatrix}\varepsilon_h\\ \varepsilon_v\end{pmatrix}\end{pmatrix} \tag{11}$$

so that $\varepsilon_{uh}=\varepsilon_u\varepsilon_h$, etc. Note that each amplitude of $|\theta)$ is multiplied by $|\chi)$, so that $|\Psi)$ contains two copies of $|\chi)$. A product state is thus recognized by the fact that the upper and lower beams have the same polarization, expressed in Eq. (11) by the fact that the upper and lower Jones vectors are proportional. This is what makes $|\Psi)$ factorizable.

3.2. Entangled States

In general, the polarizations of the upper and lower beams need not be the same, so that the state of the beam pair need not be factorizable, $|\Psi)\neq|\theta)\otimes|\chi)$. Consider for example a beam pair such that the upper and lower beams carry equal intensities, but have orthogonal polarization, the upper beam being *h*-polarized and the lower beam *v*-polarized:

$$|\Psi)=\frac{1}{\sqrt{2}}[\,|u,h)+|l,v)] \tag{12}$$

(see Fig. 3).

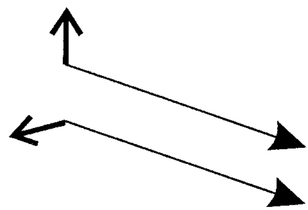

Fig. 3. Example where the polarization and position cebits are entangled: two light beams with the same intensity (amplitude) and orthogonal polarization.

If we ask for the orientation of one of the cebits, e.g., the polarization cebit, we should remember that we ask for the polarization *of the entire beam pair*. Obviously, the beam pair does not have one single polarization. If we send the beam pair through a large enough polarizer, we can find orientations where the polarizer transmits the upper beam and blocks the lower beam, or vice versa. We cannot find an orientation where both beams are blocked. Similarly, if we try to measure the position cebit and combine both beams on a beam splitter, as in Fig. 2, we will not succeed in getting all the light out of one output port of the beam splitter. The destructive interference necessary to have one dark output cannot occur, because the two beams have orthogonal polarizations.

The beam pair is thus neither in a state of pure polarization, nor in a state of pure position. The two cebits are in a state which in quantum mechanics would be called *entangled*. What we observe, e.g., using a rotatable polarizer, is that the results of measurements on the two cebits are *correlated*. If we rotate the polarizer so that it transmits h-polarization (polarization cebit "up"), we find that it transmits only the upper beam (position cebit "up"). Similarly, v-polarization correlates with the "down" position.

3.3. Joint Measurements of Two Cebits

What we have just described is in fact analogous to the joint measurement of two spins in quantum mechanics. The procedure can be generalized somewhat as follows. We place a polarizing beam splitter in each beam, upper and lower, with a photodetector at each beam splitter output; see Fig. 4. The four measured intensities, now labeled I_{++}, I_{+-}, I_{-+}, I_{--}, correspond to the projections onto the basis theses $|u, h\rangle$, $|u, v\rangle$, $|l, h\rangle$, $|l, v\rangle$. The sum of the four intensities $I_{\pm\pm}$ equals unity, assuming a normalized $|\Psi\rangle$.

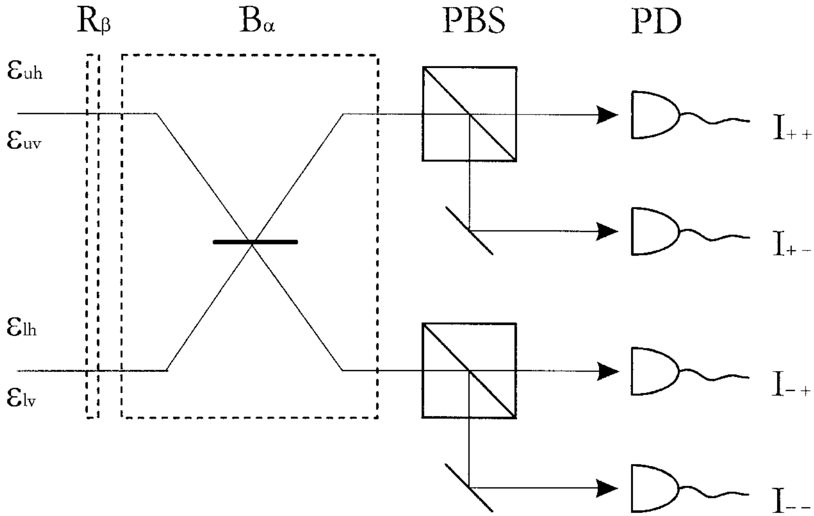

Fig. 4. Joint measurement on two cebits, position and polarization. The phase α of beam splitter B_α sets the basis for the position measurement, and the angle β of the polarization rotator R_β sets the basis for the polarization measurement; PBS is a polarizing beam splitter, and PD a photodetector.

The corresponding projection operators are the direct product of the single-cebit projection operators, e.g., $I_{++} = (\Psi|\, P_{++}\, |\Psi)$, where

$$P_{++} = P_+ \otimes P_+ = \begin{pmatrix} P_+ & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Here $A \otimes B$ is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} \tag{14}$$

where a_{ij} are the matrix elements of A so that $a_{ij}B$ are 2×2 submatrices of $A \otimes B$. In terms of the Pauli matrices, using $P_\pm = \frac{1}{2}[I_2 \pm \sigma_z]$, we get

$$I_{++} = \tfrac{1}{4}[1 + (\rho_z) + (\tau_z) + (\rho_z \tau_z)] \tag{15}$$

etc. The second and third terms are essentially single-cebit properties, with $\rho_z \equiv \sigma_z \otimes I_2$ and $\tau_z \equiv I_2 \otimes \sigma_z$. These matrices may be compared to single-spin operators for the case of two spins-$\frac{1}{2}$ in quantum mechanics. Consequently, (ρ_z) is the expectation value of a position component, irrespective of polarization. Similarly, (τ_z) measures polarization irrespective of position.

More interesting is the fourth term in Eq. (15), since $\rho_z \tau_z = \sigma_z \otimes \sigma_z$ is a joint operator of the two cebits. This is obviously the term where correlations between the cebits appear. The correlations can be obtained in "pure" form by properly adding and subtracting the four photocurrents, $(\rho_z \tau_z) = I_{++} - I_{+-} - I_{-+} + I_{--}$.

The correlations can be obtained in a different basis by combining the previously defined operations B_α and R_β; see Fig. 4. This amounts to preceding the pair of polarizing beam splitters by a unitary operation U, such that $|\Psi') = U|\Psi)$, where $U = B_\alpha \otimes R_\beta$. The resulting signals are

$$I_{++} = \tfrac{1}{4}[1 + (\rho_\alpha) + (\tau_\beta) + (\rho_\alpha \tau_\beta)] \tag{16}$$

etc., so that

$$(\rho_\alpha \tau_\beta) = I_{++} - I_{+-} - I_{-+} + I_{--} \tag{17}$$

Here ρ_α is defined as $\rho_\alpha \equiv [B_\alpha^\dagger \sigma_z B_\alpha] \otimes I_2$ and similarly $\tau_\beta \equiv I_2 \otimes [R_\beta^\dagger \sigma_z R_\beta]$, so that $\rho_\alpha \tau_\beta = [B_\alpha^\dagger \sigma_z B_\alpha] \otimes [R_\beta^\dagger \sigma_z R_\beta]$. Note that ρ_α and τ_β are commuting observables. In terms of the optical implementation of Fig. 4 this means that it makes no difference if we reverse the order of the polarization rotator R_β and the beam splitter B_α.

4. LIMITS OF THE CLASSICAL ANALOGY

4.1. Bell's Inequality and Nonlocality

The quantity $(\rho_\alpha \tau_\beta)$ is the analog of the correlation coefficient $E(\vec{a}, \vec{b})$ in the experiments by Aspect *et al.*[6] This begs the question of whether our two entangled cebits could perhaps violate a Bell-type inequality. Following Ref. 6 we could define a quantity

$$S = (\rho_\alpha \tau_\beta) - (\rho_\alpha \tau_{\beta'}) + (\rho_{\alpha'} \tau_\beta) + (\rho_{\alpha'} \tau_{\beta'}) \tag{18}$$

and investigate the inequality $|S| \leqslant 2$. One easily verifies that S can take the value $2\sqrt{2}$. For example, one chooses an input state consisting of two beams with orthogonal circular polarizations, $|\Psi) = \frac{1}{2}[|u, h) + i|u, v) + i|l, h) + |l, v)]$. The correlation coefficient in this case is given by

$$(\rho_\alpha \tau_\beta) = \sin[2(\alpha + \beta)] \tag{19}$$

The value $S = 2\sqrt{2}$ is obtained by choosing the following detection angles: $\alpha = 0$, $\alpha' = \pi/4$, $\beta = \pi/8$, $\beta' = -\pi/8$.

It seems then that the measured intensities in the classical case can be mapped one-to-one on the coincidence count rates in an EPR type experiment. However, there is one key ingredient missing in the pair of classical light beams: *nonlocality*. Bell's inequality and the experiments testing it lose their significance without nonlocality. In our classical analog, in order to measure one cebit, one always needs the entire beam pair. It is not possible to send one cebit to one place to be measured and the other cebit to another—preferably distant—place. The cebits cannot be spatially separated. Therefore, although all the expressions for the measured intensities are remarkably similar to the coincidence count rates expected in a true EPR experiment, the classical analog does not test local realism.

4.2. Quantum Logic and Nonlocality

The same issue, the fact that the two cebits cannot be spatially separated, provides a reason why it is impossible to use the cebits as qubits and thus build a quantum computer. For example, one could not build up a network of quantum logic gates, whereby two output ports of one gate are sent into the inputs of two other gates. Nonlocality thus appears to be an essential ingredient in the functioning of quantum logic.

One might perhaps hope to work around the nonlocality requirement and design a logic network such that the beams are always kept together. One would first have to represent a useful number of cebits by adding more light beams. The procedure to form the product of classical Hilbert spaces could in principle be repeated. However, each additional cebit would double the number of light beams required, so that N cebits are represented by 2^{N-1} polarized light beams. This reflects the fact that the dimensionality of the quantum mechanical Hilbert space grows exponentially with the number of spins. The quantum mechanical state vector contains 2^N complex amplitudes, which are represented in the cebit analogy by 2^N complex electric field amplitudes. The classical analogy therefore quickly becomes impractical as the number of spins increases. We would essentially be simulating a quantum computer by a classical analog computer.

In spite of all this, it is quite instructive to consider the classical analog of what is perhaps the most popular quantum gate, the so-called controlled NOT, or C-NOT, gate. This gate takes two input qubits, sometimes called the control bit and signal bit. The signal bit is flipped whenever the control bit is set (logical 1). The control bit remains "unchanged." Translated into cebit language, we could then arrange things such that the polarization cebit is flipped ($|h\rangle \leftrightarrow |v\rangle$), whenever the position cebit is "up." In other words, whenever the light is in the upper beam, change the polarization. This can simply be done by placing a half-wave plate in the upper beam; see Fig. 5.

4.3. The Single-Photon Limit

The conclusion so far is that at first sight the classical cebits seem to be very similar to true quantum mechanical qubits. Nevertheless, the cebits are different in an important way, such that they are useless for, say, quantum computers. This important difference is that in our cebits we have essentially entangled different degrees of freedom of the same particle, in this case a photon. On the other hand, in quantum computers, or EPR experiments, we need entanglement of more than one particle.

The concept of single photons may seem to contradict our starting point of a classical light beam. However, no important changes will occur if we gradually reduce the overall intensity of the beam pair. Obviously, at some point we may notice that the photocurrents are no longer continuous, but consist of a slower and slower succession of "clicks," corresponding to the detection of single photons. Each time one of the four detectors clicks, a decision has been made on the value of both the cebits that were carried by the photon. If, e.g., the "+ − " detector clicks, a photon has been recorded with position cebit "up" and polarization cebit "down" (in the chosen bases). Position and polarization are different degrees of freedom that are carried by the same photon. Since both cebits are stored in a single photon, it is impossible to measure the two cebits in different places. The photon can be recorded only once.

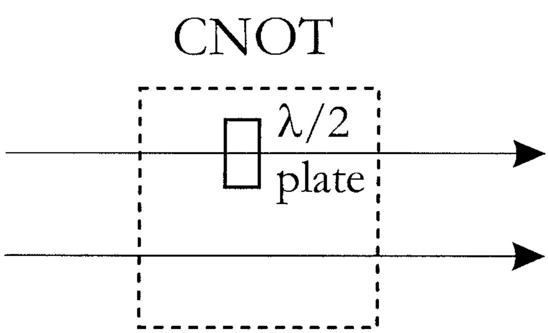

Fig. 5. A classical analog of a controlled NOT gate, using cebits. The position cebit controls the change of the polarization cebit.

5. CONCLUSION

We have seen that the classical analogy of entanglement using cebits can be pushed a long way. However, entangled cebits are not useful to build a quantum computer and neither do they violate Bell's inequality. This shows that there is a profound difference between two kinds of entanglement: entanglement of more than one particle and entanglement of different degrees of freedom of the same particle. The construct using a pair of classical light beams shows that the latter type is not truly, or exclusively, quantum mechanical in nature.

In this context it is interesting to note that in recent experiments reporting so-called Schrödinger's cat states,[2, 3] as well as in recent experimental work on quantum logic gates,[14, 16] both types of entanglement are encountered. In Refs. 2 and 16 two degrees of freedom of a single ion were entangled: its spin and its center-of-mass motion. In Refs. 3, 14, and 15 the entanglement existed between two separate quantum systems: an atom and a cavity field.

Given the profound difference between these two kinds of entanglement, it would be prudent to distinguish them also by different names. One solution would be to distinguish "nonlocal" and "classical" entanglement, for the multi-particle and single-particle case, respectively. The term "classical entanglement" seems justified even though a single particle is, strictly speaking, a quantum system. Single-photon entanglement is what remained when we took the low-intensity limit of a classical electromagnetic wave in Sec. 4. The choice to perform this thought experiment with photons rather than some other particle is irrelevant as far as the entanglement properties are concerned.

It has even been argued that the word "entanglement" should be reserved for multi-particle entanglement.[10] However, this would not appreciate the typical entanglement-like features present in the single particle case, including the example of classical light beams presented here.

The merits of the classical analogy are twofold. First, by exploring how far the analogy can be pushed, it lets one identify and appreciate those features of entanglement that are exclusively quantum mechanical. It shows in particular the fundamental role that nonlocality plays in quantum mechanics. Second, the classical light beams can be used as a tool for visualizing abstract concepts in low-dimensional Hilbert spaces. It provides a reliable guide to the intuition, provided one bears in mind the limits of the analogy.

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