

GENERALIZED PROJECTION OPERATORS IN GEOMETRIC ALGEBRA

T. A. Bouma
University of Amsterdam

(Received: October 15, 2001; Accepted: December 07, 2001)

Abstract. Given an automorphism and an anti-automorphism of a semigroup of a Geometric Algebra, then for each element of the semigroup a (generalized) projection operator exists that is defined on the entire Geometric Algebra. A single fundamental theorem holds for all (generalized) projection operators. This theorem makes previous projection operator formulas [2] equivalent to each other. The class of generalized projection operators includes the familiar subspace projection operation by implementing the automorphism ‘grade involution’ and the anti-automorphism ‘inverse’ on the semigroup of invertible versors. This class of projection operators is studied in some detail as the natural generalization of the subspace projection operators. Other generalized projection operators include projections onto *any* invertible element, or a weighted projection onto *any* element. This last projection operator even implies a possible projection operator for the zero element.

1. Introduction

A new class of generalized projection operators on a Geometric Algebra \mathcal{G} is presented. It is indexed by a nonempty set of generators, G , and two functions from G to \mathcal{G} denoted: $A \mapsto \overline{A}$ and $A \mapsto A^\dagger$ such that G is closed under the geometric product, $\overline{AB} = \overline{A} \overline{B}$, and $(AB)^\dagger = B^\dagger A^\dagger$. Each class of projection operators includes a function P_A from \mathcal{G} to \mathcal{G} for each element A of G defined by

$$P_A(X) = \frac{1}{2}(X - \overline{A}XA^\dagger). \quad (1)$$

The paper begins with the statement and proof of the Fundamental Theorem of Projection Operators and then examines projection operators for specific

choices of G , $A \mapsto \bar{A}$, and $A \mapsto A^\dagger$. The body of the paper relates the fundamental theorem to familiar projection operators and to novel projection operators. The paper ends with a summary of future work.

2. Fundamental Theorem of Projection Operators

The Fundamental Theorem of Projection Operators (FToPO) states for any projection operators P_A and P_B

$$2P_A \circ P_B = P_A + P_B - P_{AB} \quad (2)$$

Proof:

$$2P_A(P_B(X)) = \frac{1}{2}(X - \bar{B}XB^\dagger - \bar{A}(X - \bar{B}XB^\dagger)A^\dagger)$$

$$2P_A(P_B(X)) = \frac{1}{2}(X - \bar{B}XB^\dagger - \bar{A}XA^\dagger + \bar{A}(\bar{B}XB^\dagger)A^\dagger)$$

$$2P_A(P_B(X)) = \frac{1}{2}(X - \bar{B}XB^\dagger - \bar{A}XA^\dagger + (\bar{A}\bar{B})X(AB)^\dagger)$$

$$2P_A(P_B(X)) = \frac{1}{2}(X - \bar{B}XB^\dagger + X - \bar{A}XA^\dagger - X + (\bar{A}\bar{B})X(AB)^\dagger)$$

$$2P_A(P_B(X)) = P_B(X) + P_A(X) - P_{AB}(X)$$

This theorem allows projection operators to be treated directly rather than as derivative objects. This can help for applications like those in reference [1] where projection operators are used as fundamental objects of computation in Geometric Algebra.

3. Familiar Projection Operators

The most familiar projection is to let the set G be the set of invertible versions (the semigroup generated by the invertible blades), \bar{A} be the grade involution, and A^\dagger be the inverse operation.

3.1. FAMILIAR PROJECTIONS

If A is an invertible blade and x is a vector, $P_A(x)$ is the projection of x onto the subspace characterized by A .

Proof:

$$P_A(x) = \frac{1}{2}(x - \bar{A}xA^\dagger)$$

$$P_A(x) = \frac{1}{2}(xAA^\dagger - \bar{A}xA^\dagger)$$

$$P_A(x) = \frac{1}{2}(xA - \bar{A}x)A^\dagger$$

$$P_A(x) = (x \rfloor A)A^\dagger$$

So this class of projections is indeed familiar.

3.2. FAMILIAR IDENTITIES

Here are seven formulas for the vector domain portion of the projection operators P_A and P_B of two invertible blades A and B from reference [2].

1. $AB = A \wedge B \Rightarrow P_B \circ P_A = 0$
2. $AB = A \wedge B \Rightarrow P_A \circ P_B = 0$
3. $AB = A \wedge B \Rightarrow P_{AB} = P_B + P_A$
4. $AB = A \rfloor B \Rightarrow P_{AB} = P_B - P_A$
5. $AB = A \rfloor B \Rightarrow P_A \circ P_B = P_A$
6. $AB = A \rfloor B \Rightarrow P_B \circ P_A = P_A$
7. $P_A \circ P_A = P_A$

Using the FToPO it is easy to see that $1. \Rightarrow 2. \Rightarrow \dots \Rightarrow 6. \Rightarrow 7.$

1. $\Rightarrow 2.$ from the FToPO and since $P_{AB} = P_{BA}$. The latter is clear since the reverse of an invertible blade is a nonzero scalar multiple of itself.
2. $\Rightarrow 3.$ from the FToPO.

3. \Rightarrow 4. because for invertible blades A and B $AB = A \rfloor B \Rightarrow A^2 B = A(A \rfloor B) = A \wedge (A \rfloor B)$ therefore $P_B = P_{A^2 B} = P_A + P_{A \rfloor B}$.
4. \Rightarrow 5. from the FToPO.
5. \Rightarrow 6. from the FToPO and since $P_{AB} = P_{BA}$.
6. \Rightarrow 7. because for an invertible blade A , $AA = A \rfloor A$.

3.3. VERSORS AND BLADES

The full generality of the FToPO interrelates the composition of projections onto blades with projections onto versors. Projecting onto a versor is a new operation, with a simple motivation, and that motivation reproduces the familiar projection onto a blade when the versor in question is, in fact, a blade.

Formula (1) clearly shows that $P_A(x)$ is the average of two objects, namely x and $-\bar{A}xA^\dagger$. If A is a versor then the second object is always a vector. Specifically, let $A = a_1 a_2 \dots a_r$ be a versor. Now define $x_0 = x$ and inductively define $x_{i+1} = (-a_{r-i})x_i(a_{r-i}^\dagger)$ then inductively it is clear that each x_i is a vector. Expand $-\bar{A}xA^\dagger$ to get

$$\begin{aligned}
 -\bar{A}xA^\dagger &= -(-a_1)(-a_2)\dots(-a_r)x(a_r^\dagger)\dots(a_2^\dagger)(a_1^\dagger) \\
 &= -(-a_1)(-a_2)\dots(-a_r x a_r^\dagger)\dots(a_2^\dagger)(a_1^\dagger) \\
 &= -(-a_1)(-a_2)\dots(-a_r x_0 a_r^\dagger)\dots(a_2^\dagger)(a_1^\dagger) \\
 &= -(-a_1)(-a_2)\dots(x_1)\dots(a_2^\dagger)(a_1^\dagger) \\
 &= -(-a_1)(-a_2 x_{r-2} a_2^\dagger)(a_1^\dagger) \\
 &= -(-a_1)(x_{r-1})(a_1^\dagger) \\
 &= -(-a_1 x_{r-1} a_1^\dagger) \\
 &= -(x_r)
 \end{aligned}$$

As a family of versors $A(t)$ approaches the blade B , the vector $v(t) = A(t)x(A(t))^\dagger$ becomes a vector whose rejection from B remains the same as x , while the projection of $v(t)$ swings around to become diametrically opposite the projection of x . Thus, $\frac{1}{2}(x - v(t))$ smoothly becomes the projection of x onto B .

4. Novel Projection Operators

The extension of projection to versors is required to fully utilize the FToPO for blades and the interpretation of the projection of versors is truly an explanation of the older, more familiar projection onto blades. However, there are also more formal extensions of the idea of projection, of which two are explored here.

4.1. INVERSE PROJECTION OPERATION

The simplest formal extension of the familiar projection is to let the set G be the set of all invertible elements, \bar{A} be the grade involution, and A^\dagger be the inverse operation. This clearly is just an enlargement of the domain, G , of objects that can be projected onto.

If $P_W(x)$ is a vector for each vector x , then W is a versor. To see this assume that for each vector x , $P_W(x) = \frac{1}{2}(x - \bar{W}xW^\dagger)$ is a vector. Then isomorphically embed the problem into a nontrivial, nondegenerate Geometric Algebra using a LIFT as described in the appendix, and define $f(x) = \bar{W}xW^\dagger$. Clearly f is a vector-valued linear function of a vector variable (i.e. f is a linear transformation). Furthermore f is actually an orthogonal transformation of the enlarged vector space.

$$\begin{aligned}
 (f(x))^2 &= -f(x)\overline{f(x)} \\
 &= -(\bar{W}xW^\dagger)(\overline{\bar{W}xW^\dagger}) \\
 &= -(\bar{W}xW^\dagger)(W\bar{x}\bar{W}^\dagger) \\
 &= -\bar{W}x\bar{x}\bar{W}^\dagger = \bar{W}x^2\bar{W}^\dagger \\
 &= x^2(\bar{W})(\bar{W}^\dagger) = x^2(\bar{W}W^\dagger) = x^2\bar{1} \\
 &= x^2
 \end{aligned}$$

Since the enlarged space is nontrivial and nondegenerate reference [2] guarantees that there exists a nonzero versor B that performs the same transformation, i.e. there exists a nonzero versor B such that $\bar{B}xB^\dagger = \bar{W}xW^\dagger$ for each vector x . A short computation now shows that $x\rfloor(B^\sim W) = 0$ for each vector x , where B^\sim denotes the reverse of B . Note that $BB^\sim = B^\sim B$ is a scalar and fix an arbitrary vector x .

$$\begin{aligned}
\overline{B}xB^\dagger &= \overline{W}xW^\dagger \\
\overline{B^\sim}(\overline{B}xB^\dagger)(B^\sim)^\dagger &= \overline{B^\sim}(\overline{W}xW^\dagger)(B^\sim)^\dagger \\
\overline{(B^\sim B)}x(B^\sim B)^\dagger &= \overline{(B^\sim W)}x(B^\sim W)^\dagger \\
x &= \overline{(B^\sim W)}x(B^\sim W)^\dagger \\
x(B^\sim W) &= \overline{(B^\sim W)}x \\
\frac{1}{2}(x(B^\sim W) - \overline{(B^\sim W)}x) &= 0 \\
x \rfloor (B^\sim W) &= 0
\end{aligned}$$

So by Lemma (3) of the appendix, $\alpha = B^\sim W$ is a scalar. Now $W = \frac{\alpha}{B\overline{B^\sim}}B$ and since W is a scalar multiple of B it is a versor too.

The contrapositive implies that if A is an invertible element that is not a versor, then there exists a vector x such that $P_A(x)$ is not a vector. This makes the projection, P_A , onto a nonversor A a problematic operation. The computation is well-defined, but what is interpretation of a projection that sends a vector to a nonvector? The interpretation of such a transformation is an outstanding issue.

4.2. REVERSE PROJECTION OPERATION

The most general nontrivial projection operator is to let the set \mathcal{G} be the set of all elements, \bar{A} be the grade involution, and A^\dagger be the reverse.

As discussed in the previous section the interpretation of the projection onto nonversors is problematic. If A is an invertible versor then $\bar{A}xA^\dagger$ is proportional to $\bar{A}xA^{-1}$, so in the case where A is an invertible versor the two projections are not very different. The inverse projection operation is the average, while the reverse projection operation is the weighted average of x and $\bar{A}xA^\dagger$.

The class of reverse projection operators is defined for all multivectors, so there is even a projection operator for the zero element, and in fact $P_O = \frac{1}{2}$. Projection operators for noninvertible elements are actually quite interesting, for instance if an element, A , is idempotent ($A^2 = A$) then $P_A \circ P_A = \frac{1}{2}P_A$ follows from the FToPO easily.

The uses for the reverse projection operator are still unknown. When $A^\dagger = A^{-1}$ the reverse projection operator is the same as the inverse projection operator. When $A^\dagger \neq A^{-1}$ then the reverse projection operator is a weighted average that depends on the scale of A . So possibly the reverse projection operator has used as a statistical projection where the scale of A determines the certainty of the element, or possibly the actual *operation* on the elements will not be useful, but instead the algebraic properties of the projection operator itself will provide a meaningful (and useful) measure of the scale of a multivector.

5. Conclusion

This paper is a short introduction to a new class of projection operators in a Geometric Algebra. Even without taking projections onto new elements the Fundamental Theorem of Projection Operators (FToPO) unifies and generalizes the standard identities of projections onto subspaces. The outright generalizations of projection operators fall into three different potentially useful cases, each of which calls for application or interpretation. The first case is the projection onto versors, which the author believes in the *natural* generalization of the projection onto blades. The second case is the projection onto the zero element, which is simple enough that it can be appended to any other class of projection operators and preserve the FToPO, and can thereby introduce scale to projection operators. The third case is the class of weighted projection operators, P_A , that are sensitive to the scale of their generators, A .

Each case is a call for further work. The projection onto versors have a clear interpretation and only need applications demonstrate its worth. The zero element was given a geometric interpretation in reference [1], and that interpretation should be reconciled with the projection operator presented here. The weighted projection operators need both a solid interpretation and applications and therefore will probably not be well understood for some time to come. Since the class of reverse projection operators is a weighted projection operator that has a projection operator P_A for each element A of the Geometric Algebra, there is at least the hope that the reverse projection operators can help elucidate the geometric properties of arbitrary elements of a Geometric Algebra.

Appendix

The appendix contains two results used in the earlier proofs.

Lift

As taken from [1], a LIFT ('linear injective function' transformation) from one Geometric Algebra to another Geometric Algebra is defined as a linear injective map that preserves the outer product and the scalars. In more detail, given two geometric algebras, \mathcal{G}_1 and \mathcal{G}_2 and a linear injective function, f , from the vectors of \mathcal{G}_1 to the vectors of \mathcal{G}_2 then \underline{f} is a LIFT between the two algebras, where \underline{f} is the outermorphism of f .

This paper uses a LIFT to isomorphically embed a degenerate Geometric Algebra into a nondegenerate, nontrivial geometric algebra. If the Geometric Algebra is trivial it is just the scalars, and it is embedded into a Geometric Algebra over one-dimensional Euclidean vector space. If the algebra is degenerate then it is isomorphically embedded into a nondegenerate algebra as described in reference [1].

Contraction Lemma

Let \mathcal{G} be a Geometric Algebra over a nondegenerate finite dimensional vector space $\mathcal{R}^{p,q}$ then

$$x \rfloor A = 0 \quad \forall x \in \mathcal{R}^{p,q} \Rightarrow A \in \mathcal{R} \quad (3)$$

Proof: Since $\mathcal{R}^{p,q}$ is nondegenerate it has an orthogonal basis of invertible vectors e_1, \dots, e_n . The set $\{e_I : I \subset \{1, 2, \dots, n\}\}$ is a basis for \mathcal{G} where $e_\emptyset = 1$ and $e_I = \prod_{i_k \in I} e_{i_k}$. If $A = \sum \alpha^I e_I$ then $e_i \rfloor A = 0$ implies that $\alpha^I = 0$ when $i \in I$. Since $e_i \rfloor A = 0$ for each e_i it is clear that $\alpha^I = 0$ for all $I \neq \emptyset$, therefore A is a scalar.

References

- [1] Bouma T. A., L. Dorst and H. G. J. Pijls: Geometric Algebra for Subspace Operations, (Submitted for Publication), available on/line at <http://xxx.lanl.gov/abs/math.LA/0104102>.
- [2] Hestenes, D. and G. Sobczyk: "Clifford Algebras to Geometric Calculus" D. Reidel, Dordrecht, 1984.