Hyperbolic Hypergeometric Functions

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The cover shows an annotated version of the Clebsch graph. The Clebsch graph is a strongly regular graph, since each of its 16 vertices have 5 neigbours, each pair of neighbours has no common neighbour, while each pair of non-adjacent vertices share two common neighbours. The annotation gives a method to find symmetries of a very-well poised ${}_8W_7$. Given any vertex, annotated with v, and its 5 neighbours, annotated with p, q, r, s and t we can form the very-well poised hypergeometric series (using the standard notation from [16], see also Subsection 4.5.1)

$$\frac{(v;q)_\infty}{(\sqrt{pqrstv}, pv, qv, rv, sv, tv; q)_\infty} {}_8W_7\left(\frac{\sqrt{pqrstv}}{q}; p, q, r, s, t; q, v\right). \qquad \text{(i)}$$

We will explain in a moment which choice of root of $pqrstv$ to take. In whatever vertex one constructs this hypergeometric series, for whatever order of the 5 adjecent vertices, the value of the the resulting series is always the same. The equalities between the different instances of the hypergeometric series are instances of [16, III.23 and III.24].

The annotation of the Clebsch graph can be constructed by giving arbitrary values to one vertex (suppose it gets value s) and its five neighbours (which get values $a_1, a_2, \ldots, a_5$) and subsequently ensuring that the product of the values of each square in the graph equals $s\sqrt{a_1a_2a_3a_4a_5s}$. One must always take the same root of $\sqrt{a_1a_2a_3a_4a_5s}$ and this choice determines naturally the choice of root of $\sqrt{pqrstv}$ in (i).

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Contents

Chapter 1

Introduction

In this chapter we introduce the basic concepts of this thesis and give an impression of some of the methods used to study them. Moreover we describe the connections to other parts of mathematics and thereby put the results of this thesis in a broader perspective.

The main focus of this thesis is to extend some results about hypergeometric functions and the algebraic theory related to it. Hypergeometric functions have been studied for centuries by many, often famous, mathematicians including Euler, Gauß and Riemann. The continued interest in hypergeometric theory is due to its many applications and the beautiful structures arising from it.

For me, a striking example of the wide applicability of hypergeometric functions is that in the many PhD-defences I have attended on all areas of mathematics Professor Koornwinder always seemed to be able to ask a question about some hypergeometric function occurring in the work of the PhD candidate. Indeed applications of hypergeometric functions are found in subjects ranging from combinatorics and numerical analysis to dynamical systems and mathematical physics. Related to the last subject is their application in the theory of integrable systems and in representation theory. Our interest in the subject stems mainly from these last two applications.

To appreciate the kind of generalization of hypergeometric functions we consider in this thesis it is necessary to first understand something about the classical hypergeometric theory. Therefore we discuss classical hypergeometric theory in Section 1.1. Subsequently, in Section 1.2, we consider basic hypergeometric theory, a well-known generalization of classical hypergeometric theory. In Section 1.3 we consider the process of doubling, which is the essential concept which gives the connection between basic and hyperbolic hypergeometric theory. Hyperbolic hypergeometric theory is the main topic of this thesis. This is followed by a short overview of the theory of Macdonald operators and their doubled version in Section 1.4. This gives an important example of the connections between hypergeometric theory, integrable systems, and representation theory. This leads us to an overview of the different chapters in this thesis in Section 1.5.

1.1 Classical hypergeometric theory

In this section we will introduce the Gauß hypergeometric function ${}_2F_1$ as a fundamental example of a hypergeometric function. We introduce the ${}_2F_1$ as a solution to the hypergeometric differential equation. This exposition smoothly leads to some essential structures of this thesis. In particular it indicates why gamma functions are so important in this thesis. Moreover this approach to the Gauß hypergeometric function is natural, as it is quite often due to the fact that ${}_2F_1$ satisfies this differential equation that it appears in applications such as mathematical physics and representation theory. For a general introduction to hypergeometric functions and proofs of the results mentioned here see for example [2].

The hypergeometric differential equation is given by

$$x(1-x)y'' + (c-(a+b+1)x)y' - aby = 0, \tag{1.1.1}$$

for a function $y = y(x)$ and parameters $a, b, c \in \mathbb{C}$. A general second order differential equation with three regular singular points can be transformed into the hypergeometric differential equation, which has regular singular points in 0, 1 and ∞.

This equation has an analytic solution in a neighbourhood of zero, the Gauß hypergeometric function, which is denoted by ${}_2F_1(a, b; c; z)$ and is for non-integer c uniquely determined by the normalization ${}_2F_1(a, b; c; 0) = 1$. Most hypergeometric functions appearing in this thesis are in some way a generalization of ${}_2F_1$. On the other hand, specializations of the ${}_2F_1$ include the logarithm and the inverse trigonometric functions (arcsin, arccos and arctan). Moreover, several classes of classical orthogonal polynomials, including the Chebyshev, Gegenbauer, Jacobi and Legendre polynomials, can be expressed in terms of ${}_2F_1$.

One can find several expressions for the hypergeometric function ${}_2F_1$. For instance one can give an explicit power series expansion for ${}_2F_1$ at $z = 0$. We instead give two integral representations, as in this thesis most hypergeometric functions are defined by generalizations of one of these two integrals.

First of all we consider the Euler integral representation, given for $\Re(c) > \Re(b) > 0$ and $z \in \mathbb{C}$ with a branch cut from 1 to ∞ by

$${}_2F_1(a, b; c, z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \tag{1.1.2}$$

The branches of t^{b-1}, $(1-t)^{c-b-1}$ and $(1-zt)^{-a}$ are determined by the convention, employed throughout this chapter, that $a^z = \exp(z \log(a))$ for the principal branch of the logarithm. Here Γ denotes the classical Euler Gamma function. One can show directly that this integral satisfies (1.1.1) by interchanging the order of differentiation and integration and using integration by parts. For $z = 0$ this integral reduces to the beta integral

$$\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},$$

where $\Re(a), \Re(b) > 0$, which gives the normalization factor. Not obvious from this expression for the ${}_2F_1$ is the $a \leftrightarrow b$ symmetry of the ${}_2F_1$ (which is obvious from (1.1.1)).

The second integral representation we consider is the Barnes' integral representation given for $a, b \notin \mathbb{N}$ and $z \in \mathbb{C}$ with a branch cut along the positive real axis by

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s \frac{ds}{2\pi i}, \tag{1.1.3} $$

where the integration contour is indented to separate the poles of the integrand at $s = -a - n$ and $s = -b - n$ from those at $s = n$ (for $n \in \mathbb{Z}_{\geq 0}$). Again a direct calculation involving interchanging differentiation and integration again shows that the integral on the right hand side of (1.1.3) satisfies (1.1.1). This calculation uses the fundamental first order difference equation

$$ \Gamma(z+1) = z\Gamma(z) \tag{1.1.4} $$

satisfied by the Gamma function.

In the Barnes' integral representation we see that we have an integral representation of ${}_2F_1$, where the integrand is expressed in terms of gamma functions. Almost all integrals occurring in this thesis are of this form (with appropriate generalizations of the Gamma function).

A property which makes the Gamma function suitable as building block for hypergeometric functions is the fact that it solves the first order difference equation (1.1.4). Indeed, the fact that the right hand side of (1.1.3) solves (1.1.1) depends only on the fact that Γ solves (1.1.4) and satisfies good analytic and asymptotic properties.

One can define four classes of different kinds of hypergeometric functions involving four different types of gamma functions used as building blocks. These generalizations of the Gamma function all satisfy first order difference equations. Apart from classical hypergeometric functions we have basic or q-hypergeometric functions with the q-gamma function as building block, elliptic hypergeometric functions related to the elliptic gamma function and, the kind of hypergeometric functions occurring most often in this thesis, hyperbolic hypergeometric functions related to the hyperbolic gamma function. We describe the connections between these different kinds of gamma functions in more detail in Chapter 2.

1.2 Basic Hypergeometric Theory

The hyperbolic hypergeometric functions are closely related to the much better known basic hypergeometric theory, so now we first consider basic hypergeometric functions. A standard reference for results on basic hypergeometric functions is [16], which contains in particular the results mentioned here.

In basic hypergeometric theory there exists an extra parameter q, often chosen such that $|q| < 1$. Basic hypergeometric functions are a generalization of the classical hypergeometric functions in the sense that taking the (formal) limit $q \to 1$ generally returns us to the classical hypergeometric setting. Many of the results for classical hypergeometric functions generalize to the basic hypergeometric level. For example, where classical hypergeometric functions are related to representations of (the universal enveloping algebra of) a Lie algebra, q-hypergeometric functions are related to representations of quantum groups[1].

The infinite q-shifted factorials, or q-Pochhammer symbols, are defined for $|q| < 1$ by the infinite product

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1 - aq^r). \tag{1.2.1}$$

Observe that this product converges for $|q| < 1$, but diverges for other values of q. The q-gamma function Γ_q is now defined (for $|q| < 1$, $q \notin (-1, 0]$) as

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}.$$

This is a generalization of the Gamma function in the sense that $\lim_{q\to1}\Gamma_q(x) = \Gamma(x)$. Observe moreover that it satisfies the difference equation $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$ as analogue of (1.1.4).

The q-analogue of the ${}_2F_1$ function is the ${}_2\phi_1$ which has an integral representation similar to the Barnes' integral (1.1.3)

$${}_2\phi_1(q^a, q^b; q^c; q, z) = \frac{\Gamma_q(c)}{\Gamma_q(a)\Gamma_q(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma_q(a+s)\Gamma_q(b+s)}{\Gamma_q(c+s)\Gamma_q(s+1)} \frac{-\pi}{\sin(\pi s)}(-z)^s \frac{ds}{2\pi i} \tag{1.2.2}$$

for generic $a, b, c, z \in \mathbb{C}$ with a suitably indented contour. If we recall the reflection equation for the Gamma function, $\Gamma(-s)\Gamma(s+1) = -\pi/\sin(\pi s)$, this integral converges formally to the integral of (1.1.3) as $q \to 1$.

We defined ${}_2\phi_1$ for variables q^a, q^b and q^c, although it is not immediately clear that for a and a' with $q^a = q^{a'}$ the value of ${}_2\phi_1$ is the same. However a careful consideration of the definition of the q-gamma function shows this is indeed the case. In a definition using q-shifted factorials instead of the q-gamma function this problem does not arise, hence in many expressions q-shifted factorials are used instead of Γ_q. There also exists a q-analogue of the Euler integral representation (1.1.2), but in that integral the limits are less obvious so we omit this integral representation here.

The ${}_2\phi_1$ is a solution to an analog of the Gauß hypergeometric differential equation (1.1.1). Let us define the q-differential operator $\mathcal{D}_q$ as

$$(\mathcal{D}_q f)(z) = \frac{f(qz) - f(z)}{qz - z}.$$

[1]Funnily enough, the fact that the q of q-hypergeometric and the q of quantum (groups) are the same letter was purely coincidental. However they turned out to be intimately related, as we will see in Chapter 3.

If $q \to 1$ this operator converges to ordinary differentiation, i.e. $\lim_{q\to 1} \mathcal{D}_q f(z) = \frac{d}{dz} f(z)$ for a function f which is differentiable at z. The function ${}_2\phi_1(a,b;c;q,z)$ is now a solution to the difference equation

$$z(c-abqz)\mathcal{D}_q^2 f + \left(\frac{1-c}{1-q} + \frac{(1-a)(1-b)-(1-abq)}{1-q}\right)\mathcal{D}_q f - \frac{(1-a)(1-b)}{(1-q)^2} f = 0. \quad (1.2.3)$$

Substituting $a \to q^a$, $b \to q^b$ and $c \to q^c$ and using that $\lim_{q\to 1}(1-q^x)/(1-q) = x$ we indeed see that the limit $q \to 1$ of this equation is the hypergeometric differential equation (1.1.1). In this way q-difference equations may be regarded as q-analogues of differential equations. Applications of difference equations include combinatorial problems involving recursion, and certain mathematical physical systems. An important example of the last kind of application is described in Section 1.4. In this thesis we study many such difference equations.

As mentioned before, q-hypergeometric functions are often defined only for $|q| < 1$. Indeed the q-gamma function itself is not well-defined for q on the unit circle (and neither are infinite q-shifted factorials). Only when for special values of the parameters the q-hypergeometric functions become polynomials (for example ${}_2\phi_1(q^{-n}, b; c; q, z)$ is a polynomial of degree n in z) they can obviously be extended to (generic) q on the unit circle. However the q-difference equations for q on the unit circle do arise in applications (both in physics and in representation theory), hence one would like to study these equations for $|q| = 1$. Hyperbolic hypergeometric functions provide us with a method to deal with such extensions.

1.3 Doubling the q-shifted factorials

In this section we explain the concept of doubling, using as the fundamental example the doubling of q-shifted factorials. The process of doubling allows us in some sense to "extend" the q-shifted factorials (and thus the q-gamma function) to the range $|q| = 1$. For simplicity, in this section we first focus only on first order difference equations.

The meromorphic function $g(x) = 1/(x;q)_\infty$ for $x \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ satisfies the first order q-difference equation

$$g(qx) = (1-x)g(x), \quad (1.3.1)$$

for $0 < |q| < 1$. In some sense q-shifted factorials also satisfy a second first order difference equation, but this second difference equation is hidden due to the fact that the step-direction in the difference equation (1.3.1) is written multiplicatively (i.e. we multiply the argument of the function by the constant q, instead of adding a constant to the argument).

To reveal the second difference equation we identify $\mathbb{C}^* \simeq \mathbb{C}/\omega_2\mathbb{Z}$ for an appropriate $\omega_2 \in \mathbb{H} = \{z \mid \Im(z) > 0\}$ by $x = \exp(2\pi i z/\omega_2)$. We subsequently write

$q = \exp(2\pi i\omega_1/\omega_2)$ for some $\omega_1 \in \mathbb{H}$ (with $\omega_1/\omega_2 \in \mathbb{H}$). The meromorphic function on $z \in \mathbb{C}$

$$f(z) = 1/(\exp(2\pi iz/\omega_2); \exp(2\pi i\omega_1/\omega_2))_\infty \tag{1.3.2}$$

now satisfies the pair of difference equations

$$\begin{aligned} f(z+\omega_1) &= (1-\exp(2\pi iz/\omega_2))f(z) \\ f(z+\omega_2) &= f(z). \end{aligned} \tag{1.3.3}$$

Clearly the two systems (1.3.1) and (1.3.3) are equivalent under this change of variables.

Let us now consider, for $\omega_1, \omega_2 \in \mathbb{H}$ a general pair of first order difference equations for meromorphic functions f on $\mathbb{C}$,

$$\begin{aligned} f(z+\omega_1) &= a(z;\omega_1,\omega_2)f(z), \\ f(z+\omega_2) &= b(z;\omega_1,\omega_2)f(z), \end{aligned} \tag{1.3.4}$$

where a and b are non-zero meromorphic functions on $z \in \mathbb{C}$. In crucial examples of (1.3.4) in this thesis it is natural to assume the condition $\omega_1, \omega_2 \in \mathbb{H}$, which avoids the case $\omega_1/\omega_2 \in \mathbb{R}_{<0}$. Furthermore, when a and b depend meromorphically on ω_1 and ω_2, it is natural to consider solutions to (1.3.4) depending meromorphically on ω_1 and ω_2. In this introduction we simplify the exposition by insisting on $\omega_1, \omega_2 \in \mathbb{H}$ and discussing such systems only for fixed ω_1 and ω_2.

Definition 1.3.1. *We call a pair of difference equations* (1.3.4) *consistent for given $\omega_1, \omega_2 \in \mathbb{H}$ if a and b satisfy the cocycle condition*

$$a(z+\omega_2;\omega_1,\omega_2)b(z;\omega_1,\omega_2) = b(z+\omega_1;\omega_1,\omega_2)a(z;\omega_1,\omega_2). \tag{1.3.5}$$

If (1.3.4) has a non-zero meromorphic solution f, then the cocycle relation should hold, as both sides of (1.3.5) calculate the quotient $f(z+\omega_1+\omega_2)/f(z)$ of the solution in two different ways.

Observe that, for fixed $\omega_1, \omega_2 \in \mathbb{H}$, the solution space of (1.3.4) is at most one-dimensional over the field k_{ω_1,ω_2} of meromorphic ω_1- and ω_2-periodic functions on $\mathbb{C}$. For $\omega_1/\omega_2 \notin \mathbb{R}_{>0}$, this field k_{ω_1,ω_2} is the field of elliptic functions on the elliptic curve $\mathbb{C}/(\omega_1\mathbb{Z}+\omega_2\mathbb{Z})$, while for $\omega_1/\omega_2 \in \mathbb{R}_{>0}\backslash\mathbb{Q}_{>0}$ this field is just $\mathbb{C}$. Finally for $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$ the field k_{ω_1,ω_2} consists of meromorphic functions on the complex torus $\mathbb{C}/\rho\mathbb{Z}$, where ρ is an element in $\omega_1\mathbb{Z}+\omega_2\mathbb{Z}$ with smallest modulus.

Note that the system (1.3.3) is consistent. Indeed, it is a special case of a general method to obtain a consistent system from a q-difference equation on $\mathbb{C}^*$ which we describe now. Consider an arbitrary q-difference equation for meromorphic functions g on $\mathbb{C}^*$,

$$g(qx) = A_{\omega_1,\omega_2}(x)g(x), \tag{1.3.6}$$

where $\omega_1, \omega_2 \in \mathbb{H}$ and $q = q_{\omega_1,\omega_2} := \exp(2\pi i\omega_1/\omega_2)$ and for some non-zero meromorphic functions A_{ω_1,ω_2} on $\mathbb{C}^*$. As before, under the substitution $x = \exp(2\pi iz/\omega_2)$, the equation (1.3.6) is equivalent to the system

$$\begin{aligned} f(z+\omega_1) &= a(z;\omega_1,\omega_2)f(z), \\ f(z+\omega_2) &= f(z), \end{aligned} \tag{1.3.7}$$

for meromorphic functions f on $\mathbb{C}$, where $a(z;\omega_1,\omega_2) = A_{\omega_1,\omega_2}(\exp(2\pi iz/\omega_2))$. The system (1.3.7) is consistent as $a(z;\omega_1,\omega_2)$ is ω_2-periodic by construction. Note that, given a solution g to (1.3.6), the function $f(z) = g(\exp(2\pi iz/\omega_2))$ solves (1.3.7).

A second consistent system related to the q-difference equation (1.3.6) is given by the following "doubled" system

$$\begin{aligned} f(z+\omega_1) &= a(z;\omega_1,\omega_2)f(z), \\ f(z+\omega_2) &= a(z;\omega_2,\omega_1)f(z), \end{aligned} \tag{1.3.8}$$

where the second equation is obtained from the first one by interchanging $\omega_1 \leftrightarrow \omega_2$. This system is again consistent as $a(z;\omega_1,\omega_2)$ is ω_2-periodic, while $a(z;\omega_2,\omega_1)$ is ω_1-periodic. If the original equation (1.3.6) has a solution g for given ω_1 and ω_2 and a solution $\tilde{g}$ when the role of ω_1 and ω_2 is interchanged in (1.3.7), then

$$f(z) = g(\exp(2\pi iz/\omega_2))\tilde{g}(\exp(2\pi iz/\omega_1)) \tag{1.3.9}$$

gives a solution to the doubled system (1.3.8). Note that the associated parameters q_{ω_1,ω_2} and $q^{-1}_{\omega_2,\omega_1}$ of (1.3.6) for g and $\tilde{g}$ are related by modular inversion.

The doubled system related to the q-difference equation (1.3.1) satisfied by the q-shifted factorials thus becomes

$$\begin{aligned} f(z+\omega_1) &= (1-\exp(2\pi iz/\omega_2))f(z), \\ f(z+\omega_2) &= (1-\exp(2\pi iz/\omega_1))f(z). \end{aligned} \tag{1.3.10}$$

Observe that for $\Im(\omega_1/\omega_2) < 0$ the function

$$f(z) = (\exp(2\pi i(z-\omega_1)/\omega_2); \exp(-2\pi i\omega_1/\omega_2))_\infty \tag{1.3.11}$$

solves (1.3.3). Together with the solution (1.3.2) for $\Im(\omega_1/\omega_2) > 0$ this thus gives a solution

$$\frac{(\exp(2\pi i(z-\omega_2)/\omega_1); \exp(-2\pi i\omega_2/\omega_1))_\infty}{(\exp(2\pi iz/\omega_2); \exp(2\pi i\omega_1/\omega_2))_\infty}, \tag{1.3.12}$$

to the doubled system (1.3.10) for $\Im(\omega_1/\omega_2) > 0$. This is a concrete example of the construction of solutions to a doubled systems from solutions of the underlying q-difference equation. The same function with $\omega_1 \leftrightarrow \omega_2$ solves (1.3.10) for $\Im(\omega_2/\omega_1) > 0$.

The fact that a system (1.3.4) is consistent does not imply that there exists a non-zero solution. In particular consider the case $\omega_1/\omega_2 \in \mathbb{R}_{>0}$, which corresponds to $|q| = 1$. In this case there exist sequences $n_k, m_k \in \mathbb{N}$ with $n_k\omega_1 - m_k\omega_2 \to 0$ and $n_k \to \infty$ as $k \to \infty$. For a nontrivial, meromorphic, solution f to (1.3.4) for these particular values of ω_1, ω_2 we must have

$$1 = \lim_{k\to\infty} \frac{f(z+n_k\omega_1 - m_k\omega_2)}{f(z)} = \lim_{k\to\infty} \frac{\prod_{j=0}^{n_k-1} a(z+j\omega_1 - m_k\omega_2;\omega_1,\omega_2)}{\prod_{j=1}^{m_k} b(z-j\omega_2;\omega_1,\omega_2)}. \tag{1.3.13}$$

In particular in the case of the system (1.3.3), associated to the q-shifted factorial, this would become

$$1 = \lim_{k\to\infty} \prod_{j=0}^{n_k-1} (1 - \exp(2\pi i(z + j\omega_1)/\omega_2)),$$

but the right hand side does not converge to 1 for $\omega_1/\omega_2 \in \mathbb{R}_{>0}$. Hence (1.3.3) has no non-trivial solution if $\omega_1/\omega_2 \in \mathbb{R}_{>0}$, even though it is consistent. On the other hand, we have seen that there exist solutions for arbitrary $\omega_1, \omega_2 \in \mathbb{H}$ with $\omega_1/\omega_2 \notin \mathbb{R}_{>0}$ (given by (1.3.2) and (1.3.11)).

However, it turns out that the doubled system (1.3.10) does have a non-trivial solution when $\omega_1/\omega_2 \in \mathbb{R}_{>0}$. In fact, the solution (1.3.12) as a meromorphic function of $(z, \omega_1, \omega_2) \in \{\mathbb{C} \times \mathbb{H}^2 \mid \Im(\omega_1/\omega_2) > 0\}$ admits a unique meromorphic continuation to $\mathbb{C}\times\mathbb{H}^2$, which we denote by $G(z; \omega_1, \omega_2)$. For any $\omega_1, \omega_2 \in \mathbb{H}$ (thus including $\omega_1/\omega_2 \in \mathbb{R}_{>0}$!) the function $G(z)$ satisfies the doubled system (1.3.10). To arrive at these results we will construct in Chapter 2, following closely [63], a non-trivial solution $\Gamma_h(z; \omega_1, \omega_2)$, which is called the hyperbolic gamma function, to the consistent system of difference equations

$$\begin{aligned} \Gamma_h(z + \omega_1) &= 2\sin(\pi z/\omega_2)\Gamma_h(z), \\ \Gamma_h(z + \omega_2) &= 2\sin(\pi z/\omega_1)\Gamma_h(z), \end{aligned} \tag{1.3.14}$$

which is obviously well-defined for all $\omega_1, \omega_2 \in \mathbb{H}$. The relation between G and Γ_h is then given by

$$G(z) = \exp(\pi i z^2/2\omega_1\omega_2 - \pi i(\omega_1 + \omega_2)z/2\omega_1\omega_2)\Gamma_h(z).$$

Note that the system of difference equations (1.3.14) for Γ_h turns into the system of difference equations (1.3.10) satisfied by G.

Note that if $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$ the condition (1.3.13) reduces to an algebraic equation. Indeed if $\omega_1/\omega_2 = m/n$ we can use the sequences $n_k = kn$ and $m_k = km$, after which the product within the limit on the right hand side of (1.3.13) becomes the k'th power of the same expression (using the cocycle relation (1.3.5)). This leads us to the concept of superconsistency.

Definition 1.3.2. *The system* (1.3.4) *is called superconsistent for* $\omega_1, \omega_2 \in \mathbb{H}$ *such that* $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$ *if it is consistent and if we have*

$$\prod_{j=0}^{n-1} a(z + j\omega_1; \omega_1, \omega_2) = \prod_{j=0}^{m-1} b(z + j\omega_2; \omega_1, \omega_2), \tag{1.3.15}$$

if $\omega_1/\omega_2 = m/n$ *with* $m, n \in \mathbb{N}$ *and* $gcd(m, n) = 1$.

Note that if the system (1.3.4) is superconsistent then (1.3.15) holds for all m, n with $\omega_1/\omega_2 = m/n$, not only those with $gcd(m, n) = 1$. Note that if the system (1.3.4) has a non-zero solution for $\omega_1, \omega_2 \in \mathbb{H}$ with $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$, then it must be superconsistent, as both sides of (1.3.15) then represent $f(z+n\omega_1)/f(z) = f(z+m\omega_2)/f(z)$.

The system (1.3.3) is not superconsistent for such ω_1, ω_2 as (1.3.15) reduces to

$$\prod_{j=0}^{n-1}(1-\exp(2\pi i(z+j\omega_1)/\omega_2)) = 1,$$

which clearly does not hold.

On the other hand, the doubled system (1.3.10) is an example of a superconsistent system (as it should, since there exists a non-trivial solution to it). Indeed super-consistency follows for $\omega_1/\omega_2 = n/1$ from

$$\prod_{j=0}^{n-1}(1-\exp(2\pi i(z+j\omega_2)/\omega_1)) = 1-\exp(2\pi iz/\omega_2) \tag{1.3.16}$$

for all $z \in \mathbb{C}$. Setting $x = \exp(2\pi iz/\omega_1)$ and writing $\zeta = \exp(2\pi i/n)$ for the n^{th} primitive root of unity, this becomes

$$\prod_{j=0}^{n-1}(1-x\zeta^j) = 1-x^n. \tag{1.3.17}$$

This equation holds as both sides are degree n polynomials in x with constant term 1, which are zero at all n^{th} roots of unity. The relation (1.3.15) for arbitrary $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$ also follows from (1.3.17). We conclude that (1.3.10) is superconsistent for all $\omega_1, \omega_2 \in \mathbb{H}$ with $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$.

To indicate that the fact that a doubled system of difference equations is superconsistent for all $\omega_1, \omega_2 \in \mathbb{H}$ with $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$ is rather special, we mention the following result. The system of difference equations in the proposition corresponds to a doubled system related to a q-difference equation (1.3.6) for a function $A_{\omega_1,\omega_2}(x)$ which is independent of ω_1 and ω_2. The proof is given in Section 2.3.

Proposition 1.3.3. *Suppose the pair of difference equations*

$$G(z+\omega_1) = c(z/\omega_2)G(z), \qquad G(z+\omega_2) = c(z/\omega_1)G(z), \tag{1.3.18}$$

where c is a non-zero meromorphic function on $\mathbb{C}$ independent of ω_1 and ω_2, is consistent for all $\omega_1, \omega_2 \in \mathbb{H}$ and super-consistent for $\omega_1, \omega_2 \in \mathbb{H}$ with $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$. Then $c(z) = \exp(A(z-1/2))(2\sin(\pi z))^B$ for some constants $A \in \mathbb{C}$ and $B \in \mathbb{Z}$.

For such $c(z)$ there exists a non-trivial solution to (1.3.18) which is given by

$$G(z) = \exp(Az^2/2\omega_1\omega_2 - A(\omega_1+\omega_2)z/2\omega_1\omega_2)\Gamma_h(z)^B.$$

For higher order difference equations the process of doubling can easily be extended. Indeed, consider the general m'th order q-difference equation, for meromorphic functions g on $\mathbb{C}^*$,

$$\sum_{j=0}^{m} A_{j,\omega_1,\omega_2}(x)g(q^jx) = 0, \tag{1.3.19}$$

where $q = \exp(2\pi i\omega_1/\omega_2)$, for some meromorphic functions A_{j,ω_1,ω_2} on $\mathbb{C}^*$. A general discussion of such linear q-difference equations is given in, for example, [1]. Under the substitution $x = \exp(2\pi iz/\omega_2)$ the equation (1.3.19) is equivalent to the system

$$\begin{aligned} \sum_{j=0}^{m} a_j(z;\omega_1,\omega_2)f(z+j\omega_1) = 0, \\ f(z+\omega_2) - f(z) = 0, \end{aligned} \tag{1.3.20}$$

where $a_j(z;\omega_1,\omega_2) = A_{j,\omega_1,\omega_2}(\exp(2\pi iz/\omega_2))$. The related doubled system is given in analogy to (1.3.8) by

$$\begin{aligned} \sum_{j=0}^{m} a_j(z;\omega_1,\omega_2)f(z+j\omega_1) = 0, \\ \sum_{j=0}^{m} a_j(z;\omega_2,\omega_1)f(z+j\omega_2) = 0. \end{aligned} \tag{1.3.21}$$

Let us call a pair of difference equations

$$(B_{\omega_1,\omega_2}f)(z) = 0, \qquad (C_{\omega_1,\omega_2}f)(z) = 0, \tag{1.3.22}$$

where B_{ω_1,ω_2}, respectively C_{ω_1,ω_2}, is a difference operator with period ω_1, respectively ω_2, consistent if B_{ω_1,ω_2} and C_{ω_1,ω_2} commute. This reduces to Definition 1.3.1 for first order difference equations. Note that both systems (1.3.20) and (1.3.21) are consistent, as $a_j(z;\omega_1,\omega_2)$ is again ω_2-periodic (and thus $a_j(z;\omega_2,\omega_1)$ is ω_1-periodic). As for systems of first order difference equations, given a solution to (1.3.20) for $\omega_1,\omega_2 \in \mathbb{H}$ and a second solution for ω_1 and ω_2 interchanged, we can construct a solution to the doubled system (1.3.21).

For q-difference equations for which there do not exist solutions for $|q| = 1$, we may still hope that its doubled version does have a non-trivial solution for $\omega_1/\omega_2 \in \mathbb{R}_{>0}$. An important example is given by Ruijsenaars' [65] R-function, discussed in Chapter 3, which is an eigenfunction to a doubled version of the Askey-Wilson second order q-difference operator and which is well defined for $\omega_1/\omega_2 \in \mathbb{R}_{>0}$ (i.e. $|q| = 1$). The Askey-Wilson q-difference operator is an important q-analogue of the Gauß hypergeometric differential equation (1.1.1). It will play an important role in Chapters 3 and 4.

The process of doubling also has a natural extension for multivariate q-difference operators, an important example of which is given in the next section.

1.4 Macdonald operators

Now we continue by discussing the connection of (basic) hypergeometric theory to integrable systems and representation theory by looking at the Macdonald operators of type A_{n-1}. This provides us also with an interesting example for which we can describe the process of doubling in a more complicated setting. The discussion here follows the lecture notes [33] of Kirillov Jr.

There exist Macdonald operators associated to any root system but to simplify our exposition, here we only consider the operators associated to the root system of type A_{n-1}. For this root system the operators were first given by Ruijsenaars in [62] and are also known as Macdonald-Ruijsenaars operators. We belief that the doubling described in this section can be performed as well for Macdonald operators on a general root system. The restriction to the root system A_{n-1} implies we can proceed without requiring any previous knowledge on root systems. To each root system there is associated a certain finite group, called the Weyl group, which is generated by reflections in Euclidean space. The Weyl group for type A_{n-1} is just the symmetric group S_n in n letters. The significant role this group plays in the following discussion thus reflects the fact that we are considering the root system of type A_{n-1}.

Given constants $0 < |q| < 1$ and $t \in (0,1]$ we define the function $\Delta_{q,t}$ by

$$\Delta_{q,t}(x) = \prod_{1\leq j\neq k\leq n} \frac{(x_j/x_k;q)_\infty}{(tx_j/x_k;q)_\infty}. \tag{1.4.1}$$

Define the space $\mathcal{M}^{S_n}$ to be the space of S_n symmetric meromorphic functions on $(\mathbb{C}^*)^n$. Note that in particular $\Delta_{q,t} \in \mathcal{M}^{S_n}$. We can write

$$\Delta_{q,t} = \Delta^+_{q,t}\overline{\Delta^+_{q,t}}, \tag{1.4.2}$$

if $q \in (0,1)$ and $|x_j| = 1$ for all j, using

$$\Delta^+_{q,t}(x) = \prod_{1\leq j<k\leq n} \frac{(x_j/x_k;q)_\infty}{(tx_j/x_k;q)_\infty}.$$

This shows $\Delta_{q,t}$ is positive real if all x-variables are on the unit circle.

We define the translation operator T_j for $j = 1,\dots n$ by

$$T_j f(x) = f(x_1, x_2, \dots, qx_j, \dots, x_n),$$

and for a subset $J \in \{1,2,\dots,n\}$ we extend this definition to $T_J = \prod_{j\in J} T_j$ (note that the T_j commute with each other, so this is well-defined). We can now give the following explicit formula for n q-difference operators D_r, $r = 1,\dots,n$

$$\begin{aligned} D_r f(x) &= \sum_{\sigma\in S_n} \sigma\left(\frac{T_{\{1,2,\dots,r\}}\Delta^+_{q,t} f}{\Delta^+_{q,t}}\right) \\ &= r!(n-r)! \sum_{\substack{J\subset\{1,2,\dots,n\}\\ |J|=r}} \left(\prod_{j\in J, k\notin J} \frac{x_j - tx_k}{x_j - x_k}\right) T_J f(x). \end{aligned} \tag{1.4.3}$$

From this definition it is clear D_r sends $\mathcal{M}^{S_n}$ to $\mathcal{M}^{S_n}$. Observe that the Macdonald operators D_r make sense for arbitrary $q \in \mathbb{C}^*$ using the second expression in

(1.4.3). In [62] Ruijsenaars showed these operators can be interpreted as quantum Hamiltonians of a quantum integrable system, which can be interpreted as a relativistic analogue of a quantum Calogero-Moser system, in particular he showed that the D_r commute.

The natural question now is whether we can find joint eigenfunctions of these operators. In this general setting this question is still subject to research. However in the polynomial setting much more is known. Let us therefore consider the action of D_r on the space $\mathbb{C}[x, x^{-1}]^{S_n}$ of symmetric Laurent polynomials in $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$. If $p \in \mathbb{C}[x, x^{-1}]^{S_n}$ we see that $\prod_{j<k}(x_j - x_k)D_r p$ is an anti-symmetric Laurent polynomial, hence it is divisible by $\prod_{j<k}(x_j - x_k)$. Therefore the Macdonald operators map $\mathbb{C}[x, x^{-1}]^{S_n}$ to itself.

We can define a sesqui-linear form on the space $\mathbb{C}[x, x^{-1}]^{S_n}$ by

$$\langle f, g\rangle_{q,t} = \int_{\mathbb{T}^n} f(x)\overline{g(x)}\Delta_{q,t}(x) \prod_{j=1}^{n} \frac{dx_j}{2\pi i x_j}, \tag{1.4.4}$$

where the contour is the product of n unit circles $\mathbb{T} = \{x \in \mathbb{C} \mid |x| = 1\}$ traversed in the positive direction. For $q \in (0, 1)$, $\langle \cdot, \cdot\rangle_{q,t}$ forms a non-degenerate complex inner product as the weight function $\Delta_{q,t}$ then is positive real on $\mathbb{T}^n$.

The famous Macdonald conjectures [46], which have now been proven [9][2], provide evaluation formulas for the constant term $\langle 1, 1\rangle_{q,t}$.

A direct calculation shows that the Macdonald operators are symmetric with respect to this inner product, i.e. $\langle D_r f, g\rangle = \langle f, D_r g\rangle$ for $f, g \in \mathbb{C}[x, x^{-1}]^{S_n}$ and $r = 1, \ldots, n$. For generic q and t the joint spectrum of the Macdonald operators, acting on $\mathbb{C}[x, x^{-1}]^{S_n}$, is simple (i.e. the eigenspaces are one dimensional). The corresponding eigenfunctions are the Macdonald polynomials. It follows immediately that the Macdonald polynomials form an orthogonal basis of the space $\mathbb{C}[x, x^{-1}]^{S_n}$ with respect to the sesqui-linear form (1.4.4).

The Macdonald polynomials are a generalization of the Schur functions and generalize simultaneously the Jack polynomials, which are multivariate generalizations of Jacobi polynomials, and the Hall-Littlewood polynomials, see [47]. Koornwinder polynomials [41] are generalizations of Macdonald polynomials, which in the univariate case reduce to Laurent polynomial eigenfunctions (the Askey-Wilson polynomials [3]) of the Askey-Wilson q-difference operator, mentioned already in the previous section.

The Macdonald operators appear in certain representation theoretic contexts. They essentially realize the action of the center of the affine Hecke algebra in Cherednik's polynomial representation. More pertinent to this thesis, as the rank 1 case of this situation is studied in Chapter 3, they can be found as the radial part of the action of the center in suitable representations of the quantized universal enveloping algebra of a semisimple Lie algebra.

Let us define the translation operators $T_{j,\omega}$ for $j = 1, 2, \ldots, n$ and $\omega \in \mathbb{H}$ by

$$(T_{j,\omega} f)(z_1, \ldots, z_n) = f(z_1, \ldots, z_j + \omega, \ldots, z_n),$$

[2]Therefore some people [69] refer to them as Macdonald constant term ex-conjectures

and by $T_{J,\omega} = \prod_{j\in J} T_{j,\omega}$ for $J \subset \{1, 2, \ldots, n\}$. Substituting $x = \exp(2\pi i z/\omega_2)$ and $t = \exp(2\pi i\tau/\omega_2)$ for a suitable $\omega_2 \in \mathbb{H}$ and writing $q = \exp(2\pi i\omega_1/\omega_2)$ for some $\omega_1 \in \mathbb{H}$ in the definition (1.4.3) of the Macdonald operators, we see that the Macdonald operators become

$$\begin{aligned} &D_r(\omega_1, \omega_2)f(z) \\ &= r!(n-r)! \sum_{\substack{J\subset\{1,2,\ldots,n\}\\ |J|=r}} \left(\prod_{j\in J, k\notin J} \frac{e^{2\pi i z_j/\omega_2} - e^{2\pi i(\tau+z_k)/\omega_2)}}{e^{2\pi i z_j/\omega_2} - e^{2\pi i z_k/\omega_2}} \right) T_{J,\omega_1} f(z), \quad (1.4.5) \end{aligned}$$

for S_n-invariant meromorphic functions f on $(\mathbb{C}/\omega_2\mathbb{Z})^n$.

We can apply the process of doubling described in Section 1.3 on the Macdonald operators to obtain their hyperbolic version. We can extend the domain of these operators to the space $\mathcal{N}^{S_n}$ of S_n-invariant meromorphic functions on $\mathbb{C}^n$. The doubled system of difference operators then consists of the $2n$ operators $D_r(\omega_1, \omega_2)$ and $D_r(\omega_2, \omega_1)$ for $r = 1, \ldots, n$, viewed as difference operators on $\mathcal{N}^{S_n}$.

Due to the construction of doubling, as discussed in the previous section in the univariate case, the operators $D_r(\omega_1, \omega_2)$ and $D_{r'}(\omega_2, \omega_1)$ commute for all $r, r' \in \{1, 2, \ldots, n\}$. Moreover, as the original Macdonald operators commute, we see that $D_r(\omega_1, \omega_2)$ and $D_{r'}(\omega_1, \omega_2)$ commute for all $r, r' \in \{1, 2, \ldots, n\}$. Hence we have $2n$ commuting difference operators (i.e. a consistent set of operators) on $\mathcal{N}^{S_n}$ and it is natural to wonder about the spectral analysis of these operators.

The Macdonald polynomials give us for generic ω_1 and ω_2 eigenfunctions to the doubled system of Macdonald operators $D_r(\omega_1, \omega_2)$ and $D_r(\omega_2, \omega_1)$ analogous to (1.3.9). In particular, interesting results are to be expected when considering the spectral analysis of the doubled system of Macdonald operators in the case $\omega_1/\omega_2 \in \mathbb{R}_{>0}$. In the univariate case the spectral analysis of the doubled system associated to the Askey-Wilson q-difference operator for $\omega_1/\omega_2 \in \mathbb{R}_{>0}$ (involving the R function) is performed by Ruijsenaars [65] and [68].

Let us now give a hyperbolic version of the weight function $\Delta_{q,t}$ (1.4.2). Observe that $\Delta_{q,t}$ satisfies the first order difference equations

$$\frac{\Delta_{q,t}(x_1, \ldots, qx_j, \ldots, x_n)}{\Delta_{q,t}(x_1, \ldots, x_j, \ldots, x_n)} = \prod_{k\neq j} \frac{(1 - x_k/qx_j)(1 - tx_j/x_k)}{(1 - x_j/x_k)(1 - tx_k/qx_j)},$$

for $j = 1, \ldots, n$. If we double this system of q-difference equations analogous to the previous section we obtain

$$\begin{aligned} &\frac{f(z_1, \ldots, z_j + \omega_1, \ldots, z_n)}{f(z_1, \ldots, z_j, \ldots, z_n)} \\ &\quad = \prod_{k\neq j} \frac{\left(1 - \exp\left(\frac{2\pi i}{\omega_2}(z_k - z_j - \omega_1)\right)\right)\left(1 - \exp\left(\frac{2\pi i}{\omega_2}(\tau + z_j - z_k)\right)\right)}{\left(1 - \exp\left(\frac{2\pi i}{\omega_2}(z_j - z_k)\right)\right)\left(1 - \exp\left(\frac{2\pi i}{\omega_2}(\tau + z_k - z_j - \omega_1)\right)\right)}, \end{aligned}$$

where $j = 1, \dots, n$ and similar equations with ω_1 and ω_2 interchanged. A meromorphic solution to these $2n$ equations on $\mathbb{C}^n$ is given in terms of the hyperbolic gamma function Γ_h by

$$\Delta_\tau = \prod_{1\leq j\neq k\leq n} \frac{\Gamma_h(\tau + z_j - z_k)}{\Gamma_h(z_j - z_k)}. \tag{1.4.6}$$

Like the original weight function $\Delta_{q,t}$, we can split Δ_τ in two parts, given by

$$\Delta_\tau^+ = \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau + z_j - z_k)}{\Gamma_h(z_j - z_k)}, \qquad \Delta_\tau^- = \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau + z_k - z_j)}{\Gamma_h(z_k - z_j)}, \tag{1.4.7}$$

such that $\Delta_\tau = \Delta_\tau^+\Delta_\tau^-$. The Macdonald operators in logarithmic variables also have a representation like the first definition of (1.4.3), now involving Δ_τ^+.

We would like to define a sesqui-linear form in this setting and a natural candidate would be

$$\langle f, g\rangle_{\tau,\omega_1,\omega_2} = \int_{\substack{\mathbb{R}^n \\ \sum z_j = 0}} f(z)\overline{g(z)}\Delta_\tau(z) \prod_{j=1}^{n-1} dz_j.$$

Unfortunately it is unclear on which space of functions $\langle\cdot,\cdot\rangle_{\tau,\omega_1,\omega_2}$ forms a well-defined sesqui-linear form. Formally the proposed sesqui-linear form satisfies some nice properties, for example the Macdonald operators are formally symmetric and moreover the weight function is positive real if $\omega_1/\omega_2 \in \mathbb{R}_{>0}$. Under certain conditions on τ, ω_1 and ω_2 the constant term $\langle 1, 1\rangle_{\tau,\omega_1,\omega_2}$ is convergent and there exists an explicit evaluation formula for the constant term [79]. The hyperbolic version of the constant term identity related to the sesqui-linear form corresponding to the Koornwinder polynomials is precisely the evaluation integral (5.3.7) studied in Chapter 5, while the univariate constant term identity plays an important role in Chapter 4.

For $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$ (i.e. the case that q is a root of unity, in which the Macdonald polynomials are not well-defined), certain algebraic relations exist between the $D_r(\omega_1, \omega_2)$ and $D_{r'}(\omega_2, \omega_1)$. For example if $2\omega_1 = \omega_2$ a direct calculation shows

$$D_1(\omega_1, \omega_2)^2 - (n-1)(n-1)!D_2(\omega_1, \omega_2) = (n-1)!D_1(\omega_2, \omega_1).$$

This implies that the eigenvalues of an eigenfunction to all of these operators must satisfy the same algebraic equations.

We expect that the process of doubling allows one to find solutions to integrable systems, such as the Macdonald operators discussed here, for $|q| = 1$. Integrable systems governed by q-difference equations appear in several contexts, and the condition $|q| = 1$ often corresponds to distinct physical properties of the system under consideration. For example it corresponds to the massless region in the XXZ model [26].

As for the construction of these doubled operators in the setting of quantum groups, the doubling procedure is directly related to the notion of the modular double [13] of $\mathcal{U}_q(\mathfrak{sl}_n)$. Instead of looking at one copy of $\mathcal{U}_q(\mathfrak{sl}_n)$ one considers $\mathcal{U}_q(\mathfrak{sl}_n) \otimes \mathcal{U}_{\tilde{q}}(\mathfrak{sl}_n)$ for parameters q and $\tilde{q}$ related by modular inversion, as in the process of doubling the q-difference operators. For $\mathcal{U}_q(\mathfrak{sl}_2)$ the harmonic analysis of the correspond modular double is studied in Chapter 3. A key feature of the harmonic analysis of the modular double is that it is allowed to set $|q| = 1$ in the results.

1.5 Overview of the thesis

In this section we will give a short overview of the different chapters of this thesis and the aspects of the theory which are presented there.

In Chapter 2 we consider in a uniform way the rational (i.e. classical), trigonometric, hyperbolic and elliptic gamma functions as they play an important role in this thesis.

In Chapter 3 we discuss the representation theory of the modular double [13] of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$. As shortly mentioned in the previous section this modular double $\mathcal{D}$ is the tensor product of two copies of the quantum group associated to related parameters q and $\tilde{q}$, in particular $\mathcal{D} = \mathcal{U}_q \otimes \mathcal{U}_{\tilde{q}}$ for $q = \exp(\pi i \omega_1/\omega_2)$ and $\tilde{q} = \exp(\pi i \omega_2/\omega_1)$. Representing this modular double as multiplication difference operators as in [32] we can mimic the method used by Koornwinder [42], Noumi and Mimachi [53] and Koelink [35] to obtain the Askey-Wilson polynomials as zonal spherical functions on $\mathcal{U}_q(\mathfrak{su}_2)$. In this way we obtain a hyperbolic solution to the Askey-Wilson difference equations, Ruijsenaars' R-function [65]. In this derivation we use a $*$-structure related to the noncompact real form $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{sl}_2(\mathbb{C})$, i.e. we consider $|q| = 1$. Using this representation we can rederive several basic properties of the R-function.

Chapter 4 deals with the similarities between hypergeometric functions on different levels, in particular between basic, hyperbolic and elliptic hypergeometric functions. On each of the three levels we begin with the appropriate analogue of the Nassrallah-Rahman integral, [16, (6.4.1)], which is a generalization of Macdonald's constant term identity for the root system BC_1. On the elliptic level this integral is Spiridonov's beta integral [71], on the hyperbolic level it was given by Rains [61]. Subsequently we consider the transformations and contiguous relations of a far-reaching generalization of the ${}_2F_1$, for the three different types of hypergeometric function theory, which depend on 7 free parameters.

On the hyperbolic and trigonometric level we moreover take a limit of two parameters to infinity to obtain degenerations and consider the transformations and contiguous relations these degenerations satisfy. On the hyperbolic level the degenerations correspond to the R-function and we obtain a new integral representation for R, while we also (re)obtain many of the properties of R. On the trigonometric level the degenerations correspond to very-well-poised ${}_8W_7$'s, and we reobtain many classical results.

Finally we express the (hyperbolic hypergeometric) R-function as a sum of products of basic hypergeometric series (for two different parameters q and $\tilde{q}$ related by modular inversion). This expression is the analogue of the construction (1.3.9) of a solution to the doubled system of difference equations from solutions to the original q-difference equation for the Askey-Wilson q-difference equation. This expression exemplifies the intricate relation between hyperbolic and basic hypergeometric theory, as discussed in Section 1.3.

In the final chapter, Chapter 5, we consider some multivariate hyperbolic hypergeometric integrals which are generalizations of the hyperbolic hypergeometric integrals considered in Chapter 4. Starting with the BC_n type hyperbolic hypergeometric functions from [61] we derive a family of interesting degenerations. In this chapter we do not only consider first level degenerations, as in Chapter 4, but we obtain the entire degeneration scheme until we are left with a Gaussian integral. Examples of the integrals in this scheme are hyperbolic versions of the Macdonald constant term identities for root systems of type BC_n.

Like in Chapter 4 the degenerations of these top level integrals fall in two broad categories, which correspond to generalizations of the Barnes' (1.1.3), respectively Euler's (1.1.2), integral representations of the ${}_2F_1$. Using their expression as degenerations, and the evaluation and/or transformation formulas of the top-level integrals, we obtain the evaluations and/or transformations for the degenerations. The most interesting situation occurs when we consider the multivariate generalization of the hyperbolic hypergeometric integral with symmetries corresponding to a Weyl group of type E_7. This corresponds on the univariate basic hypergeometric level to very-well-poised ${}_{10}\phi_9$'s.

In this case we obtain both transformations between different kinds of integrals and symmetries of the degenerations. The symmetry groups of these degenerations show resemblance to the sequence of symmetry groups in the work of Kajiwara et al. [28] on hypergeometric solutions to the different kinds of q-Painlevé equations. The degeneration scheme itself is reminiscent of the q-Askey scheme [34] of families of orthogonal polynomials.

Several interesting open questions remain which are not considered in this thesis. A continuation of this research would be to find more hyperbolic analogues of the algebraic structures, associated to basic hypergeometric functions (such as quantized universal enveloping algebras of semisimple Lie algebras and affine Hecke algebras). In particular one would expect the doubled system of Macdonald operators, as discussed in Section 1.2, to appear in these structures. In Chapter 3 we find that the modular double of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is the proper "hyperbolic" version of $\mathcal{U}_q(\mathfrak{sl}_2)$. We would like to extend this result to other semisimple Lie algebras. Through harmonic analysis of the modular doubles of these quantum groups we would subsequently expect to find representation theoretic interpretations of multivariate hyperbolic hypergeometric functions. This should also give more insight in these modular doubles, which, apart from the modular double of $U_q(\mathfrak{sl}_n)$ [32], have not yet been studied much. Indeed the definition is not yet completely clear for all semi simple Lie algebras; it has been suggested in [32] that the second tensor copy in the modular double of $\mathcal{U}_q(\mathfrak{g})$ should be taken as the quantized universal enveloping algebra associated to the Langlands dual of $\mathfrak{g}$. Moreover it would be interesting to define a similar concept of modular double for Hecke algebras, and related to that, a theory of non-symmetric hyperbolic hypergeometric functions.

Finally one might look for a theory of hyperbolic hypergeometric series. This theory should include expressions of the form (1.3.9) if the functions g and $\tilde{g}$ are basic hypergeometric series associated to $q = \exp(2\pi i\omega_1/\omega_2)$ and $\tilde{q} = \exp(2\pi i\omega_2/\omega_1)$ respectively. In particular we would like to have extensions of the result in Section 4.6, which gives a representation of a specific hyperbolic hypergeometric integral as a sum of products of two basic hypergeometric series.

Chapter 2

Gamma functions

2.1 Introduction

In this chapter we consider the different gamma functions corresponding to the different kind of hypergeometric theories (rational, trigonometric, hyperbolic and elliptic). In particular we consider Euler's classical Gamma function Γ, the q-gamma function Γ_q, the hyperbolic gamma function Γ_h and the elliptic gamma function Γ_e. We try to give a uniform exposition of the main properties of these different gamma functions and give some relations between the different gamma functions. A uniform theory for these four gamma functions was first given by Ruijsenaars [63]. Our exposition is intended to highlight the similarities of the gamma functions. It does not contain any new results (except for the proof of Proposition 1.3.3) and is included for the convenience of the reader, because the gamma functions play such a prominent role throughout the thesis.

Throughout this chapter we use the convention for $a \in \mathbb{C}\backslash\mathbb{R}_{\leq 0}$ that $a^z = \exp(z\log(a))$, where we take the branch of the logarithm which is positive on the positive real line and has a branch cut along the negative real line.

2.2 Gamma functions

We begin the discussion by giving a definition of all gamma functions involved.

Definition 2.2.1. *i) The Gamma function is defined for* $\Re(z) > 0$ *by the integral*

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-z}dz. \tag{2.2.1}$$

ii) The q-gamma function is defined for $|q| < 1$, $q \notin (-1, 0]$ *as an infinite product by*

$$\Gamma_q(z) = (1-q)^{1-z}\prod_{r=0}^{\infty}\frac{1-q^{r+1}}{1-q^{r+z}}.$$

iii) Let $\omega_1, \omega_2 \in \mathbb{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$*. The hyperbolic gamma function is defined for* $0 < \Im(z) < \Im(\omega_1 + \omega_2)$ *by*

$$\Gamma_h(z) = \exp\left(i \int_0^\infty \left(\frac{2z - \omega_1 - \omega_2}{2t\omega_1\omega_2} - \frac{\sin(t(2z - \omega_1 - \omega_2))}{2\sin(\omega_1 t)\sin(\omega_2 t)}\right)\frac{dt}{t}\right). \quad (2.2.2)$$

iv) The elliptic gamma function is defined for $|p|, |q| < 1$ *as the infinite product*

$$\Gamma_e(z; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - p^{j+1}q^{k+1}/z}{1 - p^j q^k z}.$$

Instead of the q-gamma function we will sometimes use the q-shifted factorials given by (1.2.1). Note that the integrand in the definition of Γ_h does not have a singularity near zero, as the singularities of the two terms in the integrand cancel each other. Moreover at infinity the integrand converges to zero as $1/t^2$ and thus the integral converges.

Both $\Gamma(z)$ and $\Gamma_h(z)$ have unique analytic extensions to a meromorphic function on $z \in \mathbb{C}$. The fact that such an extension exists can be shown using the difference equations satisfied by both functions, which we consider momentarily.

In Figure 2.1 we give a diagram indicating the relations between the different gamma functions. The solid lines represent limit transitions (which we will make explicit later), while the dotted line indicates the concrete connection between the hyperbolic gamma function and the q-gamma function, as discussed already in Section 1.3.

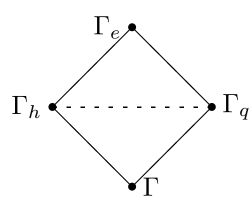


Figure 2.1: The different gamma functions and their relations

There exist many different conventions on how to express the hyperbolic gamma function. Indeed we use two different conventions in this thesis. The function $G(z)$ (not to be confused with the solution G to the doubled system (1.3.10) from Section 1.3) used in Chapters 3 and 4 was introduced in [63] and is related to Γ_h via

$$\Gamma_h(z; \omega_1, \omega_2) = G(-i\omega_1, -i\omega_2; z - \frac{1}{2}\omega_1 - \frac{1}{2}\omega_2). \quad (2.2.3)$$

In particular this implies that the condition $\Im(\omega_1), \Im(\omega_2) > 0$ in this chapter and Chapter 5 corresponds to the condition $\Re(\omega_1), \Re(\omega_2) > 0$ in Chapters 3 and 4. Moreover the hyperbolic gamma function is related to the double Sine [43] function S_2 (itself a quotient of two copies of Barnes double gamma function) by

$$\Gamma_h(z; \omega_1, \omega_2) = S_2(z \mid \omega_1, \omega_2)^{-1}.$$

Finally, as discussed in Section 1.3, for $\omega_1/\omega_2 \in \mathbb{R}_{>0}$ we can express the hyperbolic gamma function as a quotient of two q-shifted factorials by Shintani's [70] product formula, which gives the link to the τ-shifted factorials of [77]

$$\Gamma_h(z;\omega_1,\omega_2) = \exp\left(\pi i \frac{(2z-\omega_1-\omega_2)^2}{8\omega_1\omega_2} - \pi i \frac{(\omega_1^2+\omega_2^2)}{24\omega_1\omega_2}\right) \times \frac{(\exp(-2\pi i(z-\omega_2)/\omega_1);\exp(2\pi i\omega_2/\omega_1))_\infty}{(\exp(-2\pi i z/\omega_2);\exp(-2\pi i\omega_1/\omega_2))_\infty}. \quad (2.2.4)$$

Let us now consider the first order difference equations satisfied by the different gamma functions.

Proposition 2.2.2. *i) For $\Re(z) > 0$ we have*

$$\Gamma(z+1) = z\Gamma(z). \quad (2.2.5)$$

ii) The q-gamma function, respectively the infinite q-shifted factorials satisfy

$$\Gamma_q(z+1) = \frac{1-q^z}{1-q}\Gamma_q(z), \qquad (1-z)(zq;q)_\infty = (z;q)_\infty.$$

iii) The hyperbolic gamma function satisfies for $0 < \Im(z) < \Im(\omega_1)$

$$\Gamma_h(z+\omega_2) = 2\sin\left(\frac{\pi z}{\omega_1}\right)\Gamma_h(z), \quad (2.2.6)$$

and a similar equation with $\omega_1 \leftrightarrow \omega_2$.

iv) The elliptic gamma function satisfies

$$\Gamma_e(pz) = \theta(z;q)\Gamma_e(z)$$

and a similar equation with $p \leftrightarrow q$, where $\theta(z;q) = (z,q/z;q)_\infty$ is a renormalization of the Jacobi theta function (in view of Jacobi's triple product identity).

Proof. The proofs of the difference equation for the q-gamma function (and thus also the q-shifted factorials) and the elliptic gamma function follow directly from their definition. The difference equation for the Gamma function is well-known and dates back to Euler; a proof can be given by using integration by parts in the integral expression for the Gamma function.

A proof of the difference equation for the hyperbolic gamma function is given in [63]; here we give a slightly modified proof. By symmetry it suffices to prove one of the two difference equations, let us therefore prove (2.2.6). Observe that the conditions on z imply we can use (2.2.2) for both $\Gamma_h(z)$ and $\Gamma_h(z+\omega_2)$. Thus we find

$$\begin{aligned}\frac{\Gamma_h(z+\omega_2)}{\Gamma_h(z)} &= \exp\Bigg(i\int_0^\infty \frac{2z+\omega_2-\omega_1}{2t^2\omega_1\omega_2} - \frac{\sin(t(2z+\omega_2-\omega_1))}{2t\sin(\omega_1 t)\sin(\omega_2 t)}dt \\ &\qquad - i\int_0^\infty \frac{2z-\omega_1-\omega_2}{2t^2\omega_1\omega_2} - \frac{\sin(t(2z-\omega_1-\omega_2))}{2t\sin(\omega_1 t)\sin(\omega_2 t)}dt\Bigg) \\ &= \exp\left(i\int_0^\infty \frac{1}{t^2\omega_1} - \frac{\sin(t(2z+\omega_2-\omega_1))-\sin(t(2z-\omega_1-\omega_2))}{2t\sin(\omega_1 t)\sin(\omega_2 t)}dt\right) \\ &= \exp\left(i\int_0^\infty \frac{1}{t^2\omega_1} - \frac{\cos(t(2z-\omega_1))}{t\sin(\omega_1 t)}dt\right),\end{aligned}$$

using the trigonometric identity $\sin(a)-\sin(b) = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})$ in the final equality. The result now follows from the integral evaluation

$$\int_0^\infty \frac{1}{t^2\omega_1} - \frac{\cos(t(2z-\omega_1))}{t\sin(\omega_1 t)}dt = -i\log(2\sin(\pi z/\omega_1)),$$

where we take the principal branch of the logarithm. This is a rather tricky integral evaluation[1]. However, the integral in question is the sum of the integrals

$$\int_0^\infty \frac{1}{t^2\omega_1} - \frac{1}{t\sin(\omega_1 t)}dt = -i\log(2),$$

which is a rescaled version of [17, 3.529, no. 1], and

$$\int_0^\infty \frac{1-\cos(t(2z-\omega_1))}{t\sin(\omega_1 t)}dt = -i\log(\sin(\pi z/\omega_1)),$$

which follows from [17, 3.529, no. 2] by an analytic extension in the variables. □

Using the difference equation for $\Gamma(z)$ we can extend its definition as a meromorphic function first to the region $\Re(z) > -1$, by defining $\Gamma(z) = \Gamma(z+1)/z$ for $-1 < \Re(z) \leq 0$, and subsequently by induction we can define $\Gamma(z)$ as a meromorphic function for $z \in \mathbb{C}$. Similarly the difference equation in the direction ω_1 for the hyperbolic gamma function allows us to extend its definition as a meromorphic function in z, first to $\Im(2\omega_1+\omega_2) > \Im(z) > -\Im(\omega_1)$, and subsequently to $z \in \mathbb{C}$. Clearly these extensions are the unique meromorphic extensions of these functions to the whole complex plane. Moreover the difference equations (2.2.5) and (2.2.6) are valid as identities between meromorphic functions on $\mathbb{C}$.

Let us now consider the location of the poles and zeros of the gamma functions. Observe that the definition (2.2.1) as an analytic function for $\Re(z) > 0$ together with the difference equation (2.2.5) implies that Γ can have poles only in the non-positive integers. Similarly the locations of the poles and zeros of the other gamma functions are not surprising given the fact they satisfy the difference equations of Proposition 2.2.2.

[1]Indeed, neither Maple (10) nor Mathematica (5.2) is able to calculate this integral.

Proposition 2.2.3. *i)* $\Gamma(z)$ *has no zeros, and poles only at* $\mathbb{Z}_{\leq 0}$*. These poles are simple.*

ii) For fixed values of $0 < |q| < 1$ *the function* $(z;q)_\infty$ *has no poles, and zeros at* $z = q^{-n}$ $(n \in \mathbb{Z}_{\geq 0})$*. The zeros are all simple.*

iii) For fixed values of $\omega_1, \omega_2 \in \mathbb{H}$ *the hyperbolic gamma function* $\Gamma_h(z;\omega_1,\omega_2)$ *has zeros at* $z = \omega_1\mathbb{Z}_{\geq 1} + \omega_2\mathbb{Z}_{\geq 1}$ *and poles at* $\omega_1\mathbb{Z}_{\leq 0} + \omega_2\mathbb{Z}_{\leq 0}$*. For generic values of* ω_1 *and* ω_2 *these poles and zeros are all simple, in fact the multiplicity of the pole at* $z = z_0$ *is equal to the number of ways to write* $z_0 = k\omega_1 + l\omega_2$ *for* $k, l \in \mathbb{Z}_{\leq 0}$*. In particular the pole at* $z = 0$ *is always simple. A similar result on the multiplicity of the zeros holds.*

iv) For fixed $0 < |p|, |q| < 1$ *the elliptic gamma function has an essential singularity at* $z = 0$*, zeros at* $z = p^{j+1}q^{k+1}$ *for* $j, k \in \mathbb{Z}_{\geq 0}$ *and poles at* $z = p^{-j}q^{-k}$ *for* $j, k \in \mathbb{Z}_{\geq 0}$*. These zeros and poles are all simple for generic values of* p *and* q*.*

Proof. The result for the Gamma function is well known and a proof can for example be found in [81]. The results for the q-shifted factorials and the elliptic gamma function follow directly from their product representation. A proof of the result for the hyperbolic gamma function is given in [63] and basically only uses that Γ_h is defined as an analytic function without zeros in the strip $0 < \Im(z) < \Im(\omega_1 + \omega_2)$, together with the difference equation (2.2.6). □

Due to the difference equation (2.2.5) the Gamma function Γ is quasi-periodic (i.e. satisfies a non-trivial first order difference equation) in one direction. The q-gamma function is quasi-periodic in one direction and periodic in another (namely $2\pi i/\log(q)$). The hyperbolic gamma function is quasi-periodic in two directions. The elliptic gamma function is also quasi-periodic in two directions, and moreover satisfies a "hidden" periodicity in the log of the argument z (as in the transformation of a q-difference equation to a consistent system of ordinary difference equations discussed in Section 1.3).

In Figure 2.1 the three horizontal levels thus correspond to functions which have the same number of (quasi)-periods, 3 for Γ_e, 2 for Γ_h and Γ_q, and 1 for Γ itself. Limit transitions between these functions can be obtained by sending one of the periods to infinity (thus reducing the number of periods in the limit by one). In particular, sending $q \to 1$ in the q-gamma function yields the Gamma function, and amounts to sending the period $2\pi i/\log(q)$ to ∞. Moreover setting $p = 0$ in the elliptic gamma function gives the q-shifted factorial, and corresponds to sending $\log(p) \to -\infty$. Let us now explicitly give these limits, where we are only concerned with pointwise limits (though stronger results do exist). As the q-gamma function and q-shifted factorials differ only by a normalization, in each formula we choose the version which is simplest.

Proposition 2.2.4. *i) For* $z \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$ *we have*

$$\lim_{q \nearrow 1} \Gamma_q(z) = \Gamma(z).$$

ii) For $z \in \mathbb{C}\backslash\mathbb{Z}_{\leq 0}$ *and* $\omega_1, \omega_2 \in \mathbb{H}$ *we have*

$$\lim_{v \searrow 0} \Gamma_h(z\omega_1; \omega_1, \omega_2/v) \left(\frac{2\pi v \omega_1}{\omega_2}\right)^{\frac{1}{2}-z} = \frac{\Gamma(z)}{\sqrt{2\pi}}.$$

iii) For $|q| < 1$ *and* $z \in \mathbb{C}\backslash q^{\mathbb{Z}_{\leq 0}}$ *we have*

$$\Gamma_e(z; q, 0) = \frac{1}{(z;q)_\infty}.$$

iv) For $z \in \mathbb{C}\backslash\left(\omega_1\mathbb{Z}_{\leq 0} + \omega_2\mathbb{Z}_{\leq 0}\right)$ *and* $\omega_1, \omega_2 \in \mathbb{H}$ *we have*

$$\lim_{r \searrow 0} \Gamma_e(e^{2irz}; e^{i\omega_1 r}, e^{i\omega_2 r}) e^{\pi^2 i(2z-\omega_1-\omega_2)/12r\omega_1\omega_2} = \Gamma_h(z; \omega_1, \omega_2).$$

Proof. A proof of the limit between the q-gamma function and the Gamma function can be found in [40, Appendix B]. The transition between the elliptic gamma function and the q-shifted factorials is trivial as it only consists of setting $p = 0$ in the product expansion of the elliptic gamma function and observing most terms become equal to 1. The limits involving the hyperbolic gamma function are due to Ruijsenaars [63, Propositions III.6 and III.12]. □

An important role in this thesis will be played by the reflection equation and its generalizations.

Proposition 2.2.5. *We have*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.2.7}$$

as an equality between meromorphic functions. Moreover we have for $\omega_1, \omega_2 \in \mathbb{H}$

$$\Gamma_h(z)\Gamma_h(\omega_1+\omega_2-z) = 1, \tag{2.2.8}$$

and for $0 < |p|, |q| < 1$

$$\Gamma_e(z)\Gamma_e(pq/z) = 1,$$

as equations between meromorphic functions.

Proof. The reflection equation is well known for $\Gamma(z)$ and for a proof we once again refer to [81]. The reflection equation for the hyperbolic and elliptic gamma function can be directly obtained from their definitions as an integral, respectively as an infinite product (for the hyperbolic gamma function this only shows the result for $0 < \Im(z) < \Im(\omega_1 + \omega_2)$ but by analytic continuation the result must then hold for all z). However we can also give a nice different proof using only the the difference equations and the analytic properties of the gamma functions.

Assume $\omega_1/\omega_2 \notin \mathbb{R}_{>0}$. Observe, using the difference equations for the hyperbolic gamma function, that $\Gamma_h(z)\Gamma_h(\omega_1+\omega_2-z)$ is an elliptic function with periods ω_1 and ω_2. Moreover, in the fundamental domain $[0,1)\omega_1 + [0,1)\omega_2$ the product $\Gamma_h(z)\Gamma_h(\omega_1+\omega_2-z)$ has at most a simple pole at 0. But there do not exist elliptic functions with exactly one simple pole in a fundamental domain and entire elliptic functions are constant, thus we find $\Gamma_h(z)\Gamma_h(\omega_1+\omega_2-z)=c$ for some constant c. Now observe that $\Gamma_h(\omega_1/2+\omega_2/2)=1$ by (2.2.2), hence $c=1$. By analytic continuation (in ω_1 and ω_2) this result also holds for $\omega_1/\omega_2\in\mathbb{R}_{>0}$. For the elliptic gamma function a similar proof exists. □

Observe that the reflection equation (2.2.8) for the hyperbolic gamma function together with Shintani's product representation (2.2.4) is equivalent to Jacobi's theta inversion formula.

One might wonder why we did not include a reflection equation for the q-gamma function. What we would like to consider as the generalization of the reflection equation in the basic hypergeometric setting is

$$(z;q)_\infty(q/z;q)_\infty=\theta(z;q). \tag{2.2.9}$$

As we defined θ as this product this equation has become a triviality. However one could consider the Jacobi triple product (which relates θ with the Jacobi theta function) as the reflection equation on the basic hypergeometric level. Indeed, a proof similar to the above proof of the reflection equation for the hyperbolic gamma function exists for the Jacobi triple product identity.

Observe that the right hand side of (2.2.7) is basically given by the quotient $\Gamma_h(z)/\Gamma_h(z+\omega_1)$. Similarly the theta function, the right hand side of (2.2.9) is equal to the quotient $\Gamma_e(pz)/\Gamma_e(z)$. In this sense the hyperbolic gamma function is related to the Gamma function as the elliptic gamma function is related to the q-gamma function.

We are also interested in the asymptotics of these gamma functions (and in particular of the hyperbolic gamma function). For Euler's Gamma function Γ this is of course Stirling's formula [81, 12.33]. There exist no asymptotic formulas for the elliptic gamma function.

Proposition 2.2.6. *i) For $b\in\mathbb{R}$, $\phi\in(-\pi,\pi)$ we write $z=\exp(i\phi)(a+bi)$ (for $a\in\mathbb{R}$), then we have*

$$\lim_{a\to\infty}\frac{\Gamma(z)}{z^{z-1/2}\exp(-z)\sqrt{2\pi}}=1.$$

The convergence is uniform on compacta of the domain $(b,\phi)\in\mathbb{R}\times(-\pi,\pi)$.

ii) For $b\in\mathbb{R}$, $\phi\in(\pi/2,3\pi/2)$ and $0<|q|<1$ we have (writing again $z=\exp(i\phi)(a+bi)$)

$$\lim_{a\to\infty}(e^z;q)_\infty=1,$$

uniformly on compacta of the domain.

iii) Let $\phi_-=\min(\arg(\omega_1),\arg(\omega_2))$ and $\phi_+=\max(\arg(\omega_1),\arg(\omega_2))$, where we take the principal branch of the argument. For $\omega_1,\omega_2\in\mathbb{H}$, $b\in\mathbb{R}$ and

$\phi \in (\phi_+ - \pi, \phi_-)$ we have (writing once more $z = \exp(i\phi)(a+bi)$)

$$\lim_{a\to\infty} \frac{\Gamma_h(z;\omega_1,\omega_2)}{\exp\left(2\pi i\left(\frac{(2z-\omega_1-\omega_2)^2}{16\omega_1\omega_2} - \frac{\omega_1^2+\omega_2^2}{48\omega_1\omega_2}\right)\right)} = 1, \tag{2.2.10}$$

uniformly on compacta of the domain. Similarly for $\phi \in (\phi_+, \phi_- + \pi)$ we have (with $z = \exp(i\phi)(a+bi)$)

$$\lim_{a\to\infty} \frac{\Gamma_h(z;\omega_1,\omega_2)}{\exp\left(-2\pi i\left(\frac{(2z-\omega_1-\omega_2)^2}{16\omega_1\omega_2} + \frac{\omega_1^2+\omega_2^2}{48\omega_1\omega_2}\right)\right)} = 1,$$

uniformly on compacta of the domain.

Proof. For Stirling's formula (the asymptotics of $\Gamma(z)$) we refer to [81]. For the asymptotics of the q-shifted factorials we observe that for $\phi \in (\pi/2, 3\pi/2)$ we have $e^{z\exp(i\phi)} \to 0$ while the function $a \mapsto (a;q)_\infty$ is continuous at the origin and has value $(0;q)_\infty = 1$.

Proofs for the asymptotics of Γ_h (in order of increasing strength) can be found in [63, Proposition III.4], [65, Theorem A.1] and [61, Theorem 2.2]. □

2.3 Proof of Proposition 1.3.3

Let us begin by repeating the statement of Proposition 1.3.3. Recall Definition 1.3.2 of a super-consistent pair of difference equations.

Proposition 2.3.1. *Suppose the pair of difference equations*

$$G(z+\omega_1) = c(z/\omega_2)G(z), \qquad G(z+\omega_2) = c(z/\omega_1)G(z), \tag{2.3.1}$$

where c is a non-zero meromorphic function on $\mathbb{C}$ independent of ω_1 and ω_2, is consistent for all $\omega_1, \omega_2 \in \mathbb{H}$ and super-consistent for all $\omega_1, \omega_2 \in \mathbb{H}$ with $\omega_1/\omega_2 \in \mathbb{Q}_{>0}$. Then $c(z) = \exp(A(z-1/2))(2\sin(\pi z))^B$ for some constants $A \in \mathbb{C}$ and $B \in \mathbb{Z}$.

For such $c(z)$ there exists a non-trivial solution to (2.3.1) *which is given by*

$$G(z) = \exp(Az^2/2\omega_1\omega_2 - A(\omega_1+\omega_2)z/2\omega_1\omega_2)\Gamma_h(z)^B.$$

Proof. The consistency of the system of difference equations (2.3.1) implies

$$c((z+\omega_2)/\omega_2)c(z/\omega_1) = c((z+\omega_1)/\omega_1)c(z/\omega_2).$$

Inserting new variables $w = z/\omega_1$ and $v = z/\omega_2$ we obtain

$$c(w+1)c(v) = c(v+1)c(w), \tag{2.3.2}$$

or $c(w+1)/c(w) = c(v+1)/c(v)$. As we can choose ω_1, ω_2 and z such that (2.3.2) holds for arbitrary v and w (as long as $v/w \notin \mathbb{R}_{\leq 0}$), we obtain that $c(w+1)/c(w)$ is a constant function of w.

Superconsistency of the system (2.3.1) for $\omega_1 = 1$ and $\omega_2 = n$ implies

$$c(z) = \prod_{j=0}^{n-1} c((z+j)/n). \tag{2.3.3}$$

Since $c(z)$ is meromorphic we can take the logarithmic derivative of both sides of this equation to obtain

$$\frac{c'(z)}{c(z)} = \frac{1}{n}\sum_{j=0}^{n-1} \frac{c'((z+j)/n)}{c((z+j)/n)}.$$

Introducing $\gamma(z) = c'(z)/c(z)$ we see

$$n\gamma(nz) = \sum_{j=0}^{n-1} \gamma(z + \frac{j}{n}). \tag{2.3.4}$$

Note that $\gamma(z+1) = c'(z+1)/c(z+1) = c'(z)/c(z) = \gamma(z)$, so γ is a 1-periodic meromorphic solution to (2.3.4) for all n.

We will show that 1-periodic meromorphic solutions to (2.3.4) for all n are of the form $A + B\cot(\pi z)$, for $A, B \in \mathbb{C}$. First of all we observe that (2.3.4) is a linear equation in γ, that constant functions satisfy the equation, and that taking the logarithmic derivative in the equation

$$\sin(\pi z) = 2^{n-1}\prod_{j=0}^{n-1} \sin\left(\pi\frac{z+j}{n}\right)$$

(which is equivalent to (1.3.16) after expansion of the sines into exponentials) differentiated to z implies that $\cot(\pi z)$ satisfies (2.3.4). Thus indeed $A + B\cot(\pi z)$ are 1-periodic solutions to (2.3.4) for all $n \in \mathbb{N}$ and all $A, B \in \mathbb{C}$.

For the moment we restrict ourselves to solutions γ to (2.3.4), restricted to the real line. Suppose a 1-periodic solution γ to (2.3.4) with $n = 2$ has a pole in $z_0 \in (0,1)$. Then we see that γ also has a pole in at least one of $z_0/2$ and $(z_0+1)/2$. And inductively we see that γ has a pole in a point of the set $\{(z_0+j)2^{-n} \mid 0 \leq j \leq 2^n - 1\}$ for all $n \geq 1$. If however γ has an infinite number of poles in the interval $(0,1)$, the function $1/\gamma$ has infinitely many zeros in $(0,1)$, thus it has an accumulation point of zeros and therefore $1/\gamma(z) = 0$ identically, which is impossible. Thus one of the poles from one of these sets must equal a pole of a previous set, which implies that $(z_0+p_1)2^{-n_1} = (z_0+p_2)2^{-n_2}$ for some p_1, p_2, n_1, n_2 (with $n_1 \neq n_2$). This can be rewritten as $z_0 = p/(2^n - 1)$ for some $p, n \in \mathbb{N}$, in particular $z_0 \in \mathbb{Q}$.

So a 1-periodic solution γ to (2.3.4) for all n can only have a finite number of rational poles in $(0,1)$. Suppose $z_0 = p/q$ for some $p, q \in \mathbb{N}$ with $gcd(p,q) = 1$ and maximal q is such a pole. Since γ satisfies (2.3.4) for $n = q$ we see that γ also has a pole in at least one of $\frac{\frac{p}{q}+j}{q} = \frac{p+jq}{q^2}$, for $j \in \{0, 1, \ldots, q-1\}$. Since

$gcd(p + jq, q^2) = 1$ and $q^2 > q$ (note that $q \neq 1$), this is a pole with a bigger denominator, which contradicts the maximality of q. Therefore γ cannot have any poles in the interval $(0, 1)$.

Any 1-periodic solution γ to (2.3.4) for all n therefore only has real poles at the integers. Suppose γ has a pole of order $k \geq 1$ at zero. Expand $\gamma(z)$ around zero to obtain $\gamma(z) = cz^{-k} + \sum_{j>-k} c_j z^j$ with $c \neq 0$. Then (2.3.4) for $n = 2$ implies $2c(2z)^{-k} + 2\sum_{j>-k} c_j (2z)^j = cz^{-k} + \sum_{j>-k} d_j z^j$ for some constants d_j (note that there is no pole at $z = 1/2$), thus we obtain $2^{1-k} = 1$, or $k = 1$.

Now we can subtract the function $c\pi \cot(\pi z)$ (a solution which has a pole of order one in zero) from γ, to obtain a 1-periodic solution γ_2 to (2.3.4) for all n, without any poles in $\mathbb{R}$. As γ_2 is continuous on the compact set $[0, 1]$ its absolute value attains its maximum in a point $z_1 \in [0, 1]$. Applying (2.3.4) with $n = 2$ and $z = z_1$ and using that $|\gamma_2|$ is maximized in z_1 we see that $\gamma_2(z_1) = \gamma_2(z_1/2) = \gamma_2(z_1/2 + 1/2)$. Inductively we therefore obtain that γ_2 attains the same value at all points of the form $z_1 2^{-n} + k2^{-n}$ (for $n \in \mathbb{N}$, $0 \leq k \leq n - 1$). These points form a dense set, and thus γ_2 has to be a constant function on $[0, 1]$. Hence γ_2 is constant on $\mathbb{C}$.

This shows that any 1-periodic meromorphic solution to (2.3.4) for all n is indeed of the form $A + B \cot(\pi z)$. For the original function c this implies that $c(z) = K \exp(Az) \sin(\pi z)^B$, for some constants A, B and K. Since we only look for meromorphic functions c, we insist on $B \in \mathbb{Z}$. Returning to (2.3.3) and observing once again that $c(z) = 2 \sin(\pi z)$ is a solution, we obtain the condition $K = 2^B \exp(-A/2)$. Thus the only solutions to (2.3.2) and (2.3.3) are $c(z) = \exp(Az - A/2)(2 \sin(\pi z))^B$ for some constants $A \in \mathbb{C}$ and $B \in \mathbb{Z}$. Note that for these functions c there indeed exist solutions to (2.3.1), namely

$$G(z) = \exp(Az^2/2\omega_1\omega_2 - A(\omega_1 + \omega_2)z/2\omega_1\omega_2)\Gamma_h(z)^B,$$

where $\Gamma_h(z)$ denotes the hyperbolic gamma function, which follows directly from the difference equations satisfied by the hyperbolic gamma function (Proposition 2.2.2). $\square$

Chapter 3

Ruijsenaars' hypergeometric function and the modular double of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$

This chapter appeared in Advances in Mathematics [5] up to some minor cosmetic changes.

3.1 Introduction

The main goal of this chapter is to construct a solution of two commuting Askey-Wilson second order difference equations using representation theory of the modular double of the quantized universal enveloping algebra $\mathcal{U}_q$ of $\mathfrak{sl}_2(\mathbb{C})$. Furthermore we relate this solution to Ruijsenaars' hypergeometric function from [65].

By [49] there exist three inequivalent $*$-structures on $\mathcal{U}_q$, one associated to the real form $\mathfrak{su}(2)$ of $\mathfrak{sl}_2(\mathbb{C})$, one associated to $\mathfrak{su}(1,1)$, and one to $\mathfrak{sl}_2(\mathbb{R})$. Koornwinder [42], Noumi & Mimachi [53], and Koelink [35] have shown that the Askey-Wilson polynomials arise as matrix coefficients of $*$-unitary irreducible representations of $\mathcal{U}_q(\mathfrak{su}(2))$. To prove these results they used the fact that the Askey-Wilson second order difference operator arises as the radial part of the quantum Casimir in $\mathcal{U}_q$ when calculated with respect to Koornwinder's [42] twisted primitive elements. In [36] and [78] Koelink and Stokman constructed the trigonometric Askey-Wilson functions as matrix coefficients of $*$-unitary irreducible representations of $\mathcal{U}_q(\mathfrak{su}(1,1))$. In this chapter we consider matrix coefficients of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$-representations.

An essential tool is the embedding of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$ in Faddeev's [13] modular double of $\mathcal{U}_q$. The modular double consists of two commuting copies of the quantized universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ with deformation parameters $q = e^{\pi i w_1/w_2}$ and $\tilde{q} = e^{\pi i w_2/w_1}$ ($w_1, w_2 \in \mathbb{R}_{>0}$), respectively. Kharchev et al. [32] made the crucial observation that the algebraic version π_λ of the principal series representation of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$ on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$ can be extended to a representation of the modular double on the same space. In the same article they construct generalized Whittaker functions as matrix coefficients of π_λ.

We construct joint eigenvectors to the action under π_λ of two commuting twisted primitive elements (one for each copy of the quantized universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{R})$ inside the modular double) in terms of Ruijsenaars' [63] hyperbolic gamma function. The action of the two commuting quantum Casimir elements in the modular double shows that the corresponding matrix coefficients, for which we have an explicit integral representation, satisfy Askey-Wilson second order difference equations in step directions iw_1 and iw_2. By construction these matrix coefficients are invariant under interchanging of w_1 and w_2. We show duality of this solution S in its spectral variable λ and its geometric variable. Consequently it satisfies another two Askey-Wilson second order difference equations in its spectral variable.

In a series [65], [67], [68] of papers, Ruijsenaars considered a solution R of the same Askey-Wilson difference equations. These equations arose in his study of relativistic quantum integrable systems. Ruijsenaars defined the hypergeometric function R as a Barnes' type integral with integrand expressed in terms of the hyperbolic gamma function. Subsequently he established for R duality, D_4-symmetry in the parameters, asymptotic behaviour and the reduction to Askey-Wilson polynomials. We use these properties to show equality of R to S, which is not apparent from their explicit integral representations.

The structure of this chapter is as follows. In Sections 3.2 and 3.3 we recall some properties of the hyperbolic gamma function and of Ruijsenaars' hypergeometric function R, respectively. In Section 3.4 we define the modular double of $\mathcal{U}_q$ and its principal series representation on meromorphic functions. In Section 3.5 we consider the corresponding eigenvalue problem of two commuting twisted primitive elements. Using the matrix coefficients of the principal series representation we construct a solution S to the Askey-Wilson difference equations in Section 3.6, and we establish the duality of S. In Section 3.7 we show by a direct calculation that S reduces to the Askey-Wilson polynomials for certain discrete values of the spectral parameter. Finally, in Section 3.8 we show that S equals Ruijsenaars' hypergeometric function R.

3.1.1 Notational conventions

If $\pm$ appears inside the argument of functions we mean a product, e.g.

$$f(z \pm a) = f(z+a)f(z-a). \tag{3.1.1}$$

Otherwise it means that all sign combinations are possible.

Whenever we use a square root, we always mean the branch which has a cut along the negative real line and maps the positive real line to itself.

3.2 The hyperbolic gamma function

Both Ruijsenaars' and our solution to the Askey-Wilson second order difference equations are expressible in terms of the hyperbolic gamma function, which was introduced in [63]. Let us therefore recall some basic properties of this function, see [63] and the appendices of [65] for more details. Note that in this chapter we use the notation G for the hyperbolic gamma function, which is related to Γ_h through (2.2.3).

Let us first define for $w_1, w_2 \in \mathbb{C}_+ = \{z \in \mathbb{C} \mid \Re(z) > 0\}$,

$$g(w_1, w_2; z) = \int_0^\infty \left(\frac{\sin(2yz)}{2\sinh(w_1 y)\sinh(w_2 y)} - \frac{z}{w_1 w_2 y} \right) \frac{dy}{y}. \tag{3.2.1}$$

Notice that the integrand has no pole at 0. To ensure convergence of the integral at infinity however, we must impose the condition $|\Im(z)| < \Re(w)$, where w is defined by

$$w = \frac{w_1 + w_2}{2}.$$

The hyperbolic gamma function $G(z) = G(w_1, w_2; z)$ for $|\Im(z)| < \Re(w)$ is now defined by

$$G(w_1, w_2; z) = e^{ig(w_1, w_2; z)}. \tag{3.2.2}$$

The hyperbolic gamma function G owes its name to the fact that it satisfies the difference equations

$$\begin{aligned} G(z + iw_1/2) &= 2\cosh(\pi z/w_2) G(z - iw_1/2), \\ G(z + iw_2/2) &= 2\cosh(\pi z/w_1) G(z - iw_2/2). \end{aligned} \tag{3.2.3}$$

In these equations we suppress the w_1 and w_2 dependence of G, which we continue to do whenever this does not cause confusion. These two difference equations allow for an analytic continuation of G to a meromorphic function on $\mathbb{C}$. The hyperbolic gamma function can also be expressed in terms of Barnes' double gamma function, or Kurokawa's double sine function. Details can be found in [65, Appendix A].

Let us first note a few symmetries of the hyperbolic gamma function, which are all obvious from (3.2.1):

$$G(w_1, w_2; z) = G(w_2, w_1; z), \tag{3.2.4}$$

$$G(w_1, w_2; z) = G(w_1, w_2; -z)^{-1}, \tag{3.2.5}$$

$$G(w_1, w_2; z) = \overline{G(\bar{w}_1, \bar{w}_2, -\bar{z})}, \tag{3.2.6}$$

$$G(\mu w_1, \mu w_2; \mu z) = G(w_1, w_2; z) \qquad (\mu \in \mathbb{R}_{>0}). \tag{3.2.7}$$

The pole and zero locations of G are easily derived from the difference equations (3.2.3), since G has no poles or zeros in the strip $z \in \mathbb{R} \times i(-\Re(w), \Re(w))$ in view of (3.2.2). The zeros of G are contained in the set

$$\Lambda_+ = iw + iw_1\mathbb{Z}_{\geq 0} + iw_2\mathbb{Z}_{\geq 0} \tag{3.2.8}$$

and the poles in $-\Lambda_+$. The pole at $z = -iw$ is simple, and its residue equals

$$\frac{i}{2\pi}\sqrt{w_1 w_2}. \tag{3.2.9}$$

If w_1/w_2 is irrational all other poles are also simple and their residues can be calculated from (3.2.9) and the difference equations (3.2.3), see [63, Proposition III.3].

For later purposes it is convenient to call an infinite sequence of points in $\mathbb{C}$ increasing (respectively decreasing) if it is contained in a set of the form $a + \Lambda_+$ (respectively $a - \Lambda_+$) for some $a \in \mathbb{C}$. In this terminology, G has one increasing zero-sequence and one decreasing pole-sequence.

We also need an estimate for $G(z)$ as $\Re(z) \to \infty$ and $\Im(z)$ stays bounded. In fact we only need it for the quotient of two hyperbolic gamma functions, which is easily derived from the estimate of the hyperbolic gamma itself as described in [63, Proposition III.4] and [67, (3.3)]. For $a, b \in \mathbb{C}$ and $w_1, w_2 \in (0, \infty)$ the resulting estimate reads

$$\frac{G(z-a)}{G(z-b)} = \exp\left(\frac{\pi}{2iw_1w_2}(2z(b-a) + a^2 - b^2 + f(z))\right), \tag{3.2.10}$$

where $f(z)$ satisfies for $\Re(z) > \max(w_1, w_2) + \max(\Re(a), \Re(b))$,

$$|f(z)| < C(w_1, w_2, \Im(z), a, b)e^{-\pi\Re(z)/\max(w_1,w_2)}, \tag{3.2.11}$$

with C depending continuously on $(0, \infty)^2 \times \mathbb{R} \times \mathbb{C}^2$.

We also use the description of G as a quotient

$$G(z) = \frac{E(z)}{E(-z)}, \tag{3.2.12}$$

where E is an entire function with zeros at Λ_+ which are all simple if w_1/w_2 is irrational. For a precise definition of E, see [65, Appendix A].

We will occasionally meet functions defined by an integral of the form

$$M(u, d) = \int_{\mathbb{R}} \prod_{j=1}^{n} \frac{G(w_1, w_2; z - u_j)}{G(w_1, w_2; z - d_j)} dz \tag{3.2.13}$$

for $w_1, w_2 > 0$ and for parameters u_j and d_j satisfying $|\Im(u_j)|, |\Im(d_j)| < w$ and $\Im(\sum_{j=1}^{n}(u_j - d_j)) > 0$. These conditions ensure that the integral is well defined (the contour meets no poles and it decreases exponentially at $\pm\infty$). In [65, Appendix B] it is shown that

$$M(u, d) \prod_{j,k=1}^{n} E(-iw + u_j - d_k)$$

has a unique analytic extension to the set $\{(w_1, w_2, u, d) \in \mathbb{C}_+^2 \times \mathbb{C}^{2n} \mid \Im(\sum(u_j - d_j)/w_1w_2) > 0\}$. Hence $M(u, d)$ is a meromorphic function which can only have poles when some $E(-iw + u_j - d_k)$ is zero.

3.3 Ruijsenaars' hypergeometric function

Ruijsenaars [65] introduced a generalization R of the hypergeometric function as a Barnes' type integral. We recall several properties of R from [65] and [67] which we will need to relate R to the formal matrix coefficients we are going to define in subsequent sections.

We define Ruijsenaars' hypergeometric function in terms of a parameter set γ_μ ($\mu = 0, 1, 2, 3$), which is related to Ruijsenaars' original c-parameters by [67, (1.11)]. Dual parameters $\hat{\gamma}_\mu$ are defined as

$$\begin{pmatrix} \hat{\gamma}_0 \\ \hat{\gamma}_1 \\ \hat{\gamma}_2 \\ \hat{\gamma}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}. \tag{3.3.1}$$

We denote the set of parameters $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ by γ and the set of dual parameters by $\hat{\gamma}$. Note that taking dual parameters is an involution, $\hat{\hat{\gamma}} = \gamma$.

Ruijsenaars' hypergeometric function R is now defined for generic parameters $w_1, w_2 \in \mathbb{C}_+, \gamma \in \mathbb{C}^4$ by

$$R(\gamma; x, \lambda) = \\ \frac{1}{\sqrt{w_1 w_2}} \int_{\mathcal{C}} \frac{G(z \pm x + i\gamma_0) G(z \pm \lambda + i\hat{\gamma}_0)}{G(\pm x + i\gamma_0) G(\pm \lambda + i\hat{\gamma}_0) G(z + iw)} \prod_{j=1}^{3} \frac{G(i\gamma_0 + i\gamma_j + iw)}{G(z + i\gamma_0 + i\gamma_j + iw)} dz. \tag{3.3.2}$$

Note that we use the convention (3.1.1) in this expression. The integral is taken over a contour $\mathcal{C}$, which is a deformation of $\mathbb{R}$ separating the poles of the numerator from the zeros of the denominator (equivalently, $\mathcal{C}$ separates the increasing pole sequences of the integrand from the downward pole sequences). R has an analytic extension to a meromorphic function on $(w_1, w_2, \gamma, x, \lambda) \in \mathbb{C}_+^2 \times \mathbb{C}^6$, with possible poles for fixed values of w_1, w_2, and γ at

$$x \in \pm(\Lambda_+ - i\gamma_j), \qquad \lambda \in \pm(\Lambda_+ - i\hat{\gamma}_j) \qquad (j = 0, 1, 2, 3). \tag{3.3.3}$$

Recall that Λ_+ is defined by (3.2.8).

We now look at the Askey-Wilson second order difference equations which R satisfies. The equations are obtained from [65, Theorem 3.1] by not only replacing the c-variables by γ, but also multiplying the equations by a constant. These descriptions of the Askey-Wilson difference equations are more convenient for the representation theoretic approach we consider in the following sections.

Let us define the function A by

$$A(w_1,w_2,\gamma;x) = -\frac{e^{\pi i w_1/w_2 + 2\pi i\hat{\gamma}_0/w_2}}{\sinh\left(\frac{2\pi x}{w_2}\right)\sinh\left(\frac{2\pi(x+iw)}{w_2}\right)}\prod_{j=0}^{3}\cosh\left(\frac{\pi}{w_2}(x+\frac{iw_1}{2}+i\gamma_j)\right)$$
$$= \frac{1}{(1-e^{4\pi x/w_2})(1-e^{4\pi(x+iw)/w_2})}\prod_{j=0}^{3}(1-e^{2\pi(iw+i\gamma_j+x)/w_2}). \tag{3.3.4}$$

The Askey-Wilson second order difference operator $\mathcal{L}^x_\gamma$ is defined by

$$\mathcal{L}^x_\gamma f(x) = A(w_1,w_2,\gamma;x)(f(x+iw_1)-f(x)) + A(w_1,w_2,\gamma;-x)(f(x-iw_1)-f(x)). \tag{3.3.5}$$

Here the superscript x is added to emphasize that the operator acts on the x-variable (in a moment we will also consider the operator $\mathcal{L}$ acting on the spectral variable λ). We write $\widetilde{\mathcal{L}}^x_\gamma$ for the Askey-Wilson operator (3.3.5) with w_1 and w_2 interchanged.

Ruijsenaars' hypergeometric function R is an eigenfunction of four Askey-Wilson second order difference operators with eigenvalues expressible in terms of

$$v(w_1,w_2,\gamma;\lambda) = -2e^{\pi i w_1/w_2+2\pi i\hat{\gamma}_0/w_2}(\cosh(2\pi\lambda/w_2)+\cosh(\pi i w_1/w_2+2\pi i\hat{\gamma}_0/w_2)). \tag{3.3.6}$$

Specifically, R satisfies the difference equations

$$\begin{aligned}
\mathcal{L}^x_\gamma R(\gamma;x,\lambda) &= v(w_1,w_2,\gamma;\lambda)R(\gamma;x,\lambda),\\
\widetilde{\mathcal{L}}^x_\gamma R(\gamma;x,\lambda) &= v(w_2,w_1,\gamma;\lambda)R(\gamma;x,\lambda),\\
\mathcal{L}^\lambda_{\hat{\gamma}} R(\gamma;x,\lambda) &= v(w_1,w_2,\hat{\gamma};x)R(\gamma;x,\lambda),\\
\widetilde{\mathcal{L}}^\lambda_{\hat{\gamma}} R(\gamma;x,\lambda) &= v(w_2,w_1,\hat{\gamma};x)R(\gamma;x,\lambda).
\end{aligned} \tag{3.3.7}$$

Actually the last three of these equations follow from the first by various symmetries of R. The second difference equation can be derived from the first (and the fourth from the third) by using the fact that R is invariant under the exchange of w_1 and w_2,

$$R(w_1,w_2,\gamma;x,\lambda) = R(w_2,w_1,\gamma;x,\lambda).$$

This symmetry can be directly seen from the definition (3.3.2) of R and the corresponding symmetry (3.2.4) of the hyperbolic gamma function. The third difference equation can be obtained from the first by using the duality of R under the exchange of x and λ,

$$R(\gamma;x,\lambda) = R(\hat{\gamma};\lambda,x). \tag{3.3.8}$$

This duality is also a direct consequence of the definition of R using the fact that $\gamma_0 + \gamma_j = \hat{\gamma}_0 + \hat{\gamma}_j$ for $j = 1, 2, 3$.

There are more symmetries of R directly visible from the definition. Since the hyperbolic gamma function is scale invariant it follows that R is scale invariant as well,

$$R(\nu w_1, \nu w_2, \nu\gamma; \nu x, \nu\lambda) = R(w_1, w_2, \gamma; x, \lambda)$$

for $\nu \in (0, \infty)$, where $\nu\gamma$ denotes the scaled parameter set $(\nu\gamma_0, \nu\gamma_1, \nu\gamma_2, \nu\gamma_3)$. Furthermore it is immediately clear that R is symmetric under permutations of γ_1, γ_2, and γ_3. This symmetry can be extended to a D_4-symmetry in the four parameters γ (where the Weyl group of type D_4 acts on the parameters by permutations and an even number of sign flips). To formulate this result we need the c-function

$$c(\gamma; y) = \frac{1}{G(2y + iw)} \prod_{j=0}^{3} G(y - i\gamma_j)$$

and the normalization constant

$$N(\gamma) = \prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + iw). \tag{3.3.9}$$

The D_4-symmetry [67, Theorem 1.1] of R then reads

$$\frac{R(\gamma; x, \lambda)}{c(\gamma; x)c(\hat{\gamma}; \lambda)N(\gamma)} = \frac{R(w(\gamma); x, \lambda)}{c(w(\gamma); x)c(\widehat{w(\gamma)}; \lambda)N(w(\gamma))} \tag{3.3.10}$$

for all elements w of the Weyl group of type D_4. Notice that both the c-function and N are invariant under the action of the S_3-subgroup which permutes γ_1, γ_2, and γ_3.

Finally we recall the limit behaviour of R, cf. [67, Theorem 1.2]. Set $\alpha = 2\pi/w_1 w_2$. For $w_1, w_2 \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{R}^4$, and $w_1 \neq w_2$ there exists an open neighbourhood $U \subset \mathbb{C}$ of $\mathbb{R}$, such that the asymptotics of R for fixed $\lambda \in U$ are given by

$$R(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)| - \hat{\gamma}_0 - w)|\Re(x)|}) \tag{3.3.11}$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compacta. In fact, Ruijsenaars gives a precise expression for the leading term of R as $\Re(x) \to \pm\infty$ when $\lambda \in \mathbb{R}$. These results easily extend to λ in some open neighbourhood U of $\mathbb{R}$.

3.4 The modular double of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$

In this section we consider a slightly extended version of Faddeev's [13] modular double of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ and define an algebraic version of its principal series representation on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$. We define an inner product on some suitable subspace of $\mathcal{M}$, which is compatible to the $*$-structure on $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ associated to the real form $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{sl}_2(\mathbb{C})$, cf. [49].

Throughout Sections 3.4, 3.5 and 3.6 we assume that w_1 and w_2 are positive real numbers such that neither w_1/w_2 nor w_2/w_1 is an integer, unless specifically stated otherwise. We define

$$q = \exp(\pi i w_1/w_2), \qquad \tilde{q} = \exp(\pi i w_2/w_1),$$

which both lie on the unit circle (but they are not ± 1). For complex numbers β we define

$$q^\beta = e^{\beta\pi i w_1/w_2}, \qquad \tilde{q}^\beta = e^{\beta\pi i w_2/w_1}.$$

Definition 3.4.1. *The quantized universal enveloping algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$ of $\mathfrak{sl}_2(\mathbb{C})$ is the unital associative algebra over $\mathbb{C}$ generated by $K^{\pm 1}$, E, and F, subject to the relations*

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1,\\ KE &= q^2 EK,\\ KF &= q^{-2}FK,\\ EF - FE &= \frac{K - K^{-1}}{q - q^{-1}}. \end{aligned}$$

If w_1/w_2 is irrational, then the center of $\mathcal{U}_q$ is generated by the quantum Casimir element Ω, defined as

$$\Omega = qK + q^{-1}K^{-1} + (q - q^{-1})^2 FE.$$

By simply replacing q by $\tilde{q}$ (or interchanging w_1 and w_2) we obtain the quantum universal enveloping algebra $\mathcal{U}_{\tilde{q}}$. The generators of $\mathcal{U}_{\tilde{q}}$ are denoted by $\tilde{K}^{\pm 1}$, $\tilde{E}$, and $\tilde{F}$. The following concept of modular double was introduced by Faddeev [13].

Definition 3.4.2. *The modular double $\mathcal{Q}$ is $\mathcal{U}_q \otimes \mathcal{U}_{\tilde{q}}$ endowed with its standard tensor product algebra structure.*

For elements $X \in \mathcal{U}_q$ (respectively $\tilde{X} \in \mathcal{U}_{\tilde{q}}$) we also write X (respectively $\tilde{X}$) for its image under the natural embedding of $\mathcal{U}_q$ (respectively $\mathcal{U}_{\tilde{q}}$) in $\mathcal{Q}$. In particular, $X\tilde{X} = \tilde{X}X$ in $\mathcal{Q}$ for elements $X \in \mathcal{U}_q$ and $\tilde{X} \in \mathcal{U}_{\tilde{q}}$.

We now define an extension of the modular double by formally adjoining complex powers of K and $\tilde{K}$ to $\mathcal{Q}$. Let $\mathcal{A} = \bigoplus_{x\in\mathbb{C}} \mathbb{C}\hat{x}$ be the group algebra of the additive group $\hat{\mathbb{C}} = (\mathbb{C}, \oplus)$, where $\oplus$ is the translated addition $\hat{x} \oplus \hat{y} = \widehat{x+y+iw}$ (this translation in addition will make formulas simpler later on). The unit of $\hat{\mathbb{C}}$ is $\widehat{-iw}$.

Lemma 3.4.3. *There exists a unique left $\mathcal{A}$-action by algebra automorphisms on the modular double $\mathcal{Q}$ satisfying*

$$\begin{aligned} \hat{x} \cdot K^{\pm 1} &= K^{\pm 1}, & \hat{x} \cdot \tilde{K}^{\pm 1} &= \tilde{K}^{\pm 1},\\ \hat{x} \cdot E &= -q e^{2\pi x/w_2} E, & \hat{x} \cdot \tilde{E} &= -\tilde{q} e^{2\pi x/w_1} \tilde{E},\\ \hat{x} \cdot F &= -q e^{-2\pi x/w_2} F, & \hat{x} \cdot \tilde{F} &= -\tilde{q} e^{-2\pi x/w_1} \tilde{F}. \end{aligned}$$

Proof. Observe that e.g. the action of $\hat{x}$ on E can be rewritten as

$$\hat{x} \cdot E = e^{2\pi(x+iw)/w_2} E.$$

The lemma now follows by direct calculations. □

Definition 3.4.4. *The extended modular double $\mathcal{D} = \mathcal{Q} \rtimes \mathcal{A}$ is the crossed product of the modular double $\mathcal{Q}$ and the algebra $\mathcal{A}$ under its action on $\mathcal{Q}$ as defined in Lemma 3.4.3.*

Hence $\mathcal{D}$ is the vector space $\mathcal{Q} \otimes \mathcal{A}$ endowed with the unique algebra structure such that the natural embeddings of $\mathcal{Q}$ and $\mathcal{A}$ in $\mathcal{D}$ are algebra morphisms and such that

$$\hat{x}Q = (\hat{x} \cdot Q)\hat{x}, \quad \forall x \in \mathbb{C},\ \forall Q \in \mathcal{Q},$$

where we identified $\hat{x}$ (respectively Q) with their images under the natural embeddings of $\mathcal{A}$ (respectively $\mathcal{Q}$) in $\mathcal{D}$.

Now we define representations π_λ of the extended modular double $\mathcal{Q}$ on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$ depending on a complex representation label λ, cf. [32]. These representations may be viewed as algebraic versions of the principal series representations of $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))$. We define these representations in terms of the operators T_y and S_y on $\mathcal{M}$, which act by

$$T_y f(z) = f(z+y), \qquad S_y f(z) = e^{2\pi i z/y} f(z) \qquad (y \in \mathbb{C}).$$

Lemma 3.4.5. *For $\lambda \in \mathbb{C}$ the assignments*

$$\begin{aligned}
\pi_\lambda(K) &= T_{iw_1}, \quad \pi_\lambda(\tilde{K}) = T_{iw_2}, \quad \pi_\lambda(\hat{x}) = T_{x+iw},\\
\pi_\lambda(E) &= \frac{q^{1/2}}{q-q^{-1}} S_{iw_2}\left(q^{-1/2}e^{\pi\lambda/w_2} + q^{1/2}e^{-\pi\lambda/w_2}T_{iw_1}\right),\\
\pi_\lambda(F) &= -\frac{q^{1/2}}{q-q^{-1}} S_{-iw_2}\left(q^{-1/2}e^{\pi\lambda/w_2} + q^{1/2}e^{-\pi\lambda/w_2}T_{-iw_1}\right),\\
\pi_\lambda(\tilde{E}) &= \frac{\tilde{q}^{1/2}}{\tilde{q}-\tilde{q}^{-1}} S_{iw_1}\left(\tilde{q}^{-1/2}e^{\pi\lambda/w_1} + \tilde{q}^{1/2}e^{-\pi\lambda/w_1}T_{iw_2}\right),\\
\pi_\lambda(\tilde{F}) &= -\frac{\tilde{q}^{1/2}}{\tilde{q}-\tilde{q}^{-1}} S_{-iw_1}\left(\tilde{q}^{-1/2}e^{\pi\lambda/w_1} + \tilde{q}^{1/2}e^{-\pi\lambda/w_1}T_{-iw_2}\right),
\end{aligned}$$

uniquely define a representation π_λ of $\mathcal{D}$ on $\mathcal{M}$.

Observe that the action of the generators of $\mathcal{U}_{\tilde{q}}$ are obtained from the action of the generators of $\mathcal{U}_q$ by interchanging w_1 and w_2.

Proof. The defining relations of $\mathcal{D}$ are easily checked using $T_xT_y = T_{x+y} = T_yT_x$, $S_xS_{-x} = 1$, and the equation

$$T_xS_y = e^{2\pi ix/y}S_yT_x.$$

□

Remark 3.4.6. Denote $v = (w_1 - w_2)/2$, then $\pi_\lambda(\widehat{iv}) = \pi_\lambda(K)$ and $\pi_\lambda(\widehat{-iv}) = \pi_\lambda(\tilde{K})$. The extension of the modular double $\mathcal{Q}$ by $\mathcal{A}$ and the extension of the representation $\pi_\lambda|_{\mathcal{Q}}$ to π_λ thus have the effect of introducing non-integral powers of T_{iw_1} and T_{iw_2} in the image of π_λ. The introduction of this extension is not an essential part of the analysis later on and is only included for simplification. Using only integral powers of K and $\tilde{K}$ we can simulate the action of $\hat{x}$ for x in some dense subset of $\mathbb{R}$, cf. [32, Proposition 1.6].

A simple calculation shows that $\pi_\lambda(\Omega)$ acts as a scalar,

$$\pi_\lambda(\Omega)f = -2\cosh(2\pi\lambda/w_2)f, \qquad f \in \mathcal{M}. \tag{3.4.1}$$

Since π_λ is an algebraic version of the principal series representation with representation label $\lambda \in \mathbb{C}$, this is as expected.

Definition 3.4.7. *We say that $f \in \mathcal{M}$ has exponential growth with growth rate $\epsilon \in \mathbb{R}$ if there exists a compact set $K_f \in \mathbb{R}$ such that all poles of f are contained in $K_f \times i\mathbb{R} = \{x + iy \mid x \in K_f, y \in \mathbb{R}\}$ and if $|f(x+iy)| = \mathcal{O}(\exp(\epsilon|x|))$ for $x \to \pm\infty$, uniformly for y in compacta of $\mathbb{R}$.*

On the space of meromorphic functions which have negative exponential growth and which have no poles on $\mathbb{R}$, we define a sesquilinear form by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(z)\overline{g(z)}dz. \tag{3.4.2}$$

Observe that this expression is already well defined under the milder asymptotic condition that the sum of the two exponential growths of f and g is negative. Note furthermore that (3.4.2) can be rewritten as

$$\langle f, g\rangle = \int_{\mathbb{R}} f(z)\bar{g}(z)dz, \tag{3.4.3}$$

where $\bar{g}(z) := \overline{g(\bar{z})}$ now is analytic at $z \in \mathbb{R}$.

Following [32] we define an antilinear anti-algebra involution $*$ on the extended modular double $\mathcal{D}$ by

$$K^* = K,\ E^* = -E,\ F^* = -F,\ \tilde{K}^* = \tilde{K},\ \tilde{E}^* = -\tilde{E},\ \tilde{F}^* = -\tilde{F},\ \hat{x}^* = \widehat{-\bar{x}}. \tag{3.4.4}$$

If we restrict this involution to $\mathcal{U}_q$ (respectively $\mathcal{U}_{\tilde{q}}$) we obtain the $*$-structure on $\mathcal{U}_q$ (respectively $\mathcal{U}_{\tilde{q}}$) corresponding to the noncompact real form $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{sl}_2(\mathbb{C})$, cf. [49].

The following lemma relates the sesquilinear form (3.4.2) to the $*$-structure (3.4.4) on $\mathcal{D}$.

Lemma 3.4.8. *Let $\lambda \in \mathbb{C}$ and $f, g \in \mathcal{M}$. If the poles of f and g are outside the strip $\mathbb{R} \times i[-w_1, w_1]$ and if the sum of the exponential growth rates of f and g is smaller than $-2\pi/w_2$, then*

$$\langle \pi_\lambda(X)f, g\rangle = \langle f, \pi_{\bar{\lambda}}(X^*)g\rangle$$

for $X \in \mathcal{U}_{q,1} := span_{\mathbb{C}}\{1, E, F, K, K^{-1}, FK, EK^{-1}\}$.

Proof. In view of (3.4.3) the proof follows by a change of variables and some contour shifting using Cauchy's theorem. □

A similar lemma holds for the dual algebra $\mathcal{U}_{\tilde{q}}$.

3.5 Twisted primitive elements and matrix coefficients

Koornwinder [42] introduced twisted primitive elements to obtain the Askey-Wilson polynomials as matrix coefficients of finite dimensional $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$-representations. We recall the definition of twisted primitive elements and show that they act as first order difference operators under the representations π_λ. We construct eigenvectors to these operators in terms of the hyperbolic gamma function and we consider the corresponding formal matrix coefficients ψ of π_λ. In subsequent sections we relate ψ to Ruijsenaars' hypergeometric function.

Let $\rho \in \mathbb{C}$ and set

$$\nu_\rho = q^{2\rho/w_1} + q^{-2\rho/w_1} = 2\cos(2\pi\rho/w_2).$$

The twisted primitive element $Y_\rho \in \mathcal{U}_q \subset \mathcal{D}$ is defined as

$$Y_\rho = iq^{-1/2}E + iq^{-1/2}FK - \frac{\nu_\rho}{q-q^{-1}}(K-1). \tag{3.5.1}$$

Analogously we define the twisted primitive element $\tilde{Y}_\rho \in \mathcal{U}_{\tilde{q}}$ by interchanging w_1 and w_2, viz.

$$\tilde{Y}_\rho = i\tilde{q}^{-1/2}\tilde{E} + i\tilde{q}^{-1/2}\tilde{F}\tilde{K} - \frac{\tilde{\nu}_\rho}{\tilde{q}-\tilde{q}^{-1}}(\tilde{K}-1),$$

where $\tilde{\nu}_\rho = 2\cos(2\pi\rho/w_1)$.

Denoting

$$\mu_\tau(\rho) = \frac{\nu_\rho - \nu_\tau}{q-q^{-1}}, \quad \tilde{\mu}_\tau(\rho) = \frac{\tilde{\nu}_\rho - \tilde{\nu}_\tau}{\tilde{q}-\tilde{q}^{-1}},$$

we now have the following lemma.

Lemma 3.5.1. *The function*

$$H^\lambda_{\tau,\rho}(z) = \frac{G(z+\lambda/2-3iw/2 \pm i\tau)}{G(z-\lambda/2-iw/2 \pm i\rho)}$$

satisfies

$$\begin{aligned}\pi_\lambda(Y_\rho)H^\lambda_{\tau,\rho} &= \mu_\tau(\rho)H^\lambda_{\tau,\rho},\\ \pi_\lambda(\tilde{Y}_\rho)H^\lambda_{\tau,\rho} &= \tilde{\mu}_\tau(\rho)H^\lambda_{\tau,\rho}.\end{aligned} \tag{3.5.2}$$

Proof. Since $H^{\lambda}_{\tau,\rho}$ is invariant under the exchange of w_1 and w_2, it is sufficient to prove only the first eigenvalue equation. A calculation shows that $\pi_\lambda(Y_\rho)f = \mu_\tau(\rho)f$ is equivalent to the first order difference equation

$$f(z+iw_1/2) = \frac{\cosh(\frac{\pi}{w_2}(z+\lambda/2-3iw/2\pm i\tau))}{\cosh(\frac{\pi}{w_2}(z-\lambda/2-iw/2\pm i\rho))} f(z-iw_1/2). \tag{3.5.3}$$

(The exact calculation can be found in Appendix 3.A.) Using the difference equation (3.2.3) for the hyperbolic gamma function it immediately follows that $H^{\lambda}_{\tau,\rho}$ satisfies the difference equation (3.5.3). □

Remark 3.5.2. For any one of the two equations (3.5.2) there are infinitely many solutions (we can e.g. multiply a solution to the first equation by any iw_1-periodic function). The crucial step in finding common solutions to both difference equations is to rewrite the first difference equation in the specific form (3.5.3). Indeed, the resulting solution $H^{\lambda}_{\tau,\rho}$ in terms of hyperbolic gamma functions is invariant under interchanging w_1 and w_2, hence it automatically satisfies the second difference equation. This is the main difference between our analysis and the one in [78].

Now let us consider the adjoint Y^*_σ, which is

$$Y^*_\sigma = iq^{1/2}E + iq^{-3/2}FK + \frac{\nu_{\bar\sigma}}{q-q^{-1}}(K-1).$$

Since $\mu_\upsilon(\sigma)^* = -\mu_{\bar\upsilon}(\bar\sigma)$, we are interested in solutions to the equation $\pi_{\bar\lambda}(Y^*_\sigma)f = -\mu_{\bar\upsilon}(\bar\sigma)f$ and the corresponding equation $\pi_{\bar\lambda}(\tilde Y^*_\sigma)f = -\tilde\mu_{\bar\upsilon}(\bar\sigma)f$ for the second component of the modular double.

Lemma 3.5.3. *The function*

$$F^{\lambda}_{\upsilon,\sigma}(z) = \frac{G(z+\bar\lambda-iw/2\pm i\bar\upsilon)}{G(z-\bar\lambda+iw/2\pm i\bar\sigma)}$$

satisfies

$$\begin{aligned}\pi_{\bar\lambda}(Y^*_\sigma)F^{\lambda}_{\upsilon,\sigma} &= -\mu_{\bar\upsilon}(\bar\sigma)F^{\lambda}_{\upsilon,\sigma},\\ \pi_{\bar\lambda}(\tilde Y^*_\sigma)F^{\lambda}_{\upsilon,\sigma} &= -\tilde\mu_{\bar\upsilon}(\bar\sigma)F^{\lambda}_{\upsilon,\sigma}.\end{aligned}$$

Proof. The proof is similar to the proof of the previous lemma. □

We will need a few results on the analytic properties of the two functions $H^{\lambda}_{\tau,\rho}$ and $F^{\lambda}_{\upsilon,\sigma}$.

Lemma 3.5.4. *The possible pole locations of $H^{\lambda}_{\tau,\rho}$ and $F^{\lambda}_{\upsilon,\sigma}$ are at*

$$-\lambda/2\pm i\tau+iw-\Lambda_+, \qquad \lambda/2\pm i\rho+iw+\Lambda_+$$

and

$$-\bar\lambda/2\pm i\bar\upsilon-\Lambda_+, \qquad \bar\lambda/2\pm i\bar\sigma+\Lambda_+$$

respectively. Furthermore, $H^{\lambda}_{\tau,\rho}$ and $F^{\lambda}_{\upsilon,\sigma}$ have exponential growth with growth rates $\pi(2\Im(\lambda)-2w)/w_1w_2$ and $\pi(-2\Im(\lambda)-2w)/w_1w_2$, respectively.

Proof. The proof follows directly from the zero/pole locations and asymptotics of the hyperbolic gamma function (see Section 3.2). □

Define

$$\xi = \max(|\Re(\rho)|, |\Re(\sigma)|, |\Re(\tau)|, |\Re(\upsilon)|) \tag{3.5.4}$$

and

$$\zeta = w/2 - \xi - |\Im(\lambda/2)|. \tag{3.5.5}$$

We assume that the parameters $\rho, \sigma, \upsilon, \tau$ and the variable λ are such that $\zeta > 0$. For $|\Im(x)| < \zeta$ define

$$\psi(\rho, \sigma, \tau, \upsilon; x, \lambda) = \langle \pi_\lambda(\hat{x}) H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma} \rangle, \tag{3.5.6}$$

which is well defined since the exponential growth of the integrand equals $-2\pi(w_1 + w_2)/w_1 w_2 < 0$ and the pole sequences of $\pi_\lambda(\hat{x})H^\lambda_{\tau,\rho}$ and $F^\lambda_{\upsilon,\sigma}$ all stay away from the real line due to the condition $|\Im(x)| < \zeta$. Note that the increasing pole sequences of $\pi_\lambda(\hat{x})H^\lambda_{\tau,\rho}$ and $F^\lambda_{\upsilon,\sigma}$ are all located above the real line and the decreasing pole sequences are all located below the real line due to the shifted addition in $\hat{\mathbb{C}}$. Observe furthermore that the matrix coefficient ψ is invariant under the exchange of w_1 and w_2, cf. Remark 3.5.2. The function ψ will be related to Ruijsenaars' hypergeometric function R in Section 3.8.

Using (3.2.5), (3.2.6), and (3.4.3) we can write ψ as

$$\psi(\rho, \sigma, \tau, \upsilon; x, \lambda) = \int_{\mathbb{R}} \frac{G(z + x + \lambda/2 - iw/2 \pm i\tau)G(z - \lambda/2 - iw/2 \pm i\sigma)}{G(z + x - \lambda/2 + iw/2 \pm i\rho)G(z + \lambda/2 + iw/2 \pm i\upsilon)} dz, \tag{3.5.7}$$

which is of the form (3.2.13). It follows from the discussion at the end of Section 3.2 that

$$\begin{aligned}\Psi(\gamma; x, \lambda) = {} & E(x \pm i\gamma_0)E(x \pm i\gamma_1)E(-x \pm i\gamma_2)E(-x \pm i\gamma_3) \\ & \times E(\lambda \pm i\hat{\gamma}_0)E(\lambda \pm i\hat{\gamma}_1)E(-\lambda \pm i\hat{\gamma}_2)E(-\lambda \pm i\hat{\gamma}_3)\psi(\gamma; x, \lambda)\end{aligned} \tag{3.5.8}$$

has an entire extension to

$$\mathcal{O} = \{(w_1, w_2, \rho, \sigma, \tau, \upsilon, x, \lambda) \in \mathbb{C}^2_+ \times \mathbb{C}^6\}. \tag{3.5.9}$$

Hence ψ is meromorphic on the same domain $\mathcal{O}$.

3.6 The Askey-Wilson difference equations

We show that the formal matrix coefficient ψ (see Section 3.5) satisfies a second order difference equation with step size iw_1 using a radial part calculation of the Casimir Ω with respect to twisted primitive elements. As a consequence a renormalization S (3.6.10) of ψ satisfies an Askey-Wilson second order difference equation. Since S, like ψ, is invariant under exchanging w_1 and w_2, we obtain a second difference equation with step size iw_2. We furthermore show that S satisfies a duality in the geometric and spectral variables, and we derive various obvious symmetries of S.

Let us start by establishing a correspondence between the set of parameters ρ, σ, τ, and υ and Ruijsenaars' parameter set γ by

$$\gamma_0 = -\rho + \sigma, \quad \gamma_1 = \rho + \sigma, \quad \gamma_2 = -\tau - \upsilon, \quad \gamma_3 = \tau - \upsilon. \tag{3.6.1}$$

Observe that $\hat{\gamma}_0$ (see (3.3.1)) becomes

$$\hat{\gamma}_0 = \frac{1}{2}(\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3) = \sigma - \upsilon.$$

We will also use the abbreviation γ for the parameters $(\rho, \sigma, \tau, \upsilon)$. In particular we write $\psi(\gamma; x, \lambda)$ for (3.5.6). Later we show that (3.6.1) is the parameter correspondence which relates ψ to Ruijsenaars' hypergeometric function.

Now we perform a radial part calculation of the Casimir element Ω with respect to the twisted primitive elements (see (3.5.1)). The result is stated in terms of the function A, see (3.3.4).

Lemma 3.6.1. *We have*

$$\hat{x}\Omega = \hat{x}\Omega(x) \mod \hat{x}\mathcal{U}_{q,2}(Y_\rho - \mu_\tau(\rho)) + (Y_\sigma - \mu_\upsilon(\sigma))\hat{x}\mathcal{U}_{q,2},$$

where $\mathcal{U}_{q,2} := span_{\mathbb{C}}\{1, K^{-1}, F\}$ *and* $\Omega(x)$ *is defined as the element*

$$\Omega(x) = B(x)K + C(x) + D(x)K^{-1}$$

with coefficients

$$\begin{aligned}
B(x) &= q^{-1}A(\gamma_0, -\gamma_0, \gamma_1, -\gamma_1; x),\\
C(x) &= q^{-1+2\hat{\gamma}_0/w_1}\left[-A(\gamma; x) - A(\gamma; -x) + 1 + q^{2-4\hat{\gamma}_0/w_1}\right],\\
D(x) &= q^{-1}A(\gamma_2, -\gamma_2, \gamma_3, -\gamma_3; -x).
\end{aligned}$$

Proof. The proof involves a radial part calculation similar to the one performed in [78, Proposition 3.3]. In fact we can use the calculation in [78] using an embedding ϕ of the extended quantum universal enveloping algebra $\mathcal{U}_q \rtimes \hat{\mathbb{C}}$ into the one in [78], given by

$$\begin{aligned}
\phi(K) &= K^2, & \phi(\hat{x}) &= \left(\widehat{\frac{2x+iw_2}{iw_1}}\right)K,\\
\phi(E) &= -iKX^+, & \phi(F) &= iX^-K^{-1}.
\end{aligned}$$

A direct calculation gives $\phi(\Omega) = (q - q^{-1})^2\Omega + 2$ and $\phi(Y_\rho) = Y_{2\rho/w_1}$ (on the right hand side we use the Ω and Y from [78], which have a slightly different definition). Note that in [78] the radial part is calculated modulo a larger vector space. However, it is easily verified that the present smaller space suffices for the proof. $\square$

Using this radial part calculation we can prove that ψ (3.5.6) satisfies a gauge transformed Askey-Wilson second order difference equation.

Lemma 3.6.2. *The function $\psi(x) = \psi(\gamma; x, \lambda)$ satisfies the difference equation*

$$-2\cosh(2\pi\lambda/w_2)\psi(x) = B(x)\psi(x+iw_1) + C(x)\psi(x) + D(x)\psi(x-iw_1), \quad (3.6.2)$$

and a similar equation with w_1 and w_2 interchanged. These equations hold as identities between meromorphic functions on the domain $\mathcal{O}$ (see (3.5.9)).

Proof. Observe that by the symmetry of ψ in w_1 and w_2 we only have to prove the difference equation (3.6.2).

We first prove the lemma under restricted parameter conditions, which allow us to use the expression (3.5.6) of ψ as a matrix coefficient of the $\mathcal{D}$-representation π_λ. Using analytic continuation we can subsequently remove these parameter constraints, cf. the discussion at the end of Section 3.5.

Let us assume that $w_1, w_2 > 0$ and that

$$w_2 > 7w_1 + 4\xi + 2|\Im(\lambda)| + 4|\Im(x)| \quad (3.6.3)$$

holds. Then $|\Im(x)| < \zeta$, so ψ is defined by (3.5.6) (recall that ξ and ζ are defined by (3.5.4) and (3.5.5), respectively). By (3.4.1),

$$-2\cosh(2\pi\lambda/w_2)\psi(x) = \langle \pi_\lambda(\hat{x}\Omega)H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle \quad (3.6.4)$$

holds. By Lemma 3.6.1 there exist $X, Z \in \mathcal{U}_{q,2}$ such that

$$\hat{x}\Omega = \hat{x}\Omega(x) + \hat{x}X(Y_\rho - \mu_\tau(\rho)) + (Y_\sigma - \mu_\upsilon(\sigma))\hat{x}Z. \quad (3.6.5)$$

Since $\pi_\lambda(Y_\rho - \mu_\tau(\rho))H^\lambda_{\tau,\rho} = 0$, we have

$$\langle \pi_\lambda(\hat{x}X(Y_\rho - \mu_\tau(\rho)))H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle = 0. \quad (3.6.6)$$

The exponential growth of $\pi_\lambda(\hat{x}Z)H^\lambda_{\tau,\rho}$ is at most the exponential growth of $H^\lambda_{\tau,\rho}$ plus $2\pi/w_2$ (due to the possible occurrence of an S_{iw_2} factor in $\pi_\lambda(Z)$). The sum of the exponential growths of $\pi_\lambda(\hat{x}Z)H^\lambda_{\tau,\rho}$ and $F^\lambda_{\upsilon,\sigma}$ is at most $-2\pi/w_1$, hence strictly smaller than $-2\pi/w_2$, since the restrictions on the parameters imply that $w_2 > w_1$. Moreover the condition (3.6.3) implies that neither $\pi_\lambda(\hat{x}Z)H^\lambda_{\tau,\rho}$ nor $F^\lambda_{\upsilon,\sigma}$ has any poles in the strip $\mathbb{R}\times i[-w_1, w_1]$. Using Lemma 3.4.8 and the fact that $Y_\sigma \in \mathcal{U}_{q,1}$, we thus obtain

$$\langle \pi_\lambda((Y_\sigma - \mu_\upsilon(\sigma))\hat{x}Z)H_{\tau,\rho}, F_{\upsilon,\sigma}\rangle = \langle \pi_\lambda(\hat{x}Z)H_{\tau,\rho}, \pi_{\bar{\lambda}}(Y^*_\sigma + \mu_{\bar{\upsilon}}(\bar{\sigma}))F_{\upsilon,\sigma}\rangle = 0. \quad (3.6.7)$$

Combining (3.6.4), (3.6.5), (3.6.6), and (3.6.7) now yields

$$-2\cosh(2\pi\lambda/w_2)\psi(x) = \langle \pi_\lambda(\hat{x}\Omega)H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle = \langle \pi_\lambda(\hat{x}\Omega(x))H^\lambda_{\tau,\rho}, F^\lambda_{\upsilon,\sigma}\rangle. \quad (3.6.8)$$

Furthermore, by Lemma 3.6.1 (remember that $\widehat{x+iw_1}$ and $\hat{x}K$ act in the same way under π_λ) we have

$$\langle \pi_\lambda(\hat{x}\Omega(x))H_{\tau,\rho}, F_{\upsilon,\sigma}\rangle = B(x)\psi(x+iw_1) + C(x)\psi(x) + D(x)\psi(x-iw_1). \quad (3.6.9)$$

The lemma for the restricted parameter conditions follows now directly from (3.6.8) and (3.6.9). □

Using the function

$$\Delta(\gamma;x)=\frac{G(x+i\gamma_2)G(x+i\gamma_3)}{G(x-i\gamma_0)G(x-i\gamma_1)},$$

we can define a renormalization S of ψ as

$$S(\gamma;x,\lambda)=\frac{N(\gamma)\psi(\gamma;x,\lambda)}{\sqrt{w_1w_2}\Delta(\gamma;x)\Delta(\hat{\gamma};\lambda)}. \tag{3.6.10}$$

The function N (3.3.9) is a convenient normalization factor when matching S to R in Section 3.8.

Lemma 3.6.3. *$S(\gamma;x,\lambda)$ is meromorphic on $\mathcal{O}$ with possible poles at*

$$\lambda=\pm(\nu-i\hat{\gamma}_k),\qquad x=\pm(\nu-i\gamma_k),\qquad i\gamma_0+i\gamma_l=-\nu-iw$$

for $\nu\in\Lambda_+$, $k=0,1,2,3$, and $l=1,2,3$.

Proof. Using (3.5.8) and (3.2.12) we can express S as

$$S(\gamma;x,\lambda)=\frac{\Psi(\gamma;x,\lambda)N(\gamma)}{\prod_{k=0}^{3}E(\pm x+i\gamma_k)E(\pm\lambda+i\hat{\gamma}_k)}.$$

From this expression we can easily read off that the possible pole hyperplanes are as stated in the lemma (they have to be either poles of $N(\gamma)$ or zeros of one of the E-functions in the denominator). □

Theorem 3.6.4. *The function $S(\gamma;x,\lambda)$ is a simultaneous eigenfunction of the two Askey-Wilson type second order difference operators $\mathcal{L}_\gamma^x$ and $\widetilde{\mathcal{L}}_\gamma^x$ (see (3.3.5)) with eigenvalues $v(\lambda;w_1,w_2,\gamma)$ and $v(\lambda;w_2,w_1,\gamma)$ respectively, where v is defined by (3.3.6).*

Proof. Note that Δ satisfies the first order difference equation

$$\Delta(x+iw_1/2)=\frac{\cosh(\frac{\pi}{w_2}(x+i\gamma_2))\cosh(\frac{\pi}{w_2}(x+i\gamma_3))}{\cosh(\frac{\pi}{w_2}(x-i\gamma_0))\cosh(\frac{\pi}{w_2}(x-i\gamma_1))}\Delta(x-iw_1/2).$$

The desired eigenvalue equation (3.3.5) for $\mathcal{L}_\gamma^x$ now follows immediately from Lemma 3.6.2.

To prove the result for the operator $\widetilde{\mathcal{L}}_\gamma^x$ we note that S is symmetric in w_1 and w_2, while interchanging w_1 and w_2 transforms $\mathcal{L}$ to $\tilde{\mathcal{L}}$. We could also prove the second difference equation by repeating the argument for the first difference equation using the component $\mathcal{U}_{\tilde{q}}$ of the modular double. □

We continue the analysis of the eigenfunction S by proving its duality in the geometric variable x and the spectral variable λ, similar to the duality (3.3.8) for Ruijsenaars' hypergeometric function R. The duality transformation $\gamma\to\hat{\gamma}$ of the parameters (see (3.3.1)) is equivalent to interchanging ρ and υ under the parameter correspondence (3.6.1): $(\rho,\sigma,\tau,\upsilon)\to(\upsilon,\sigma,\tau,\rho)$.

Theorem 3.6.5 (Duality). *We have*

$$S(\gamma; x, \lambda) = S(\hat{\gamma}; \lambda, x)$$

as meromorphic functions on $\mathcal{O}$.

Proof. Assume that $w_1, w_2 > 0$ and $w/2 > \xi + |\Im(x)| + |\Im(\lambda)|$, where ξ is as in (3.5.4). Note that these restrictions on the parameters are invariant under the exchange $(x, \gamma) \leftrightarrow (\lambda, \hat{\gamma})$. Then we can use the integral representation (3.5.7) for both $\psi(\gamma; x, \lambda)$ and $\psi(\hat{\gamma}; \lambda, x)$ to compute

$$\begin{aligned}\psi(\gamma; x, \lambda) &= \int_{\mathbb{R}} \frac{G(z + x + \lambda/2 - iw/2 \pm i\tau)G(z - \lambda/2 - iw/2 \pm i\sigma)}{G(z + x - \lambda/2 + iw/2 \pm i\rho)G(z + \lambda/2 + iw/2 \pm i\upsilon)} dz \\ &= \int_{\mathbb{R}} \frac{G(z + x/2 + \lambda - iw/2 \pm i\tau)G(z - x/2 - iw/2 \pm i\sigma)}{G(z + x/2 + iw/2 \pm i\rho)G(z - x/2 + \lambda + iw/2 \pm i\upsilon)} dz \\ &= \psi(\hat{\gamma}; \lambda, x),\end{aligned}$$

where we used the change of integration variable $z \to z + (\lambda - x)/2$ and a contour shift in the second equality. This contour shift is allowed since the integrand converges to zero exponentially at $\pm\infty$, and the conditions on the parameters ensure that there are no poles picked up by shifting the contour back to $\mathbb{R}$.

Since Ψ (see (3.5.8)) is entire on $\mathcal{O}$, it follows that $\psi(\gamma; x, \lambda) = \psi(\hat{\gamma}; \lambda, x)$ holds as identity between meromorphic functions on $\mathcal{O}$. The desired duality for S now follows from $N(\gamma) = N(\hat{\gamma})$ and $\hat{\hat{\gamma}} = \gamma$. □

Corollary 3.6.6. *The function* $S(\gamma; x, \lambda)$ *is a simultaneous eigenfunction of the Askey-Wilson second order difference operators* $\mathcal{L}_\gamma^x$, $\widetilde{\mathcal{L}}_\gamma^x$, $\mathcal{L}_{\hat{\gamma}}^\lambda$, *and* $\widetilde{\mathcal{L}}_{\hat{\gamma}}^\lambda$ *with eigenvalues* $v(\lambda; w_1, w_2, \gamma)$, $v(\lambda; w_2, w_1, \gamma)$, $v(x; w_1, w_2, \hat{\gamma})$, *and* $v(x; w_2, w_1, \hat{\gamma})$ *respectively.*

Proof. The fact that S is an eigenfunction of $\mathcal{L}_\gamma^x$ and $\widetilde{\mathcal{L}}_\gamma^x$ was proved in Theorem 3.6.4. The proof for the other two difference operators follows from this fact and duality (Theorem 3.6.5). □

It is immediately clear from the integral representation (3.5.7) that ψ is invariant under sign flips of the parameters ρ, σ, τ, and υ. This leads to the following symmetries for S.

Lemma 3.6.7. *Let* W_n *be the Weyl group of type* D_n, *which acts on* n*-tuples by permutations and even numbers of sign changes. Let* $V = W_2 \times W_2 \subset W_4$ *be the Weyl group of type* $D_2 \times D_2$, *where the first (respectively second) component acts on the parameters* (γ_0, γ_1) *(respectively* (γ_2, γ_3)*) of the four-tuple* $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$. *For an element* $v \in V$ *we have*

$$\frac{S(\gamma; x, \lambda)}{c(\gamma; x)c(\hat{\gamma}; \lambda)N(\gamma)} = \frac{S(v(\gamma); x, \lambda)}{c(v(\gamma); x)c(\widehat{v(\gamma)}; \lambda)N(v(\gamma))}$$

as meromorphic functions on $\mathcal{O}$.

Proof. Note that the action of $V \simeq \mathbb{Z}_2^{\times 4}$ on the parameters $(\rho, \sigma, \tau, \upsilon)$ is by sign flips of ρ, σ, τ, and υ. Under the conditions $\zeta > 0$ and $|\Im(x)| < \zeta$ it follows from the integral representation (3.5.7) of ψ that ψ is invariant under the action of V on γ (note that the parameter restrictions are V-invariant).

Observe that

$$c(\gamma; x)\Delta(\gamma; x) = \frac{G(x \pm i\gamma_2)G(x \pm i\gamma_3)}{G(2x + iw)}$$

is also V-invariant. Since the action of V commutes with taking dual parameters (which is obvious in the parameters $\rho, \sigma, \tau, \upsilon$, since V acts by flipping signs while taking dual parameters amounts to interchanging ρ and υ) we have a similar result for $c(\hat{\gamma}; \lambda)\Delta(\hat{\gamma}; \lambda)$. Combining these results and using (3.6.10) now yields the desired symmetry of S for the restricted parameter set. These extra conditions on the parameters can be removed by analytic continuation (compare with the proof of Theorem 3.6.5). □

Remark 3.6.8. The symmetries described in Lemma 3.6.7 should be compared to the D_4 symmetry (3.3.10) of R. Note that for R only an $S_3 \subset W_4$ symmetry holds trivially from its integral representation (3.3.2), where S_3 acts by permuting γ_1, γ_2, and γ_3.

Let us now consider the asymptotics of S, compare with the asymptotics (3.3.11) of R.

Lemma 3.6.9. *Let $w_1, w_2 \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{C}^4$, and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ such that $\zeta > 0$, where ζ is given by* (3.5.5)*. Then*

$$S(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)| - \Re(\hat{\gamma}_0) - w)|\Re(x)|})$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compact subsets of $(-\zeta, \zeta)$, where $\alpha = 2\pi/w_1 w_2$.

Proof. Under the parameter restrictions as stated in the lemma, S does not have x-independent poles (see Lemma 3.6.3) and the integral representation (3.5.7) for ψ holds.

In view of (3.6.10) and the asymptotics

$$\frac{1}{\Delta(\gamma; x)} = \mathcal{O}(e^{\mp\alpha\hat{\gamma}_0 x}) \tag{3.6.11}$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compacta, it suffices to prove

$$\psi(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)| - w)|\Re(x)|}) \tag{3.6.12}$$

for $\Re(x) \to \pm\infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. The asymptotic formula (3.6.11) follows directly from the estimates (3.2.10) and (3.2.11) for the hyperbolic gamma function.

Note that it suffices to prove (3.6.12) for $\Re(x) \to \infty$ since

$$\psi(\gamma; x, \lambda) = \psi(\check{\gamma}; -x, -\lambda) \tag{3.6.13}$$

where $\breve{\gamma} = (\sigma, \rho, \upsilon, \tau)$ (in the γ_μ notation, $\breve{\gamma} = (-\gamma_0, \gamma_1, \gamma_2, -\gamma_3)$). Equation (3.6.13) follows by the change of integration variable $z \to -z$ in (3.5.7) and a subsequent contour shift.

To prove (3.6.12) for $\Re(x) \to \infty$ we consider the integral representation (3.5.7) of ψ. We define

$$\epsilon = \max(w_1, w_2) + \frac{1}{2}|\Re(\lambda)| + \max(|\Im(\rho)|, |\Im(\sigma)|, |\Im(\tau)|, |\Im(\upsilon)|) \tag{3.6.14}$$

and we consider the division of $\mathbb{R}$ in five intervals

$$\begin{aligned} &I_1 = (-\infty, -\Re(x) - \epsilon), \qquad I_2 = (-\Re(x) - \epsilon, -\Re(x) + \epsilon), \\ &I_3 = (-\Re(x) + \epsilon, -\epsilon), \qquad I_4 = (-\epsilon, \epsilon), \qquad I_5 = (\epsilon, \infty), \end{aligned} \tag{3.6.15}$$

for $\Re(x) > 2\epsilon$. We write the integral (3.5.7) defining ψ as the sum of five integrals over I_j ($j = 1, 2, \ldots, 5$) and we bound the integral over each I_j separately. The intervals are chosen in such a way that the estimates (3.2.10) and (3.2.11) for the hyperbolic gamma function can be used to bound the integrand over the intervals I_1, I_3, and I_5. To estimate the integrals over the remaining intervals I_2 and I_4 we use the fact that their lengths are finite and independent of $\Re(x)$. For each interval I_j we show that the integral over I_j is $\mathcal{O}(e^{\alpha(|\Im(\lambda)|-w)|\Re(x)|})$ as $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compact subsets of $(-\zeta, \zeta)$. As a consequence ψ is also of this order. Details are given in Appendix 3.B. □

3.7 Reduction to Askey-Wilson polynomials

Using an indirect method, Ruijsenaars [65, Theorem 3.2] proved that R reduces to the Askey-Wilson polynomials [3] when the spectral parameter is specialized to certain specific discrete values. We now show by a direct calculation that S (3.6.10) reduces to the Askey-Wilson polynomials for the same discrete spectral values.

Let us first introduce some standard notations for basic hypergeometric series, see [16]. For $q \in \mathbb{C}$ we write

$$\begin{aligned} (a; q)_n &= \prod_{k=0}^{n-1} (1 - aq^k), \\ (a_1, a_2, \ldots, a_k; q)_n &= \prod_{j=1}^{k} (a_j; q)_n. \end{aligned}$$

The q-hypergeometric series is defined by

$${}_{s+1}\phi_s \left[\begin{matrix} a_1, \ldots, a_{s+1} \\ b_1, \ldots, b_s \end{matrix} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{s+1}; q)_k}{(b_1, \ldots, b_s, q; q)_k} z^k$$

provided that either $|q| < 1$ or that the series terminates. The Askey-Wilson polynomials [3] are defined as

$$r_n(x; a, b, c, d \mid q) = {}_4\phi_3 \left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{2\pi x/w_2}, ae^{-2\pi x/w_2} \\ ab, ac, ad \end{matrix} ; q, q \right].$$

Notice that the q^{-n} term in the above expression causes the series to terminate. This implies that r_n is a polynomial of degree n in $\cosh(2\pi x/w_2)$. Finally, if we use the parameter correspondence

$$a = -e^{2\pi i\gamma_0/w_2}q, \quad b = -e^{2\pi i\gamma_1/w_2}q, \quad c = -e^{2\pi i\gamma_2/w_2}q, \quad d = -e^{2\pi i\gamma_3/w_2}q, \tag{3.7.1}$$

and if we define

$$\lambda_n = iw + i\hat{\gamma}_0 + inw_1, \tag{3.7.2}$$

then the Askey-Wilson polynomials satisfy the Askey-Wilson second order difference equation

$$\mathcal{L}^x_\gamma r_n(x; a, b, c, d \mid q^2) = v(\lambda_n; w_1, w_2, \gamma) r_n(x; a, b, c, d \mid q^2).$$

Here we use the Askey-Wilson operator $\mathcal{L}^x_\gamma$ (3.3.5) and eigenvalue v (3.3.6).

Ruijsenaars has shown in [65] by an indirect method that

$$R(w_1, w_2, \gamma; x, \lambda_n) = r_n(x; a, b, c, d \mid q^2) \tag{3.7.3}$$

for $n \in \mathbb{Z}_{\geq 0}$, under the parameter correspondence (3.7.1). Similarly we have

Theorem 3.7.1. *Under the parameter correspondence* (3.7.1) *we have*

$$S(w_1, w_2, \gamma; x, \lambda_n) = r_n(x; a, b, c, d \mid q^2)$$

for $n \in \mathbb{Z}_{\geq 0}$.

Proof. Without loss of generality we assume that the parameters w_1, w_2, γ, x are generic.

For generic λ we can express ψ as an integral

$$\psi(\gamma; x, \lambda) = \int_{\mathcal{C}} I(\gamma; x, \lambda, z) dz \tag{3.7.4}$$

with $I(z) = I(\gamma; x, \lambda, z)$ given by

$$I(z) = \frac{G(z + x + \lambda/2 - iw/2 \pm i\tau) G(z - \lambda/2 - iw/2 \pm i\sigma)}{G(z + x - \lambda/2 + iw/2 \pm i\rho) G(z + \lambda/2 + iw/2 \pm iv)}$$

and with contour $\mathcal{C}$ a deformation of $\mathbb{R}$ separating the upward pole sequences of I from the downward pole sequences of I. When $\lambda \to \lambda_n$, the pole $z_k := \lambda/2 - iw/2 - i\sigma - ikw_1$ from a downward pole sequence of I will collide with the pole $-\lambda/2 + iw/2 - iv + i(n-k)w_1$ from an upward pole sequence of I for

$0 \leq k \leq n$. In order to compute the limit $\lambda \to \lambda_n$ in (3.7.4), we therefore first shift the contour $\mathcal{C}$ over the poles at z_k $(0 \leq k \leq n)$ while picking up poles. In the resulting integral the colliding poles are on the same side of the integration contour, hence the limit $\lambda \to \lambda_n$ can be taken.

To calculate the residues of I at z_k we first remark that k consecutive applications of the difference equation (3.2.3) yield

$$\frac{G(z)}{G(z-ikw_1)} = e^{k\pi z/w_2} q^{-k^2/2} (-e^{-2\pi z/w_2} q; q^2)_k.$$

Using this equation we can write

$$I(z) = \frac{G(z+ikw_1+x+\lambda/2-iw/2 \pm i\tau)G(z+ikw_1-\lambda/2-iw/2 \pm i\sigma)}{G(z+ikw_1+x-\lambda/2+iw/2 \pm i\rho)G(z+ikw_1+\lambda/2+iw/2 \pm iv)}$$
$$\times q^{2k} \frac{(-e^{-\frac{2\pi}{w_2}(z+ikw_1+x-\frac{1}{2}\lambda+\frac{1}{2}iw \pm i\rho)}q, -e^{-\frac{2\pi}{w_2}(z+ikw_1+\frac{1}{2}\lambda+\frac{1}{2}iw \pm iv)}q; q^2)_k}{(-e^{-\frac{2\pi}{w_2}(z+ikw_1+x+\frac{1}{2}\lambda-\frac{1}{2}iw \pm i\tau)}q, -e^{-\frac{2\pi}{w_2}(z+ikw_1-\frac{1}{2}\lambda-\frac{1}{2}iw \pm i\sigma)}q, q^2)_k}.$$

Using the fact that the residue of the hyperbolic gamma function at $z = -iw$ equals (3.2.9), we obtain that the residue Res_k of I at z_k equals

$$Res_k = \frac{i\sqrt{w_1w_2}}{2\pi} \frac{G(x+\lambda-iw-i\sigma \pm i\tau)G(-iw-2i\sigma)}{G(x-i\sigma \pm i\rho)G(\lambda-i\sigma \pm iv)}$$
$$\times q^{2k} \frac{(-e^{-\frac{2\pi}{w_2}(x-i\sigma \pm i\rho)}q, -e^{-\frac{2\pi}{w_2}(\lambda-i\sigma \pm iv)}q; q^2)_k}{(e^{-\frac{2\pi}{w_2}(x+\lambda-i\sigma \pm i\tau)}q^2, e^{\frac{2\pi}{w_2}2i\sigma}q^2, q^2; q^2)_k}.$$

Now we can rewrite S as

$$S(\gamma; x, \lambda) = \frac{N(\gamma)}{\sqrt{w_1w_2}\Delta(\gamma; x)\Delta(\hat{\gamma}; \lambda)} \left(-2\pi i \sum_{k=0}^{n} Res_k + \int_{\mathcal{C}'} I(z)dz \right),$$

where the contour $\mathcal{C}'$ is chosen in such a way that all upward pole sequences and the poles z_k $(0 \leq k \leq n)$ are above $\mathcal{C}'$, while all poles in downward pole sequences except z_k $(0 \leq k \leq n)$ are below $\mathcal{C}'$. In this expression the integral $\int_{\mathcal{C}'} I(z)dz$ has an analytic extension to $\lambda = \lambda_n$. Furthermore $S(\gamma; x, \lambda)$ is analytic at $\lambda = \lambda_n$, while $\Delta(\hat{\gamma}; \lambda)$ and Res_k $(0 \leq k \leq n)$ have simple poles at $\lambda = \lambda_n$. Hence we obtain

$$S(\gamma; x, \lambda_n) = \lim_{\lambda \to \lambda_n} -\frac{2\pi i N(\gamma)}{\sqrt{w_1w_2}\Delta(\gamma; x)\Delta(\hat{\gamma}; \lambda)} \sum_{k=0}^{n} Res_k$$
$$= e^{\frac{2n\pi}{w_2}(x-iw-i\gamma_0)} \frac{(e^{-\frac{2\pi}{w_2}(x-iw+\gamma_{2/3})}q^{-2n}; q^2)_n}{(e^{-\frac{2\pi}{w_2}(i\gamma_0+i\gamma_{2/3})}q^{-2n}; q^2)_n}$$
$$\times \; {}_4\phi_3 \left[\begin{matrix} q^{-2n}, e^{-\frac{2\pi}{w_2}(x-iw-i\gamma_{0/1})}, e^{-\frac{2\pi i}{w_2}(\gamma_2+\gamma_3)}q^{-2n} \\ e^{-\frac{2\pi}{w_2}(x-iw+i\gamma_{2/3})}q^{-2n}, e^{\frac{2\pi i}{w_2}(\gamma_0+\gamma_1)}q^2 \end{matrix} ; q^2, q^2 \right],$$

where the notation $\gamma_{0/1}$ (respectively $\gamma_{2/3}$) means that there are two terms, one with γ_0 and another with γ_1 (respectively, γ_2 and γ_3). Inserting the parameter correspondence (3.7.1) we obtain

$$\begin{aligned} S(\gamma; x, \lambda_n) &= e^{2\pi n x/w_2} a^{-n} \frac{(e^{-2\pi x/w_2} c^{-1} q^{-2n+2}, e^{-2\pi x/w_2} d^{-1} q^{-2n+2}; q^2)_n}{(a^{-1} c^{-1} q^{-2n+2}, a^{-1} d^{-1} q^{-2n+2}; q^2)_n} \\ &\quad \times {}_4\phi_3 \left[\begin{matrix} q^{-2n}, e^{-2\pi x/w_2} a, e^{-2\pi x/w_2} b, c^{-1} d^{-1} q^{-2n+2} \\ e^{-2\pi x/w_2} c^{-1} q^{-2n+2}, e^{-2\pi x/w_2} d^{-1} q^{-2n+2}, ab \end{matrix} ; q^2, q^2 \right]. \end{aligned}$$

Using Sears' transformation [16, (III.15)] of a terminating balanced ${}_4\phi_3$ series with parameters specialized to

$$\begin{aligned} &a = ae^{-2\pi x/w_2}, && b = be^{-2\pi x/w_2}, && c = c^{-1} d^{-1} q^{-2n+2}, \\ &d = ab, && e = e^{-2\pi x/w_2} c^{-1} q^{-2n+2}, && f = e^{-2\pi x/w_2} d^{-1} q^{-2n+2} \end{aligned}$$

now yields the desired result. □

3.8 Equality to Ruijsenaars' hypergeometric function

We have already seen in previous sections that Ruijsenaars' hypergeometric function R and the renormalized formal matrix coefficient S have several properties in common. They satisfy the same Askey-Wilson second order difference equations, they have the same duality property, they specialize in the same way to the Askey-Wilson polynomials and their possible pole locations coincide. These common properties suffice to show that R and S are equal.

Theorem 3.8.1. *We have*

$$R(w_1, w_2, \gamma; x, \lambda) = S(w_1, w_2, \gamma; x, \lambda). \tag{3.8.1}$$

This theorem is equivalent to the following identity between hyperbolic integrals.

Corollary 3.8.2. *For $w_1, w_2, \Re(\gamma_j) > 0$ and $|x|, |\lambda|, |\gamma_j| < w/6$ we have*

$$\begin{aligned} &\int_{\mathbb{R}} \frac{G(z + x + \lambda/2 - iw/2 \pm i(\gamma_3 - \gamma_2)/2) G(z - \lambda/2 - iw/2 \pm i(\gamma_0 + \gamma_1)/2)}{G(z + x - \lambda/2 + iw/2 \pm i(\gamma_0 - \gamma_1)/2) G(z + \lambda/2 + iw/2 \pm i(\gamma_2 + \gamma_3)/2)} dz \\ &\quad = \frac{G(x + i\gamma_2) G(x + i\gamma_3) G(\lambda + i\hat{\gamma}_2) G(\lambda + i\hat{\gamma}_3)}{G(x + i\gamma_0) G(x - i\gamma_1) G(\lambda + i\hat{\gamma}_0) G(\lambda - i\hat{\gamma}_1)} \\ &\qquad \times \int_{\mathcal{C}} \frac{G(z \pm x + i\gamma_0) G(z \pm \lambda + i\hat{\gamma}_0)}{G(z + iw) \prod_{j=1}^3 G(z + i\gamma_0 + i\gamma_j + iw)} dz, \end{aligned}$$

where the contour $\mathcal{C}$ is the real line with a downward indentation at the origin.

Proof. The proof consists of inserting the integral representations of R and S in (3.8.1). See (3.3.2) for the integral representation of R, and (3.5.7), (3.6.10) for the integral representation of S. □

In order to prove Theorem 3.8.1 we first consider the Casorati-determinant of S and R in the iw_1 direction.

Lemma 3.8.3. *The Casorati-determinant*

$$\delta(\gamma; z, \lambda) = S(\gamma; z + iw_1/2, \lambda)R(\gamma; z - iw_1/2, \lambda) - S(\gamma; z - iw_1/2, \lambda)R(\gamma; z + iw_1/2, \lambda)$$

of S and R in the iw_1 direction is identically zero.

Proof. We suppress the λ and γ dependence of $\delta(z)$ whenever this does not cause confusion. We prove the lemma for generic parameters $w_1, w_2 \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{R}^4$, and $\lambda \in U \setminus \mathbb{R}$, under the condition $w_2 > 2\xi + 2|\Im(\lambda)| + 3w_1$, where U is an open subset such that the asymptotics (3.3.11) of R hold for $\lambda \in U$.

A simple calculation involving the Askey-Wilson difference equations satisfied by R and S (see (3.3.7) and Theorem 3.6.6 respectively) shows that

$$\delta(z + iw_1/2) = \frac{A(\gamma; -z)}{A(\gamma; z)}\delta(z - iw_1/2),$$

where A is defined by (3.3.4). Since the function

$$T(z) = \sinh(2\pi z/w_2) \prod_{j=0}^{3} \frac{G(z - i\gamma_j - iw_1/2)}{G(z + i\gamma_j + iw_1/2)}$$

satisfies the same difference equation, we conclude that

$$m(z) = \frac{\delta(z)}{T(z)}$$

is an iw_1-periodic function.

We now show that $m(z)$ is an entire function in z. Let us look at the possible poles of the Casorati-determinant $\delta(z)$. By Lemma 3.6.9 the possible poles of S are located at

$$\pm(\Lambda_+ - i\gamma_j), \qquad (j = 0, 1, 2, 3).$$

From (3.3.3) the possible poles of R are located at the same points. Hence $\delta(z)$ can only have poles at

$$\pm(\Lambda_+ - i\gamma_j) \pm iw_1/2, \qquad (j = 0, 1, 2, 3)$$

Here all sign combinations are possible. Furthermore, using the pole and zero locations (3.2.8) of the hyperbolic gamma function, we can easily see that the possible zeros of $T(z)$ are located at

$$\pm(\Lambda_+ + i\gamma_j + iw_1/2), \qquad riw_2, \qquad (j = 0, 1, 2, 3;\ r \in \mathbb{Z}).$$

By the assumption that the parameters are generic, we conclude that m has no pole sequences of the form $p + ikw_1$ ($k \in \mathbb{Z}$). By the iw_1-periodicity of m it now follows that m cannot have any poles.

In the limit $\Re(z) \to \infty$ we have

$$\frac{1}{T(z)} = \mathcal{O}(e^{\alpha(\hat{\gamma}_0 + w_1)z})$$

uniformly for $\Im(z)$ in compacta, in view of the estimates (3.2.10) and (3.2.11) for the hyperbolic gamma function. Here $\alpha = 2\pi/w_1w_2$ as before.

Furthermore, using the asymptotics for S (see Lemma 3.6.9) and for R (see (3.3.11)) we have for $\Re(z) \to \infty$

$$\delta(z) = \mathcal{O}(e^{2\alpha(|\Im(\lambda)| + |\hat{\gamma}_0| - w)|\Re(z)|})$$

uniformly for $\Im(z)$ in compact subsets of $(-\zeta + w_1/2, \zeta - w_1/2)$. Observe that the interval $(-\zeta + w_1/2, \zeta - w_1/2)$ is nonempty due to the conditions on the parameters.

Combining these two asymptotic estimates we obtain

$$m(z) = \frac{\delta(z)}{T(z)} = \mathcal{O}(e^{\alpha(2|\Im(\lambda)| - w_2)|\Re(z)|}) \to 0 \tag{3.8.2}$$

for $\Re(z) \to \infty$, uniformly for $\Im(z)$ in compacta of $(-\zeta + w_1/2, \zeta - w_1/2)$.

The asymptotics of $m(z)$ for $\Re(z) \to -\infty$ can be obtained in a similar way and is also given by (3.8.2). Combining the asymptotics with the fact that $m(z)$ is analytic and iw_1-periodic we conclude that $m(z)$ is bounded on $\mathbb{C}$ since $\zeta - w_1/2 > w_1/2$.

For these parameters we conclude by Liouville's theorem that $m(z)$ is constant. In fact, by the asymptotic expansion (3.8.2), m is identically zero. We can now extend this result to all values of the parameters by analytic continuation, which proves the lemma. □

Proof of Theorem 3.8.1. Consider the quotient

$$Q(\gamma; x, \lambda) = \frac{R(\gamma; x, \lambda)}{S(\gamma; x, \lambda)}.$$

By Lemma 3.8.3, Q is an iw_1-periodic meromorphic function in x. Since Q is symmetric in w_1 and w_2 (for both R and S are invariant under interchanging w_1 and w_2), Q is also iw_2-periodic. If we choose $w_1, w_2 > 0$ such that $w_1/w_2 \notin \mathbb{Q}$, then the set $\{kw_1 + lw_2 \mid k, l \in \mathbb{Z}\}$ is dense on the real line, hence $Q(\gamma; x, \lambda)$ is constant as meromorphic function in x. Analytic continuation (in w_1, w_2, and γ) allows us to extend this result to all possible values of w_1 and w_2 in $\mathbb{C}_+$ and $\gamma \in \mathbb{C}^4$.

By the duality properties of R and S (see (3.3.8) and Theorem 3.6.5 respectively), we have

$$Q(\gamma; x, \lambda) = Q(\hat{\gamma}; \lambda, x).$$

This implies that Q is also constant as function in λ.

In particular we have

$$Q(w_1, w_2, \gamma; x, \lambda) = \frac{S(w_1, w_2, \gamma; x, \lambda_0)}{R(w_1, w_2, \gamma; x, \lambda_0)}$$

with λ_0 given by (3.7.2). By Theorem 3.7.1 we have $S(w_1, w_2, \gamma; x, \lambda_0) \equiv 1$, and by (3.7.3) we have $R(w_1, w_2, \gamma; x, \lambda_0) \equiv 1$. Hence $Q \equiv 1$, as desired. □

3.A Eigenfunction of $\pi_\lambda(Y_\rho)$

In this appendix we give the explicit calculation to rewrite the eigenvalue equation $\pi_\lambda(Y_\rho)f = \mu_\tau(\rho)f$ as the first order difference equation (3.5.3).

Using the explicit expression (3.5.1) of Y_ρ, the eigenvalue equation becomes

$$iq^{-1/2}\pi_\lambda(E)f + iq^{-1/2}\pi_\lambda(FK)f - \frac{\nu_\rho}{q-q^{-1}}(\pi_\lambda(K-1))f = \frac{\nu_\rho - \nu_\tau}{q-q^{-1}}f.$$

By the explicit definition (Lemma 3.4.5) of π_λ we obtain

$$\begin{aligned}&\frac{i}{q-q^{-1}}e^{2\pi z/w_2}\left(q^{-1/2}e^{\pi\lambda/w_2}f(z) + q^{1/2}e^{-\pi\lambda/w_2}f(z+iw_1)\right)\\ &-\frac{i}{q-q^{-1}}e^{-2\pi z/w_2}\left(q^{-1/2}e^{\pi\lambda/w_2}f(z+iw_1) + q^{1/2}e^{-\pi\lambda/w_2}f(z)\right)\\ &\qquad -\frac{\nu_\rho}{q-q^{-1}}\left(f(z+iw_1) - f(z)\right) = \frac{\nu_\rho-\nu_\tau}{q-q^{-1}}f(z).\end{aligned}$$

Multiplying by $q - q^{-1}$ and rearranging the terms yields

$$\begin{aligned}&\left(ie^{2\pi z/w_2}q^{-1/2}e^{\pi\lambda/w_2} - ie^{-2\pi z/w_2}q^{1/2}e^{-\pi\lambda/w_2} + \nu_\tau\right)f(z)\\ &\quad = \left(-ie^{2\pi z/w_2}q^{1/2}e^{-\pi\lambda/w_2} + ie^{-2\pi z/w_2}q^{-1/2}e^{\pi\lambda/w_2} + \nu_\rho\right)f(z+iw_1),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&f(z+iw_1)\\ &\quad = \frac{\cosh(\pi i/2 + 2\pi z/w_2 - \pi i w_1/(2w_2) + \pi\lambda/w_2) + \cosh(2\pi i\tau/w_2)}{\cosh(-\pi i/2 + 2\pi z/w_2 + \pi i w_1/(2w_2) - \pi\lambda/w_2) + \cosh(2\pi i\rho/w_2)}f(z).\end{aligned}$$

Replacing the variable z by $z - iw_1/2$ we can now rewrite the latter equation as

$$\begin{aligned}f(z+iw_1/2) &= \frac{\cosh\left(\frac{\pi i}{2} + \frac{2\pi z}{w_2} - \frac{3\pi i w_1}{2w_2} + \frac{\pi\lambda}{w_2}\right) + \cosh\left(\frac{2\pi i\tau}{w_2}\right)}{\cosh\left(-\frac{\pi i}{2} + \frac{2\pi z}{w_2} - \frac{\pi i w_1}{2w_2} - \frac{\pi\lambda}{w_2}\right) + \cosh\left(\frac{2\pi i\rho}{w_2}\right)}f(z-iw_1/2)\\ &= \frac{\cosh(\frac{\pi}{w_2}(z+\lambda/2 - 3iw_1/4 + iw_2/4 \pm i\tau))}{\cosh(\frac{\pi}{w_2}(z-\lambda/2 - iw_1/4 - iw_2/4 \pm i\rho))}f(z-iw_1/2)\\ &= \frac{\cosh(\frac{\pi}{w_2}(z+\lambda/2 - 3iw/2 \pm i\tau))}{\cosh(\frac{\pi}{w_2}(z-\lambda/2 - iw/2 + i\rho))}f(z-iw_1/2),\end{aligned}$$

where we used the $i\pi$-antiperiodicity of the hyperbolic cosine in the last equality.

3.B The limit behaviour of ψ

In this appendix we give the details on the calculation of the limit behaviour of ψ, cf. the proof of Lemma 3.6.9. Throughout this section we assume that $w_1, w_2 \in \mathbb{R}_{>0}$, $\lambda \notin \mathbb{R}$, $\gamma \in \mathbb{C}^4$, and that $\zeta > 0$ (with ζ given by (3.5.5)). We prove that

$$\psi(\gamma; x, \lambda) = \mathcal{O}(e^{\alpha(|\Im(\lambda)|-w)|\Re(x)|}) \tag{3.B.1}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. As explained in the proof of Lemma 3.6.9, we prove (3.B.1) by splitting $\mathbb{R}$ in five intervals and bounding the integral representation (3.5.7) of ψ over each interval.

3.B.1 Preparations

Let us first define a function K by

$$K(z, \lambda, a, b) = \frac{G(z + \lambda/2 - iw/2 \pm ia)}{G(z - \lambda/2 + iw/2 \pm ib)}.$$

The integral representation (3.5.7) for ψ can then be written as

$$\psi(\gamma; x, \lambda) = \int_{\mathbb{R}} K(z + x, \lambda, \tau, \rho) K(z, -\lambda, \sigma, \upsilon) dz. \tag{3.B.2}$$

The behaviour of K in the limit $z \to \pm\infty$ is controlled by

$$K_{\pm}(z, \lambda, a, b) = e^{\mp i\alpha(z(\lambda - iw) - a^2/2 + b^2/2)}.$$

Explicitly, for fixed a, b, and λ we have

$$K(z, \lambda, a, b) = K_{\pm}(z, \lambda, a, b) e^{g(z,\lambda,a,b)} \tag{3.B.3}$$

for $\pm\Re(z) > \max(w_1, w_2) + |\Re(\lambda)|/2 + \max(|\Im(a)|, |\Im(b)|)$, where

$$|g(z, \lambda, a, b)| < C(\Im(z)) e^{-\alpha \min(w_1, w_2)|\Re(z)|/2}, \tag{3.B.4}$$

with C depending continuously on $\Im(z)$, cf. (3.2.10) and (3.2.11).

3.B.2 General estimation scheme

Let ϵ and the intervals I_j ($j \in \{1, \ldots, 5\}$) be defined as in (3.6.14) and (3.6.15). We only consider the asymptotics for $\Re(x) \to \infty$. Assume that $\Re(x) > 2\epsilon$, causing the intervals to form a partition of the real line. We write the integral (3.B.2) defining ψ as

$$\psi(x) = \sum_{j=1}^{5} \psi_j(x), \tag{3.B.5}$$

where

$$\psi_j(x) = \int_{I_j} K(z+x,\lambda,\tau,\rho)K(z,-\lambda,\sigma,\upsilon)dz$$

for $j \in \{1,2,\ldots,5\}$. We bound these integrals using (3.B.3) (if one of them is applicable for the interval at hand).

For $j=1$ we have

$$\begin{aligned}\psi_1(x) &= \int_{-\infty}^{-\Re(x)-\epsilon} K_-(z+x,\lambda,\tau,\rho)K_-(z,-\lambda,\sigma,\upsilon)e^{g_1(z+x,x)}dz \\ &= e^{i\alpha\lambda x}e^{-\alpha w\bar{x}}e^{i\alpha(\rho^2+\upsilon^2-\tau^2-\sigma^2)/2}\int_{-\infty}^{-\epsilon} e^{2\alpha wz+g_1(z+i\Im(x),x)}dz \\ &= \mathcal{O}(e^{-\alpha(\Im(\lambda)+w)\Re(x)})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta,\zeta)$. Here $g_1(z,x) = g(z,\lambda,\tau,\rho) + g(z-x,-\lambda,\sigma,\upsilon)$ which satisfies an equation like (3.B.4) for $z < -\epsilon$

$$|g_1(z+i\Im(x),x)| < Ce^{-\alpha\min(w_1,w_2)|\Re(z)|/2}.$$

where the constant C is independent of $\Im(x)$, because $\Im(x)$ is bounded. In particular, $g_1(z+i\Im(x),x)$ is uniformly bounded for $z \in (-\infty,-\epsilon)$ and $x \in \{z \in \mathbb{C} \mid \Re(z) \geq 2\epsilon, |\Im(z)| < \zeta\}$.

Likewise we have for $j=5$,

$$\begin{aligned}\psi_5(x) &= \int_{\epsilon}^{\infty} K_+(z+x,\lambda,\tau,\rho)K_+(z,-\lambda,\sigma,\upsilon)e^{g_5(z,x)}dz \\ &= e^{-\alpha x(w+i\lambda)}e^{i\alpha(\sigma^2+\tau^2-\rho^2-\upsilon^2)/2}\int_{\epsilon}^{\infty} e^{-2\alpha wz+g_5(z,x)}dz \\ &= \mathcal{O}(e^{\alpha(\Im(\lambda)-w)\Re(x)})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta,\zeta)$. Here g_5 is a function which satisfies a bound like (3.B.4) for $z > \epsilon$, cf. the previous paragraph.

For $j=3$ we need to be a bit more careful. First observe that

$$\begin{aligned}\psi_3(x) &= \int_{-\Re(x)+\epsilon}^{-\epsilon} K_+(z+x,\lambda,\tau,\rho)K_-(z,-\lambda,\sigma,\upsilon)e^{g_3(z,x)}dz \\ &= e^{-\alpha x(w+i\lambda)}e^{i\alpha(\tau^2+\upsilon^2-\rho^2-\sigma^2)/2}\int_{-\Re(x)+\epsilon}^{-\epsilon} e^{-2i\alpha\lambda z+g_3(z,x)}dz,\end{aligned}$$

where $g_3 = g(z+x,\lambda,\tau,\rho) + g(z,-\lambda,\sigma,\upsilon)$ is bounded on $z \in (-\Re(x)+\epsilon,-\epsilon)$ by $Ce^{-\alpha\min(w_1,w_2)\min(-z,z-\Re(x))/2}$, and hence by the constant C itself. Therefore we have

$$\begin{aligned}|\psi_3(x)| &\leq e^{\alpha(\Im(\lambda)-w)\Re(x)+\alpha\Im(x)\Re(\lambda)}e^{\alpha\Im(\tau^2+\upsilon^2-\rho^2-\sigma^2)/2}\int_{-\Re(x)+\epsilon}^{-\epsilon} e^{2\alpha\Im(\lambda)z+C}dz \\ &= \mathcal{O}(e^{\alpha(|\Im(\lambda)|-w)\Re(x)})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. Here we get the final approximation by evaluating the integral and using that $\Im(\lambda) \neq 0$.

For $j = 4$ we cannot use (3.B.3) for the entire integrand. However we still have

$$\begin{aligned}\psi_4(x) &= \int_{-\epsilon}^{\epsilon} K_+(z + x, \lambda, \tau, \rho) K(z, -\lambda, \sigma, \upsilon) e^{g_4(z,x)} dz \\ &= e^{i\alpha x(iw-\lambda)} \int_{-\epsilon}^{\epsilon} K_+(z, \lambda, \tau, \rho) K(z, -\lambda, \sigma, \upsilon) e^{g_4(z,x)} dz \\ &= \mathcal{O}(e^{\alpha(\Im(\lambda)-w)\Re(x)})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$. Here g_4 is a function satisfying the bound $g_4(z, x) < Ce^{-\alpha \min(w_1,w_2)(\Re(x)-\epsilon)/2} \leq C$, for $z \in [-\epsilon, \epsilon]$ and $\Re(x) > 2\epsilon$.

Finally for $j = 2$ we have in a similar way

$$\begin{aligned}\psi_2(x) &= \int_{-\epsilon}^{\epsilon} K(z + i\Im(x), \lambda, \tau, \rho) K_-(z - \Re(x), -\lambda, \sigma, \upsilon) e^{g_2(z,x)} dz \\ &= e^{-i\alpha\Re(x)(-\lambda-iw)} \int_{-\epsilon}^{\epsilon} K(z + \Im(x), \lambda, \tau, \rho) K_-(z, -\lambda, \sigma, \upsilon) e^{g_2(z,x)} dz \\ &= \mathcal{O}(e^{-\alpha(\Im(\lambda)+w)x})\end{aligned}$$

for $\Re(x) \to \infty$, uniformly for $\Im(x)$ in compacta of $(-\zeta, \zeta)$, where g_2 is a bounded function, cf. the previous paragraph.

By (3.B.5) we conclude that the asymptotics (3.B.1) for ψ holds, as desired.

Chapter 4

Properties of generalized univariate hypergeometric functions

This chapter is joint work with Jasper Stokman and Eric Rains and presents the contents of an article accepted for publication in Communications in Mathematical Physics [6] with some cosmetic changes.

4.1 Introduction

The Gauß hypergeometric function, one of the cornerstones in the theory of classical univariate special functions, has been generalized in various fundamental directions. A theory on multivariate root system analogues of the Gauß hypergeometric function, due to Heckman and Opdam, has emerged, forming the basic tools to solve trigonometric and hyperbolic quantum many particle systems of Calogero-Moser type and generalizing the Harish-Chandra theory of spherical functions on Riemannian symmetric spaces (see [23] and references therein). A further important development has been the generalization to q-special functions, leading to the theory of Macdonald polynomials [48], which play a fundamental role in the theory of relativistic analogues of the trigonometric quantum Calogero-Moser systems (see e.g. [64]) and in harmonic analysis on quantum compact symmetric spaces (see e.g. [54], [44]). In this chapter, we focus on far-reaching generalizations of the Gauß hypergeometric function within the classes of elliptic, hyperbolic and trigonometric *univariate* special functions.

Inspired by results on integrable systems, Ruijsenaars [63] defined gamma functions of rational, trigonometric, hyperbolic and elliptic type. Correspondingly there are four types of special function theories, with the rational (resp. trigonometric) theory being the standard theory on hypergeometric (resp. q-

hypergeometric) special functions, while the hyperbolic theory is well suited to deal with unimodular base q. The theory of elliptic special functions, initiated by Frenkel and Turaev in [15], is currently in rapid development. The starting point of our analysis is the definition of the various generalized hypergeometric functions as an explicit hypergeometric integral of elliptic, hyperbolic and trigonometric type depending on seven auxiliary parameters (besides the bases). The elliptic and hyperbolic analogue of the hypergeometric function are due to Spiridonov [73], while the trigonometric analogue of the hypergeometric function is essentially an integral representation of the function Φ introduced and studied extensively by Gupta and Masson in [19]. Under a suitable parameter discretization, the three classes of generalized hypergeometric functions reduce to Rahman's [58] (trigonometric), Spiridonov's [73] (hyperbolic), and Spiridonov's and Zhedanov's [75], [73] (elliptic) families of biorthogonal rational functions.

Spiridonov [73] gave an elementary derivation of the symmetry of the elliptic hypergeometric function with respect to a twisted action of the Weyl group of type E_7 on the parameters using the elliptic analogue [71] of the Nassrallah-Rahman [51] beta integral. In this chapter we follow the same approach to establish the E_6-symmetry (respectively E_7-symmetry) of the trigonometric (respectively hyperbolic) hypergeometric function, using now the Nassrallah-Rahman beta integral (respectively its hyperbolic analogue from [77]). The E_6-symmetry of Φ has recently been established in [45] by different methods. Spiridonov [73] also gave elementary derivations of contiguous relations for the elliptic hypergeometric function using the fundamental addition formula for theta functions (see (4.3.6)), entailing a natural elliptic analogue of the Gauß hypergeometric differential equation. Following the same approach we establish contiguous relations and generalized Gauss hypergeometric equations for the hyperbolic and trigonometric hypergeometric function. For Φ it again leads to simple proofs of various results from [19].

Although the elliptic hypergeometric function is the most general amongst the generalized hypergeometric functions under consideration (rigorous limits between the different classes of special functions have been obtained in the recent paper [61] by Rains), it is also the most rigid in its class, in the sense that it does not admit natural degenerations within the class of elliptic special functions itself (there is no preferred limit point on an elliptic curve). On the other hand, for the hyperbolic and trigonometric hypergeometric functions various interesting degenerations within their classes are possible, as we point out in this chapter. It leads to many nontrivial identities and results, some of which are new and some are well known. In any case, it provides new insight in identities, e.g. as being natural consequences of symmetry breaking in the degeneration process, and it places many identities and classes of univariate special functions in a larger framework. For instance, viewing the trigonometric hypergeometric function as a degeneration of the elliptic hypergeometric function, we show that the breaking of symmetry (from E_7 to E_6) leads to a second important integral representation of Φ.

Moreover we show that Ruijsenaars' [65] relativistic analogue R of the hypergeometric function is a degeneration of the hyperbolic hypergeometric function, and that the D_4-symmetry [67] of R and the four Askey-Wilson second-order difference equations [65] satisfied by R are direct consequences of the E_7-symmetry and the contiguous relations of the hyperbolic hypergeometric function. Similarly, the Askey-Wilson function [37] is shown to be a degeneration of the trigonometric hypergeometric function. In this chapter we aim at deriving the symmetries of (degenerate) hyperbolic and trigonometric hypergeometric functions directly from appropriate hyperbolic and trigonometric beta integral evaluations using the above mentioned techniques of Spiridonov [73].

We hope that the general framework proposed in this chapter will shed light on the fundamental, common structures underlying various quantum relativistic Calogero-Moser systems and various quantum noncompact homogeneous spaces. In the present univariate setting, degenerations and specializations of the generalized hypergeometric functions play a key role in solving rank one cases of quantum relativistic integrable Calogero-Moser systems and in harmonic analysis on various quantum SL_2 groups. On the elliptic level, the elliptic hypergeometric function provides solutions of particular cases of van Diejen's [10] very general quantum relativistic Calogero-Moser systems of elliptic type (see e.g. [73]), while elliptic biorthogonal rational functions have been identified with matrix coefficients of the elliptic quantum SL_2 group in [38]. On the hyperbolic level, the Ruijsenaars' R-function solves the rank one case of a quantum relativistic Calogero-Moser system of hyperbolic type (see [68]) and arises as a matrix coefficient of the modular double of the quantum SL_2 group (see Chapter 3). On the trigonometric level, similar results are known for the Askey-Wilson function, which is a degeneration of the trigonometric hypergeometric function (see [37] and [36]). For higher rank only partial results are known, see e.g. [39], [59] (elliptic) and [76] (trigonometric).

The outline of the chapter is as follows. In Section 2 we discuss the general pattern of symmetry breaking when integrals with E_7-symmetry are degenerated. In Section 3 we introduce Spiridonov's [73] elliptic hypergeometric function. We shortly recall Spiridonov's [73] techniques to derive the E_7-symmetry and the contiguous relations for the elliptic hypergeometric function. In Section 4 these techniques are applied for the hyperbolic hypergeometric function and its top level degenerations. We show that a reparametrization of the top level degeneration of the hyperbolic hypergeometric function is Ruijsenaars' [65] relativistic hypergeometric function R. Key properties of R, such as a new integral representation, follow from the symmetries and contiguous relations of the hyperbolic hypergeometric function. In Section 5 these techniques are considered on the trigonometric level. We link the top level degeneration of the trigonometric hypergeometric function to the Askey-Wilson function. Moreover, we show that the techniques lead to elementary derivations of series representations and three term recurrence relations of the various trigonometric integrals. The trigonometric integrals are contour integrals over indented unit circles in the complex plane, which can be re-expressed as integrals over the real line with indentations by "unfolding" the trigonometric integral. We show that this provides a link with Agarwal type integral representations of basic hypergeometric series (see [16, Chapter 4]). Finally, in Section 6 we extend the techniques from [77] to connect the hyperbolic and trigonometric theory. It leads to an explicit expression of the hyperbolic hypergeometric function as a bilinear sum of trigonometric hypergeometric functions. In the top level degeneration, it explicitly relates Ruijsenaars' relativistic hypergeometric function to the Askey-Wilson function.

4.1.1 Notation

We denote $\sqrt{\cdot}$ for the branch of the square root $z \mapsto z^{\frac{1}{2}}$ on $\mathbb{C} \setminus \mathbb{R}_{<0}$ with positive values on $\mathbb{R}_{>0}$.

4.2 Weyl groups and symmetry breaking

The root system of type E_7 and its parabolic root sub-systems plays an important role in this chapter. In this section we describe our specific choice of realization of the root systems and Weyl groups, and we explain the general pattern of symmetry breaking which arises from degenerating integrals with Weyl group symmetries.

Degeneration of integrals with Weyl group symmetries in general causes symmetry breaking since the direction of degeneration in parameter space is not invariant under the symmetry group. All degenerations we consider are of the following form. For a basis Δ of a given irreducible, finite root system R in Euclidean space $(V, \langle \cdot, \cdot \rangle)$ with associated Weyl group W we denote

$$V^+(\Delta) = \{v \in V \mid \langle v, \alpha \rangle \geq 0 \quad \forall\, \alpha \in \Delta\}$$

for the associated positive Weyl chamber. We will study integrals $I(u)$ meromorphically depending on a parameter $u \in \mathcal{G}$. The parameter space will be some complex hyperplane $\mathcal{G}$ canonically isomorphic to the complexification $V_{\mathbb{C}}$ of V, from which it inherits a W-action. The integrals under consideration will be W-invariant under an associated twisted W-action. We degenerate such integrals by taking limits in parameter space along distinguished directions $v \in V^+(\Delta)$. The resulting degenerate integrals will thus inherit symmetries with respect to the isotropy subgroup

$$W_v = \{\sigma \in W \mid \sigma v = v\},$$

which is a standard parabolic subgroup of W with respect to the given basis Δ, generated by the simple reflections s_α, $\alpha \in \Delta \cap v^\perp$ (since $v \in V^+(\Delta)$).

All symmetry groups we will encounter are parabolic subgroups of the Weyl group W of type E_8. We use in this article the following explicit realization of the root system $R(E_8)$ of type E_8. Let ϵ_k be the kth element of the standard orthonormal basis of $V = \mathbb{R}^8$, with corresponding scalar product denoted by $\langle \cdot, \cdot \rangle$. We also denote $\langle \cdot, \cdot \rangle$ for its complex bilinear extension to $\mathbb{C}^8$. We write $\delta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \cdots + \epsilon_8)$. We realize the root system $R(E_8)$ of type E_8 in $\mathbb{R}^8$ as

$$R(E_8) = \{v = \sum_{j=1}^{8} c_j \epsilon_j + c\delta \mid \langle v, v \rangle = 2,\ c_j, c \in \mathbb{Z} \text{ and } \sum_{j=1}^{8} c_j \text{ even}\}.$$

For later purposes, it is convenient to have explicit notations for the roots in $R(E_8)$. The roots are $\pm\alpha_{jk}^+$ ($1 \le j < k \le 8$), α_{jk}^- ($1 \le j \neq k \le 8$), β_{jklm} ($1 \le j < k < l < m \le 8$), $\pm\gamma_{jk}$ ($1 \le j < k \le 8$) and $\pm\delta$, where

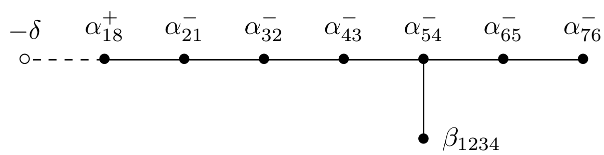
$$\begin{aligned}
\alpha_{jk}^+ &= \epsilon_j + \epsilon_k,\\
\alpha_{jk}^- &= \epsilon_j - \epsilon_k,\\
\beta_{jklm} &= \frac{1}{2}(\epsilon_j + \epsilon_k + \epsilon_l + \epsilon_m - \epsilon_n - \epsilon_p - \epsilon_q - \epsilon_r),\\
\gamma_{jk} &= \frac{1}{2}(-\epsilon_j - \epsilon_k + \epsilon_l + \epsilon_m + \epsilon_n + \epsilon_p + \epsilon_q + \epsilon_r)
\end{aligned}$$

and with (j,k,l,m,n,p,q,r) a permutation of $(1,2,3,4,5,6,7,8)$.

The canonical action of the associated Weyl group W on $\mathbb{C}^8$ is determined by the reflections $s_\gamma u = u - \langle u, \gamma\rangle\gamma$ for $u \in \mathbb{C}^8$ and $\gamma \in R(E_8)$. It is convenient to work with two different choices $\overline{\Delta}_1$, $\overline{\Delta}_2$ of bases for $R(E_8)$, namely

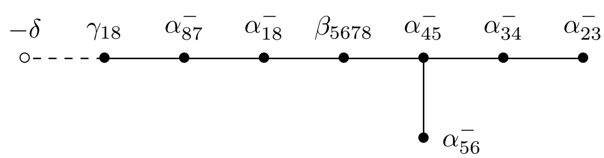
$$\begin{aligned}
\overline{\Delta}_1 &= \{\alpha_{76}^-, \beta_{1234}, \alpha_{65}^-, \alpha_{54}^-, \alpha_{43}^-, \alpha_{32}^-, \alpha_{21}^-, \alpha_{18}^+\},\\
\overline{\Delta}_2 &= \{\alpha_{23}^-, \alpha_{56}^-, \alpha_{34}^-, \alpha_{45}^-, \beta_{5678}, \alpha_{18}^-, \alpha_{87}^-, \gamma_{18}\},
\end{aligned}$$

with corresponding (affine) Dynkin diagrams



(4.2.1)

and



(4.2.2)

respectively, where the open node corresponds to the simple affine root, which we have labeled by the negative of the highest root of $R(E_8)$ with respect to the given basis (which in both cases is given by $\delta \in V^+(\overline{\Delta}_j)$). The reason for considering two different basis is the following: we will see that degenerating an elliptic hypergeometric integral with $W(E_7)$-symmetry to the trigonometric level in the direction of the basis element $\alpha_{18}^+ \in \overline{\Delta}_1$, respectively the basis element $\gamma_{18} \in \overline{\Delta}_2$, leads to two essentially different trigonometric hypergeometric integrals with $W(E_6)$-symmetry. The two integrals can be easily related since they arise as degeneration of the same elliptic hypergeometric integral. This leads directly to highly nontrivial trigonometric identities, see Section 4.5 for details.

This remark in fact touches on the basic philosophy of this chapter: it is the symmetry breaking in the degeneration of hypergeometric integrals which lead to various nontrivial identities. It forms an explanation why there are so many more nontrivial identities on the hyperbolic, trigonometric and rational level when compared to the elliptic level.

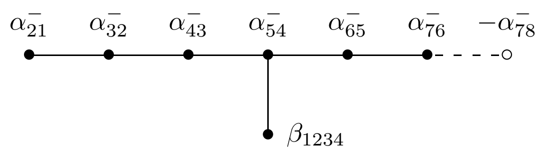
Returning to the precise description of the relevant symmetry groups, we will mainly encounter stabilizer subgroups of the isotropy subgroup W_δ. Observe that W_δ is a standard parabolic subgroup of W with respect to both bases $\overline{\Delta}_j$ since $\delta \in V^+(\overline{\Delta}_j)$ $(j = 1, 2)$, with associated simple reflections s_α, $\alpha \in \Delta_1 := \overline{\Delta}_1 \setminus \{\alpha_{18}^+\}$, respectively s_α, $\alpha \in \Delta_2 := \overline{\Delta}_2 \setminus \{\gamma_{18}\}$. Hence W_δ is isomorphic to the Weyl group of type E_7, and we accordingly write

$$W(E_7) := W_\delta.$$

We realize the corresponding standard parabolic root system $R(E_7) \subset R(E_8)$ as

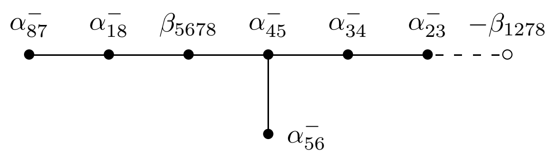
$$R(E_7) = R(E_8) \cap \delta^\perp \subseteq \delta^\perp \subset \mathbb{R}^8.$$

Both Δ_1 and Δ_2 form a basis of $R(E_7)$, and the associated (affine) Dynkin diagrams are given by



(4.2.3)

and



(4.2.4)

respectively (where we have used that α_{78}^-, respectively β_{1278}, is the highest root of $R(E_7)$ with respect to the basis Δ_1, respectively Δ_2). Note that the root system $R(E_7)$ consists of the roots of the form α_{jk}^- and β_{jklm}.

The top level univariate hypergeometric integrals which we will consider in this article depend meromorphically on a parameter $u \in \mathcal{G}_c$ with $\mathcal{G}_c \subset V_{\mathbb{C}} = \mathbb{C}^8$ $(c \in \mathbb{C})$ the complex hyperplane

$$\mathcal{G}_c = \frac{c}{2}\delta + \delta^\perp = \{u = (u_1, u_2, \ldots, u_8) \in \mathbb{C}^8 \mid \sum_{j=1}^{8} u_j = 2c\}.$$

The action on $\mathbb{C}^8$ of the isotropy subgroup $W(E_7) = W_{-\delta} \subset W$ preserves the hyperplane $\delta^\perp$ and fixes δ, hence it canonically acts on $\mathcal{G}_c$. We extend it to an action of the associated affine Weyl group $W_a(E_7)$ of $R(E_7)$ as follows. Denote L for the ($W(E_7)$-invariant) root lattice $L \subset \delta^\perp$ of $R(E_7)$, defined as the $\mathbb{Z}$-span of all $R(E_7)$-roots. The affine Weyl group $W_a(E_7)$ is the semi-direct product group $W_a(E_7) = W(E_7) \ltimes L$. The action of $W(E_7)$ on $\mathcal{G}_c$ can then be extended to an action of the affine Weyl group $W_a(E_7)$ depending on an extra parameter $z \in \mathbb{C}$ by letting $\gamma \in L$ act as the shift

$$\tau_\gamma^z u = u - z\gamma, \qquad u \in \mathcal{G}_c.$$

We suppress the dependence on z whenever its value is implicitly clear from context.

We also use a multiplicative version of the $W(E_7)$-action on $\mathcal{G}_c$. Consider the action of the group C_2 of order two on $\mathbb{C}^8$, with the non-unit element of C_2 acting by multiplication by -1 of each coordinate. We define the parameter space $\mathcal{H}_c$ for a parameter $c \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ as

$$\mathcal{H}_c = \{t = (t_1, \ldots, t_8) \in \mathbb{C}^8 \mid \prod_{j=1}^{8} t_j = c^2\}/C_2.$$

Note that this is well defined because if t satisfies $\prod t_i = c^2$, then so does $-t$. We sometimes abuse notation by simply writing $t = (t_1, \ldots, t_8)$ for the element $\pm t$ in $\mathcal{H}_c$ if no confusion can arise.

We view the parameters $t \in \mathcal{H}_{\exp(c)}$ as the exponential parameters associated to $u \in \mathcal{G}_c$. Modding out by the action of the 2-group C_2 allows us to put a $W_a(E_7)$-action on $\mathcal{H}_{\exp(c)}$, which is compatible to the $W_a(E_7)$-action on $\mathcal{G}_c$ as defined above. Concretely, consider the surjective map $\psi_c : \mathcal{G}_c \to \mathcal{H}_{\exp(c)}$ defined by

$$\psi_c(u) = \pm(\exp(u_1), \ldots, \exp(u_8)), \qquad u \in \mathcal{G}_c.$$

For $u \in \mathcal{G}_c$ we have $\psi_c^{-1}(\psi_c(u)) = u + 2\pi i L$, where L is the root lattice of $R(E_7)$ as defined above. Since L is $W(E_7)$-invariant, we can now define the action of $W_a(E_7)$ on $\mathcal{H}_{\exp(c)}$ by $\sigma\psi_c(u) = \psi_c(\sigma u)$, $\sigma \in W_a$ (for any auxiliary parameter $z \in \mathbb{C}$).

Regardless of whether we view the action of the affine Weyl group additively or multiplicatively, we will use the abbreviated notations $s_{jk} = s_{\alpha_{jk}^-}$, $w = s_{\beta_{1234}}$ and $\tau_{jk}^z = \tau_{\alpha_{jk}^-}^z$ throughout the article. Note that s_{jk} $(j \neq k)$ acts by interchanging the jth and kth coordinate. Furthermore, $W(E_7)$ is generated by the simple reflections s_α $(\alpha \in \Delta_1)$, which are the simple permutations $s_{j,j+1}$ $(j = 1, \ldots, 6)$ and w. The multiplicative action of w on $\mathcal{H}_c$ is explicitly given by $w(\pm t) = \pm(st_1, st_2, st_3, st_4, s^{-1}t_5, s^{-1}t_6, s^{-1}t_7, s^{-1}t_8)$ where $s^2 = c/t_1t_2t_3t_4 = t_5t_6t_7t_8/c$. Finally, note that the longest element v of the Weyl group $W(E_7)$ acts by multiplication with -1 on the root system $R(E_7)$, and hence it acts by $vu = c/2 - u$ on $\mathcal{G}_c$ and by $v(\pm t) = \pm(c^{\frac{1}{2}}/t_1, \ldots, c^{\frac{1}{2}}/t_8)$ on $\mathcal{H}_c$.

4.3 The univariate elliptic hypergeometric function

4.3.1 The elliptic gamma function

We will use notations which are consistent with [16]. We fix throughout this section two bases $p, q \in \mathbb{C}$ satisfying $|p|, |q| < 1$. The q-shifted factorial is defined by

$$\left(a; q\right)_\infty = \prod_{j=0}^{\infty}(1 - aq^j).$$

We write $\big(a_1,\ldots,a_m;q\big)_\infty=\prod_{j=1}^m\big(a_j;q\big)_\infty$, $\big(az^{\pm 1};q\big)_\infty=\big(az,az^{-1};q\big)_\infty$ etc. as shorthand notations for products of q-shifted factorials. The renormalized Jacobi theta-function is defined by

$$\theta(a;q)=\big(a,q/a;q\big)_\infty.$$

The elliptic gamma function [63], defined by the infinite product

$$\Gamma_e(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},$$

is a meromorphic function in $z\in\mathbb{C}^\times=\mathbb{C}\setminus\{0\}$ which satisfies the difference equation

$$\Gamma_e(qz;p,q)=\theta(z;p)\Gamma_e(z;p,q), \tag{4.3.1}$$

satisfies the reflection equation

$$\Gamma_e(z;p,q)=1/\Gamma_e(pq/z;p,q),$$

and is symmetric in p and q,

$$\Gamma_e(z;p,q)=\Gamma_e(z;q,p).$$

For products of theta-functions and elliptic gamma functions we use the same shorthand notations as for the q-shifted factorial, e.g.

$$\Gamma_e(a_1,\ldots,a_m;p,q)=\prod_{j=1}^m\Gamma_e(a_j;p,q).$$

In this section we call a sequence of points a downward (respectively upward) sequence of points if it is of the form ap^jq^k (respectively $ap^{-j}q^{-k}$) with $j,k\in\mathbb{Z}_{\geq 0}$ for some $a\in\mathbb{C}$. Observe that the elliptic gamma function $\Gamma_e(az;p,q)$, considered as a meromorphic function in z, has poles at the upward sequence $a^{-1}p^{-j}q^{-k}$ $(j,k\in\mathbb{Z}_{\geq 0})$ of points and has zeros at the downward sequence $a^{-1}p^{j+1}q^{k+1}$ $(j,k\in\mathbb{Z}_{\geq 0})$ of points.

4.3.2 Symmetries of the elliptic hypergeometric function

The fundamental starting point of our investigations is Spiridonov's [71] elliptic analogue of the classical beta integral,

$$\frac{(q;q)_\infty(p;p)_\infty}{2}\int_{\mathcal{C}}\frac{\prod_{j=1}^6\Gamma_e(t_jz^{\pm 1};p,q)}{\Gamma_e(z^{\pm 2};p,q)}\frac{dz}{2\pi iz}=\prod_{1\leq j<k\leq 6}\Gamma_e(t_jt_k;p,q) \tag{4.3.2}$$

for generic parameters $t\in\mathbb{C}^6$ satisfying the balancing condition $\prod_{j=1}^6 t_j=pq$, where the contour $\mathcal{C}$ is chosen as a deformation of the positively oriented unit circle $\mathbb{T}$ separating the downward sequences $t_j p^{\mathbb{Z}_{\geq 0}} q^{\mathbb{Z}_{\geq 0}}$ $(j = 1, \ldots, 6)$ of poles of the integrand from the upward sequences $t_j^{-1} p^{\mathbb{Z}_{\leq 0}} q^{\mathbb{Z}_{\leq 0}}$ $(j = 1, \ldots, 6)$. Note here that the factor $1/\Gamma_e(z^{\pm 2}; p, q)$ of the integrand is analytic on $\mathbb{C}^\times$. Moreover, observe that we can take the positively oriented unit circle $\mathbb{T}$ as contour if the parameters satisfy $|t_j| < 1$ $(j = 1, \ldots, 6)$. Several elementary proofs of (4.3.2) are now known, see e.g. [71], [72] and [59].

We define the integrand $I_e(t; z) = I_e(t; z; p, q)$ for the univariate elliptic hypergeometric function as

$$I_e(t; z; p, q) = \frac{\prod_{j=1}^{8} \Gamma_e(t_j z^{\pm 1}; p, q)}{\Gamma_e(z^{\pm 2}; p, q)},$$

where $t = (t_1, t_2, \ldots, t_8) \in \left(\mathbb{C}^\times\right)^8$. For parameters $t \in \mathbb{C}^8$ with $\prod_{j=1}^{8} t_j = p^2 q^2$ and $t_i t_j \notin p^{\mathbb{Z}_{\leq 0}} q^{\mathbb{Z}_{\leq 0}}$ for $1 \leq i, j \leq 8$ (possibly equal), we can define the elliptic hypergeometric function $S_e(t) = S_e(t; p, q)$ by

$$S_e(t; p, q) = \int_{\mathcal{C}} I_e(t; z; p, q) \frac{dz}{2\pi i z},$$

where the contour $\mathcal{C}$ is a deformation of $\mathbb{T}$ which separates the downward sequences $t_j p^{\mathbb{Z}_{\geq 0}} q^{\mathbb{Z}_{\geq 0}}$ $(j = 1, \ldots, 8)$ of poles of $I_e(t; \cdot)$ from the upward sequences $t_j^{-1} p^{\mathbb{Z}_{\leq 0}} q^{\mathbb{Z}_{\leq 0}}$ $(j = 1, \ldots, 8)$. If the parameters satisfy $|t_j| < 1$ this contour can again be taken as the positively oriented unit circle $\mathbb{T}$.

The elliptic hypergeometric function S_e extends uniquely to a meromorphic function on $\{t \in \mathbb{C}^8 : \prod t_j = p^2 q^2\}$. In fact, for a particular value τ of the parameters for which the integral is not defined, we first deform for t in a small open neighborhood of τ the contour $\mathcal{C}$ to include those upward poles which collide at $t = \tau$ with downward poles. The resulting expression is an integral which is analytic at an open neighborhood of τ plus a sum of residues depending meromorphically on the parameters t. This expression yields the desired meromorphic extension of $S_e(t)$ at τ. For further detailed analysis of meromorphic dependencies of integrals like S_e, see e.g. [65] and [59].

Since $I_e(t; -z) = I_e(-t; z)$ where $-t = (-t_1, \ldots, -t_8)$, we have $S_e(t) = S_e(-t)$, hence we can and will view S_e as a meromorphic function $S_e : \mathcal{H}_{pq} \to \mathbb{C}$. Furthermore, $S_e(t)$ is the special case $I\!I^1_{\mathrm{BC}_1}$ of Rains' [59] multivariate elliptic hypergeometric integrals $I\!I^m_{\mathrm{BC}_n}$, and it coincides with Spiridonov's [73, §5] elliptic analogue $V(\cdot)$ of the Gauss hypergeometric function.

Remark 4.3.1. Note that $S_e(t; p, q)$ reduces to the elliptic beta integral (4.3.2) when e.g. $t_1 t_6 = pq$. More generally, for e.g. $t_1 t_6 = p^{m+1} q^{n+1}$ $(m, n \in \mathbb{Z}_{\geq 0})$ it follows from [74, Thm. 11] that $S_e(t; p, q)$ essentially coincides with the two-index elliptic biorthogonal rational function R_{nm} of Spiridonov [74, App. A], which is the product of two very-well-poised terminating elliptic hypergeometric ${}_{12}V_{11}$ series (the second one with the role of the bases p and q reversed).

Next we determine the explicit $W(E_7)$-symmetries of $S_e(t)$ in terms of the $W(E_7)$ action on $t \in \mathcal{H}_{pq}$ from Section 4.2. This result was previously obtained by Rains [59] and by Spiridonov [73]. We give here a proof which is similar to Spiridonov's [73, §5] proof.

Theorem 4.3.2. *The elliptic hypergeometric function $S_e(t)$ $(t \in \mathcal{H}_{pq})$ is invariant under permutations of $(t_1, \ldots, t_8)$ and it satisfies*

$$S_e(t;p,q) = S_e(wt;p,q) \prod_{1 \le j < k \le 4} \Gamma_e(t_j t_k;p,q) \prod_{5 \le j < k \le 8} \Gamma_e(t_j t_k;p,q) \tag{4.3.3}$$

as meromorphic functions in $t \in \mathcal{H}_{pq}$, where (recall) $w = s_{\beta_{1234}}$.

Proof. The permutation symmetry is trivial. To prove (4.3.3) we first prove it for parameters $t \in \mathbb{C}^8$ satisfying $\prod_{j=1}^8 t_j = p^2q^2$ and satisfying the additional restraints $|t_j| < 1$ $(j = 1, \ldots, 8)$, $|t_j| > |pq|^{\frac{1}{3}}$ $(j = 5, \ldots, 8)$ and $|\prod_{j=5}^8 t_j| < |pq|$ (which defines a non-empty open subset of parameters of $\{t \in \mathbb{C} \mid \prod_{j=1}^8 t_j = p^2q^2\}$ since $|p|, |q| < 1$). For these special values of the parameters we consider the double integral

$$\int_{\mathbb{T}^2} \frac{\prod_{j=1}^4 \Gamma_e(t_j z^{\pm 1};p,q)\Gamma_e(sx^{\pm 1}z^{\pm 1};p,q) \prod_{j=5}^8 \Gamma_e(t_j s^{-1} x^{\pm 1};p,q)}{\Gamma_e(z^{\pm 2}, x^{\pm 2};p,q)} \frac{dz}{2\pi i z} \frac{dx}{2\pi i x},$$

where s is chosen to balance both the z as the x integral, so $s^2 \prod_{j=1}^4 t_j = pq = s^{-2} \prod_{j=5}^8 t_j$. By the additional parameter restraints we have $|s| < 1$ and $|t_j/s| < 1$ for $j = 5, \ldots, 8$, hence the integration contour $\mathbb{T}$ separates the downward pole sequences of the integrand from the upward ones for both integration variables. Using the elliptic beta integral (4.3.2) to integrate this double integral either first over the variable z, or first over the variable x, now yields (4.3.3). Analytic continuation then implies the identity (4.3.3) as meromorphic functions on $\mathcal{H}_{pq}$. □

An interesting equation for $S_e(t)$ arises from Theorem 4.3.2 by considering the action of the longest element v of $W(E_7)$, using its decomposition

$$v = s_{45}s_{36}s_{48}s_{37}s_{34}s_{12}ws_{37}s_{48}ws_{35}s_{46}w \tag{4.3.4}$$

as products of permutations and w.

Corollary 4.3.3. *We have*

$$S_e(t;p,q) = S_e(vt;p,q) \prod_{1 \le j < k \le 8} \Gamma_e(t_j t_k;p,q) \tag{4.3.5}$$

as meromorphic functions in $t \in \mathcal{H}_{pq}$.

Remark 4.3.4. Corollary 4.3.3 is the special case $n = m = 1$ of [59, Thm. 3.1], see also [73, §5, (iii)] for a proof close to our present derivation.

4.3.3 Contiguous relations

For sake of completeness we recall here Spiridonov's [73, §6] derivation of certain contiguous relations cq. difference equations for the elliptic hypergeometric function $S_e(t)$ (most notably, Spiridonov's elliptic hypergeometric equation). The starting point is the fundamental theta function identity [16, Exercise 2.16],

$$\frac{1}{y}\theta(ux^{\pm 1}, yz^{\pm 1}; p) + \frac{1}{z}\theta(uy^{\pm 1}, zx^{\pm 1}; p) + \frac{1}{x}\theta(uz^{\pm 1}, xy^{\pm 1}; p) = 0, \tag{4.3.6}$$

which holds for arbitrary $u, x, y, z \in \mathbb{C}^\times$. For the $W_a(E_7)$-action on $\mathcal{H}_{pq}$ we take in this subsection $\tau_{ij} = \tau_{ij}^{-\log(q)}$, which multiplies t_i by q and divides t_j by q. Note that the q-difference operators τ_{ij} are already well defined on $\{t \in \mathbb{C}^8 \mid \prod_{j=1}^8 t_j = p^2q^2\}$.

Using the difference equation (4.3.1) of the elliptic gamma function and using (4.3.6), we have

$$\frac{\theta(q^{-1}t_8t_7^{\pm 1}; p)}{\theta(t_6t_7^{\pm 1}; p)} I_e(\tau_{68}t; z) + (t_6 \leftrightarrow t_7) = I_e(t; z),$$

where $(t_6 \leftrightarrow t_7)$ means the same term with t_6 and t_7 interchanged. For generic $t \in \mathbb{C}^8$ with $\prod_{j=1}^8 t_j = p^2q^2$ we integrate this equality over $z \in \mathcal{C}$, with $\mathcal{C}$ a deformation of $\mathbb{T}$ which separates the upward and downward pole sequences of all three integrands at the same time. We obtain

$$\frac{\theta(q^{-1}t_8t_7^{\pm 1}; p)}{\theta(t_6t_7^{\pm 1}; p)} S_e(\tau_{68}t) + (t_6 \leftrightarrow t_7) = S_e(t) \tag{4.3.7}$$

as meromorphic functions in $t \in \mathcal{H}_{pq}$. This equation is also the $n = 1$ instance of [60, Thm. 3.1]. Note that in both terms on the left hand side the same parameter t_8 is divided by q, while two different parameters (t_6 and t_7) are multiplied by q. We can obtain a different equation (i.e. not obtainable by applying an S_8 symmetry to (4.3.7)) by substituting the parameters vt in (4.3.7), where $v \in W(E_7)$ is the longest Weyl group element, and by using (4.3.5). The crux is that $\tau_{68}vt = v\tau_{86}t$. We obtain

$$\frac{\theta(t_7/qt_8; p)}{\theta(t_7/t_6; p)} \prod_{j=1}^{5} \theta(t_jt_6/q; p) S_e(\tau_{86}t) + (t_6 \leftrightarrow t_7) = \prod_{j=1}^{5} \theta(t_jt_8; p) S_e(t) \tag{4.3.8}$$

for $t \in \mathcal{H}_{pq}$. We arrive at Spiridonov's [73, §6] elliptic hypergeometric equation for $S_e(t)$.

Theorem 4.3.5 ([71]). *We have*

$$A(t)S_e(\tau_{87}t; p, q) + (t_7 \leftrightarrow t_8) = B(t)S_e(t; p, q) \tag{4.3.9}$$

as meromorphic functions in $t \in \mathcal{H}_{pq}$*, where* A *and* B *are defined by*

$$\begin{aligned} A(t) &= \frac{1}{t_8\theta(t_7/qt_8, t_8/t_7;p)} \prod_{j=1}^{6} \theta(t_j t_7/q;p) \\ B(t) &= \frac{\theta(t_7t_8/q;p)}{t_6\theta(t_7/qt_6, t_8/qt_6;p)} \prod_{j=1}^{5} \theta(t_j t_6;p) \\ &\quad - \frac{\theta(t_6/t_8, t_6t_8;p)}{t_6\theta(t_7/qt_6, t_7/qt_8, t_8/t_7;p)} \prod_{j=1}^{5} \theta(t_j t_7/q;p) \\ &\quad - \frac{\theta(t_6/t_7, t_6t_7;p)}{t_6\theta(t_7/t_8, t_8/qt_6, t_8/qt_7;p)} \prod_{j=1}^{5} \theta(t_j t_8/q;p). \end{aligned}$$

Remark 4.3.6. Note that B has an S_6-symmetry in $(t_1, t_2, \ldots, t_6)$ even though it is not directly apparent from its explicit representation.

Proof. This follows by taking an appropriate combination of three contiguous relations for $S_e(t)$. Specifically, the three contiguous relations are (4.3.7) and (4.3.8) with t_6 and t_8 interchanged, and (4.3.7) with t_7 and t_8 interchanged. □

By combining contiguous relations for $S_e(t)$ and exploring the $W(E_7)$-symmetry of $S_e(t)$, one can obtain various other contiguous relations involving $S_e(\tau_x t)$, $S_e(\tau_y t)$, and $S_e(\tau_z t)$ for suitable root lattice vectors $x, y, z \in L$. A detailed analysis of such procedures is undertaken for three term transformation formulas on the trigonometric setting by Lievens and Van der Jeugt [45] (see also Section 4.5).

Remark 4.3.7. Interchanging the role of the bases p and q and using the symmetry of $S_e(t;p,q)$ in p and q, we obtain contiguous relations for $S_e(t;p,q)$ with respect to multiplicative p-shifts in the parameters.

4.4 Hyperbolic hypergeometric integrals

4.4.1 The hyperbolic gamma function

The hyperbolic gamma function G used in this section is related to Γ_h through (2.2.3). We fix throughout this section $\omega_1, \omega_2 \in \mathbb{C}$ satisfying $\Re(\omega_1), \Re(\omega_2) > 0$, and we write

$$\omega = \frac{\omega_1 + \omega_2}{2}.$$

Ruijsenaars' [63] hyperbolic gamma function is defined by

$$G(z;\omega_1,\omega_2) = \exp\left(i \int_0^\infty \left(\frac{\sin(2zt)}{2\sinh(\omega_1 t)\sinh(\omega_2 t)} - \frac{z}{\omega_1\omega_2 t} \right) \frac{dt}{t} \right)$$

for $z \in \mathbb{C}$ satisfying $|\Im(z)| < \Re(\omega)$. There exists a unique meromorphic extension of $G(\omega_1, \omega_2; z)$ to $z \in \mathbb{C}$ satisfying

$$\begin{aligned} G(z;\omega_1,\omega_2) &= G(z;\omega_2,\omega_1),\\ G(z;\omega_1,\omega_2) &= G(-z;\omega_1,\omega_2)^{-1},\\ G(z+i\omega_1;\omega_1,\omega_2;) &= -2is((z+i\omega)/\omega_2)G(z;\omega_1,\omega_2),\end{aligned} \tag{4.4.1}$$

where we use the shorthand notation $s(z) = \sinh(\pi z)$. The second equation here is called the reflection equation. In this section we will omit the ω_1, ω_2 dependence of G if no confusion is possible, and we formulate all results only with respect to $i\omega_1$-shifts. We use similar notations for products of hyperbolic gamma functions as for q-shifted factorials and elliptic gamma functions, e.g.

$$G(z_1, \ldots, z_n; \omega_1, \omega_2) = \prod_{j=1}^{n} G(z_j; \omega_1, \omega_2).$$

The hyperbolic gamma function G is a degeneration of the elliptic gamma function Γ_e,

$$\begin{aligned} \lim_{r\searrow 0} \Gamma_e\bigl(\exp(2\pi irz); \exp(-2\pi\omega_1 r), \exp(-2\pi\omega_2 r)\bigr) \exp\left(\frac{\pi(z-i\omega)}{6ir\omega_1\omega_2}\right)& \\ = G(z-i\omega;\omega_1,\omega_2)& \end{aligned} \tag{4.4.2}$$

for $\omega_1, \omega_2 > 0$, see [63, Prop. III.12].

In this section we call a sequence of points a downward (respectively upward) sequence of points if it is of the form $a+i\mathbb{Z}_{\leq 0}\omega_1+i\mathbb{Z}_{\leq 0}\omega_2$ (respectively $a+i\mathbb{Z}_{\geq 0}\omega_1+i\mathbb{Z}_{\geq 0}\omega_2$) for some $a \in \mathbb{C}$. Recall from [63] that the hyperbolic gamma function $G(\omega_1, \omega_2; z)$, viewed as meromorphic function in $z \in \mathbb{C}$, has poles at the downward sequence $-i\omega + i\mathbb{Z}_{\leq 0}\omega_1 + i\mathbb{Z}_{\leq 0}\omega_2$ of points and has zeros at the upward sequence $i\omega + i\mathbb{Z}_{\geq 0}\omega_1 + i\mathbb{Z}_{\geq 0}\omega_2$ of points. The pole of $G(z; \omega_1, \omega_2)$ at $z = -i\omega$ is simple and

$$\operatorname*{Res}_{z=-i\omega}\bigl(G(z;\omega_1,\omega_2)\bigr) = \frac{i}{2\pi}\sqrt{\omega_1\omega_2}. \tag{4.4.3}$$

All contours in this section will be chosen as deformations of the real line $\mathbb{R}$ separating the upward pole sequences of the integrand from the downward ones.

We will also need to know the asymptotic behavior of $G(z)$ as $\Re(z) \to \pm\infty$ (uniformly for $\Im(z)$ in compacta of $\mathbb{R}$). For our purposes it is sufficient to know that for any $a, b \in \mathbb{C}$ we have

$$\lim_{\Re(z)\to\infty} \frac{G(z-a;\omega_1,\omega_2)}{G(z-b;\omega_1,\omega_2)} \exp\left(\frac{\pi iz}{\omega_1\omega_2}(b-a)\right) = \exp\left(\frac{\pi i}{2\omega_1\omega_2}(b^2-a^2)\right), \tag{4.4.4}$$

where the corresponding $o(\Re(z))$-tail as $\Re(z) \to \infty$ can be estimated uniformly for $\Im(z)$ in compacta of $\mathbb{R}$, and that for periods satisfying $\omega_1\omega_2 \in \mathbb{R}_{>0}$,

$$|G(u+x;\omega_1,\omega_2)| \leq M \exp\left(\pi\Im\Bigl(\frac{u}{\omega_1\omega_2}\Bigr)|x|\right), \qquad \forall\, x \in \mathbb{R} \tag{4.4.5}$$

for some constant $M > 0$, provided that the line $u + \mathbb{R}$ does not hit a pole of G. See [65, Appendix A] for details and for more precise asymptotic estimates.

4.4.2 Symmetries of the hyperbolic hypergeometric function

The univariate hyperbolic beta integral [77, (1.10)] is

$$\int_{\mathcal{C}} \frac{G(i\omega \pm 2z;\omega_1,\omega_2)}{\prod_{j=1}^{6} G(u_j \pm z;\omega_1,\omega_2)} dz = 2\sqrt{\omega_1\omega_2} \prod_{1\leq j<k\leq 6} G(i\omega - u_j - u_k;\omega_1,\omega_2) \quad (4.4.6)$$

for generic $u_1,\ldots,u_6 \in \mathbb{C}$ satisfying the additive balancing condition $\sum_{j=1}^{6} u_j = 4i\omega$. Note that this integral converges since the asymptotic behaviour of the integrand at $z = \pm\infty$ is $\mathcal{O}(\exp(-4\pi|z|\omega/\omega_1\omega_2))$ in view of the reflection equation (4.4.1), the limit (4.4.4) and the fact that $\Re\left(\frac{\omega}{\omega_1\omega_2}\right) > 0$ due to the imposed conditions $\Re(\omega_j) > 0$ on the periods ω_j $(j = 1,2)$.

We can now define the integrand of the hyperbolic hypergeometric function $I_h(u;z) = I_h(u;z;\omega_1,\omega_2)$ as

$$I_h(u;z;\omega_1,\omega_2) = \frac{G(i\omega \pm 2z;\omega_1,\omega_2)}{\prod_{j=1}^{8} G(u_j \pm z;\omega_1,\omega_2)}$$

for arbitrary parameters $u \in \mathbb{C}^8$. The hyperbolic hypergeometric function $S_h(u) = S_h(u;\omega_1,\omega_2)$ is defined by

$$S_h(u;\omega_1,\omega_2) = \int_{\mathcal{C}} I_h(u;z;\omega_1,\omega_2) dz \quad (4.4.7)$$

for generic parameters $u \in \mathcal{G}_{2i\omega}$ (see Section 4.2 for the definition of $\mathcal{G}_{2i\omega}$). The asymptotic behaviour of $I_h(u;z)$ at $z = \pm\infty$ is again $\mathcal{O}(\exp(-4\pi|z|\omega/\omega_1\omega_2))$, so the integral absolutely converges. It follows from (4.4.3) and the analytic difference equations for the hyperbolic gamma function that $S_h(u)$ has a unique meromorphic extension to $u \in \mathcal{G}_{2i\omega}$, cf. the analysis for the elliptic hypergeometric function $S_e(t)$. We thus can and will view $S_h(u)$ as a meromorphic function in $u \in \mathcal{G}_{2i\omega}$. Note furthermore that the real line can be chosen as integration contour in the definition of $S_h(u)$ if $u \in \mathcal{G}_{2i\omega}$ satisfies $\Im(u_j - i\omega) < 0$ for all j. The hyperbolic hypergeometric function $S_h(u)$ $(u \in \mathcal{G}_{2i\omega})$ coincides with Spiridonov's [73, §5] hyperbolic analogue $s(\cdot)$ of the Gauss hypergeometric function.

Using (4.4.2) and the reflection equation of G, we can obtain the hyperbolic hypergeometric function $S_h(vu;\omega_1,\omega_2) = S_h(i\omega - u_1,\ldots,i\omega - u_8;\omega_1,\omega_2)$ $(u \in \mathcal{G}_{2i\omega})$ formally as the degeneration $r \downarrow 0$ of the elliptic hypergeometric function $S_e(t;p,q)$ with $p = \exp(-2\pi\omega_1 r)$, $q = \exp(-2\pi\omega_2 r)$ and $t = \psi_{2i\omega}(2\pi i r u) \in \mathcal{H}_{\exp(-4\pi r\omega)} = \mathcal{H}_{pq}$. This degeneration, which turns out to preserve the $W(E_7)$-symmetry (see below), can be proven rigorously, see [61]. This entails in particular a derivation of the hyperbolic beta integral (4.4.6) as rigorous degeneration of the elliptic beta integral (4.3.2) (see [77, §5.4] for the formal analysis).

Next we give the explicit $W(E_7)$ symmetries of $S_h(u)$ in terms of the $W(E_7)$ action on $u \in \mathcal{G}_{2i\omega}$ from Section 4.2.

Theorem 4.4.1. *The hyperbolic hypergeometric function* $S_h(u)$ $(u \in \mathcal{G}_{2i\omega})$ *is invariant under permutations of* $(u_1, \ldots, u_8)$ *and it satisfies*

$$\begin{aligned} S_h(u;\omega_1,\omega_2) &= S_h(wu;\omega_1,\omega_2) \\ &\quad \times \prod_{1\le j<k\le 4} G(i\omega - u_j - u_k;\omega_1,\omega_2) \prod_{5\le j<k\le 8} G(i\omega - u_j - u_k;\omega_1,\omega_2) \end{aligned}$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$.

Proof. The proof is analogous to the proof in the elliptic case (Theorem 4.3.2). For the w-symmetry we consider for suitable $u \in \mathcal{G}_{2i\omega}$ the double integral

$$\int_{\mathbb{R}^2} \frac{G(i\omega \pm 2z, i\omega \pm 2x)}{\prod_{j=1}^4 G(u_j \pm z) G(i\omega + s \pm x \pm z) \prod_{k=5}^8 G(u_k - s \pm x)} dz dx$$

with $s = i\omega - \frac{1}{2}(u_1 + u_2 + u_3 + u_4) = -i\omega + \frac{1}{2}(u_5 + u_6 + u_7 + u_8)$. We impose the conditions $\Im(s) < 0$ and

$$\Im\big(u_j - i\omega\big) < 0 \quad (j = 1, \ldots, 4), \qquad \Im(u_k - i\omega) < \Im(s) \quad (k = 5, \ldots, 8) \tag{4.4.8}$$

on $u \in \mathcal{G}_{2i\omega}$ to ensure that the upward and downward pole sequences of the integrand of the double integral are separated by $\mathbb{R}$. Next we show that the parameter restraints

$$-\Re\Big(\frac{\omega}{\omega_1\omega_2}\Big) < \Im\Big(\frac{s}{\omega_1\omega_2}\Big) < 0 \tag{4.4.9}$$

on $u \in \mathcal{G}_{2i\omega}$ suffice to ensure absolute convergence of the double integral. Using the reflection equation and asymptotics (4.4.5) of G we obtain the estimate

$$\frac{1}{|G(i\omega + s \pm x \pm z)|} \le M \exp\left(-2\pi\Im\Big(\frac{s + i\omega}{\omega_1\omega_2}\Big)\big(|z + x| + |z - x|\big)\right), \quad \forall\,(x, z) \in \mathbb{R}^2$$

for some constant $M > 0$. It follows that the factor $G(i\omega + s \pm x \pm z)^{-1}$ of the integrand is absolutely and uniformly bounded if $\Im\big((i\omega + s)/\omega_1\omega_2\big) \ge 0$, i.e. if $\Im\big(s/\omega_1\omega_2\big) \ge -\Re\big(\omega/\omega_1\omega_2\big)$. The asymptotic behaviour of the remaining factors of the integrand (which breaks up in factors only depending on x or on z) can easily be determined by (4.4.5), leading finally to the parameter restraints (4.4.9) for the absolute convergence of the double integral.

It is easy to verify that the parameter subset of $\mathcal{G}_{2i\omega}$ defined by the additional restraints $\Im(s) < 0$, (4.4.8) and (4.4.9) is non-empty (by e.g. constructing parameters $u \in \mathcal{G}_{2i\omega}$ with small associated balancing parameter s). Using Fubini's Theorem and the hyperbolic beta integral (4.4.6), we now reduce the double integral to a single integral by either evaluating the integral over x, or by evaluating the integral over z. Using furthermore that

$$wu = (u_1 + s, u_2 + s, u_3 + s, u_4 + s, u_5 - s, u_6 - s, u_7 - s, u_8 - s)$$

for $u \in \mathcal{G}_{2i\omega}$, it follows that the resulting identity is the desired w-symmetry of S_h for the restricted parameter domain. Analytic continuation now completes the proof. □

The symmetry of $S_h(u)$ ($u \in \mathcal{G}_{2i\omega}$) with respect to the action of the longest Weyl group element $v \in W(E_7)$ is as follows.

Corollary 4.4.2. *The hyperbolic hypergeometric function S_h satisfies*

$$S_h(u;\omega_1,\omega_2) = S_h(vu;\omega_1,\omega_2) \prod_{1\le j<k\le 8} G(i\omega - u_j - u_k;\omega_1,\omega_2) \tag{4.4.10}$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$.

Proof. This follows from Theorem 4.4.1 and (4.3.4). □

4.4.3 Contiguous relations

Contiguous relations for the hyperbolic hypergeometric function S_h can be derived in nearly exactly the same manner as we did for the elliptic hypergeometric function S_e (see Section 4.3.3 and [73, §6]). We therefore only indicate the main steps. Using the $p = 0$ case of (4.3.6) we have

$$s(x \pm v)s(y \pm z) + s(x \pm y)s(z \pm v) + s(x \pm z)s(v \pm y) = 0,$$

where $s(x \pm v) = s(x+v)s(x-v)$. In this subsection we write $\tau_{jk} = \tau_{jk}^{i\omega_1}$ ($1 \le j \neq k \le 8$), which acts on $u \in \mathcal{G}_{2i\omega}$ by subtracting $i\omega_1$ from u_j and adding $i\omega_1$ to u_k. We now obtain in analogy to the elliptic case the difference equation

$$\frac{s((u_8 + i\omega \pm (u_7 - i\omega))/\omega_2)}{s((u_6 - i\omega \pm (u_7 - i\omega))/\omega_2)} S_h(\tau_{68}u) + (u_6 \leftrightarrow u_7) = S_h(u)$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$. Using (4.4.10) we subsequently obtain

$$\frac{s((u_7 - u_8 + 2i\omega)/\omega_2)}{s((u_7 - u_6)/\omega_2)} \prod_{j=1}^{5} s((u_j + u_6)/\omega_2) S_h(\tau_{86}u) + (u_6 \leftrightarrow u_7) = \prod_{j=1}^{5} s((u_j + u_8 - 2i\omega)/\omega_2) S_h(u)$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$. Combining these contiguous relations and simplifying we obtain

$$A(u)S_h(\tau_{87}u) - (u_7 \leftrightarrow u_8) = B(u)S_h(u), \qquad u \in \mathcal{G}_{2i\omega}, \tag{4.4.11}$$

where

$$A(u) = s\Big(\frac{2i\omega - u_7 + u_8}{\omega_2}\Big) \prod_{j=1}^{6} s\Big(\frac{u_j + u_7}{\omega_2}\Big),$$

$$\begin{aligned} B(u) = &\frac{s\big(\frac{u_8 \pm u_7}{\omega_2}\big) s\big(\frac{2i\omega + u_8 - u_7}{\omega_2}\big) s\big(\frac{2i\omega - u_8 + u_7}{\omega_2}\big)}{s\big(\frac{2i\omega + u_8 - u_6}{\omega_2}\big) s\big(\frac{2i\omega + u_7 - u_6}{\omega_2}\big)} \prod_{j=1}^{5} s\Big(\frac{-2i\omega + u_j + u_6}{\omega_2}\Big) \\ &- \frac{s\big(\frac{2i\omega - u_8 + u_7}{\omega_2}\big) s\big(\frac{u_7 - u_6}{\omega_2}\big) s\big(\frac{-2i\omega + u_6 + u_7}{\omega_2}\big)}{s\big(\frac{2i\omega + u_8 - u_6}{\omega_2}\big)} \prod_{j=1}^{5} s\Big(\frac{u_j + u_8}{\omega_2}\Big) \\ &+ \frac{s\big(\frac{2i\omega + u_8 - u_7}{\omega_2}\big) s\big(\frac{u_8 - u_6}{\omega_2}\big) s\big(\frac{-2i\omega + u_6 + u_8}{\omega_2}\big)}{s\big(\frac{2i\omega + u_7 - u_6}{\omega_2}\big)} \prod_{j=1}^{5} s\Big(\frac{u_j + u_7}{\omega_2}\Big). \end{aligned}$$

This leads to the following theorem.

Theorem 4.4.3. *We have*

$$A(u)(S_h(\tau_{87}u) - S_h(u)) - (u_7 \leftrightarrow u_8) = B_2(u)S_h(u) \tag{4.4.12}$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$, where $A(u)$ is as above and with $B_2(u)$ defined by

$$\begin{aligned} B_2(u) = &\frac{s((u_7 \pm u_8)/\omega_2)s((u_7 - u_8 \pm 2i\omega)/\omega_2)}{4} \\ &\times \left(\sum_{j=7}^{8} s(2(i\omega + u_j)/\omega_2) - \sum_{j=1}^{6} s(2(i\omega - u_j)/\omega_2) \right). \end{aligned} \tag{4.4.13}$$

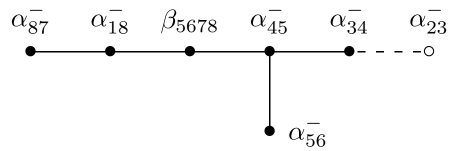
Proof. It follows from (4.4.11) that (4.4.12) holds with $B_2(u) = B(u) - A(u) - A(s_{78}u)$. The alternative expression (4.4.13) for B_2 was obtained by Mathematica. Observe though that part of the zero locus of $B_2(u)$ ($u \in \mathcal{G}_{2i\omega}$) can be predicted in advance. Indeed, the left hand side of (4.4.12) vanishes if $u_7 = u_8$ (both terms then cancel each other), and it vanishes if $u_7 = u_8 \pm i\omega$ (one term vanishes due to a s-factor, while the other term vanishes since either $S_h(\tau_{87}u) = S_h(u)$ or $S_h(\tau_{78}u) = S_h(u)$). The zero of $B_2(u)$ at $u_7 = -u_8$ can be predicted from the fact that all hyperbolic hypergeometric functions S_h in (4.4.12) can be evaluated for $u_7 = -u_8$ using the hyperbolic beta integral (4.4.6). □

4.4.4 The degeneration to the hyperbolic Barnes integral

In this subsection we degenerate the hyperbolic hypergeometric function $S_h(u)$ ($u \in \mathcal{G}_{2i\omega}$) along the highest root β_{1278} of $R(E_7)$ with respect to the basis Δ_2 of $R(E_7)$ (see (4.2.4) for the associated Dynkin diagram). The resulting degenerate integral $B_h(u)$ thus inherits symmetries with respect to the standard maximal parabolic subgroup

$$W_2(D_6) := W(E_7)_{\beta_{1278}} \subset W(E_7),$$

which is isomorphic to the Weyl group of type D_6 and is generated by the simple reflections s_α ($\alpha \in \Delta_2 \setminus \{\alpha_{23}^-\}$). The corresponding Dynkin diagram is



Concretely, for generic parameters $u \in \mathcal{G}_{2i\omega}$ we define $B_h(u) = B_h(u;\omega_1,\omega_2)$ by

$$B_h(u;\omega_1,\omega_2) = 2\int_{\mathcal{C}} \frac{\prod_{j=3}^{6} G(z-u_j;\omega_1,\omega_2)}{\prod_{j=1,2,7,8} G(z+u_j;\omega_1,\omega_2)}dz.$$

This integral converges absolutely since the asymptotic behaviour of the integrand at $z = \pm\infty$ is $\exp(-4\pi\omega|z|/\omega_1\omega_2)$. We may take the real line as integration contour if $u \in \mathcal{G}_{2i\omega}$ satisfies $\Im(u_j - i\omega) < 0$ for all j. Observe that the integral $B_h(u)$ has a unique meromorphic extension to $u \in \mathcal{G}_{2i\omega}$. We call $B_h(u)$ the hyperbolic Barnes integral since it is essentially Ruijsenaars' [65] hyperbolic generalization of the Barnes integral representation of the Gauss hypergeometric function, see Subsection 4.4.6.

Remark 4.4.4. The parameter space of the hyperbolic Barnes integral B_h is in fact the quotient space $\mathcal{G}_{2i\omega}/\mathbb{C}\beta_{1278}$. Indeed, for $\xi \in \mathbb{C}$ we have

$$B_h(u + \xi\beta_{1278}) = B_h(u)$$

as meromorphic functions in $u \in \mathcal{G}_{2i\omega}$, which follows by an easy application of (4.4.4) and Cauchy's Theorem.

Proposition 4.4.5. *For $u \in \mathcal{G}_{2i\omega}$ satisfying $\Im(u_j - i\omega) < 0$ $(j = 1, \ldots, 8)$ we have*

$$\lim_{r\to\infty} S_h(u - r\beta_{1278}) \exp\left(\frac{2\pi r\omega}{\omega_1\omega_2}\right) \exp\left(\frac{\pi i}{2\omega_1\omega_2}\Big(\sum_{j=1,2,7,8} u_j^2 - \sum_{j=3}^{6} u_j^2\Big)\right) = B_h(u).$$

Proof. The conditions on the parameters $u \in \mathcal{G}_{2i\omega}$ allow us to choose the real line as integration contour in the integral expression of $S_h(u - r\beta_{1278})$ ($r \in \mathbb{R}$) as well as in the integral expression of $B_h(u)$. Using that the integrand $I_h(u;z)$ of $S_h(u)$ is even in x, using the reflection equation for the hyperbolic gamma function, and by a change of integration variable, we have

$$\begin{aligned} S_h(u - r\beta_{1278}) &= \int_{-\infty}^{\infty} \frac{G(i\omega \pm 2z)}{\prod_{j=1,2,7,8} G(u_j - \frac{r}{2} \pm z)\prod_{j=3}^{6} G(u_j + \frac{r}{2} \pm z)}dz \\ &= 2\int_{0}^{\infty} \frac{G(i\omega \pm 2z)}{\prod_{j=1,2,7,8} G(u_j - \frac{r}{2} \pm z)\prod_{j=3}^{6} G(u_j + \frac{r}{2} \pm z)}dz \\ &= 2e^{-\frac{2\pi r\omega}{\omega_1\omega_2}} \int_{-\frac{r}{2}}^{\infty} k_1(2z+r)k_2(z+r)L(z)dz, \end{aligned}$$

where

$$
\begin{aligned}
L(z) &= \frac{\prod_{j=3}^{6} G(z-u_j)}{\prod_{j=1,2,7,8} G(z+u_j)},\\
k_1(z) &= \frac{G(z+i\omega)}{G(z-i\omega)} e^{-\frac{2\pi\omega z}{\omega_1\omega_2}} = \big(1-e^{-2\pi z/\omega_1}\big)\big(1-e^{-2\pi z/\omega_2}\big),\\
k_2(z) &= \frac{\prod_{j=1,2,7,8} G(z-u_j)}{\prod_{j=3}^{6} G(z+u_j)} e^{\frac{4\pi\omega z}{\omega_1\omega_2}}.
\end{aligned}
$$

Here the second expression of k_1 follows from the analytic difference equations satisfied by G. The pointwise limits of k_1 and k_2 are

$$
\lim_{z\to\infty} k_1(z) = 1 \qquad \lim_{z\to\infty} k_2(z) = e^{\frac{\pi i}{2\omega_1\omega_2}(\sum_{j=3}^{6} u_j^2 - \sum_{j=1,2,7,8} u_j^2)}.
$$

Moreover, observe that $k_1(z)$ is uniformly bounded for $z \in \mathbb{R}_{\geq 0}$ by 4, and that $k_2(z)$, being a continuous function on $\mathbb{R}_{\geq 0}$ with finite limit at infinity, is also uniformly bounded for $z \in \mathbb{R}_{\geq 0}$.

Denote by $\chi_{(-r/2,\infty)}(z)$ the indicator function of the interval $(-r/2,\infty)$. By Lebesgue's theorem of dominated convergence we now conclude that

$$
\begin{aligned}
\lim_{r\to\infty} S_h(u - r\beta_{1278}) e^{\frac{2\pi r\omega}{\omega_1\omega_2}} &= 2\lim_{r\to\infty}\int_{-\frac{r}{2}}^{\infty} k_1(2z+r)k_2(z+r)L(z)dz\\
&= 2\int_{-\infty}^{\infty}\lim_{r\to\infty}\chi_{(-r/2,\infty)}(z)k_1(2z+s)k_2(z+s)L(z)dz\\
&= 2e^{\frac{\pi i}{2\omega_1\omega_2}\left(\sum_{j=3}^{6} u_j^2 - \sum_{j=1,2,7,8} u_j^2\right)}\int_{-\infty}^{\infty} L(z)dz\\
&= e^{\frac{\pi i}{2\omega_1\omega_2}(\sum_{j=3}^{6} u_j^2 - \sum_{j=1,2,7,8} u_j^2)} B_h(u),
\end{aligned}
$$

as desired. □

In the following corollary we use Proposition 4.4.5 to degenerate the hyperbolic beta integral (4.4.6). The resulting integral evaluation formula is an hyperbolic analogue of the nonterminating Saalschütz formula [16, (2.10.12)], see Subsection 4.5.4.

Corollary 4.4.6. *For generic $u \in \mathbb{C}^6$ satisfying $\sum_{j=1}^{6} u_j = 4i\omega$ we have*

$$
\int_{\mathcal{C}} \frac{G(z-u_4, z-u_5, z-u_6)}{G(z+u_1, z+u_2, z+u_3)} dz = \sqrt{\omega_1\omega_2}\prod_{j=1}^{3}\prod_{k=4}^{6} G(i\omega - u_j - u_k). \tag{4.4.14}
$$

Proof. Substitute the parameters $u' = (u_1, u_2, u_4, u_5, u_6, 0, u_3, 0)$ in Proposition 4.4.5 with $u_j \in \mathbb{C}$ satisfying $\Im(u_j - i\omega) < 0$ and $\sum_{j=1}^{6} u_j = 4i\omega$. Then $B_h(u')$ is the left hand side of (4.4.14), multiplied by 2. On the other hand, by Proposition 4.4.5 and (4.4.6) we have

$$\begin{aligned} B_h(u') &= \lim_{r\to\infty} S_h(u'-r\beta_{1278}) \exp\left(\frac{2\pi r\omega}{\omega_1\omega_2}+\frac{\pi i}{2\omega_1\omega_2}(\sum_{j=1}^{3}u_j^2-\sum_{j=4}^{6}u_j^2)\right) \\ &= 2\sqrt{\omega_1\omega_2}\prod_{j=1}^{3}\prod_{k=4}^{6}G(i\omega-u_j-u_k) \\ &\qquad \times \lim_{r\to\infty}\frac{\prod_{1\le j<k\le 3}G(i\omega-u_j-u_k+r)}{\prod_{4\le j<k\le 6}G(u_j+u_k-i\omega+r)} \\ &\qquad\qquad \times \exp\left(\frac{2\pi r\omega}{\omega_1\omega_2}+\frac{\pi i}{2\omega_1\omega_2}(\sum_{j=1}^{3}u_j^2-\sum_{j=4}^{6}u_j^2)\right) \\ &= 2\sqrt{\omega_1\omega_2}\prod_{j=1}^{3}\prod_{k=4}^{6}G(i\omega-u_j-u_k), \end{aligned}$$

where the last equality follows from a straightforward but tedious computation using (4.4.4). The result for arbitrary generic parameters $u\in\mathbb{C}^6$ satisfying $\sum_{j=1}^{6}u_j=4i\omega$ now follows by analytic continuation. □

Next we determine the explicit $W_2(D_6)$-symmetries of $B_h(u)$.

Proposition 4.4.7. *The hyperbolic Barnes integral $B_h(u)$ ($u\in\mathcal{G}_{2i\omega}$) is invariant under permutations of (u_1,u_2,u_7,u_8) and of (u_3,u_4,u_5,u_6) and it satisfies*

$$B_h(u)=B_h(wu)\prod_{j=1,2}\prod_{k=3,4}G(i\omega-u_j-u_k)\prod_{j=5,6}\prod_{k=7,8}G(i\omega-u_j-u_k) \quad (4.4.15)$$

as meromorphic functions in $u\in\mathcal{G}_{2i\omega}$

Proof. The permutations symmetry is trivial. The symmetry (4.4.15) can be proven by degenerating the corresponding symmetry of S_h, see Theorem 4.4.1. We prove here the w-symmetry by considering the double integral

$$\int_{\mathbb{R}^2}\frac{G(z-u_3,z-u_4,x-u_5+s,x-u_6+s,z-x-i\omega-s)}{G(z+u_1,z+u_2,x+u_7-s,x+u_8-s,z-x+i\omega+s)}dzdx$$

with $s=i\omega-\frac{1}{2}(u_1+u_2+u_3+u_4)=-i\omega+\frac{1}{2}(u_5+u_6+u_7+u_8)$, where we impose on $u\in\mathcal{G}_{2i\omega}$ the additional conditions

$$-\Re\left(\frac{\omega}{\omega_1\omega_2}\right)<\Im\left(\frac{s}{\omega_1\omega_2}\right)<\Re\left(\frac{\omega}{\omega_1\omega_2}\right) \quad (4.4.16)$$

to ensure the absolute convergence of the double integral (this condition is milder than the corresponding condition (4.4.9) for S_h due to the missing factors $G(i\omega\pm$

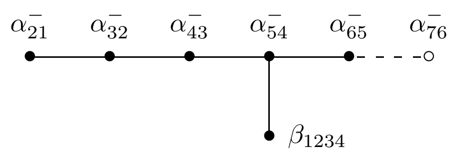
$2z, i\omega \pm 2x$) in the numerator of the integrand), and the conditions (4.4.8) to ensure that the upward and downward pole sequences are separated by $\mathbb{R}$. Using Fubini's Theorem and the hyperbolic Saalschütz summation (4.4.14), similarly as in the proof of Theorem 4.4.1, yields (4.4.15). □

4.4.5 The degeneration to the hyperbolic Euler integral

In this subsection we degenerate the hyperbolic hypergeometric function $S_h(u)$ ($u \in \mathcal{G}_{2i\omega}$) along the highest root α^-_{78} of $R(E_7)$ with respect to the basis Δ_1 of $R(E_7)$ (see (4.2.3) for the associated Dynkin diagram). The resulting degenerate integral $E_h(u)$ thus inherits symmetries with respect to the standard maximal parabolic subgroup

$$W_1(D_6) := W(E_7)_{\alpha^-_{78}} \subset W(E_7),$$

which is isomorphic to the Weyl group of type D_6 and is generated by the simple reflections s_α ($\alpha \in \Delta_1 \setminus \{\alpha^-_{76}\}$). The corresponding Dynkin diagram is



By the conditions $\Re(\omega_j) > 0$ on the periods ω_j ($j = 1, 2$) we have that $\Re\big(\frac{\omega}{\omega_1\omega_2}\big) > 0$. For generic parameters $u = (u_1, \ldots, u_6) \in \mathbb{C}^6$ satisfying

$$\Im\Big(\frac{1}{\omega_1\omega_2}\sum_{j=1}^{6} u_j\Big) > \Re\Big(\frac{2\omega}{\omega_1\omega_2}\Big) \tag{4.4.17}$$

we now define $E_h(u) = E_h(u; \omega_1, \omega_2)$ by

$$E_h(u; \omega_1, \omega_2) = \int_{\mathcal{C}} \frac{G(i\omega \pm 2z; \omega_1, \omega_2)}{\prod_{j=1}^{6} G(u_j \pm z; \omega_1, \omega_2)} dz.$$

It follows from the asymptotics (4.4.4) of the hyperbolic gamma function that the condition (4.4.17) on the parameters ensures the absolute convergence of $E_h(u)$. Furthermore, $E_h(u)$ admits a unique meromorphic continuation to parameters $u \in \mathbb{C}^6$ satisfying (4.4.17) (in fact, it will be shown later that $E_h(u)$ has a unique meromorphic continuation to $u \in \mathbb{C}^6$ by relating E_h to the hyperbolic Barnes integral B_h). Observe furthermore that $E_h(u)$ reduces to the hyperbolic beta integral (4.4.6) when the parameters $u \in \mathbb{C}^6$ satisfy the balancing condition $\sum_{j=1}^{6} u_j = 4i\omega$. We call $E_h(u)$ the hyperbolic Euler integral since its trigonometric analogue is a natural generalization of the Euler integral representation of the Gauss hypergeometric function, see Subsection 4.5.4 and [16, §6.3].

Proposition 4.4.8. *For $u \in \mathcal{G}_{2i\omega}$ satisfying $\Im(u_j - i\omega) < 0$ $(j = 1, \ldots, 8)$, $\Im\big((u_7 + u_8)/\omega_1\omega_2\big) \geq 0$ and (4.4.17), we have*

$$\lim_{r\to\infty} S_h(u - r\alpha_{78}^{-}) \exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7+u_8)(2r - u_7 + u_8)\right) = E_h(u_1, u_2, u_3, u_4, u_5, u_6). \quad (4.4.18)$$

Remark 4.4.9. Proposition 4.4.8 is trivial when $u_7 = -u_8$ due to the reflection equation for G. The resulting limit is the hyperbolic beta integral (4.4.6) (since the balancing condition reduces to $\sum_{j=1}^{6} u_j = 4i\omega$).

Proof. The assumptions on the parameters ensure that the integration contours in S_h and E_h can be chosen as the real line. We denote the integrand of the Euler integral by

$$J(z) = \frac{G(iw \pm 2z)}{\prod_{j=1}^{6} G(u_j \pm z)}$$

and we set

$$H(z) = \frac{G(z - u_7)}{G(z + u_8)} \exp\left(-\frac{\pi i z}{\omega_1\omega_2}(u_7 + u_8)\right).$$

This allows us to write

$$I_h(u - r\alpha_{78}^{-}; z) \exp\left(-\frac{2\pi i r}{\omega_1\omega_2}(u_7 + u_8)\right) = J(z)H(r+z)H(r-z),$$

where (recall) $I_h(u; z)$ is the integrand of the hyperbolic hypergeometric function $S_h(u)$. Observe that H is a continuous function on $\mathbb{R}$ satisfying

$$\lim_{z\to\infty} H(z) = \exp\left(\frac{\pi i}{2\omega_1\omega_2}(u_8^2 - u_7^2)\right),$$
$$\lim_{z\to-\infty} H(z) \exp\left(\frac{2\pi i z}{\omega_1\omega_2}(u_7 + u_8)\right) = \exp\left(\frac{\pi i}{2\omega_1\omega_2}(u_7^2 - u_8^2)\right)$$

by (4.4.4) and by the reflection equation for the hyperbolic gamma function. Moreover, H is uniformly bounded on $\mathbb{R}$ in view of the parameter condition $\Im\big(u_7 + u_8/\omega_1\omega_2\big) \geq 0$ on the parameters, and we have

$$\lim_{r\to\infty} H(r+z)H(r-z) = \exp\left(\frac{\pi i}{\omega_1\omega_2}(u_8^2 - u_7^2)\right)$$

for fixed $z \in \mathbb{R}$.

By Lebesgue's theorem of dominated convergence we conclude that

$$\begin{aligned}\lim_{r\to\infty} S_h\big(u - r\alpha_{78}^-\big)\exp\left(-\frac{2\pi i r}{\omega_1\omega_2}(u_7+u_8)\right) &= \lim_{r\to\infty}\int_{\mathbb{R}} J(z)H(r+z)H(r-z)dz \\ &= \int_{\mathbb{R}} J(z)\lim_{r\to\infty} H(r+z)H(r-z)dz \\ &= E_h(u_1,\ldots,u_6)\exp\left(\frac{\pi i}{\omega_1\omega_2}(u_8^2-u_7^2)\right),\end{aligned}$$

as desired. □

As a corollary of Proposition 4.4.8 we obtain the hyperbolic beta integral of Askey-Wilson type, initially independently proved in [68] and in [77].

Corollary 4.4.10. *For generic* $u = (u_1, u_2, u_3, u_4) \in \mathbb{C}^4$ *satisfying the convergence condition* $\Im\big(\frac{1}{\omega_1\omega_2}\sum_{j=1}^4 u_j\big) > \Re\big(\frac{2\omega}{\omega_1\omega_2}\big)$ *we have*

$$\int_{\mathcal{C}} \frac{G(i\omega \pm 2z)}{\prod_{j=1}^4 G(u_j \pm z)}\,dz = 2\sqrt{\omega_1\omega_2}\,G(u_1+u_2+u_3+u_4-3i\omega)\prod_{1\le j<k\le 4} G(i\omega-u_j-u_k). \tag{4.4.19}$$

Proof. Apply Proposition 4.4.8 under the additional condition $u_5 = -u_6$ on the associated parameters $u \in \mathcal{G}_{2i\omega}$. Using the reflection equation for the hyperbolic gamma function we see that the right hand side of (4.4.18) becomes the hyperbolic Askey-Wilson integral. On the other hand, $S_h(u - r\alpha_{78}^-)$ can be evaluated by the hyperbolic beta integral (4.4.6), resulting in

$$\begin{aligned}&\int_{\mathcal{C}} \frac{G(i\omega \pm 2z)}{\prod_{j=1}^4 G(u_j \pm z)}\,dz \\ &= 2\sqrt{\omega_1\omega_2}\,G(i\omega-u_7-u_8)\prod_{1\le j<k\le 4} G(i\omega-u_j-u_k) \\ &\times \lim_{r\to\infty}\exp\left(-\frac{\pi i}{\omega_1\omega_2}(u_7+u_8)(2r-u_7+u_8)\right)\prod_{j=1}^4 \frac{G(i\omega-u_j-u_7+r)}{G(-i\omega+u_j+u_8+r)} \\ &= 2\sqrt{\omega_1\omega_2}\,G(u_1+u_2+u_3+u_4-3i\omega)\prod_{1\le j<k\le 4} G(i\omega-u_j-u_k),\end{aligned}$$

where we used the balancing condition on u and the asymptotics (4.4.4) of the hyperbolic gamma function to obtain the last equality. The additional parameter restrictions which we have imposed in order to be able to apply Proposition 4.4.8 can now be removed by analytic continuation. □

Since both the Euler and Barnes integrals are limits of the hyperbolic hypergeometric function we can connect them according to the following theorem.

Theorem 4.4.11. *We have*

$$\begin{aligned}B_h(u) = E_h(u_2 - s, u_7 - s, u_8 - s, u_3 + s, u_4 + s, u_5 + s)\\ \times \prod_{j=3}^{5} G(i\omega - u_1 - u_j) \prod_{j=2,7,8} G(i\omega - u_6 - u_j).\end{aligned} \quad (4.4.20)$$

as meromorphic functions in $\{u \in \mathcal{G}_{2i\omega} \,|\, \Im\big((u_1+u_6)/\omega_1\omega_2\big) < \Re\big(2\omega/\omega_1\omega_2\big)\}$*, where*

$$s = \frac{1}{2}(u_2 + u_6 + u_7 + u_8) - i\omega = i\omega - \frac{1}{2}(u_1 + u_3 + u_4 + u_5).$$

Proof. This theorem can be proved by degenerating a suitable E_7-symmetry of S_h using Proposition 4.4.5 and Proposition 4.4.8. We prove the theorem here directly by analyzing the double integral

$$\frac{1}{\sqrt{\omega_1\omega_2}} \int_{\mathbb{R}^2} \frac{G(i\omega \pm 2z) \prod_{j=3}^{5} G(x - u_j)}{G(i\omega + s + x \pm z) G(x + u_1) \prod_{j=2,7,8} G(u_j - s \pm z)} \, dz dx$$

for $\Re(\omega_1), \Re(\omega_2) > 0$, $u \in \mathcal{G}_{2i\omega}$ and $s = \frac{1}{2}(u_2 + u_6 + u_7 + u_8) - i\omega$, where we impose the additional parameter restraints $\omega_1\omega_2 \in \mathbb{R}_{>0}$ and

$$|\Im(s)| < \Re(\omega), \qquad \Im(u_6 + s) < 0$$

to ensure absolute convergence of the double integral (which follows from a straightforward analysis of the integrand using (4.4.4) and (4.4.5), cf. the proof of Theorem 4.4.1), and

$$\Im(s) < 0, \qquad \Im(i\omega - u_j) > 0 \quad (j = 1, 3, 4, 5), \qquad \Im(i\omega - u_k + s) > 0 \quad (k = 2, 7, 8)$$

to ensure pole sequence separation by the integration contours. Note that these parameter restraints imply the parameter condition $\Im(u_1 + u_6) < 2\Re(\omega)$ needed for the hyperbolic Euler integral in the right hand side of (4.4.20) to be defined. Integrating the double integral first over x and using the integral evaluation formula (4.4.14) of Barnes type, we obtain an expression of the double integral as a multiple of $E_h(u_2 - s, u_3 + s, u_4 + s, u_5 + s, u_7 - s, u_8 - s)$. Integrating first over z and using the hyperbolic Askey-Wilson integral (4.4.19), we obtain an expression of the double integral as a multiple of $B_h(u)$. The resulting identity is (4.4.20) for a restricted parameter domain. Analytic continuation now completes the proof. □

Corollary 4.4.12. *The hyperbolic Euler integral* $E_h(u)$ *has a unique meromorphic continuation to* $u \in \mathbb{C}^6$ *(which we also denote by* $E_h(u)$*).*

From the degeneration from S_h to E_h (see Proposition 4.4.8) it is natural to interpret the parameter domain $\mathbb{C}^6$ as $\mathcal{G}_{2i\omega}/\mathbb{C}\alpha_{78}^-$ via the bijection

$$\mathbb{C}^6 \ni u \mapsto \big(u_1, \ldots, u_6, 2i\omega - \sum_{j=1}^{6} u_j, 0\big) + \mathbb{C}\alpha_{78}^-. \quad (4.4.21)$$

We use this identification to transfer the natural $W_1(D_6) = W(E_7)_{\alpha_{78}^-}$-action on $\mathcal{G}_{2i\omega}/\mathbb{C}\alpha_{78}^-$ to the parameter space $\mathbb{C}^6$ of the hyperbolic Euler integral. It is generated by permutations of $(u_1, \ldots, u_6)$ and by the action of $w \in W_1(D_6)$, which is given explicitly by

$$w(u) = (u_1 + s, u_2 + s, u_3 + s, u_4 + s, u_5 - s, u_6 - s), \qquad u \in \mathbb{C}^6, \tag{4.4.22}$$

where $s = i\omega - \frac{1}{2}(u_1 + u_2 + u_3 + u_4)$. An interesting feature of $W_1(D_6)$-symmetries of the hyperbolic Euler integral (to be derived in Corollary 4.4.14), is the fact that the nontrivial w-symmetry of E_h generalizes to the following explicit integral transformation for E_h.

Proposition 4.4.13. *For periods $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re(\omega_1), \Re(\omega_2) > 0$ and $\omega_1\omega_2 \notin \mathbb{R}_{>0}$ and for parameters $s \in \mathbb{C}$ and $u = (u_1, \ldots, u_6) \in \mathbb{C}^6$ satisfying*

$$\Im\Big(\frac{s}{\omega_1\omega_2}\Big) > -\Re\Big(\frac{\omega}{\omega_1\omega_2}\Big), \qquad \Im\Big(\frac{u_1 + u_2 + u_3 + u_4}{\omega_1\omega_2}\Big), \Im\Big(\frac{u_5 + u_6 - 2s}{\omega_1\omega_2}\Big) > \Re\Big(\frac{2\omega}{\omega_1\omega_2}\Big) \tag{4.4.23}$$

and

$$\Im(u_j - i\omega) < 0 \quad (j = 1, \ldots, 4), \qquad \Im(u_k - i\omega) < \Im(s) < 0 \quad (k = 5, 6), \tag{4.4.24}$$

we have

$$\begin{aligned} \int_{\mathbb{R}} E_h(u_1, u_2, u_3, u_4, i\omega + s + x, i\omega + s - x) \frac{G(i\omega \pm 2x)}{G(u_5 - s \pm x, u_6 - s \pm x)}\, dx \\ = 2\sqrt{\omega_1\omega_2}\, \frac{G(i\omega - u_5 - u_6 + 2s)}{G(i\omega - u_5 - u_6, i\omega + 2s)}\, E_h(u). \end{aligned} \tag{4.4.25}$$

Proof. Observe that the requirement $\omega_1\omega_2 \notin \mathbb{R}_{>0}$ ensures the existence of parameters $u \in \mathbb{C}^6$ and $s \in \mathbb{C}$ satisfying the restraints (4.4.23) and (4.4.24). Furthermore, (4.4.23) ensures that

$$\Im\Big(\frac{1}{\omega_1\omega_2}\Big(\sum_{j=1}^{6} u_j\Big)\Big), \Im\Big(\frac{1}{\omega_1\omega_2}\Big(\sum_{j=1}^{4} u_j + 2i\omega + 2s\Big)\Big) > \Re\Big(\frac{2\omega}{\omega_1\omega_2}\Big),$$

hence both hyperbolic Euler integrals in (4.4.25) are defined. We derive the integral transformation (4.4.25) by considering the double integral

$$\int_{\mathbb{R}^2} \frac{G(i\omega \pm 2z, i\omega \pm 2x)}{G(i\omega + s \pm x \pm z) \prod_{j=1}^{4} G(u_j \pm z) \prod_{k=5}^{6} G(u_k - s \pm x)}\, dz dx,$$

which absolutely converges by (4.4.23). Integrating the double integral first over x using the hyperbolic Askey-Wilson integral (4.4.19) yields the right hand side of (4.4.25). Integrating first over z results in the left hand side of (4.4.25). □

Corollary 4.4.14. *The hyperbolic Euler integral $E_h(u)$ $(u \in \mathbb{C}^6)$ is symmetric in $(u_1, \ldots, u_6)$ and it satisfies*

$$E_h(u) = E_h(wu)G(i\omega - u_5 - u_6)G(\sum_{j=1}^{6} u_j - 3i\omega) \prod_{1 \leq j < k \leq 4} G(i\omega - u_j - u_k) \quad (4.4.26)$$

as meromorphic functions in $u \in \mathbb{C}^6$.

Proof. The permutation symmetry is trivial. For (4.4.26) we apply Proposition 4.4.13 with $s = i\omega - \frac{1}{2}(u_1 + u_2 + u_3 + u_4)$. The hyperbolic Euler integral in the left hand side of the integral transformation (4.4.25) can now be evaluated by the hyperbolic beta integral (4.4.6). The remaining integral is an explicit multiple of $E_h(wu)$. The resulting identity yields (4.4.26) for a restricted parameter domain. Analytic continuation completes the proof. □

Remark 4.4.15. The w-symmetry (4.4.26) of E_h can also be proved by degenerating the w-symmetry of S_h, or by relating (4.4.26) to a $W_2(D_6)$-symmetry of B_h using Theorem 4.4.11.

The longest Weyl group element $v_1 \in W_1(D_6)$ and the longest Weyl group element $v \in W(E_7)$ have the same action on $\mathcal{G}_{2i\omega}/\mathbb{C}\alpha_{78}^-$. Consequently, under the identification (4.4.21), v_1 acts on $\mathbb{C}^6$ by

$$v_1(u) = (i\omega - u_1, \ldots, i\omega - u_6), \qquad u \in \mathbb{C}^6.$$

Corollary 4.4.16. *The symmetry of the hyperbolic Euler integral $E_h(u)$ with respect to the longest Weyl group element $v_1 \in W_1(D_6)$ is*

$$E_h(u) = E_h(v_1 u)G(-3i\omega + \sum_{j=1}^{6} u_j) \prod_{1 \leq j < k \leq 6} G(i\omega - u_j - u_k)$$

as meromorphic functions in $u \in \mathbb{C}^6$.

Proof. For parameters $u \in \mathcal{G}_{2i\omega}$ such that both u and vu satisfy the parameter restraints of Proposition 4.4.8, we degenerate the v-symmetry (4.4.10) of S_h using (4.4.18). Analytic continuation completes the proof. □

The contiguous relations for S_h degenerate to the following contiguous relations for E_h.

Lemma 4.4.17. *We have*

$$\frac{\prod_{j=1}^{4} s((u_j + u_5)/\omega_2)}{s((u_5 - u_6 + 2i\omega)/\omega_2)} (E_h(\tau_{65}^{i\omega_1} u) - E_h(u)) - (u_5 \leftrightarrow u_6)$$
$$= s((u_5 \pm u_6)/\omega_2) s((2i\omega - \sum_{j=1}^{6} u_j)/\omega_2) E_h(u) \quad (4.4.27)$$

as meromorphic functions in $u \in \mathbb{C}^6$.

Proof. Use Proposition 4.4.8 to degenerate the contiguous relation (4.4.12) for the hyperbolic hypergeometric function S_h to E_h. □

4.4.6 Ruijsenaars' R-function

Motivated by the theory of quantum integrable, relativistic particle systems on the line, Ruijsenaars [65], [67], [68] introduced and studied a generalized hypergeometric R-function R, which is essentially the hyperbolic Barnes integral $B_h(u)$ with respect to a suitable reparametrization (and re-interpretation) of the parameters $u \in \mathcal{G}_{2i\omega}$. The new parameters will be denoted by $(\gamma, x, \lambda) \in \mathbb{C}^6$ with $\gamma = (\gamma_1, \ldots, \gamma_4)^T \in \mathbb{C}^4$, where x (respectively λ) is viewed as the geometric (respectively spectral) parameter, while the four parameters γ_j are viewed as coupling constants. As a consequence of the results derived in the previous subsections, we will re-derive many of the properties of the generalized hypergeometric R-function, and we obtain a new integral representation of R in terms of the hyperbolic Euler integral E_h.

Set

$$N(\gamma) = \prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + i\omega).$$

Ruijsenaars' [65] generalized hypergeometric function $R(\gamma; x, \lambda; \omega_1, \omega_2) = R(\gamma, \lambda)$ is defined by

$$R(\gamma; x, \lambda) = \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{N(\gamma)}{G(i\gamma_0 \pm x, i\hat{\gamma}_0 \pm \lambda)} B_h(u) \tag{4.4.28}$$

where $u \in \mathcal{G}_{2i\omega}/\mathbb{C}\beta_{1278}$ with

$$\begin{aligned} &u_1 = i\omega, && u_2 = i\omega + i\gamma_0 + i\gamma_1, && u_3 = -i\gamma_0 + x, \\ &u_4 = -i\gamma_0 - x, && u_5 = -i\hat{\gamma}_0 + \lambda, && u_6 = -i\hat{\gamma}_0 - \lambda, \\ &u_7 = i\omega + i\gamma_0 + i\gamma_2, && u_8 = i\omega + i\gamma_0 + i\gamma_3. && \end{aligned} \tag{4.4.29}$$

Note that $R(\gamma; x, \lambda; \omega_1, \omega_2)$ is invariant under permuting the role of the two periods ω_1 and ω_2. Observe furthermore that the map $(\gamma, x, \lambda) \to u + \mathbb{C}\beta_{1278}$, with u given by (4.4.29), defines a bijection $\mathbb{C}^6 \xrightarrow{\sim} \mathcal{G}_{2i\omega}/\mathbb{C}\beta_{1278}$.

We define the dual parameters $\hat{\gamma}$ by

$$\hat{\gamma} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \gamma. \tag{4.4.30}$$

We will need the following auxiliary function

$$c(\gamma; y) = \frac{1}{G(2y + i\omega)} \prod_{j=0}^{3} G(y - i\gamma_j).$$

The following proposition was derived by different methods in [67].

Proposition 4.4.18. *R is even in x and λ and self-dual, i.e.*

$$R(\gamma; x, \lambda) = R(\gamma; -x, \lambda) = R(\gamma; x, -\lambda) = R(\hat{\gamma}; \lambda, x).$$

Furthermore, for an element $\sigma \in W(D_4)$, where $W(D_4)$ is the Weyl-group of type D_4 acting on the parameters γ by permutations and even numbers of sign flips, we have

$$\frac{R(\gamma;x,\lambda)}{c(\gamma;x)c(\hat{\gamma};\lambda)N(\gamma)} = \frac{R(\sigma\gamma;x,\lambda)}{c(\sigma\gamma;x)c(\widehat{\sigma\gamma};\lambda)N(\sigma\gamma)}.$$

Proof. These symmetries are all direct consequences of the $W_2(D_6)$-symmetries of the hyperbolic Barnes integral B_h (see Proposition 4.4.7). Concretely, note that the $W_2(D_6)$-action on $\mathbb{C}^6 \simeq \mathcal{G}_{2i\omega}/\mathbb{C}\beta_{1278}$ is given by

$$\begin{aligned}
s_{78}(\gamma,x,\lambda) &= (\gamma_0,\gamma_1,\gamma_3,\gamma_2,x,\lambda),\\
s_{18}(\gamma,x,\lambda) &= (-\gamma_3,\gamma_1,\gamma_2,-\gamma_0,x,\lambda),\\
w(\gamma,x,\lambda) &= (\gamma_1,\gamma_0,\gamma_2,\gamma_3,x,\lambda),\\
s_{45}(\gamma,x,\lambda) &= \big(\frac{1}{2}(\gamma_0+\hat{\gamma}_0)+\frac{i}{2}(x+\lambda), \frac{1}{2}(\gamma_1+\hat{\gamma}_1)-\frac{i}{2}(x+\lambda),\\
&\qquad \frac{1}{2}(\gamma_2+\hat{\gamma}_2)-\frac{i}{2}(x+\lambda), \frac{1}{2}(\gamma_3+\hat{\gamma}_3)-\frac{i}{2}(x+\lambda),\\
&\qquad \frac{i}{2}(\hat{\gamma}_0-\gamma_0)+\frac{1}{2}(x-\lambda), \frac{i}{2}(\hat{\gamma}_0-\gamma_0)+\frac{1}{2}(\lambda-x)\big),\\
s_{34}(\gamma,x,\lambda) &= (\gamma,-x,\lambda),\\
s_{56}(\gamma,x,\lambda) &= (\gamma,x,-\lambda).
\end{aligned}$$

The fact that $R(\gamma;x,\lambda)$ is even in x and λ follows now from the $s_{34} \in W_2(D_6)$ and $s_{56} \in W_2(D_6)$ symmetry of B_h, respectively (see Proposition 4.4.7). Similarly, the duality is obtained from the action of $s_{35}s_{46}$ and using that $\gamma_0+\gamma_i = \hat{\gamma}_0+\hat{\gamma}_i$ ($i=1,2,3$), while the $W(D_4)$-symmetry in γ follows from considering the action of $s_{27} \in W_2(D_6)$ (which interchanges $\gamma_1 \leftrightarrow \gamma_2$), s_{78}, s_{18} and w. □

Remark 4.4.19. Corollary 4.4.6 implies the explicit evaluation formula

$$R(\gamma; i\omega+i\gamma_3,\lambda;\omega_1,\omega_2) = \prod_{j=1}^{2} \frac{G(i\omega+i\gamma_0+i\gamma_j)}{G(i\omega+i\gamma_j+i\gamma_3)G(i\hat{\gamma}_j \pm \lambda)}.$$

Using the $W(D_4)$-symmetry of R, this implies

$$R(\gamma; i\omega+i\gamma_0,\lambda;\omega_1,\omega_2) = 1,$$

in accordance with [65, (3.26)].

Using Proposition 4.4.7 and Theorem 4.4.11 we can derive several different integral representations of the R-function. First we derive the integral representation of R which was previously derived in Chapter 3 by relating R to matrix coefficients of representations of the modular double of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{C}))$.

Proposition 4.4.20. *We have*

$$R(\gamma;x,\lambda) = \frac{N(\gamma)}{2\sqrt{\omega_1\omega_2}} \frac{G(x-i\gamma_0, x-i\gamma_1, \lambda-i\hat{\gamma}_0, \lambda-i\hat{\gamma}_1)}{G(x+i\gamma_2, x+i\gamma_3, \lambda+i\hat{\gamma}_2, \lambda+i\hat{\gamma}_3)} B_h(\upsilon),$$

where

$$\upsilon_{1/2} = x - \frac{\lambda}{2} + \frac{i\omega}{2} \pm \frac{i}{2}(\gamma_0 - \gamma_1), \qquad \upsilon_{3/4} = -x - \frac{\lambda}{2} + \frac{i\omega}{2} \pm \frac{i}{2}(\gamma_3 - \gamma_2),$$
$$\upsilon_{5/6} = \frac{\lambda}{2} + \frac{i\omega}{2} \pm \frac{i}{2}(-\gamma_0 - \gamma_1), \qquad \upsilon_{7/8} = \frac{\lambda}{2} + \frac{i\omega}{2} \pm \frac{i}{2}(\gamma_2 + \gamma_3)$$

and $\upsilon_{j/k} = \alpha \pm \beta$ *means* $\upsilon_j = \alpha + \beta$ *and* $\upsilon_k = \alpha - \beta$.

Proof. Express $B_h(s_{36}ws_{35}s_{28}ws_{18}u)$ in terms of $B_h(u)$ using the $W_2(D_6)$-symmetries of the hyperbolic Barnes integral B_h (see Proposition 4.4.7) and specialize u as in (4.4.29). This gives the desired equality. □

Moreover we can express R in terms of the hyperbolic Euler integral E_h, which leads to a previously unknown integral representation.

Theorem 4.4.21. *We have*

$$\begin{aligned} R(\gamma; x, \lambda) &= \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{\prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + i\omega, \lambda - i\hat{\gamma}_j)}{G(\lambda + i\hat{\gamma}_0)} E_h(u) \\ &= \frac{1}{2\sqrt{\omega_1\omega_2}} \frac{\prod_{j=1}^{3} G(i\gamma_0 + i\gamma_j + i\omega, \lambda - i\hat{\gamma}_j)}{G(\lambda + i\hat{\gamma}_0)\prod_{j=0}^{3} G(i\gamma_j \pm x)} E_h(\upsilon), \end{aligned}$$

where $u \in \mathbb{C}^6$ *is given by*

$$u_j = \frac{i\omega}{2} + i\gamma_{j-1} - \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2}, \quad (j = 1, \ldots, 4), \qquad u_{5/6} = \frac{i\omega}{2} \pm x + \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2},$$

and $\upsilon \in \mathbb{C}^6$ *is given by*

$$\upsilon_j = \frac{i\omega}{2} - i\gamma_{j-1} + \frac{i\hat{\gamma}_0}{2} + \frac{\lambda}{2}, \quad (j = 1, \ldots, 4), \qquad \upsilon_{5/6} = \frac{i\omega}{2} \pm x - \frac{i\hat{\gamma}_0}{2} - \frac{\lambda}{2}.$$

Proof. To prove the first equation, express $R(\gamma; x, \lambda)$ using the $W(D_4)$-symmetry (see Proposition 4.4.18) in terms of $R(-\gamma_3, \gamma_1, \gamma_2, -\gamma_0; x, \lambda)$. Subsequently use the identity relating B_h to E_h, see Theorem 4.4.11. To obtain the second equation, apply the symmetry of E_h with respect to the longest Weyl-group element $\upsilon_1 \in W_1(D_6)$ (see Corollary 4.4.16) in the first equation and use that R is even in λ. □

The contiguous relation for E_h (Lemma 4.4.17) now becomes the following result.

Proposition 4.4.22 ([65]). *Ruijsenaars' R-function satisfies the Askey-Wilson second order difference equation*

$$\begin{aligned} A(\gamma; x; \omega_1, \omega_2)(R(\gamma; x + i\omega_1, \lambda) - R(\gamma; x, \lambda)) + (x \leftrightarrow -x) \\ = B(\gamma; \lambda; \omega_1, \omega_2)R(\gamma; x, \lambda), \quad (4.4.31) \end{aligned}$$

where

$$
\begin{aligned}
A(\gamma; x; \omega_1, \omega_2) &= \frac{\prod_{j=0}^{3} s((i\omega + x + i\gamma_j)/\omega_2)}{s(2x/\omega_2)s(2(x+i\omega)/\omega_2)},\\
B(\gamma; \lambda; \omega_1, \omega_2) &= s((\lambda - i\omega - i\hat{\gamma}_0)/\omega_2)s((\lambda + i\omega + i\hat{\gamma}_0)/\omega_2).
\end{aligned}
$$

Remark 4.4.23. As is emphasized in [65], R satisfies four Askey-Wilson second order difference equations; two equations acting on the geometric variable x (namely (4.4.31), and (4.4.31) with the role of ω_1 and ω_2 interchanged), as well as two equations acting on the spectral parameter λ by exploiting the duality of R (see Proposition 4.4.18).

For later purposes, it is convenient to rewrite (4.4.31) as the eigenvalue equation

$$
\big(\mathcal{L}_{\gamma}^{\omega_1,\omega_2} R(\gamma; \cdot\,, \lambda; \omega_1, \omega_2)\big)(x) = B(\gamma; \lambda; \omega_1, \omega_2) R(\gamma; x, \lambda; \omega_1, \omega_2)
$$

for the Askey-Wilson second order difference operator

$$
\big(\mathcal{L}_{\gamma}^{\omega,\omega_2} f\big)(x) := A(\gamma; x; \omega_1, \omega_2)\big(f(x+i\omega_1) - f(x)\big) + \big(x \leftrightarrow -x\big). \tag{4.4.32}
$$

4.5 Trigonometric hypergeometric integrals

4.5.1 Basic hypergeometric series

In this section we assume that the base q satisfies $0 < |q| < 1$. The trigonometric gamma function [63] is essentially the q-gamma function $\Gamma_q(x)$, see [16]. For ease of presentation we express all the results in terms of the q-shifted factorial $\big(z;q\big)_\infty$, which are related to $\Gamma_q(x)$ by

$$
\Gamma_q(x) = \frac{\big(q;q\big)_\infty}{\big(q^x;q\big)_\infty}(1-q)^{1-x}
$$

(with a proper interpretation of the right hand side). The q-shifted factorial is the $p = 0$ degeneration of the elliptic gamma function,

$$
\Gamma_e(z; 0, q) = \frac{1}{\big(z;q\big)_\infty}. \tag{4.5.1}
$$

while the role of the first order analytic difference equation is taken over by

$$
\big(z;q\big)_\infty = (1-z)\big(qz;q\big)_\infty.
$$

However there is no reflection equation anymore; its role is taken over by the product formula for Jacobi's (renormalized) theta function

$$
\theta(z;q) = \big(z, q/z; q\big)_\infty.
$$

As a function of z the q-shifted factorial $\big(z;q\big)_\infty$ is entire with zeros at $z=q^{-n}$ for $n\in\mathbb{Z}_{\geq 0}$. In this section we call a sequence of the form aq^{-n} ($n\in\mathbb{Z}_{\geq 0}$) an upward sequence (since they diverge to infinity for large n) and a sequence of the form aq^n ($n\in\mathbb{Z}_{\geq 0}$) a downward sequence (as the elements converge to zero for large n).

We will use standard notations for basic hypergeometric series from [16]. In particular, the ${}_{r+1}\phi_r$ basic hypergeometric series is

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_1,\ldots,a_{r+1} \\ b_1,\ldots b_r\end{matrix};q,z\right)=\sum_{n=0}^\infty \frac{\big(a_1,\ldots,a_{r+1};q\big)_n}{\big(q,b_1,\ldots,b_r;q\big)_n}z^n, \qquad |z|<1, $$

where $\big(a;q\big)_n=\prod_{j=0}^{n-1}(1-aq^j)$ and with the usual convention regarding products of such expressions. The very-well-poised ${}_{r+1}\phi_r$ basic hypergeometric series is

$$ {}_{r+1}W_r\big(a_1;a_4,a_5,\ldots,a_{r+1};q,z\big)={}_{r+1}\phi_r\left(\begin{matrix} a_1,qa_1^{\frac12},-qa_1^{\frac12},a_4,\ldots,a_{r+1} \\ a_1^{\frac12},-a_1^{\frac12},qa_1/a_4,\ldots,qa_1/a_{r+1}\end{matrix};q,z\right). $$

Finally, the bilateral basic hypergeometric series ${}_r\psi_r$ is defined as

$$ \begin{aligned} {}_r\psi_r\left(\begin{matrix} a_1,a_2,\ldots,a_r \\ b_1,b_2,\ldots,b_r\end{matrix};q,z\right)&=\sum_{n=0}^\infty\frac{\big(a_1,a_2,\ldots,a_r;q\big)_n}{\big(b_1,b_2,\ldots,b_r;q\big)_n}z^n \\ &\quad+\sum_{n=1}^\infty\frac{\big(q/b_1,q/b_2,\ldots,q/b_r;q\big)_n}{\big(q/a_1,q/a_2,\ldots,q/a_r;q\big)_n}\left(\frac{b_1\cdots b_r}{a_1\cdots a_r z}\right)^n, \end{aligned} $$

provided that $|b_1\cdots b_r/a_1\cdots a_r|<|z|<1$ to ensure absolute and uniform convergence.

We end this introductory subsection by an elementary lemma which will enable us to rewrite trigonometric integrals with compact integration cycle in terms of trigonometric integrals with noncompact integration cycle. Let $\mathbb{H}_+$ be the upper half plane in $\mathbb{C}$. In this section we choose $\tau\in\mathbb{H}_+$ such that $q=e(\tau)$ once and for all, where $e(x)$ is a shorthand notation for $\exp(2\pi i x)$. We furthermore write $\Lambda=\mathbb{Z}+\mathbb{Z}\tau$.

Lemma 4.5.1. *Let $u,v\in\mathbb{C}$ such that $u\notin v+\Lambda$. There exists an $\eta=\eta(u,v)\in\mathbb{C}$, unique up to Λ-translates, such that*

$$ \begin{aligned} \frac{\theta\big(e(u+v-\eta-x),e(x-\eta);q\big)}{\theta\big(e(u-\eta),e(v-\eta);q\big)}&=\frac{\big(e((v-u)/\tau)-1\big)\theta\big(e(x-u),e(v-x);q\big)}{\tau\big(q,q;q\big)_\infty\theta\big(e(v-u);q\big)} \\ &\quad\times\sum_{n=-\infty}^\infty\frac{1}{\big(1-e((v-x-n)/\tau)\big)\big(e((x+n-u)/\tau)-1\big)}. \end{aligned} \tag{4.5.2} $$

Proof. Set $\widetilde{q} = e(-1/\tau)$. The bilateral sum

$$\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} \frac{1}{\big(1-e((v-x-n)/\tau)\big)\big(e((x+n-u)/\tau)-1\big)} \\ &= \frac{1}{(1-e((v-x)/\tau))(e((x-u)/\tau)-1)} \\ &\qquad \times {}_2\psi_2\left(\begin{matrix} e((v-x)/\tau), e((u-x)/\tau) \\ \widetilde{q}e((v-x)/\tau), \widetilde{q}e((u-x)/\tau)\end{matrix}; \widetilde{q}, \widetilde{q}\right) \end{aligned}$$

defines an elliptic function on $\mathbb{C}/\Lambda$, with possible poles at most simple and located at $u+\Lambda$ and at $v+\Lambda$. Hence there exists a $\eta \in \mathbb{C}$ (unique up to Λ-translates) and a constant $C_\eta \in \mathbb{C}$ such that

$$f(x) = C_\eta \frac{\theta\big(e(u+v-\eta-x), e(x-\eta); q\big)}{\theta\big(e(x-u), e(v-x); q\big)}.$$

We now compute the residue of f at u in two different ways:

$$\operatorname*{Res}_{x=u}(f) = \frac{\tau}{2\pi i} \frac{1}{\big(1-e((v-u)/\tau)\big)}$$

from the bilateral series expression of f, and

$$\operatorname*{Res}_{x=u}(f) = -\frac{C_\eta}{2\pi i} \frac{\theta\big(e(u-\eta), e(v-\eta); q\big)}{\big(q,q;q\big)_\infty \theta\big(e(v-u); q\big)}$$

from the expression of f as a quotient of theta-functions. Combining both identities yields an explicit expression of the constant C_η in terms of η, resulting in the formula

$$f(x) = \frac{\tau\big(q,q;q\big)_\infty \theta\big(e(v-u);q\big)}{\big(e((v-u)/\tau)-1\big)} \frac{\theta\big(e(u+v-\eta-x), e(x-\eta); q\big)}{\theta\big(e(x-u), e(v-x), e(u-\eta), e(v-\eta); q\big)}$$

for f. Rewriting this identity yields the desired result. □

4.5.2 Trigonometric hypergeometric integrals with E_6 symmetries

We consider trigonometric degenerations of $S_e(t)$ ($t \in \mathcal{H}_{pq}$) along root vectors $\alpha \in R(E_8)$ lying in the $W(E_7) = W(E_8)_\delta$-orbit

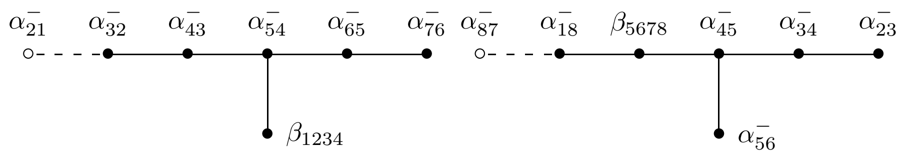
$$\mathcal{O} := W(E_7)\big(\alpha_{18}^+\big) = \{\alpha_{jk}^+, \gamma_{jk} \mid 1 \le j < k \le 8\}, \tag{4.5.3}$$

cf. Section 4.2. The degenerations relate to the explicit bijection

$$\mathcal{G}_0 \xrightarrow{\sim} \mathcal{G}_{\log(pq)}, \qquad (u_1, \ldots, u_8) \mapsto (u_1, \ldots, u_8) + \log(pq)\alpha \tag{4.5.4}$$

on the parameter spaces (in logarithmic form) of the associated integrals. We obtain two different trigonometric degenerations, depending on whether we degenerate along an orbit vector of the form $\alpha = \alpha_{jk}^+$, or of the form γ_{jk}.

Specifically, we consider the trigonometric degenerations $S_t(t)$ respectively $U_t(t)$ $(t \in \mathcal{H}_1)$ of $S_e(t)$ $(t \in \mathcal{H}_{pq})$ along the orbit vector α_{18}^+ and γ_{18} respectively. The orbit vector α_{18}^+ (respectively γ_{18}) is the additional simple root turning the basis Δ_1 (respectively Δ_2) of $R(E_7)$ into the basis $\overline{\Delta}_1$ (respectively $\overline{\Delta}_2$) of $R(E_8)$, see Section 4.2. The induced symmetry group of $S_t(t)$ $(t \in \mathcal{H}_1)$ is the isotropy subgroup $W(E_7)_{\alpha_{18}^+}$ of $W(E_7)$, while the induced symmetry group of $U_t(t)$ $(t \in \mathcal{H}_1)$ is $W(E_7)_{\gamma_{18}}$. It follows from the analysis in Section 4.2 that $W(E_7)_{\alpha_{18}^+} = W(E_7)_{\gamma_{18}}$ is a maximal, standard parabolic subgroup of $W(E_7)$ with respect to both bases Δ_1 and Δ_2, isomorphic to the Weyl group $W(E_6)$ of type E_6, with corresponding simple roots $\Delta_1' = \Delta_1 \setminus \{\alpha_{21}^-\}$ and $\Delta_2' = \Delta_2 \setminus \{\alpha_{87}^-\}$, and with corresponding Dynkin diagrams



Observe that α_{18}^- and α_{76}^- are the highest roots of the standard parabolic root system $R(E_6)$ of type E_6 in $R(E_7)$ corresponding to the bases Δ_1' and Δ_2' respectively. From now on we write

$$W(E_6) := W(E_7)_{\alpha_{18}^+} = W(E_7)_{\gamma_{18}}.$$

We first introduce the trigonometric hypergeometric integrals $S_t(t)$ and $U_t(t)$ $(t \in \mathcal{H}_1)$ explicitly. Their integrands are defined by

$$I_t(t;z) = \frac{(z^{\pm 2}, t_1^{-1}z^{\pm 1}, t_8^{-1}z^{\pm 1}; q)_\infty}{\prod_{j=2}^{7}(t_j z^{\pm 1}; q)_\infty},$$

$$J_t^\mu(t;z) = 2\frac{\theta(t_1t_8/\mu z, z/\mu; q)}{\theta(t_1/\mu, t_8/\mu; q)}\left(1 - \frac{z^2}{q}\right)\prod_{i=2}^{7}\frac{\left(z/t_i;q\right)_\infty}{\left(t_i z;q\right)_\infty}\prod_{j=1,8}\frac{1}{\left(t_j z/q, t_j/z; q\right)_\infty},$$

where $t = (t_1, \ldots, t_8) \in \left(\mathbb{C}^\times\right)^8$. For generic $t = (t_1, \ldots, t_8) \in \mathbb{C}^8$ satisfying $\prod_{j=1}^8 t_j = 1$ and generic $\mu \in \mathbb{C}$ we now define the resulting trigonometric hypergeometric integrals as

$$S_t(t) = \int_{\mathcal{C}} I_t(t;z)\frac{dz}{2\pi i z}, \qquad U_t^\mu(t) = \int_{\mathcal{C}'} J_t^\mu(t;q)\frac{dz}{2\pi i z}$$

where $\mathcal{C}$ (respectively $\mathcal{C}'$) is a deformation of the positively oriented unit circle $\mathbb{T}$ including the pole sequences $t_j q^{\mathbb{Z}_{\geq 0}}$ $(j = 2, \ldots, 7)$ of $I_t(t;z)$ and excluding their reciprocals (respectively including the pole sequences $t_j q^{\mathbb{Z}_{\geq 0}}$ $(j = 1, 8)$ of $J_t^\mu(t;z)$ and excluding the pole sequences $t_j^{-1} q^{\mathbb{Z}_{\leq 1}}$ $(j = 1, 8)$ and $t_i^{-1} q^{\mathbb{Z}_{\leq 0}}$

($i=2,\ldots,7$)). As in the elliptic and hyperbolic cases, one observes that $S_t(t)$ (respectively $U_t^\mu(t)$) admits a unique meromorphic extension to the parameter domain $\{t\in\mathbb{C}^8 \mid \prod_{j=1}^8 t_j=1\}$ (respectively $\{(\mu,t)\in\mathbb{C}^\times\times\mathbb{C}^8 \mid \prod_{j=1}^8 t_j=1\}$). We call $S_t(t)$ the trigonometric hypergeometric function.

Lemma 4.5.2. *The integral $U_t^\mu(t)$ is independent of $\mu\in\mathbb{C}^\times$.*

Proof. There are several different, elementary arguments to prove the lemma, we give here the argument based on Liouville's Theorem. Note that $U_t^{q\mu}(t)=U_t^\mu(t)$, and that the possible poles of $\mu\mapsto U_t^\mu(t)$ are at $t_jq^{\mathbb{Z}}$ ($j=1,8$). Without loss of generality we assume the generic conditions on the parameters $t\in\mathbb{C}^8$ ($\prod_{j=1}^8 t_j=1$) such that $U_t^\mu(t)$ admits the integral representation as above, and such that $t_1\notin t_8q^{\mathbb{Z}}$. The latter condition ensures that the possible poles $t_1q^{\mathbb{Z}},t_8q^{\mathbb{Z}}$ of $\mu\mapsto U_t^\mu(t)$ are at most simple. But the residue of $U_t^\mu(t)$ at $\mu=t_j$ ($j=1,8$) is zero, since it is an integral over a deformation $\mathcal{C}'$ of $\mathbb{T}$ whose integrand is analytic within the integration contour $\mathcal{C}'$ and vanishes at the origin. Hence $\mathbb{C}^\times\ni\mu\mapsto U_t^\mu(t)$ is bounded and analytic, hence constant by Liouville's Theorem. □

In view of Lemma 4.5.2, we omit the μ-dependence in the notation for $U_t^\mu(t)$. Since $I_t(-t;z)=I_t(t;-z)$ and $J_t^\mu(-t;z)=J_t^{-\mu}(t;-z)$, we may and will view S_t and U_t as meromorphic function on $\mathcal{H}_1$.

By choosing a special value of μ, we are able to derive another, "unfolded" integral representation of $U_t(t)$ as follows. Let $\mathbb{H}_+$ be the upper half plane in $\mathbb{C}$. Choose $\tau\in\mathbb{H}_+$ such that $q=e(\tau)$, where $e(x)$ is a shorthand notation for $\exp(2\pi ix)$. Recall the surjective map $\psi_0:\mathcal{G}_0\to\mathcal{H}_1$ from Section 4.2.

Corollary 4.5.3. *For generic parameters $u\in\mathcal{G}_0$ we have*

$$\begin{aligned}U_t(\psi_0(2\pi iu))&=\frac{2}{\tau\big(q,q;q\big)_\infty}\frac{\big(e((u_8-u_1)/\tau)-1\big)}{e(u_1)\theta(e(u_8-u_1);q)}\\&\times\int_{\mathcal{L}}\Bigg\{\Big(1-\frac{e(2x)}{q}\Big)\prod_{j=2}^{7}\frac{\big(e(x-u_j);q\big)_\infty}{\big(e(x+u_j);q\big)_\infty}\frac{\big(qe(x-u_1),qe(x-u_8);q\big)_\infty}{\big(q^{-1}e(x+u_1),q^{-1}e(x+u_8);q\big)_\infty}\\&\qquad\times\frac{e(x)}{\big(1-e((u_8-x)/\tau)\big)\big(1-e((x-u_1)/\tau)\big)}\Bigg\}dx\end{aligned}$$

where the integration contour $\mathcal{L}$ is some translate $\xi+\mathbb{R}$ ($\xi\in i\mathbb{R}$) of the real line with a finite number of indentations, such that $\mathcal{C}$ separates the pole sequences $-u_1+\mathbb{Z}+\mathbb{Z}_{\leq 1}\tau$, $-u_8+\mathbb{Z}+\mathbb{Z}_{\leq 1}\tau$ and $-u_j+\mathbb{Z}+\mathbb{Z}_{\leq 0}\tau$ ($j=2,\ldots,7$) of the integrand from the pole sequences $u_1+\mathbb{Z}_{\geq 0}\tau$ and $u_8+\mathbb{Z}_{\geq 0}\tau$.

Remark 4.5.4. Note that always $\xi\neq 0$ in Corollary 4.5.3. Due to the balancing condition $\sum_{j=1}^8 u_j=0$, there are no parameter choices for which $\mathcal{L}=\mathbb{R}$ can be taken as integration contour. This is a reflection of the fact that there are no parameters $t\in\mathcal{H}_1$ such that the unit circle $\mathbb{T}$ can be chosen as integration cycle in the original integral representation $U_t(t)=\int_{\mathcal{C}'}J_t^\mu(t;z)\frac{dz}{2\pi iz}$ of $U_t(t)$.

Proof. In the integral expression

$$U_t(\psi_c(2\pi iu))=\int_{\mathcal{C}'}J_t^{\mu}(\psi_0(2\pi iu);z)\frac{dz}{2\pi iz},$$

we change the integration variable to $z=e(x)$, take $\mu=e(\eta(u_1,u_8))$, and we use Lemma 4.5.1 to rewrite the quotient of theta-functions in the integrand as a bilateral sum. Changing the integration over the indented line segment with the bilateral sum using Fubini's Theorem, we can rewrite the resulting expression as a single integral over a noncompact integration cycle $\mathcal{L}$. This leads directly to the desired result. □

In the following lemma we show that $U_t(t)$ can be expressed as a sum of two nonterminating very-well-poised ${}_{10}\phi_9$ series.

Lemma 4.5.5. *As meromorphic functions in $t\in\mathcal{H}_1$, we have*

$$U_t(t)=\frac{2}{\big(q,t_1^2,\frac{t_1t_8}{q},\frac{t_8}{t_1};q\big)_\infty}\prod_{j=2}^{7}\frac{\big(\frac{t_1}{t_j};q\big)_\infty}{\big(t_1t_j;q\big)_\infty}\,{}_{10}W_9\big(\frac{t_1^2}{q};t_1t_2,t_1t_3,\ldots,t_1t_7,\frac{t_1t_8}{q};q,q\big)$$
$$+\big(t_1\leftrightarrow t_8\big).$$

Proof. For generic $t\in\mathcal{H}_1$ we shrink the contour $\mathcal{C}'$ in the integral representation of $U_t^{\mu}(t)=\int_{\mathcal{C}'}J_t^{\mu}(t;z)\frac{dz}{2\pi iz}$ to the origin while picking up the residues at the pole sequences $t_1q^{\mathbb{Z}_{\geq 0}}$ and $t_8q^{\mathbb{Z}_{\geq 0}}$ of the integrand $J_t^{\mu}(t;z)$. The resulting sum of residues can be directly rewritten as a sum of two very-well-poised ${}_{10}\phi_9$ series, leading to the desired identity (cf. the general residue techniques in [16, §4.10]). □

Remark 4.5.6. Lemma 4.5.5 yields that $U_t(t)$ is, up to an explicit rescaling factor, an integral form of the particular sum Φ of two very-well-poised ${}_{10}\phi_9$ series as e.g. studied in [19] and [45] (see [19, (1.8)], [45, (9c)]). Note furthermore that the explicit μ-dependent quotient of theta-functions in the integrand of $U_t^{\mu}(t)$ has the effect that it balances the very-well-poised ${}_{10}\phi_9$ series when picking up the residues of $J_t^{\mu}(t;z)$ at the two pole sequences $t_1q^{\mathbb{Z}_{\geq 0}}$ and $t_8q^{\mathbb{Z}_{\geq 0}}$.

In the following proposition we show that S_t (respectively U_t) is the degeneration of S_e along the root vector α_{18}^{+} (respectively γ_{18}).

Proposition 4.5.7. *Let $t=(t_1,\ldots,t_8)\in\mathbb{C}^8$ be generic parameters satisfying the balancing condition $\prod_{j=1}^{8}t_j=1$. Then*

$$\begin{aligned}S_t(t)&=\lim_{p\to 0}S_e(pqt_1,t_2,\ldots,t_7,pqt_8),\\ U_t(t)&=\lim_{p\to 0}\theta\big(t_1t_8/pq;q\big)S_e\big((pq)^{-\frac{1}{2}}t_1,(pq)^{\frac{1}{2}}t_2,\ldots,(pq)^{\frac{1}{2}}t_7,(pq)^{-\frac{1}{2}}t_8\big).\end{aligned}\tag{4.5.5}$$

Proof. For the degeneration to $S_t(t)$ we use that

$$I_e(pqt_1,t_2,\ldots,t_7,pqt_8;z)=\frac{\prod_{j=2}^{7}\Gamma_e(t_jz^{\pm 1};p,q)}{\Gamma_e(z^{\pm 2},t_1^{-1}z^{\pm 1},t_8^{-1}z^{\pm 1};p,q)}$$

in view of the reflection equation for Γ_e, which (pointwise) tends to $I_t(t;z)$ as $p\to 0$ in view of (4.5.1). A standard application of Lebesgue's dominated convergence theorem leads to the limit of the associated integrals.

The degeneration to $U_t(t)$ is more involved, since one needs to use a nontrivial symmetry argument to cancel some unwanted sequences of poles of $I_e(t;z)$. To ease the notations we set

$$t_p=\big((pq)^{-\frac{1}{2}}t_1,(pq)^{\frac{1}{2}}t_2,\ldots,(pq)^{\frac{1}{2}}t_7,(pq)^{-\frac{1}{2}}t_8\big)$$

and we denote

$$Q(z)=\frac{\theta\big((pq)^{-\frac{1}{2}}t_1t_8/\mu z,(pq)^{-\frac{1}{2}}t_1z,(pq)^{-\frac{1}{2}}t_8z,(pq)^{\frac{1}{2}}z/\mu;q\big)}{\theta\big(z^2;q\big)}.$$

By (4.3.6), we have the identity

$$Q(z)+Q(z^{-1})=\theta\big(t_1t_8/pq,t_1/\mu,t_8\mu;q\big).$$

Since the integrand $I_e(\tau_p;z)$ is invariant under $z\mapsto z^{-1}$, we can consequently write

$$\theta\big(t_1t_8/pq;q\big)S_e(t_p;z)=\frac{2}{\theta\big(t_1/\mu,t_8/\mu;q\big)}\int_{\mathcal{C}}Q(z)I_e(t_p;z)\frac{dz}{2\pi iz},$$

with $\mathcal{C}$ a deformation of the positively oriented unit circle $\mathbb{T}$ separating the downward pole sequences of the integrand from the upward pole sequences. Taking $(pq)^{\frac{1}{2}}z$ as a new integration variable and using the functional equation and reflection equation of Γ_e, we obtain the integral representation

$$\begin{aligned}\theta\big(t_1t_8/pq;q\big)S_e(t_p;z)=2\int_{\mathcal{C}}\frac{\theta\big(t_1t_8/\mu z,z/\mu;q\big)}{\theta\big(t_1/\mu,t_8/\mu;q\big)}\theta\big(z^2/q;p\big)\prod_{j=2}^{7}\\ \times\frac{\Gamma_e(t_jz;p,q)}{\Gamma_e(z/t_j;p,q)}\prod_{j=1,8}\Gamma_e\big(t_jz/q,t_j/z;p,q\big)\frac{dz}{2\pi iz},\end{aligned}\tag{4.5.6}$$

where $\mathcal{C}$ is a deformation of the positively oriented unit circle $\mathbb{T}$ which includes the pole sequences $t_1p^{\mathbb{Z}_{\geq 0}}q^{\mathbb{Z}_{\geq 0}},t_8p^{\mathbb{Z}_{\geq 0}}q^{\mathbb{Z}_{\geq 0}}$ and $t_jp^{\mathbb{Z}_{\geq 1}}q^{\mathbb{Z}_{\geq 1}}$ $(j=2,\ldots,7)$, and which excludes the pole sequences $t_1^{-1}p^{\mathbb{Z}_{\leq 0}}q^{\mathbb{Z}_{\leq 1}}$, $t_8^{-1}p^{\mathbb{Z}_{\leq 0}}q^{\mathbb{Z}_{\leq 1}}$ and $t_j^{-1}p^{\mathbb{Z}_{\leq 0}}q^{\mathbb{Z}_{\leq 0}}$ $(j=2,\ldots,7)$. We can now take the limit $p\to 0$ in (4.5.6) with p-independent, fixed integration contour $\mathcal{C}$, leading to the desired limit relation

$$\lim_{p\to 0}\theta\big(t_1t_8/pq;q\big)S_e(t_p)=U_t^{\mu}(t).$$

□

Remark 4.5.8. Observe that Lemma 4.5.5 and the proof of Proposition 4.5.7 entail independent proofs of Lemma 4.5.2.

By specializing the parameters $t \in \mathcal{H}_1$ in Proposition 4.5.7 further, we arrive at trigonometric integrals which can be evaluated by (4.3.2). The resulting trigonometric degenerations lead immediately to the trigonometric Nassrallah-Rahman integral evaluation formula [16, (6.4.1)] and Gasper's integral evaluation formula [16, (4.11.4)]:

Corollary 4.5.9. *For generic $t = (t_1, \ldots, t_6) \in \mathbb{C}^6$ satisfying the balancing condition $\prod_{j=1}^6 t_j = 1$ we have*

$$\int_{\mathcal{C}} \frac{\big(z^{\pm 2}, t_1^{-1} z^{\pm 1}; q\big)_\infty}{\prod_{j=2}^6 \big(t_j z^{\pm 1}; q\big)_\infty} \frac{dz}{2\pi i z} = \frac{2 \prod_{j=2}^6 \big(1/t_1 t_j; q\big)_\infty}{(q;q)_\infty \prod_{2 \le j < k \le 6} \big(t_j t_k; q\big)_\infty}, \tag{4.5.7}$$

where $\mathcal{C}$ is the deformation of $\mathbb{T}$ separating the pole sequences $t_j q^{\mathbb{Z}_{\ge 0}}$ $(j = 2, \ldots, 6)$ of the integrand from their reciprocals, and

$$\begin{aligned} \int_{\mathcal{C}'} \frac{\theta\big(t_5 t_6/\mu z, z/\mu; q\big)}{\theta\big(t_5/\mu, t_6/\mu; q\big)} \left(1 - \frac{z^2}{q}\right) \prod_{j=1}^4 \frac{\big(z/t_j; q\big)_\infty}{\big(t_j z; q\big)_\infty} \prod_{k=5,6} \frac{1}{\big(t_k z/q, t_k/z; q\big)_\infty} \frac{dz}{2\pi i z} \\ = \frac{\prod_{1 \le j < k \le 4} \big(1/t_j t_k; q\big)_\infty}{\big(q, t_5 t_6/q; q\big)_\infty \prod_{j=1}^4 \prod_{k=5}^6 \big(t_j t_k; q\big)_\infty}, \end{aligned} \tag{4.5.8}$$

where $\mathcal{C}'$ is the deformation of $\mathbb{T}$ separating the pole sequences $t_5 q^{\mathbb{Z}_{\ge 0}}, t_6 q^{\mathbb{Z}_{\ge 0}}$ of the integrand from the pole sequences $t_j^{-1} q^{\mathbb{Z}_{\le 0}}$ $(j = 1, \ldots, 4)$, $t_5^{-1} q^{\mathbb{Z}_{\le 1}}$ and $t_6^{-1} q^{\mathbb{Z}_{\le 1}}$.

Proof. For the first integral evaluation, take $t \in \mathcal{H}_1$ and $t_7 = t_8^{-1}$ in the degeneration from S_e to S_t, and use the elliptic Nassrallah-Rahman integral evaluation formula (4.3.2).

For the second integral evaluation, take $t \in \mathcal{H}_1$ with $t_6 = t_7^{-1}$ in the degeneration from S_e to U_t^μ and again use (4.3.2) to evaluate the elliptic integral. It leads to the second integral evaluation formula with generic parameters $(t_1, t_2, t_3, t_4, t_5, t_8) \in \mathbb{C}^6$ satisfying $t_1 \cdots t_5 t_8 = 1$. □

The second integral in Corollary 4.5.9 can be unfolded using Lemma 4.5.1 as in Corollary 4.5.3. We obtain for generic parameters $u \in \mathbb{C}^6$ satisfying $\sum_{j=1}^6 u_j = 0$,

$$\begin{aligned} \int_{\mathcal{L}} \Bigg\{ \left(1 - \frac{e(2x)}{q}\right) \prod_{j=1}^4 \frac{\big(e(x - u_j); q\big)_\infty}{\big(e(x + u_j); q\big)_\infty} \frac{\big(qe(x - u_5), qe(x - u_6); q\big)_\infty}{\big(q^{-1}e(x + u_5), q^{-1}e(x + u_6); q\big)_\infty} \\ \times \frac{e(x)}{\big(1 - e((u_6 - x)/\tau)\big)\big(1 - e((x - u_5)/\tau)\big)} \Bigg\} dx \\ = \frac{\tau t_5 \theta\big(t_6/t_5; q\big)}{\big(e((u_6 - u_5)/\tau) - 1\big)} \frac{\big(q;q\big)_\infty \prod_{1 \le j < k \le 4} \big(1/t_j t_k; q\big)_\infty}{\big(t_5 t_6/q\big)_\infty \prod_{j=1}^4 \prod_{k=5}^6 \big(t_j t_k; q\big)_\infty}, \end{aligned}$$

where $\tau \in \mathbb{H}_+$ such that $q = e(\tau)$, where $t_j = e(u_j)$ $(j = 1, \ldots, 6)$ and where the integration contour $\mathcal{L}$ is some translate $\xi + \mathbb{R}$ $(\xi \in i\mathbb{R})$ of the real line with a finite number of indentations, such that $\mathcal{C}$ separates the pole sequences $-u_5+\mathbb{Z}+\mathbb{Z}_{\leq 1}\tau$, $-u_6+\mathbb{Z}+\mathbb{Z}_{\leq 1}\tau$ and $-u_j+\mathbb{Z}+\mathbb{Z}_{\leq 0}\tau$ $(j=1,\ldots,4)$ of the integrand from the pole sequences $u_5+\mathbb{Z}_{\geq 0}\tau$ and $u_6+\mathbb{Z}_{\geq 0}\tau$. This is Agarwal's identity [16, (4.7.5)].

Furthermore, using Lemma 4.5.5 the second integral in Corollary 4.5.9 can be written as a sum of two very-well-poised ${}_8\phi_7$-series. We obtain for generic $t\in\mathbb{C}^6$ satisfying $\prod_{j=1}^6 t_j=1$,

$$\frac{1}{\big(q,t_5^2,t_5t_6/q,t_6/t_5;q\big)_\infty}\prod_{j=1}^4\frac{\big(t_5/t_j;q\big)_\infty}{\big(t_5t_j;q\big)_\infty}\,{}_8W_7\Big(\frac{t_5^2}{q};t_5t_1,t_5t_2,t_5t_3,t_5t_4,\frac{t_5t_6}{q};q,q\Big)$$
$$+\,(t_5\leftrightarrow t_6)=\frac{\prod_{1\leq j<k\leq 4}\big(1/t_jt_k;q\big)_\infty}{\big(q,t_5t_6/q;q\big)_\infty\prod_{j=1}^4\prod_{k=5}^6\big(t_jt_k;q\big)_\infty}$$

which is Bailey's summation formula [16, (2.11.7)] of the sum of two very-well-poised ${}_8\phi_7$ series.

We can now compute the (nontrivial) $W(E_6)$-symmetries of the trigonometric hypergeometric integrals S_t and U_t by taking limits of the corresponding symmetries on the elliptic level using Proposition 4.5.7. We prefer to give a derivation based on the trigonometric evaluation formulas (see Corollary 4.5.9), in analogy to our approach in the elliptic and hyperbolic cases.

Proposition 4.5.10. *The trigonometric integrals $S_t(t)$ and $U_t(t)$ $(t\in\mathcal{H}_1)$ are invariant under permutations of (t_1,t_8) and of $(t_2,\ldots,t_7)$. Furthermore,*

$$\begin{aligned}
S_t(t)&=S_t(wt)\frac{\big(1/t_1t_2,1/t_1t_3,1/t_1t_4,1/t_8t_5,1/t_8t_6,1/t_8t_7;q\big)_\infty}{\big(t_2t_3,t_2t_4,t_3t_4,t_5t_6,t_5t_7,t_6t_7;q\big)_\infty},\\
U_t(t)&=U_t(wt)\frac{\big(1/t_2t_3,1/t_2t_4,1/t_3t_4,1/t_5t_6,1/t_5t_7,1/t_6t_7;q\big)_\infty}{\big(t_1t_2,t_1t_3,t_1t_4,t_5t_8,t_6t_8,t_7t_8;q\big)_\infty}
\end{aligned}\tag{4.5.9}$$

as meromorphic functions in $t\in\mathcal{H}_1$.

Proof. In order to derive the w-symmetry of $S_t(t)$ we consider the double integral

$$\int_{\mathbb{T}^2}\frac{\big(z^{\pm 2},x^{\pm 2},t_1^{-1}z^{\pm 1},st_8^{-1}x^{\pm 1};q\big)_\infty}{\big(t_2z^{\pm 1},t_3z^{\pm 1},t_4z^{\pm 1},sz^{\pm 1}x^{\pm 1},s^{-1}t_5x^{\pm 1},s^{-1}t_6x^{\pm 1},s^{-1}t_7x^{\pm 1};q\big)_\infty}\frac{dz}{2\pi iz}\frac{dx}{2\pi ix}$$

for parameters $(t_1,\ldots,t_8)\in\mathbb{C}^8$ satisfying $\prod_{j=1}^8 t_j=1$, where $s^2t_1t_2t_3t_4=1=s^{-2}t_5t_6t_7t_8$ and where we assume the additional parameter restraints

$$|t_2|,|t_3|,|t_4|,|s|,|t_5/s|,|t_6/s|,|t_7/s|<1$$

to ensure that the integration contour $\mathbb{T}$ separates the downward sequences of poles of from the upward sequences. The desired transformation then follows by either integrating the double integral first to x, or first to z, using in each case the trigonometric Nassrallah-Rahman integral evaluation formula (see Corollary 4.5.9).

The proof of the w-symmetry of $U_t(t)$ follows the same line of arguments. For $\epsilon > 0$ we denote $\epsilon\mathbb{T}$ for the positively oriented circle in the complex plane with radius ϵ and centered at the origin. The w-symmetry

$$U_t^{\mu/s}(t) = U_t^{\mu}(wt)\frac{\big(1/t_2t_3, 1/t_2t_4, 1/t_3t_4, 1/t_5t_6, 1/t_5t_7, 1/t_6t_7; q\big)_\infty}{\big(t_1t_2, t_1t_3, t_1t_4, t_5t_8, t_6t_8, t_7t_8; q\big)_\infty}$$

for $t \in \mathcal{H}_1$, where $s^2t_1t_2t_3t_4 = 1 = s^{-2}t_5t_6t_7t_8$, by considering for $(t_1, \ldots, t_8) \in \mathbb{C}^8$ satisfying $\prod_{j=1}^{8} t_j = 1$ the double integral

$$\int_{(|qs|\mathbb{T})^2} \left\{ \frac{\theta\big(st_1t_8/\mu z, t_8z/\mu x, x/\mu; q\big)}{\theta\big(st_8/\mu, st_1/\mu, t_8/s\mu; q\big)} \left(1 - \frac{z^2}{q}\right)\left(1 - \frac{x^2}{q}\right) \frac{\big(z/t_2, z/t_3; q\big)_\infty}{\big(t_1z/q, t_1/z, t_2z\big)_\infty} \right.$$
$$\left. \times \frac{\big(z/t_4, xz/s, sx/t_5, sx/t_6, sx/t_7; q\big)_\infty}{\big(t_3z, t_4z, sxz/q, sz/x, sx/z, t_5x/s, t_6x/s, t_7x/s, t_8x/qs, t_8/sx; q\big)_\infty} \right\} \frac{dz}{2\pi iz}\frac{dx}{2\pi ix}$$

with $s^2t_1t_2t_3t_4 = 1 = s^{-2}t_5t_6t_7t_8$, where we assume the additional parameter restraints

$$0 < |s| \ll |q^{\frac{1}{2}}|, \qquad |t_1|, |t_2^{-1}|, |t_3^{-1}|, |t_4^{-1}| < |qs|,$$
$$|t_5|, |t_6|, |t_7| < |q^{-1}|, \qquad |t_8| < |qs^2|$$

to ensure a proper separation by the integration contours of the upward sequences of poles from the downward sequences. Using the second trigonometric integral evaluation formula of Corollary 4.5.9 then yields the desired result for the restricted parameter domain. Analytic continuation completes the proof. □

Remark 4.5.11. Rewriting $U_t(t)$ as a sum of two very-well-poised ${}_{10}\phi_9$ series (see Lemma 4.5.5 and Remark 4.5.6), the w-symmetry of $U_t(t)$ becomes Bailey's four-term transformation formula [16, (2.12.9)], see also [19]. The identification of the symmetry group of U_t with the Weyl group of type E_6 has been derived by different methods in [45].

Finally we relate the two trigonometric integrals S_t and U_t. We can obtain the following proposition as a degeneration of a particular $W(E_7)$-symmetry of S_e, but we prefer here to give a direct proof using double integrals.

Proposition 4.5.12. *As meromorphic function in $t \in \mathcal{H}_1$, we have*

$$S_t(t)\frac{\prod_{2\le j<k\le 5}(t_jt_k; q)_\infty (t_6t_7; q)_\infty}{(1/qt_1t_8, 1/t_1t_6, 1/t_1t_7, 1/t_8t_6, 1/t_8t_7; q)_\infty}$$
$$= U_t(t_6/s, st_2, st_3, st_4, st_5, t_1/s, t_8/s, t_7/s),$$

where $t_2t_3t_4t_5s^2 = 1 = t_1t_6t_7t_8/s^2$

Proof. For $(t_1,\ldots,t_8)\in\mathbb{C}^8$ satisfying $\prod_{j=1}^8 t_j=1$ we consider the double integral

$$\int_{z\in\eta\mathbb{T}}\int_{x\in\mathbb{T}}\frac{\theta(\mu z,t_6t_7\mu/s^2z)}{\theta(t_6\mu/s,t_7\mu/s)}\left(1-\frac{z^2}{q}\right)\prod_{j=2}^{5}\frac{1}{(t_jx^{\pm1};q)_\infty}$$
$$\times\frac{(x^{\pm2},zx^{\pm1}/s,sz/t_1,sz/t_8;q)_\infty}{(szx^{\pm1},t_7z/sq,t_7/sz,t_6z/sq,t_6/sz,t_1z/s,t_8z/s;q)_\infty}\frac{dx}{2\pi ix}\frac{dz}{2\pi iz},$$

with $s^2t_2t_3t_4t_5=1=s^{-2}t_1t_6t_7t_8$ and with $0<\eta<\min\big(|s^{-1}|,|q^{\frac12}|\big)$, where we assume the additional parameter restraints

$$|t_2|,|t_3|,|t_4|,|t_5|<1,\qquad |t_6|,|t_7|<\eta|s|,\qquad |t_1|,|t_8|<\eta^{-1}|s|$$

to ensure a proper separation by the integration contours of the upward sequences of poles from the downward sequences. Using Corollary 4.5.9, we can first integrate over x using the trigonometric Nassrallah-Rahman integral evaluation formula, or first integrate over z using the second integral evaluation formula of Corollary 4.5.9. The resulting identity gives the desired result for restricted parameter values. Analytic continuation completes the proof. □

Remark 4.5.13. **(i)** Combining Proposition 4.5.12 with Lemma 4.5.5 we obtain an expression of $S_t(t)$ as a sum of two very-well-poised ${}_{10}\phi_9$ series, which is originally due to Rahman [16, (6.4.8)].

(ii) For e.g. $t_1t_6=q^m$ $(m\in\mathbb{Z}_{\geq0})$, it follows from **(i)** (see also [58] and [16, (6.4.10)]) that the $S_t(t;p,q)$ essentially coincides with the biorthogonal rational function of Rahman [58], which is explicitly given as a terminating very-well-poised ${}_{10}\phi_9$ series.

4.5.3 Contiguous relations

The fundamental equation on this level equals

$$\frac{1}{y}(1-vx^{\pm1})(1-yz^{\pm1})+\frac{1}{z}(1-vy^{\pm1})(1-zx^{\pm1})+\frac{1}{x}(1-vz^{\pm1})(1-xy^{\pm1})=0 \quad (4.5.10)$$

where $(1-ax^{\pm1})=(1-ax)(1-ax^{-1})$. The fundamental relation (4.5.10) is the $p=0$ reduction of (4.3.6). In this section $\tau_{ij}=\tau_{ij}^{-\log(q)}$ acts as in the elliptic case by multiplying t_i by q and dividing t_j by q. Formula (4.5.10) leads as in the elliptic case to the difference equation

$$\frac{(1-t_5t_6^{\pm1}/q)}{(1-t_4t_6^{\pm1})}S_t(\tau_{45}t)+\frac{(1-t_5t_4^{\pm1}/q)}{(1-t_6t_4^{\pm1})}S_t(\tau_{65}t)=S_t(t),\qquad t\in\mathcal{H}_1. \quad (4.5.11)$$

To obtain a second difference equation between trigonometric hypergeometric functions where two times the same parameter is multiplied by q, we can mimic the approach in the elliptic case with the role of the longest Weyl group element taken over by the element $u = ws_{35}s_{46}w \in W(E_6)$. Alternatively, one can rewrite the difference equation (4.3.8) for S_e in the form

$$\frac{\theta\big(t_3/qt_4, 1/t_1t_5, 1/t_8t_5, t_2t_5/q, t_5t_6/q, t_5t_7/q; p\big)}{\theta\big(t_3/t_5; p\big)} S_e(\tau_{45}\widetilde{t}) + (t_3 \leftrightarrow t_5) \\ = \theta\big(1/qt_1t_4, 1/qt_8t_4, t_2t_4, t_4t_6, t_4t_7; p\big) S_e(\widetilde{t})$$

where $t \in \mathcal{H}_1$ and $\widetilde{t} = (pqt_1, t_2, \ldots, t_7, pqt_8)$, and degenerate it using Proposition 4.5.7. We arrive at

$$\frac{(1-t_3/qt_4)}{(1-t_3/t_5)} \prod_{j=1,8} \frac{(1-1/t_5t_j)}{(1-1/qt_4t_j)} \prod_{j=2,6,7} \frac{(1-t_5t_j/q)}{(1-t_4t_j)} S_t(\tau_{45}t) + (t_3 \leftrightarrow t_5) = S_t(t), \\ t \in \mathcal{H}_1 \quad (4.5.12)$$

Together these equations imply the following result.

Proposition 4.5.14. *We have*

$$A(t)S_t(\tau_{45}t) + (t_4 \leftrightarrow t_5) = B(t)S_t(t) \qquad (4.5.13)$$

as meromorphic functions in $t \in \mathcal{H}_1$, *where*

$$A(t) = -\frac{(1-\frac{1}{t_1t_5})(1-\frac{1}{t_8t_5})\prod_{j=2,3,6,7}(1-\frac{t_5t_j}{q})}{t_4(1-\frac{t_4t_5}{q})(1-\frac{t_5}{qt_4})(1-\frac{t_4}{t_5})}$$

$$B(t) = -\frac{(1-\frac{1}{qt_1t_6})(1-\frac{1}{qt_8t_6})(1-t_3t_6)(1-t_7t_6)(1-t_2t_6)}{t_6(1-\frac{t_4}{qt_6})(1-\frac{t_5}{qt_6})} \\ + \frac{(1-\frac{t_6}{t_4})(1-t_6t_4)\prod_{j=1,8}(1-\frac{1}{t_jt_5})\prod_{j=2,3,7}(1-\frac{t_jt_5}{q})}{t_6(1-\frac{t_5}{qt_6})(1-\frac{t_4t_5}{q})(1-\frac{t_5}{qt_4})(1-\frac{t_4}{t_5})} \\ + \frac{(1-\frac{t_6}{t_5})(1-t_6t_5)\prod_{j=1,8}(1-\frac{1}{t_jt_4})\prod_{j=2,3,7}(1-\frac{t_jt_4}{q})}{t_6(1-\frac{t_4}{qt_6})(1-\frac{t_4t_5}{q})(1-\frac{t_4}{qt_5})(1-\frac{t_5}{t_4})}.$$

Despite the apparent asymmetric expression B still satisfies $B(s_{67}t) = B(t)$.

The contiguous relation for the elliptic hypergeometric function S_e with step-size p can also be degenerated to the trigonometric level. A direct derivation is as follows. By (4.3.6) we have

$$\frac{\theta(t_8^{-1}t_7^{\pm 1}; q)}{\theta(t_6t_7^{\pm 1}; q)} I_t(t_1, t_2, \ldots, t_5, qt_8, t_7, t_6/q; z) \\ + \frac{\theta(t_8^{-1}t_6^{\pm 1}; q)}{\theta(t_7t_6^{\pm 1}; q)} I_t(t_1, t_2, \ldots, t_6, qt_8, t_7/q; z) = I_t(t; z).$$

Integrating this equation we obtain

$$\frac{\theta(t_8^{-1}t_7^{\pm 1};q)}{\theta(t_6t_7^{\pm 1};q)}S_t(t_1,t_2,\ldots,t_5,qt_8,t_7,t_6/q)+(t_6\leftrightarrow t_7)=S_t(t) \tag{4.5.14}$$

as meromorphic functions in $t\in\mathcal{H}_1$, a three term transformation for S_t. The three term transformation [19, (6.5)] is equivalent to the sum of two equations of this type (in which the parameters are chosen such that two terms coincide and two other terms cancel each other).

Remark 4.5.15. In [45] it is shown that there are essentially five different types of three term transformations for Φ (see Remark 4.5.6), or equivalently of the integrals U_t and S_t. The different types arise from a careful analysis of the three term transformations in terms of the $W(E_7)$-action on $\mathcal{H}_1$. It is likely that all five different types of three term transformations for Φ can be re-obtained by degenerating contiguous relations for S_e with step-size p (similarly as the derivation of (4.5.14)): concretely, the five prototypes are in one-to-one correspondence to the orbits of

$$\{(\alpha,\beta,\gamma)\in\mathcal{O}^3\mid\alpha,\beta,\gamma\text{ are pair-wise different}\}$$

under the diagonal action of $W(E_7)$, where $\mathcal{O}$ is the $W(E_7)$-orbit (4.5.3).

4.5.4 Degenerations with D_5 symmetries

In this section we consider degenerations of S_t and U_t with symmetries with respect to the Weyl group of type D_5. Compared to the analysis on the hyperbolic level, we introduce a trigonometric analog of the Euler and Barnes' type integrals, as well as a third, new type of integral arising as degeneration of U_t. We first introduce the degenerate integrals explicitly.

For generic $t=(t_1,\ldots,t_6)\in\left(\mathbb{C}^\times\right)^6$ we define the trigonometric Euler integral as

$$E_t(t)=\int_{\mathcal{C}}\frac{\left(z^{\pm 2},t_1^{-1}z^{\pm 1};q\right)_\infty}{\prod_{j=2}^6\left(t_jz^{\pm 1};q\right)_\infty}\frac{dz}{2\pi iz}, \tag{4.5.15}$$

where $\mathcal{C}$ is a deformation of the positively oriented unit circle $\mathbb{T}$ separating the decreasing pole sequences $t_jq^{\mathbb{Z}_{\geq 0}}$ $(j=2,\ldots,6)$ of the integrand from their reciprocals. We have $E_t(-t)=E_t(t)$, and E_t has a unique meromorphic extension to $\left(\mathbb{C}^\times\right)^6$. The resulting meromorphic function on $\left(\mathbb{C}^\times\right)^6/C_2$ is denoted also by E_t.

For generic $\mu\in\mathbb{C}^\times$ and generic $t=(t_1,\ldots,t_8)\in\mathbb{C}^8$ satisfying the balancing condition $\prod_{j=1}^8t_j=1$ we define the trigonometric Barnes integral as

$$B_t(t)=2\int_{\mathcal{C}}\frac{\theta(t_2t_7/\mu z,z/\mu;q)}{\theta(t_2/\mu,t_7/\mu;q)}\frac{(z/t_1,z/t_8;q)_\infty}{\prod_{j=3}^6(t_jz;q)_\infty(t_2/z,t_7/z;q)_\infty}\frac{dz}{2\pi iz}, \tag{4.5.16}$$

where $\mathcal{C}$ is a deformation of $\mathbb{T}$ separating the decreasing pole sequences $t_2q^{\mathbb{Z}_{\geq 0}}$ and $t_7q^{\mathbb{Z}_{\geq 0}}$ of the integrand from the increasing pole sequences $t_j^{-1}q^{\mathbb{Z}_{\leq 0}}$ $(j=3,\ldots,6)$. Analogously to the analysis of the integral $U_t(t)$, we have that the trigonometric

Barnes integral $B_t(t)$ uniquely extends to a meromorphic function in $\{(\mu,t)\in\mathbb{C}^\times\times\mathbb{C}^8 \mid \prod_{j=1}^8 t_j=1\}$ which is independent of μ (cf. Lemma 4.5.2). Furthermore, by a change of integration variable we have $B_t(-t)=B_t(t)$, hence B_t may (and will) be interpreted as meromorphic function on $\mathcal{H}_1$.

Finally, for generic $t=(t_1,\ldots,t_6)\in\big(\mathbb{C}^\times\big)^6$ we consider

$$V_t(t)=2\int_{\mathcal{C}}\frac{\theta\big(qt_2t_3t_4t_5t_6z;q\big)}{\theta\big(qt_1t_2t_3t_4t_5t_6;q\big)}\left(1-\frac{z^2}{q}\right)\prod_{j=2}^6\frac{\big(z/t_j;q\big)_\infty}{\big(t_jz;q\big)_\infty}\frac{1}{\big(t_1z/q,t_1/z;q\big)_\infty}\frac{dz}{2\pi iz}, \tag{4.5.17}$$

where $\mathcal{C}$ is a deformation of $\mathbb{T}$ separating the decreasing pole sequence $t_1q^{\mathbb{Z}_{\geq 0}}$ of the integrand from the remaining (increasing) pole sequences. As before, V_t unique extends to a meromorphic function on $\big(\mathbb{C}^\times\big)^6/C_2$.

Similarly as for $U_t(t)$, the trigonometric Barnes integral $B_t(t)$ can be unfolded. Recall that $q=e(\tau)$ with $\tau\in\mathbb{H}_+$, where $e(x)$ is a shorthand notation for $\exp(2\pi ix)$.

Lemma 4.5.16. *For generic parameters $u\in\mathcal{G}_0$ we have*

$$\begin{aligned}B_t(\psi_0(2\pi iu))=&\frac{2}{\tau\big(q,q;q\big)_\infty}\frac{\big(e((u_7-u_2)/\tau)-1\big)}{e(u_2)\theta(e(u_7-u_2);q)}\\&\times\int_{\mathcal{L}}\left\{\frac{\big(e(x-u_1),qe(x-u_2),qe(x-u_7),e(x-u_8);q\big)_\infty}{\big(e(x+u_3),e(x+u_4),e(x+u_5),e(x+u_6);q\big)_\infty}\right.\\&\left.\qquad\times\frac{e(x)}{\big(1-e((u_7-x)/\tau)\big)\big(1-e((x-u_2)/\tau)\big)}\right\}dx,\end{aligned}$$

where the integration contour $\mathcal{L}$ is some translate $\xi+\mathbb{R}$ ($\xi\in i\mathbb{R}$) of the real line with a finite number of indentations, such that $\mathcal{C}$ separates the pole sequences $-u_j+\mathbb{Z}+\mathbb{Z}_{\leq 0}\tau$ ($j=3,\ldots,6$) of the integrand from the pole sequences $u_2+\mathbb{Z}_{\geq 0}\tau$ and $u_7+\mathbb{Z}_{\geq 0}\tau$.

Proof. The proof is similar to the proof of Corollary 4.5.3. ☐

For $B_t(t)$ and $V_t(t)$ we have the following series expansions in balanced ${}_4\phi_3$'s (respectively in a very-well-poised ${}_8\phi_7$).

Lemma 4.5.17. (a) *We have*

$$B_t(t)=\frac{2\big(t_2/t_1,t_2/t_8;q\big)_\infty}{\big(q,t_7/t_2,t_2t_3,t_2t_4,t_2t_5,t_2t_6;q\big)_\infty}\,{}_4\phi_3\left(\begin{matrix}t_2t_3,t_2t_4,t_2t_5,t_2t_6\\qt_2/t_7,t_2/t_1,t_2/t_8\end{matrix};q,q\right)+(t_2\leftrightarrow t_7)$$

as meromorphic functions in $t\in\mathcal{H}_1$.
(b) *We have*

$$V_t(t)=\frac{2}{\big(q,t_1^2;q\big)_\infty}\prod_{j=2}^6\frac{\big(t_1/t_j;q\big)_\infty}{\big(t_1t_j;q\big)_\infty}\,{}_8W_7\left(\frac{t_1^2}{q};t_1t_2,t_1t_3,\ldots,t_1t_6;q,\frac{1}{t_1t_2t_3t_4t_5t_6}\right)$$

as meromorphic functions in $t \in \left(\mathbb{C}^{\times}\right)^{6}/C_2$*:* $|t_1t_2t_3t_4t_5t_6| > 1$.

Proof. This follows by a straightforward residue computation as in the proof of Lemma 4.5.5 (cf. also [16, §4.10]). For **(a)** one picks up the residues at the increasing pole sequences $t_2q^{\mathbb{Z}_{\geq 0}}$ and $t_7q^{\mathbb{Z}_{\geq 0}}$ of the integrand of $B_t(t)$; for **(b)** one picks up the residues at the single increasing pole sequence $t_1q^{\mathbb{Z}_{\geq 0}}$ of the integrand of $V_t(t)$. □

Proposition 4.5.18. *For generic* $t \in \mathcal{H}_1$ *we have*

$$\lim_{u\to 0} S_t(t_1,\ldots,t_6,t_7u,t_8/u) = E_t(t_1,\ldots,t_6),$$

$$\lim_{u\to 0} \left(t_2t_7/u;q\right)_\infty S_t(t_1u^{-\frac{1}{2}},t_2u^{-\frac{1}{2}},t_3u^{\frac{1}{2}},t_4u^{\frac{1}{2}},t_5u^{\frac{1}{2}},t_6u^{\frac{1}{2}},t_7u^{-\frac{1}{2}},t_8u^{-\frac{1}{2}}) = B_t(t),$$

$$\lim_{u\to 0} \left(t_1t_7/u;q\right)_\infty U_t(t_1,\ldots,t_6,t_7/u,t_8u) = V_t(t_1,\ldots,t_6),$$

$$\lim_{u\to 0} U_t(t_2u^{\frac{1}{2}},t_1u^{\frac{1}{2}},t_3u^{-\frac{1}{2}},t_4u^{-\frac{1}{2}},t_5u^{-\frac{1}{2}},t_6u^{-\frac{1}{2}},t_8u^{\frac{1}{2}},t_7u^{\frac{1}{2}}) = B_t(t).$$

Proof. The first limit is direct. For the second limit, we follow the same approach as in the proof of Proposition 4.5.7. Define $Q(z)$ as

$$Q(z) = \frac{\theta(t_2zs^{-\frac{1}{2}},t_7zs^{-\frac{1}{2}},\mu zs^{\frac{1}{2}},t_2t_7\mu s^{-\frac{1}{2}}z^{-1};q)}{\theta(z^2;q)}.$$

Using (4.3.6) we obtain the equation

$$Q(z) + Q(z^{-1}) = \theta(t_2t_7/s, t_2\mu, t_7\mu; q),$$

and hence, as in the proof of Proposition 4.5.7,

$$(t_2t_7/u;q)_\infty S_t(t_u) = 2\frac{(t_2t_7/u;q)_\infty}{\theta(t_2t_7/u,t_2\mu,t_7\mu;q)}\int_{\mathcal{C}} I_t(t_u;z)Q(z)\frac{dz}{2\pi iz}$$

for an appropriate contour $\mathcal{C}$, where we use the abbreviated notation

$$t_u = (t_1u^{-\frac{1}{2}},t_2u^{-\frac{1}{2}},t_3u^{\frac{1}{2}},t_4u^{\frac{1}{2}},t_5u^{\frac{1}{2}},t_6u^{\frac{1}{2}},t_7u^{-\frac{1}{2}},t_8u^{-\frac{1}{2}}).$$

Taking $u^{-\frac{1}{2}}z$ as new integration variable we obtain

$$\begin{aligned}(ut_2t_7;q)_\infty S_t(t_u) = 2\int_{\mathcal{C}} &\frac{\theta(\mu z,t_2t_7\mu/z;q)}{\theta(t_2\mu,t_7\mu;q)}\frac{(z/t_1,z/t_8;q)_\infty}{\prod_{j=3}^{6}(t_jz;q)_\infty(t_2/z,t_7/z;q)_\infty}\\ &\times(1-\frac{u}{z^2})\frac{(ut_1^{-1}/z,ut_8^{-1}/z,qut_2^{-1}/z,qut_7^{-1}/z;q)_\infty}{(qu/t_2t_7;q)_\infty\prod_{j=3}^{6}(t_ju/z;q)_\infty}\frac{dz}{2\pi iz},\end{aligned}$$

where, for u small enough, we take $\mathcal{C}$ to be a u-independent deformation of $\mathbb{T}$ separating the decreasing pole sequences $t_2q^{\mathbb{Z}_{\geq 0}}, t_7q^{\mathbb{Z}_{\geq 0}}$ and $t_juq^{\mathbb{Z}_{\geq 0}}$ $(j = 3,\ldots,6)$ of the integrand from the decreasing pole sequences $t_j^{-1}q^{\mathbb{Z}_{\leq 0}}$ $(j = 3,\ldots,6)$. The limit $u \to 0$ can be taken in the resulting integral, leading to the desired result.

To prove the third limit, we set $\mu = q/t_1t_7t_8$ in the integral expression of $U_t(t) = \int_{\mathcal{C}'} J_t^\mu(t;z)\frac{dz}{2\pi iz}$ to remove the contribution $\big(t_7z;q\big)_\infty$ in the denominator of the integrand:

$$U_t(t) = 2\int_{\mathcal{C}'} \frac{\theta(t_1t_7t_8/z;q)}{\theta(t_1t_7,t_7t_8;q)}\left(1-\frac{z^2}{q}\right) \times \prod_{j=2}^{6}\frac{\big(z/t_j;q\big)_\infty}{\big(t_jz;q\big)_\infty}\frac{\big(z/t_7,q/t_7z;q\big)_\infty}{\big(t_1z/q,t_1/z,t_8z/q,t_8/z;q\big)_\infty}\frac{dz}{2\pi iz}.$$

In the resulting integral the desired limit can be taken directly, leading to the desired result.

For the fourth limit, one easily verifies that

$$B_t(t) = \lim_{u\to 0} U_t^{\mu u^{\frac{1}{2}}}(t_2u^{\frac{1}{2}},t_1u^{\frac{1}{2}},t_3u^{-\frac{1}{2}},t_4u^{-\frac{1}{2}},t_5u^{-\frac{1}{2}},t_6u^{-\frac{1}{2}},t_8u^{\frac{1}{2}},t_7u^{\frac{1}{2}})$$

for generic $t\in\mathcal{H}_1$ after changing integration variable z to $zu^{\frac{1}{2}}$ on the right hand side. □

Proposition 4.5.18 and Corollary 4.5.9 immediately lead to the following three trigonometric integral evaluations (of which the first is the well known Askey-Wilson integral evaluation [16, (6.1.4)]).

Corollary 4.5.19. *For generic parameters $t=(t_1,t_2,t_3,t_4)\in\mathbb{C}^4$ we have*

$$\int_{\mathcal{C}}\frac{\big(z^{\pm 2};q\big)_\infty}{\prod_{j=1}^4\big(t_jz^{\pm 1};q\big)_\infty}\frac{dz}{2\pi iz} = \frac{2\big(t_1t_2t_3t_4;q\big)_\infty}{\big(q;q\big)_\infty\prod_{1\le j<k\le 4}\big(t_jt_k;q\big)_\infty},$$

$$\int_{\mathcal{C}'}\frac{\theta\big(qt_2t_3t_4z;q\big)}{\theta\big(qt_1t_2t_3t_4;q\big)}\left(1-\frac{z^2}{q}\right)\frac{\big(z/t_2,z/t_3,z/t_4;q\big)_\infty}{\big(t_1/z,t_1z/q,t_2z,t_3z,t_4z;q\big)_\infty}\frac{dz}{2\pi iz} = \frac{\big(qt_1t_2t_3t_4,1/t_2t_3,1/t_2t_4,1/t_3t_4;q\big)_\infty}{\big(q,t_1t_2,t_1t_3,t_1t_4;q\big)_\infty}$$

with $\mathcal{C}$ (respectively $\mathcal{C}'$) a deformation of $\mathbb{T}$ separating the sequences $t_jq^{\mathbb{Z}_{\ge 0}}$ ($j=1,\ldots,4$) from their reciprocals (respectively separating $t_1q^{\mathbb{Z}_{\ge 0}}$ from the sequences $t_1^{-1}q^{\mathbb{Z}_{\le 1}}$, $t_2^{-1}q^{\mathbb{Z}_{\le 0}}$, $t_3^{-1}q^{\mathbb{Z}_{\le 0}}$ and $t_4^{-1}q^{\mathbb{Z}_{\le 0}}$).

For generic $\mu\in\mathbb{C}^\times$ and $t\in\mathbb{C}^6$ satisfying $\prod_{j=1}^6 t_j = 1$ we have

$$\int_{\mathcal{C}}\frac{\theta\big(t_1t_5/\mu z,z/\mu;q\big)}{\theta\big(t_1/\mu,t_5/\mu;q\big)}\frac{\big(z/t_6;q\big)_\infty}{\big(t_1/z,t_2z,t_3z,t_4z,t_5/z;q\big)_\infty}\frac{dz}{2\pi iz} = \frac{1}{\big(q;q\big)_\infty}\prod_{j=2}^{4}\frac{\big(1/t_jt_6;q\big)_\infty}{\big(t_1t_j,t_jt_5;q\big)_\infty},$$

with $\mathcal{C}$ a deformation of $\mathbb{T}$ separating the pole sequences $t_1q^{\mathbb{Z}_{\ge 0}}$, $t_5q^{\mathbb{Z}_{\ge 0}}$ from $t_2^{-1}q^{\mathbb{Z}_{\le 0}}$, $t_3^{-1}q^{\mathbb{Z}_{\le 0}}$ and $t_4^{-1}q^{\mathbb{Z}_{\le 0}}$.

Proof. Specializing the degeneration from S_t to E_t in Proposition 4.5.18 to generic parameters $t \in \mathcal{H}_1$ under the additional condition $t_1t_2 = 1$ and using the trigonometric Nassrallah-Rahman integral evaluation (Corollary 4.5.9) leads to the Askey-Wilson integral evaluation with corresponding parameters (t_3, t_4, t_5, t_6).

Similarly, specializing the degeneration from U_t to V_t (respectively S_t to B_t) to generic parameters $t \in \mathcal{H}_1$ under the additional condition $t_2t_3 = 1$ (respectively $t_1t_3 = 1$) and using the Nassrallah-Rahman integral evaluation we obtain the second (respectively third) integral evaluation with parameters (t_1, t_4, t_5, t_6) (respectively $(t_2, t_4, t_5, t_6, t_7, t_8)$). □

Various well-known identities are direct consequences of Corollary 4.5.19. First of all, analogous to the unfolding of the integrals U_t and V_t (see Corollary 4.5.3 and Lemma 4.5.16), the left hand side of the third integral evaluation can be unfolded. We obtain for generic $u \in \mathbb{C}^6$ with $\sum_{j=1}^6 u_j = 0$,

$$\int_{\mathcal{L}} \frac{\big(qe(x-u_1), qe(x-u_5), e(x-u_6); q\big)_\infty}{\big(e(x+u_2), e(x+u_3), e(x+u_4); q\big)_\infty} \frac{e(x)}{\Big(1-e\big(\frac{u_5-x}{\tau}\big)\Big)\Big(1-e\big(\frac{x-u_1}{\tau}\big)\Big)}\,dx$$
$$= \frac{\tau t_1 \theta\big(t_5/t_1; q\big)}{\big(e((u_5-u_1)/\tau)-1\big)} \frac{\big(q, 1/t_2t_6, 1/t_3t_6, 1/t_4t_6; q\big)_\infty}{\big(t_1t_2, t_1t_3, t_1t_4, t_2t_5, t_3t_5, t_4t_5; q\big)_\infty}$$

where $\tau \in \mathbb{H}_+$ such that $q = e(\tau)$, where $t_j = e(u_j)$ $(j = 1, \ldots, 6)$ and where the integration contour $\mathcal{L}$ is some translate $\xi + \mathbb{R}$ $(\xi \in i\mathbb{R})$ of the real line with a finite number of indentations such that $\mathcal{C}$ separates the pole sequences $-u_j + \mathbb{Z} + \mathbb{Z}_{\leq 0}\tau$ $(j = 2, 3, 4)$ of the integrand from the pole sequences $u_1 + \mathbb{Z}_{\geq 0}\tau$ and $u_5 + \mathbb{Z}_{\geq 0}\tau$. This integral identity is Agarwal's [16, (4.4.6)] trigonometric analogue of Barnes' second lemma.

The left hand side of the second integral evaluation in Corollary 4.5.19 can be rewritten as a unilateral sum by picking up the residues at $t_1q^{\mathbb{Z}_{\geq 0}}$, cf. Lemma 4.5.5. The resulting identity is

$${}_6\phi_5\left(\begin{matrix} q^{-1}t_1^2, q^{\frac{1}{2}}t_1, -q^{\frac{1}{2}}t_1, t_1t_2, t_1t_3, t_1t_4 \\ q^{-\frac{1}{2}}t_1, -q^{-\frac{1}{2}}t_1, t_1/t_2, t_1/t_3, t_1/t_4 \end{matrix}; q, \frac{1}{t_1t_2t_3t_4}\right)$$
$$= \frac{\big(t_1^2, 1/t_2t_3, 1/t_2t_4, 1/t_3t_4; q\big)_\infty}{\big(1/t_1t_2t_3t_4, t_1/t_2, t_1/t_3, t_1/t_4; q\big)_\infty},$$

for generic $t \in \mathbb{C}^4$ satisfying $|t_1t_2t_3t_4| > 1$, which is the ${}_6\phi_5$ summation formula [16, (2.7.1)].

For generic $t \in \mathbb{C}^6$ satisfying $\prod_{j=1}^6 t_j = 1$ the left hand side of the third integral evaluation in Corollary 4.5.19 can be written as a sum of two unilateral series by picking up the poles of the integrand at the decreasing sequences $t_1q^{\mathbb{Z}_{\geq 0}}$ and $t_5q^{\mathbb{Z}_{\geq 0}}$

of poles of the integrand. The resulting identity is

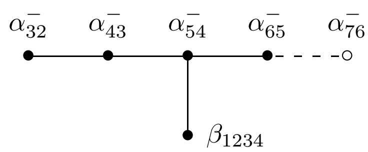
$$\frac{\left(t_1/t_6;q\right)_\infty}{\left(t_5/t_1,t_1t_2,t_1t_3,t_1t_4;q\right)_\infty}\,{}_3\phi_2\left(\begin{matrix}t_1t_2,t_1t_3,t_1t_4\\ qt_1/t_5,t_1/t_6\end{matrix};q,q\right)+\left(t_1\leftrightarrow t_5\right)$$
$$=\prod_{j=2}^{4}\frac{\left(1/t_jt_6;q\right)_\infty}{\left(t_1t_j,t_jt_5;q\right)_\infty}$$

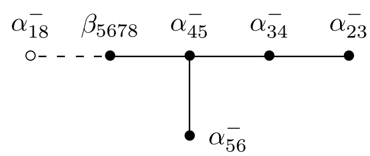
for generic $t\in\mathbb{C}^6$ satisfying $\prod_{j=1}^6 t_j=1$, which is the nonterminating version [16, (2.10.12)] of Saalschütz formula.

We now return to the three trigonometric hypergeometric integrals E_t, B_t and V_t. Recall that the symmetry group of S_t and U_t is the subgroup $W(E_6)=W(E_7)_{\alpha_{18}^+}=W(E_7)_{\gamma_{18}}$, which is a maximal standard parabolic subgroup of $W(E_7)$ with respect to both bases Δ_1 and Δ_2 of $R(E_7)$ (see Section 4.2), with corresponding sub-bases $\Delta_1'=\Delta_1\setminus\{\alpha_{12}^-\}$ and $\Delta_2'=\Delta_2\setminus\{\alpha_{87}^-\}$ respectively. The four limits of Proposition 4.5.18 now imply that the trigonometric integrals E_t, B_t and V_t have symmetry groups $W(E_6)_{\alpha_{78}^-}$ or $W(E_6)_{\beta_{1278}}$. The stabilizer subgroup $W(E_6)_{\alpha_{78}^-}$ is a standard maximal parabolic subgroup of $W(E_6)$ with respect to both bases Δ_1' or Δ_2', with corresponding sub-basis

$$\Delta(D_5)=\Delta_1'\setminus\{\alpha_{76}^-\}=\Delta_2'\setminus\{\alpha_{18}^-\}$$

and with corresponding Dynkin sub-diagrams

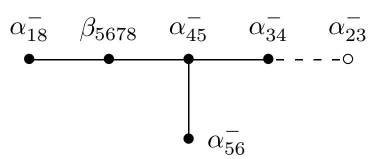




respectively. Similarly, $W(E_6)_{\beta_{1278}}$ is a standard maximal parabolic subgroup of $W(E_6)$ with respect to the basis Δ_2', with corresponding sub-basis

$$\Delta'(D_5)=\Delta_2'\setminus\{\alpha_{23}^-\}$$

and with corresponding Dynkin sub-diagram



We write

$$W(D_5)=W(E_6)_{\alpha_{78}^-},\qquad W'(D_5)=W(E_6)_{\beta_{1278}}$$

for the corresponding isotropy group, which are both isomorphic to the Weyl group of type D_5.

The isotropy group $W(D_5)$ acts on $(\mathbb{C}^\times)^6/C_2$: the simple reflections corresponding to roots of the form $\alpha_{ij}^- \in \Delta(D_5)$ act by permuting the ith and jth coordinate, while w acts by

$$w(\pm t) = \pm\big(st_1, st_2, st_3, st_4, t_5/s, t_6/s\big), \qquad s^2 = 1/t_1t_2t_3t_4.$$

With this action, the degenerations to E_t and V_t in Proposition 4.5.18 are $W(D_5)$-equivariant in an obvious sense.

We can now directly compute the $W(D_5)$-symmetries of the trigonometric integrals E_t and V_t, as well as $W'(D_5)$-symmetries of B_t, by taking limits of the corresponding symmetries for S_t and U_t using Proposition 4.5.18. This yields the following result.

Proposition 4.5.20. **a)** *The trigonometric hypergeometric integrals $E_t(t)$ and $V_t(t)$ $(t \in (\mathbb{C}^\times)^6/C_2)$ are invariant under permutations of $(t_2, \ldots, t_6)$. Furthermore,*

$$\begin{aligned} E_t(t) &= E_t(wt)\frac{(1/t_1t_2, 1/t_1t_3, 1/t_1t_4, t_1t_2t_3t_4t_5t_6; q)_\infty}{(t_2t_3, t_2t_4, t_3t_4, t_5t_6; q)_\infty}, \\ V_t(t) &= V_t(wt)\frac{(1/t_2t_3, 1/t_2t_4, 1/t_3t_4, 1/t_5t_6; q)_\infty}{(t_1t_2, t_1t_3, t_1t_4, 1/t_1t_2t_3t_4t_5t_6; q)_\infty} \end{aligned}$$

as meromorphic functions in $t \in (\mathbb{C}^\times)^6/C_2$.

b) *The trigonometric Barnes integral $B_t(t)$ $(t \in \mathcal{H}_1)$ is invariant under permutations of the pairs (t_1, t_8), (t_2, t_7) and of (t_3, t_4, t_5, t_6). Furthermore,*

$$B_t(t) = B_t(wt)\frac{\big(1/t_1t_3, 1/t_1t_4, 1/t_5t_8, 1/t_6t_8; q\big)_\infty}{\big(t_2t_4, t_2t_3, t_5t_7, t_6t_7; q\big)_\infty}$$

as meromorphic functions in $t \in \mathcal{H}_1$.

Remark 4.5.21. The w-symmetry of V_t, rewritten in series form using Lemma 4.5.17, gives the transformation formula [16, (2.10.1)] for very-well-poised ${}_8\phi_7$ basic hypergeometric series.

Similarly as in the hyperbolic theory, the w-symmetry of E_t generalizes to the following integral transformation formula for the trigonometric Euler integral E_t.

Proposition 4.5.22. *For $t \in (\mathbb{C}^\times)^6$ and $s \in \mathbb{C}^\times$ satisfying*

$$|t_2|, |t_3|, |t_4|, |s|, |t_5/s|, |t_6/s| < 1$$

we have

$$\begin{aligned} \int_{\mathbb{T}} E_t(t_1, t_2, t_3, t_4, sx, sx^{-1})\frac{\big(x^{\pm 2}; q\big)_\infty}{\big(t_5x^{\pm 1}/s, t_6x^{\pm 1}/s; q\big)_\infty}\frac{dx}{2\pi i x} \\ = \frac{2\big(t_5t_6; q\big)_\infty}{\big(q, s^2, t_5t_6/s^2; q\big)_\infty} E_t(t_1, \ldots, t_6). \end{aligned}$$

Proof. The proof is similar to the hyperbolic case (see Proposition 4.4.13), now using the double integral

$$\int_{\mathbb{T}^2} \frac{\big(z^{\pm 2}, z^{\pm 1}/t_1, x^{\pm 2}; q\big)_\infty}{\big(sz^{\pm 1}x^{\pm 1}; q\big)_\infty \prod_{j=2}^4 \big(t_j z^{\pm 1}; q\big)_\infty \prod_{k=5}^6 \big(t_k x^{\pm 1}/s; q\big)_\infty} \frac{dz}{2\pi i z} \frac{dx}{2\pi i x}.$$

□

Specializing $s^2 = 1/t_1t_2t_3t_4$ in Proposition 4.5.22 and using the trigonometric Nassrallah-Rahman integral (see Corollary 4.5.19), we re-obtain the w-symmetry of E_t (see Proposition 4.5.20).

The three trigonometric integrals E_t, B_t and V_t are interconnected as follows.

Proposition 4.5.23. *We have*

$$\begin{aligned} B_t(t) &= \frac{\big(1/t_1t_6, 1/t_8t_6; q\big)_\infty}{\big(t_2t_3, t_2t_4, t_2t_5, t_6t_7; q\big)_\infty} V_t\big(t_7/s, t_3s, t_4s, t_5s, t_1/s, t_8/s\big), \\ &= \frac{\big(1/t_1t_3, 1/t_1t_4, 1/t_1t_5, 1/t_6t_8; q\big)_\infty}{\big(t_2t_6, t_7t_6; q\big)_\infty} E_t\big(t_8/v, t_7/v, t_3v, t_4v, t_5v, t_2/v\big), \end{aligned}$$

as meromorphic functions in $t \in \mathcal{H}_1$*, where* $s^2 = t_1t_6t_7t_8 = 1/t_2t_3t_4t_5$ *and* $v^2 = t_2t_6t_7t_8 = 1/t_1t_3t_4t_5$*.*

Proof. This follows by combining Proposition 4.5.12 and Proposition 4.5.18. Concretely, to relate B_t and V_t one computes for generic $t \in \mathcal{H}_1$ and with $s^2 = 1/t_2t_3t_4t_5$,

$$\begin{aligned} B_t(t) &= \lim_{u\to 0} \big(t_2t_7/u; q\big)_\infty S_t\big(t_1u^{-\frac12}, t_2u^{-\frac12}, t_3u^{\frac12}, t_4u^{\frac12}, t_5u^{\frac12}, t_6u^{\frac12}, t_7u^{-\frac12}, t_8u^{-\frac12}\big) \\ &= \frac{\big(1/t_1t_6, 1/t_8t_6; q\big)_\infty}{\big(t_2t_3, t_2t_4, t_2t_5, t_6t_7; q\big)_\infty} \\ &\qquad \times \lim_{u\to 0} \big(t_2t_7/u; q\big)_\infty U_t\big(t_7/s, t_3s, t_4s, t_5s, t_1/s, t_8/s, t_2s/u, t_6u/s\big) \\ &= \frac{\big(1/t_1t_6, 1/t_8t_6; q\big)_\infty}{\big(t_2t_3, t_2t_4, t_2t_5, t_6t_7; q\big)_\infty} V_t\big(t_7/s, t_3s, t_4s, t_5s, t_1/s, t_8/s\big), \end{aligned}$$

where the first and third equality follows from Proposition 4.5.18 and the second equality follows from Proposition 4.5.12. To relate B_t and E_t, we first note that Proposition 4.5.12 is equivalent to the identity

$$\begin{aligned} U_t(t) = &\frac{\big(1/t_6t_7; q\big)_\infty \prod_{2\le j<k\le 5} \big(1/t_jt_k; q\big)_\infty}{\big(t_1t_8/q, t_1t_6, t_8t_6, t_1t_7, t_8t_7; q\big)_\infty} \\ &\qquad \times S_t\big(t_6/s, t_2s, t_3s, t_4s, t_5s, t_1/s, t_8/s, t_7/s\big), \end{aligned} \tag{4.5.18}$$

where $t \in \mathcal{H}_1$ and $s^2 = 1/t_2t_3t_4t_5$. For generic $t \in \mathcal{H}_1$ and with $v^2 = 1/t_1t_3t_4t_5$ we then compute

$$\begin{aligned} B_t(t) &= \lim_{u\to 0} U_t\big(t_2u^{\frac12}, t_1u^{\frac12}, t_3u^{-\frac12}, t_4u^{-\frac12}, t_5u^{-\frac12}, t_6u^{-\frac12}, t_8u^{\frac12}, t_7u^{\frac12}\big) \\ &= \frac{\big(1/t_1t_3, 1/t_1t_4, 1/t_1t_5, 1/t_6t_8; q\big)_\infty}{\big(t_2t_6, t_7t_6; q\big)_\infty} \\ &\qquad \times \lim_{u\to 0} S_t\big(t_8/v, t_7/v, t_3v, t_4v, t_5v, t_2/v, t_1vu, t_6/vu\big) \\ &= \frac{\big(1/t_1t_3, 1/t_1t_4, 1/t_1t_5, 1/t_6t_8; q\big)_\infty}{\big(t_2t_6, t_7t_6; q\big)_\infty} E_t\big(t_8/v, t_7/v, t_3v, t_4v, t_5v, t_2/v\big), \end{aligned}$$

where the first and third equality follows from Proposition 4.5.18 and the second equality follows from (4.5.18). □

Remark 4.5.24. **a)** Combining the interconnection between E_t and V_t from Proposition 4.5.23 with the expression of V_t as very-well-poised ${}_8\phi_7$ series from Lemma 4.5.17 yields the Nassrallah-Rahman integral representation [16, (6.3.7)].

b) Similarly, combining the interconnection between B_t and V_t from Proposition 4.5.23 with their series expressions from Lemma 4.5.17 yields the expression [16, (2.10.10)] of a very-well-poised ${}_8\phi_7$ series as a sum of two balanced ${}_4\phi_3$ series.

Degenerating the contiguous relations of S_t using Proposition 4.5.18 leads directly to contiguous relations for E_t, B_t and V_t. For instance, we obtain

Proposition 4.5.25. *We have*

$$A(t)E_t(t_1, t_2, t_3, qt_4, t_5/q, t_6) + (t_4 \leftrightarrow t_5) = B(t)E_t(t)$$

as meromorphic functions in $t \in \big(\mathbb{C}^\times\big)^6/C_2$, *where*

$$\begin{aligned} A(t) &= -\frac{(1-\frac{1}{t_1t_5})(1-\frac{t_2t_5}{q})(1-\frac{t_3t_5}{q})(1-\frac{t_6t_5}{q})}{t_4(1-\frac{t_4t_5}{q})(1-\frac{t_5}{qt_4})(1-\frac{t_4}{t_5})}, \\ B(t) &= \frac{q}{t_1t_4t_5} - t_2t_3t_6 + \frac{qt_4}{t_5}A(t) + \frac{qt_5}{t_4}A(s_{45}t). \end{aligned}$$

Proof. Substitute $t = (t_1, t_2, t_3, t_4, t_5, t_7u, t_6, t_8/u)$ with $\prod_{j=1}^8 t_j = 1$ in (4.5.13) and take the limit $u \to 0$. □

For later purposes, we also formulate the corresponding result for $B_t(t)$. We substitute $t = \big(t_1u^{-\frac12}, t_7u^{-\frac12}, t_3u^{\frac12}, t_4u^{\frac12}, t_5u^{\frac12}, t_2u^{-\frac12}, t_6u^{\frac12}, t_8u^{-\frac12}\big)$ for generic $t \in \mathbb{C}^8$ satisfying $\prod_{j=1}^8 t_j = 1$ in (4.5.13), multiply the resulting equation by the factor $u^{\frac12}\big(t_2t_7/u; q\big)_\infty$, and take the limit $u \to 0$. We arrive at

$$\alpha(t)B_t(t_1, t_2, t_3, qt_4, t_5/q, t_6, t_7, t_8) + (t_4 \leftrightarrow t_5) = \beta(t)B_t(t), \qquad t \in \mathcal{H}_1, \quad (4.5.19)$$

where

$$\alpha(t) = -\frac{\left(1-\frac{1}{t_1t_5}\right)\left(1-\frac{t_2t_5}{q}\right)\left(1-\frac{t_5t_7}{q}\right)\left(1-\frac{1}{t_5t_8}\right)}{t_4\left(1-\frac{t_4}{t_5}\right)\left(1-\frac{t_5}{qt_4}\right)},$$

$$\beta(t) = t_7\left(1-t_2t_3\right)\left(1-t_2t_6\right) + \frac{\left(1-t_2t_4\right)}{\left(1-\frac{t_2t_5}{q}\right)}\alpha(t) + \frac{\left(1-\frac{t_2t_4}{q}\right)}{\left(1-t_2t_5\right)}\alpha(s_{45}t).$$

The degeneration of (4.5.14) yields Bailey's [16, (2.11.1)] three term transformation formula for very-well-poised ${}_8\phi_7$'s:

Proposition 4.5.26. *We have*

$$\frac{\theta(t_1^{-1}t_2^{\pm 1};q)}{\theta(t_3t_2^{\pm 1};q)}E_t(t_3/q, qt_1, t_2, t_4, t_5, t_6) + (t_2 \leftrightarrow t_3) = E_t(t).$$

Proof. Consider (4.5.14) with t_1 and t_8, t_2 and t_7, and t_3 and t_6 interchanged. Subsequently substitute the parameters $(t_1, t_2, t_3, t_4, t_5, t_6, t_7u, t_8/u)$ with $\prod_{j=1}^{8} t_j = 1$ and take the limit $u \to 0$. $\square$

4.5.5 The Askey-Wilson function

In this subsection we relate the trigonometric hypergeometric integrals with D_5 symmetry to the nonpolynomial eigenfunction of the Askey-Wilson second order difference operator, known as the Askey-Wilson function. The Askey-Wilson function is the trigonometric analog of Ruijsenaars' R-function, and is closely related to harmonic analysis on the quantum $\mathrm{SU}(1,1)$ group.

As for the R-function, we introduce the Askey-Wilson function in terms of the trigonometric Barnes integral B_t. Besides the usual Askey-Wilson parameters we also use logarithmic variables in order to make the connection to the R-function more transparent. We write the base $q \in \mathbb{C}^\times$ with $|q| < 1$ as $q = e(\omega_1/\omega_2)$ with $\tau = \omega_1/\omega_2 \in \mathbb{H}_+$ and $e(x) = \exp(2\pi i x)$ as before. From the previous subsection it follows that the parameter space of $B_t(t)$ is $\mathcal{H}_1/\mathbb{C}^\times e(\beta_{1278})$. In logarithmic coordinates, this relates to $\mathcal{G}_0/\mathbb{C}\beta_{1278}$. We identify $\mathcal{G}_0/\mathbb{C}\beta_{1278}$ with $\mathbb{C}^6$ by assigning to the six-tuple $(\gamma, x, \lambda) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \lambda, x)$ the class in $\mathcal{G}_0/\mathbb{C}\beta_{1278}$ represented by $u = (u_1, \ldots, u_8) \in \mathcal{G}_0$ with

$$\begin{aligned} u_1 &= \left(-\gamma_0 - \gamma_1 - 2\omega\right)/\omega_2, & u_2 &= 0,\\ u_3 &= \left(\hat{\gamma}_0 + \omega - i\lambda\right)/\omega_2, & u_4 &= \left(\gamma_0 + \omega - ix\right)/\omega_2,\\ u_5 &= \left(\gamma_0 + \omega + ix\right)/\omega_2, & u_6 &= \left(\hat{\gamma}_0 + \omega + i\lambda\right)/\omega_2,\\ u_7 &= \left(-\gamma_0 - \gamma_3\right)/\omega_2, & u_8 &= \left(-\gamma_0 - \gamma_2 - 2\omega\right)/\omega_2, \end{aligned} \tag{4.5.20}$$

where $\omega = \frac{1}{2}(\omega_1 + \omega_2)$ as before. We define the corresponding six-tuple of Askey-Wilson parameters (a, b, c, d, μ, z) by

$$\begin{aligned} (a,b,c,d) &= \left(e((\gamma_0+\omega)/\omega_2), e((\gamma_1+\omega)/\omega_2), e((\gamma_2+\omega)/\omega_2), e(\gamma_3+\omega)/\omega_2)\right),\\ (\mu, z) &= \left(e(-i\lambda/\omega_2), e(-ix/\omega_2)\right). \end{aligned} \tag{4.5.21}$$

The four-tuple (a,b,c,d) represents the four parameter freedom in the Askey-Wilson theory, while z (respectively μ) plays the role of geometric (respectively spectral) parameter. Furthermore, we define the dual Askey-Wilson parameters by

$$\big(\widetilde{a},\widetilde{b},\widetilde{c},\widetilde{d}\big)=\big(e((\hat{\gamma}_0+\omega)/\omega_2),e((\hat{\gamma}_1+\omega)/\omega_2),e((\hat{\gamma}_2+\omega)/\omega_2),e((\hat{\gamma}_3+\omega)/\omega_2)\big),$$

with $\hat{\gamma}$ the dual parameters defined by (4.4.30). We furthermore associate to the logarithmic parameters $u\in\mathcal{G}_0$ (see (4.5.20)) the parameters $t=\psi_0(2\pi iu)\in\mathcal{H}_1$, so that

$$t=\psi_0(2\pi iu)=\big(1/ab,1,\widetilde{a}\mu,az,a/z,\widetilde{a}/\mu,q/ad,1/ac\big)\tag{4.5.22}$$

We define the Askey-Wilson function $\phi(\gamma;x,\lambda)=\phi\big(\gamma;x,\lambda;\omega_1,\omega_2\big)$ by

$$\phi(\gamma;x,\lambda)=\frac{\big(q,t_2t_3,t_2t_4,t_2t_5,t_2t_6;q\big)_\infty}{2}B_t(t)\tag{4.5.23}$$

with $t=\psi_0(2\pi iu)\in\mathcal{H}_1$ and u given by (4.5.20). Note the similarity to the definition of Ruijsenaars' R-function, see (4.4.28).

From the series expansion of B_t as sum of two balanced ${}_4\phi_3$, we have in terms of Askey-Wilson parameters (4.5.21),

$$\begin{aligned}\phi(\gamma;x,\lambda)=&\frac{\big(ab,ac;q\big)_\infty}{\big(q/ad;q\big)_\infty}\,{}_4\phi_3\left(\begin{matrix}az^{\pm1},\widetilde{a}\mu^{\pm1}\\ ab,ac,ad\end{matrix};q,q\right)\\&+\frac{\big(qb/d,qc/d,\widetilde{a}\mu^{\pm1},az^{\pm1};q\big)_\infty}{\big(ad/q,q\widetilde{a}\mu^{\pm1}/ad,qz^{\pm1}/d;q\big)_\infty}\,{}_4\phi_3\left(\begin{matrix}qz^{\pm1}/d,q\widetilde{a}\mu^{\pm1}/ad\\ q^2/ad,qb/d,qc/d\end{matrix};q,q\right),\end{aligned}\tag{4.5.24}$$

which shows that $\phi(\gamma;x,\lambda)$ is, up to a (z,γ)-independent rescaling factor, the Askey-Wilson function as defined in e.g. [37].

We now re-derive several fundamental properties of the Askey-Wilson function using the results of the previous subsection. Comparing the symmetries of the Askey-Wilson function $\phi(\gamma;x,\lambda)$ to the symmetries of the R-function (Proposition 4.4.18), the symmetry in the parameters γ is broken (from the Weyl group of type D_4 to the Weyl group of type D_3). The most important symmetry (self-duality) is also valid for the Askey-Wilson function and has played a fundamental role in the study of the associated generalized Fourier transform (see [37]). Self-duality of the Askey-Wilson function has a natural interpretation in terms of Cherednik's theory on double affine Hecke algebras, see [76]. Concretely, the symmetries of the Askey-Wilson function are as follows.

Proposition 4.5.27. *The Askey-Wilson function $\phi(\gamma;x,\lambda)$ is even in x and λ and is self-dual,*

$$\phi(\gamma;x,\lambda)=\phi(\gamma;-x,\lambda)=\phi(\gamma;x,-\lambda)=\phi(\hat{\gamma};\lambda,x).$$

Furthermore, $\phi(\gamma;x,\lambda)$ has a $W(D_3)$-symmetry in the parameters γ, given by

$$\begin{aligned}
\phi(\gamma_1,\gamma_0,\gamma_2,\gamma_3;x,\lambda) &= \frac{\big(e((-\hat{\gamma}_3+\omega\pm i\lambda)/\omega_2);q\big)_\infty}{\big(e((\hat{\gamma}_2+\omega\pm i\lambda)/\omega_2);q\big)_\infty}\,\phi(\gamma;x,\lambda),\\
\phi(\gamma_0,\gamma_2,\gamma_1,\gamma_3;x,\lambda) &= \phi(\gamma;x,\lambda),\\
\phi(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;x,\lambda) &= \frac{\big(e((-\gamma_3+\omega\pm ix)/\omega_2);q\big)_\infty}{\big(e((\gamma_2+\omega\pm ix)/\omega_2);q\big)_\infty}\,\phi(\gamma;x,\lambda).
\end{aligned}$$

Proof. Similarly as for the R-function (see Proposition 4.4.18), the symmetries of the Askey-Wilson function correspond to the $W(D_5)$-symmetries of B_t. Alternatively, all symmetries follow trivially from the series expansion (4.5.24) of the Askey-Wilson function, besides its symmetry with respect to $\gamma_0 \leftrightarrow \gamma_1$ and $\gamma_2 \leftrightarrow -\gamma_3$. These two symmetry relations are equivalent under duality, since

$$(\gamma_1,\gamma_0,\gamma_2,\gamma_3)\hat{} = (\hat{\gamma}_0,\hat{\gamma}_1,-\hat{\gamma}_3,-\hat{\gamma}_2),$$

so we only discuss the symmetry with respect to $\gamma_0 \leftrightarrow \gamma_1$. By Proposition 4.5.20 we have, with parameters t given by (4.5.22) and (4.5.20),

$$\begin{aligned}
\phi(\gamma_1,&\gamma_0,\gamma_2,\gamma_3;x,\lambda)\\
&= \frac{\big(q,e((\gamma_1+\omega\pm ix)/\omega_2),e((\hat{\gamma}_0+\omega\pm i\lambda)/\omega_2);q\big)_\infty}{2}B_t(ws_{35}t)\\
&= \frac{\big(e((-\hat{\gamma}_3+\omega\pm i\lambda)/\omega_2);q\big)_\infty}{\big(e((\hat{\gamma}_2+\omega\pm i\lambda)/\omega_2);q\big)_\infty}\,\phi(\gamma;x,\lambda),
\end{aligned}$$

as desired. □

Next we show that the Askey-Wilson function satisfies the same Askey-Wilson second order difference equation (with step-size $i\omega_1$) as Ruijsenaars' R-function, a result which has previously been derived from detailed studies of the associated Askey-Wilson polynomials in [25], cf. also [37].

Lemma 4.5.28. *The Askey-Wilson function $\phi(\gamma;x,\lambda)$ satisfies the second order difference equation*

$$\begin{aligned}
A(\gamma;x;\omega_1,\omega_2)(\phi(\gamma;x+i\omega_1,\lambda;\omega_1,\omega_2)-\phi(\gamma;x,\lambda;\omega_1,\omega_2))+(x\leftrightarrow -x)\\
= B(\gamma;\lambda;\omega_1,\omega_2)\phi(\gamma;x,\lambda;\omega_1,\omega_2),
\end{aligned}$$

where A and B are given by

$$\begin{aligned}
A(\gamma;x;\omega_1,\omega_2) &= \frac{\prod_{j=0}^{3}\sinh(\pi(i\omega+x+i\gamma_j)/\omega_2)}{\sinh(2\pi x/\omega_2)\sinh(2\pi(i\omega+x)/\omega_2)},\\
B(\gamma;\lambda;\omega_1,\omega_2) &= \sinh(\pi(\lambda-i\omega-i\hat{\gamma}_0)/\omega_2)\sinh(\pi(\lambda+i\omega+i\hat{\gamma}_0)/\omega_2).
\end{aligned}$$

Proof. Specialize the parameters according to (4.5.22) and (4.5.20) in (4.5.19). Subsequently express $B_t(t)$, $B_t(\tau_{45}t)$ and $B_t(\tau_{54}t)$ in terms of $\phi(\gamma;x,\lambda)$, $\phi(\gamma;x+i\omega_1,\lambda)$ and $\phi(\gamma;x-i\omega_1,\lambda)$ respectively. The resulting equation is the desired difference equation. □

Remark 4.5.29. Denoting $\Phi(z;\mu)=\Phi(a,b,c,d;z,\mu)$ for the Askey-Wilson function in the usual Askey-Wilson parameters, Lemma 4.5.28 becomes the Askey-Wilson second order difference equation

$$A(z)(\Phi(qz,\mu)-\Phi(z,\mu))+A(z^{-1})(\Phi(z/q,\mu)-\Phi(z,\mu)) \\ =\big(\widetilde{a}(\gamma+\gamma^{-1})-1-\widetilde{a}^2\big)\Phi(z,\mu),$$

where

$$A(z)=\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-qz^2)(1-z^2)}$$

and $\widetilde{a}=e((\hat{\gamma}_0+\omega)/\omega_2)$.

We have now seen that the R-function $R(\gamma;x,\lambda;\omega_1,\omega_2)$ as well as the Askey-Wilson function $\phi(\gamma;x,\lambda;\omega_1,\omega_2)$ are solutions to the eigenvalue problem

$$\mathcal{L}_\gamma^{\omega_1,\omega_2}f=B(\gamma;\lambda;\omega_1,\omega_2)f \tag{4.5.25}$$

for the Askey-Wilson second order difference operator $\mathcal{L}_\gamma^{\omega_1,\omega_2}$ (4.4.32) with step-size $i\omega_1$. These two solutions have essentially different behaviour in the $i\omega_2$-step direction: the Askey-Wilson function $\phi(\gamma;x,\lambda)$ is $i\omega_2$-periodic, while the R-function $R(\gamma;x,\lambda)$ is $\omega_1\leftrightarrow\omega_2$ invariant (hence is also an eigenfunction of the Askey-Wilson second order difference operator $\mathcal{L}_\gamma^{\omega_2,\omega_1}$ with step-size $i\omega_2$, with eigenvalue $B(\gamma;\lambda;\omega_2,\omega_1)$). On the other hand, note that $\widetilde{\tau}=-\omega_2/\omega_1\in\mathbb{H}_+$ and that

$$A(\gamma;x;\omega_2,\omega_1)=A(-\gamma;-x;-\omega_2,\omega_1),\qquad B(\gamma;\lambda;\omega_2,\omega_1)=B(-\gamma;\lambda;-\omega_2,\omega_1)$$

with $-\gamma=(-\gamma_0,-\gamma_1,-\gamma_2,-\gamma_3)$, so that the reparametrized Askey-Wilson function $\phi(-\gamma;x,\lambda;-\omega_2,\omega_1)$ (with associated modular inverted base $\widetilde{q}=e(-\omega_2/\omega_1)$) does satisfy the Askey-Wilson second order difference equation

$$\big(\mathcal{L}_\gamma^{\omega_2,\omega_1}\phi(-\gamma;\,\cdot\,,\lambda;-\omega_2,\omega_1)\big)(x)=B(\gamma;\lambda;\omega_2,\omega_1)\phi(-\gamma;x,\lambda;-\omega_2,\omega_1), \tag{4.5.26}$$

cf. [66, §6.6]. In the next section we match the hyperbolic theory to the trigonometric theory, which in particular entails an explicit expression of the R-function in terms of products of Askey-Wilson functions in base q and base $\widetilde{q}$.

Note furthermore that Proposition 4.5.27 hints at the fact that the solution space to the Askey-Wilson eigenvalue problem (4.5.25) admits a natural twisted $W(D_4)$-action on the parameters γ. In fact, the solution space to (4.5.25) is invariant under permutations of $(\gamma_0,\gamma_1,\gamma_2,\gamma_3)$. Furthermore, a straightforward computation shows that

$$g(\gamma;\,\cdot\,)^{-1}\circ\mathcal{L}_\gamma^{\omega_1,\omega_2}\circ g(\gamma;\,\cdot\,) \\ =\mathcal{L}^{\omega_1,\omega_2}_{(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2)}+B(\gamma;\lambda;\omega_1,\omega_2)-B(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;\lambda;\omega_1,\omega_2)$$

for the gauge factor

$$g(\gamma;x)=\frac{\big(e((\gamma_2+\omega\pm ix)/\omega_2);q\big)_\infty}{\big(e((-\gamma_3+\omega\pm ix)/\omega_2);q\big)_\infty},$$

which implies that for a given solution $F_\lambda(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;\,\cdot)$ to the eigenvalue problem

$$\mathcal{L}^{\omega_1,\omega_2}_{(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2)}f=B(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;\lambda;\omega_1,\omega_2)f,$$

we obtain a solution

$$\widetilde{F}_\lambda(\gamma;x):=g(\gamma;x)F_\lambda(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;x)$$

to the eigenvalue problem (4.5.25). A similar observation forms the starting point of Ruijsenaars' [67] analysis of the $W(D_4)$-symmetries of the R-function (see also Section 4.4.6).

Remark 4.5.30. A convenient way to formalize the $W(D_4)$-symmetries of the eigenvalue problem (4.5.25) (in the present trigonometric setting) is by interpreting $i\omega_2$-periodic solutions to (4.5.25), depending meromorphically on (γ,x,λ), as defining a sub-vectorbundle $\widetilde{\Gamma}^0(\omega_1,\omega_2)$ of the meromorphic vectorbundle $\Gamma^0(\omega_1,\omega_2)$ over

$$X=\big(\mathbb{C}/\mathbb{Z}\omega_2\big)^4\times\mathbb{C}/(\mathbb{Z}i\omega_1+\mathbb{Z}i\omega_2)\times\mathbb{C}/\mathbb{Z}i\omega_2$$

consisting of meromorphic functions in $(\gamma,x,\lambda)\in\big(\mathbb{C}/\mathbb{Z}\omega_2\big)^4\times\mathbb{C}/\mathbb{Z}i\omega_2\times\mathbb{C}/\mathbb{Z}i\omega_2$. The above analysis can now equivalently be reformulated as the following property of $\widetilde{\Gamma}^0(\omega_1,\omega_2)$: the sub-vectorbundle $\widetilde{\Gamma}^0(\omega_1,\omega_2)$ is $W(D_4)$-invariant with respect to the twisted $W(D_4)$-action

$$(\sigma\cdot f)(\gamma;x,\lambda):=V_\sigma(\gamma;x,\lambda)^{-1}f(\sigma^{-1}\gamma;x,\lambda),\qquad \sigma\in W(D_4)\tag{4.5.27}$$

on $\Gamma^0(\omega_1,\omega_2)$, where $V_\sigma(\gamma;x,\lambda)=h(\sigma^{-1}\gamma;x,\lambda)/h(\gamma;x,\lambda)$ $(\sigma\in W(D_4))$ is the 1-coboundary with $h(\gamma;x,\lambda)=h(\gamma;x,\lambda;\omega_1,\omega_2)$ e.g. given by

$$h(\gamma;x,\lambda;\omega_1,\omega_2)=\frac{\theta\big(e((\gamma_3-\hat{\gamma}_0+ix)/\omega_2);q\big)}{\theta\big(e((\omega-\gamma_3-ix)/\omega_2);q\big)\prod_{j=0}^{3}\big(e((\omega-\gamma_j+ix)/\omega_2);q\big)_\infty}\in\Gamma_0(\omega_1,\omega_2)^\times,\tag{4.5.28}$$

and where $W(D_4)$ acts on the γ parameters by permutations and even sign changes. By a straightforward analysis using Casorati-determinants and the asymptotically free solutions to the eigenvalue problem (4.5.25), one can furthermore show that $\widetilde{\Gamma}^0(\omega_1,\omega_2)$ is a (trivial) meromorphic vectorbundle over X of rank two (compare with the general theory on difference equations in [57]).

We end this subsection by expressing the Askey-Wilson function $\phi(\gamma;x,\lambda)$ in terms of the trigonometric integrals E_t and V_t using Proposition 4.5.23. Note its close resemblance with the hyperbolic case, cf. Theorem 4.4.21.

Lemma 4.5.31. **a)** *We have*

$$\phi(\gamma;x,\lambda) = \frac{\big(q;q\big)_\infty}{2} \frac{\big(e((\hat{\gamma}_0+\omega-i\lambda)/\omega_2), e((\hat{\gamma}_1+\omega+i\lambda)/\omega_2);q\big)_\infty}{\big(e((-\hat{\gamma}_3+\omega-i\lambda)/\omega_2);q\big)_\infty}$$
$$\times \big(e((\hat{\gamma}_2+\omega+i\lambda)/\omega_2);q\big)_\infty \frac{\prod_{j=0}^{2}\big(e((\gamma_j+\omega\pm ix)/\omega_2);q\big)_\infty}{\big(e((-\gamma_3+\omega\pm ix)/\omega_2);q\big)_\infty} E_t(t)$$

with

$$t_1 = e((-\frac{3\omega}{2}+\gamma_3-\frac{\hat{\gamma}_0}{2}+\frac{i\lambda}{2})/\omega_2), \qquad t_2 = e((\frac{\omega}{2}+\gamma_2-\frac{\hat{\gamma}_0}{2}+\frac{i\lambda}{2})/\omega_2),$$
$$t_3 = e((\frac{\omega}{2}+\gamma_1-\frac{\hat{\gamma}_0}{2}+\frac{i\lambda}{2})/\omega_2), \qquad t_4 = e((\frac{\omega}{2}+\gamma_0-\frac{\hat{\gamma}_0}{2}+\frac{i\lambda}{2})/\omega_2),$$
$$t_5 = e((\frac{\omega}{2}+\frac{\hat{\gamma}_0}{2}+ix-\frac{i\lambda}{2})/\omega_2), \qquad t_6 = e((\frac{\omega}{2}+\frac{\hat{\gamma}_0}{2}-ix-\frac{i\lambda}{2})/\omega_2).$$

b) *We have*

$$\phi(\gamma;x,\lambda) = \frac{\big(q;q\big)_\infty}{2} \frac{\big(e((\hat{\gamma}_0+\omega+i\lambda)/\omega_2), e((\hat{\gamma}_1+\omega-i\lambda)/\omega_2);q\big)_\infty}{\big(e((-\hat{\gamma}_3+\omega+i\lambda)/\omega_2);q\big)_\infty}$$
$$\times \big(e((\hat{\gamma}_2+\omega-i\lambda)/\omega_2);q\big)_\infty V_t(t)$$

with

$$t_1 = e((\frac{3\omega}{2}-\gamma_3+\frac{\hat{\gamma}_0}{2}-\frac{i\lambda}{2})/\omega_2), \qquad t_2 = e((-\frac{\omega}{2}-\gamma_2+\frac{\hat{\gamma}_0}{2}-\frac{i\lambda}{2})/\omega_2),$$
$$t_3 = e((-\frac{\omega}{2}-\gamma_1+\frac{\hat{\gamma}_0}{2}-\frac{i\lambda}{2})/\omega_2), \qquad t_4 = e((-\frac{\omega}{2}-\gamma_0+\frac{\hat{\gamma}_0}{2}-\frac{i\lambda}{2})/\omega_2),$$
$$t_5 = e((-\frac{\omega}{2}-\frac{\hat{\gamma}_0}{2}+ix+\frac{i\lambda}{2})/\omega_2), \qquad t_6 = e((-\frac{\omega}{2}-\frac{\hat{\gamma}_0}{2}-ix+\frac{i\lambda}{2})/\omega_2).$$

Proof. **a)** We rewrite $\phi(\gamma;x,\lambda)$ in terms of $\phi(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;x,-\lambda)$ using Proposition 4.5.27. The defining expression (4.5.23) of $\phi(\gamma_0,\gamma_1,-\gamma_3,-\gamma_2;x,-\lambda)$ now gives

$$\phi(\gamma;x,\lambda) = \frac{\big(q, e((\hat{\gamma}_1+\omega\pm i\lambda)/\omega_2), e((\gamma_0+\omega\pm ix)/\omega_2);q\big)_\infty}{2\big(e((-\gamma_3+\omega\pm ix)/\omega_2);q\big)_\infty}$$
$$\times \big(e((\gamma_2+\omega\pm ix)/\omega_2);q\big)_\infty B_t(\xi)$$

with

$$\xi = \big(e((-\gamma_0-\gamma_1-2\omega)/\omega_2), 1, e((\hat{\gamma}_1+\omega+i\lambda)/\omega_2), e((\gamma_0+\omega-ix)/\omega_2),$$
$$e((\gamma_0+\omega+ix)/\omega_2), e((\hat{\gamma}_1+\omega-i\lambda)/\omega_2), e((\gamma_2-\gamma_0)/\omega_2), e((\gamma_3-\gamma_0-2\omega)/\omega_2)\big).$$

With this specific ordered set ξ of parameters we apply Proposition 4.5.23 to rewrite $B_t(\xi)$ in terms of E_t, which results in the desired identity.
b) This follows from applying Proposition 4.5.23 directly to the definition (4.5.23) of $\phi(\gamma;x,\lambda)$. □

Using the expression of the Askey-Wilson function in terms of V_t and using Lemma 4.5.17, we thus obtain an expression of the Askey-Wilson function as very-well-poised ${}_8\phi_7$ series.

4.6 Hyperbolic versus trigonometric theory

4.6.1 Hyperbolic versus trigonometric gamma functions

We fix throughout this section periods $\omega_1, \omega_2 \in \mathbb{C}$ with $\Re(\omega_1) > 0$, $\Re(\omega_2) > 0$ and $\tau = \omega_1/\omega_2 \in \mathbb{H}_+$. We set

$$q = q_{\omega_1,\omega_2} = e(\omega_1/\omega_2), \qquad \widetilde{q} = \widetilde{q}_{\omega_1,\omega_2} = e(-\omega_2/\omega_1)$$

where $e(x) = \exp(2\pi i x)$ as before, so that $|q|, |\widetilde{q}| < 1$.

Shintani's [70] product expansion is

$$G(\omega_1, \omega_2; x) = e\left(-\frac{1}{48}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)\right) e\left(-\frac{x^2}{4\omega_1\omega_2}\right) \frac{\big(e((ix+\omega)/\omega_2); q\big)_\infty}{\big(e((ix-\omega)/\omega_1); \widetilde{q}\big)_\infty}, \quad (4.6.1)$$

where $\omega = \frac{1}{2}(\omega_1+\omega_2)$ as before. For a proof of (4.6.1), see [77, Prop. A.1]. In other words, the product expansion (4.6.1) expresses the hyperbolic gamma function as a quotient of two trigonometric gamma functions (one in base q, the other in the modular inverted base $\widetilde{q}$). In this section we explicitly write the base-dependence; e.g. we write $S_t(t; q)$ ($t \in \mathcal{H}_1$) to denote the trigonometric hypergeometric function $S_t(t)$ in base q.

4.6.2 Hyperbolic versus trigonometric hypergeometric integrals

We explore (4.6.1) to relate the hyperbolic integrals to their trigonometric analogs. We start with the hyperbolic hypergeometric function $S_h(u)$ ($u \in \mathcal{G}_{2i\omega}$). For $u \in \mathcal{G}_{2i\omega}$ we write

$$t_j = e\big((iu_j + \omega)/\omega_2\big), \qquad \widetilde{t}_j = e\big((iu_j - \omega)/\omega_1\big), \qquad j = 1, \ldots, 8. \qquad (4.6.2)$$

Observe that $\prod_{j=1}^8 t_j = q^2$ and $\prod_{j=1}^8 \widetilde{t}_j = \widetilde{q}^6$.

Theorem 4.6.1. *As meromorphic functions of $u \in \mathcal{G}_{2i\omega}$ we have*

$$\begin{aligned} S_h(u) = \omega_2 e\Big(\big(2\omega^2 + \sum_{j=1}^5 u_j^2 - u_6^2 + u_7^2 - u_8^2\big)/2\omega_1\omega_2\Big) \frac{(\widetilde{q}, \widetilde{q}; \widetilde{q})_\infty}{2} \frac{\prod_{j=1}^5 \theta\big(\widetilde{t}_j \widetilde{t}_7/\widetilde{q}; \widetilde{q}\big)}{\theta\big(\widetilde{t}_7/\widetilde{t}_6; \widetilde{q}\big)} \\ \times U_t\big(\widetilde{q}^{\frac{3}{2}}/\widetilde{t}_8, \widetilde{q}^{\frac{1}{2}}/\widetilde{t}_1, \ldots, \widetilde{q}^{\frac{1}{2}}/\widetilde{t}_6, \widetilde{q}^{\frac{3}{2}}/\widetilde{t}_7; \widetilde{q}\big) S_t\big(t_6/q, t_1, \ldots, t_5, t_7, t_8/q; q\big) \\ + \ (u_6 \leftrightarrow u_7), \end{aligned} \qquad (4.6.3)$$

with the parameters t_j and $\widetilde{t}_j$ given by (4.6.2).

Proof. We put several additional conditions on the parameters, which can later be removed by analytic continuity. We assume that $\omega_1, -\omega_2 \in \mathbb{H}_+$ and that $\Re(i\omega) < 0$. We furthermore choose parameters $u \in \mathcal{G}_{2i\omega}$ satisfying $\Re(u_j - i\omega) > 0$ and $\Im(u_j - i\omega) < 0$ for $j = 1, \ldots, 8$. Then

$$\begin{aligned} S_h(u) &= \int_{\mathbb{R}} \frac{G(i\omega \pm 2x)}{\prod_{j=1}^8 G(u_j \pm x)} dx \\ &= e\left(\frac{7}{24}(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1})\right) e\big((\omega^2 + \sum_{j=1}^8 u_j^2)/2\omega_1\omega_2\big) \int_{\mathbb{R}} W(x)\widetilde{W}(x)dx, \end{aligned}$$

where

$$\begin{aligned} W(x) &= \frac{\big(e(\pm 2ix/\omega_2); q\big)_\infty}{\prod_{j=1}^8 \big(t_j e(\pm ix/\omega_2); q\big)_\infty}, \\ \widetilde{W}(x) &= e\big(2x^2/\omega_1\omega_2\big) \frac{\prod_{j=1}^8 \big(\widetilde{t}_j e(\pm ix/\omega_1); \widetilde{q}\big)_\infty}{\big(\widetilde{q} e(\pm 2ix/\omega_1); \widetilde{q}\big)_\infty} \end{aligned}$$

by (4.6.1). Using Cauchy's Theorem and elementary asymptotic estimates of the integrand, we may rotate the integration contour $\mathbb{R}$ to $i\omega_2\mathbb{R}$. Since the factor $W(x)$ is $i\omega_2$-periodic, we can fold the resulting integral, interchange summation and integration by Fubini's Theorem, to obtain the expression

$$S_h(u) = e\left(\frac{7}{24}(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1})\right) e\big((\omega^2 + \sum_{j=1}^8 u_j^2)/2\omega_1\omega_2\big) \int_0^{i\omega_2} W(x)F(x)dx,$$

where

$$\begin{aligned} F(x) &= \sum_{n=-\infty}^{\infty} \widetilde{W}(x + mi\omega_2) \\ &= \widetilde{W}(x)\, {}_{10}\psi_{10}\left(\begin{matrix} \widetilde{q}e(ix/\omega_1), -\widetilde{q}e(ix/\omega_1), \{\widetilde{q}\widetilde{t}_j^{-1} e(ix/\omega_1)\}_{j=1}^8 \\ e(ix/\omega_1), -e(ix/\omega_1), \{\widetilde{t}_j e(ix/\omega_1)\}_{j=1}^8 \end{matrix}; \widetilde{q}, \widetilde{q}\right). \end{aligned}$$

At this stage we have to resort to [16, (5.6.3)], which expresses a very-well-poised ${}_{10}\psi_{10}$ bilateral series as a sum of three very-well-poised ${}_{10}\phi_9$ unilateral series. This results in the formula

$$\begin{aligned} F(x) &= e\big(2x^2/\omega_1\omega_2\big)\theta\big(\widetilde{t}_6 e(\pm ix/\omega_1), \widetilde{t}_7 e(\pm ix/\omega_1); \widetilde{q}\big) \\ &\quad \times \frac{\big(\widetilde{q};\widetilde{q}\big)_\infty \prod_{j=1}^5 \big(\widetilde{t}_j\widetilde{t}_8/\widetilde{q}, \widetilde{q}\widetilde{t}_j/\widetilde{t}_8; \widetilde{q}\big)_\infty}{\big(\widetilde{q}^2/\widetilde{t}_6\widetilde{t}_8, \widetilde{q}^2/\widetilde{t}_7\widetilde{t}_8, \widetilde{t}_8/\widetilde{t}_6, \widetilde{t}_8/\widetilde{t}_7, \widetilde{q}^3/\widetilde{t}_8^2; \widetilde{q}\big)_\infty}\, {}_{10}W_9\big(\widetilde{q}^2/\widetilde{t}_8^2; \{\widetilde{q}^2/\widetilde{t}_j\widetilde{t}_8\}_{j=1}^7; \widetilde{q}, \widetilde{q}\big) \\ &\quad + (u_8; u_6, u_7), \end{aligned}$$

where $(u_8; u_6, u_7)$ means cyclic permutation of the parameters (u_8, u_6, u_7). Note that the ${}_{10}\phi_9$ series in the expression of $F(x)$ are independent of x. Combining

Jacobi's inversion formula, the Jacobi triple product identity and the modularity

$$\frac{(q;q)_\infty}{(\widetilde{q};\widetilde{q})_\infty}=\sqrt{\frac{\omega_2}{-i\omega_1}}\,e\left(-\frac{1}{24}\big(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\big)\right) \tag{4.6.4}$$

of Dedekind's eta function, we obtain

$$\theta\big(e(u/\omega_1);\widetilde{q}\big)=e\left(-\frac{1}{24}\big(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\big)\right)e\big((u+\omega)^2/2\omega_1\omega_2\big)\theta\big(e(-u/\omega_2);q\big) \tag{4.6.5}$$

for the rescaled Jacobi theta function $\theta(\cdot)$, see e.g. [16] or [77]. As a result, we can rewrite the theta functions in the expression of $F(x)$ as theta functions in base q,

$$\begin{aligned}&e\big(2x^2/\omega_1\omega_2\big)\theta\big(\widetilde{t}_6e(\pm ix/\omega_1),\widetilde{t}_7e(\pm ix/\omega_1);\widetilde{q}\big)\\&=e\left(-\frac{1}{6}\big(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\big)\right)e\big(-(u_6^2+u_7^2)/\omega_1\omega_2\big)\theta\big(qe(\pm ix/\omega_2)/t_6,qe(\pm ix/\omega_2)/t_7;q\big).\end{aligned}$$

We thus obtain the expression

$$\begin{aligned}S_h(u)&=C(u_8;u_6,u_7)\int_0^{i\omega_2}\theta\big(qe(\pm ix/\omega_2)/t_6,qe(\pm ix/\omega_2)/t_7;q\big)W(x)dx\\&\qquad+(u_8;u_6,u_7)\\&=i\omega_2C(u_8;u_6,u_7)S_t\big(t_6/q,t_1,\ldots,t_5,t_8,t_7/q;q\big)+(u_8;u_6,u_7),\end{aligned} \tag{4.6.6}$$

where we have used that $|t_j|<1$ for $j=1,\ldots,6$, with

$$\begin{aligned}C(u_8;u_6,u_7)&=e\left(\frac{1}{8}\big(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\big)\right)e\left((\omega^2+\sum_{j=1}^5u_j^2-u_6^2-u_7^2+u_8^2)/2\omega_1\omega_2\right)\\&\times\frac{(\widetilde{q};\widetilde{q})_\infty\prod_{j=1}^5\big(\widetilde{t}_j\widetilde{t}_8/\widetilde{q},\widetilde{q}\widetilde{t}_j/\widetilde{t}_8;\widetilde{q}\big)_\infty}{\big(\widetilde{q}^2/\widetilde{t}_6\widetilde{t}_8,\widetilde{q}^2/\widetilde{t}_7\widetilde{t}_8,\widetilde{t}_8/\widetilde{t}_6,\widetilde{t}_8/\widetilde{t}_7,\widetilde{q}^3/\widetilde{t}_8^2;\widetilde{q}\big)_\infty}\\&\times{}_{10}W_9\big(\widetilde{q}^2/\widetilde{t}_8^2;\{\widetilde{q}^2/\widetilde{t}_j\widetilde{t}_8\}_{j=1}^7;\widetilde{q},\widetilde{q}\big).\end{aligned}$$

We thus have obtained an expression of $S_h(u)$ as a sum of three trigonometric hypergeometric functions S_t in base q, with coefficients expressed as very-well-poised ${}_{10}\phi_9$ series in base $\widetilde{q}$. The next step is to use a three term transformation for S_t to write $S_h(u)$ as a sum of two trigonometric hypergeometric functions S_t in base q, with coefficients now being a sum of two very-well-poised ${}_{10}\phi_9$ series.

Concretely, we consider the contiguous relation (4.3.8) for S_e with $p\leftrightarrow q$ and with parameters specialized to $(t_1,\ldots,t_5,pt_6,pt_7,t_8)$. Taking the limit $p\to0$ leads to the three term transformation

$$\begin{aligned}&S_t\big(t_6/q,t_1,\ldots,t_5,t_8,t_7/q;q\big)\\&\quad=\frac{\theta\big(t_7/t_8;q\big)}{\theta\big(t_7/t_6;q\big)}\prod_{j=1}^5\frac{\theta\big(t_jt_6;q\big)}{\theta\big(t_jt_8;q\big)}S_t\big(t_7/q,t_1,\ldots,t_6,t_8/q;q\big)+(u_6\leftrightarrow u_7)\end{aligned} \tag{4.6.7}$$

for S_t. Rewriting the coefficients in (4.6.7) in base $\widetilde{q}$ using the Jacobi inversion formula (4.6.5),

$$\frac{\theta\big(t_7/t_8;q\big)}{\theta\big(t_7/t_6;q\big)}\prod_{j=1}^{5}\frac{\theta\big(t_jt_6;q\big)}{\theta\big(t_jt_8;q\big)}=e\big((u_6^2-u_8^2)/\omega_1\omega_2\big)\frac{\theta\big(\widetilde{t}_8/\widetilde{t}_7;\widetilde{q}\big)}{\theta\big(\widetilde{t}_6/\widetilde{t}_7;\widetilde{q}\big)}\prod_{j=1}^{5}\frac{\theta\big(\widetilde{q}^2/\widetilde{t}_j\widetilde{t}_6;\widetilde{q}\big)}{\theta\big(\widetilde{q}^2/\widetilde{t}_j\widetilde{t}_8;\widetilde{q}\big)}$$

and using the resulting three term transformation in (4.6.6), we obtain

$$S_h(u)=D(u_6,u_7)S_t(t_7/q,t_1,\ldots,t_6,t_8/q;q)+(u_6\leftrightarrow u_7)$$

with

$$D(u_6,u_7)=i\omega_2\Big(e\big((u_6^2-u_8^2)/\omega_1\omega_2\big)\frac{\theta\big(\widetilde{t}_8/\widetilde{t}_7;\widetilde{q}\big)}{\theta\big(\widetilde{t}_6/\widetilde{t}_7;\widetilde{q}\big)}\prod_{j=1}^{5}\frac{\theta\big(\widetilde{q}^2/\widetilde{t}_j\widetilde{t}_6;\widetilde{q}\big)}{\theta\big(\widetilde{q}^2/\widetilde{t}_j\widetilde{t}_8;\widetilde{q}\big)}C(u_8;u_6,u_7)$$
$$+C(u_6;u_7,u_8)\Big).$$

The coefficient $D(u_6,u_7)$ is a sum of two very-well-poised ${}_{10}\phi_9$ series in base $\widetilde{q}$, which can be expressed in terms of the trigonometric integral U_t (in base $\widetilde{q}$) by direct computations using Lemma 4.5.5. This yields the desired result. □

Remark 4.6.2. **i)** Note that the $W(E_6)$-symmetry of the trigonometric integrals S_t and U_t is upgraded to a $W(E_7)$-symmetry in Theorem 4.6.1 since the second term in the right hand side of (4.6.3) is the first term with the role of u_6 and u_7 interchanged.

ii) Specializing the parameters in Theorem 4.6.1 to $u\in\mathcal{G}_{2i\omega}$ with $u_1=-u_6$ (so that $t_1t_6=q$ and $\widetilde{t}_1\widetilde{t}_6=\widetilde{q}$), the left hand side of the identity can be evaluated by the hyperbolic Nassrallah-Rahman integral evaluation (4.4.6). For the right hand side of the identity, the second term vanishes because $\theta\big(\widetilde{t}_1\widetilde{t}_6/\widetilde{q};\widetilde{q}\big)=0$ under the particular parameter specialization. The remaining product of two trigonometric integrals can be evaluated by Corollary 4.5.9. The equality of both sides of the resulting identity can be reconfirmed using (4.6.1) and (4.6.4). It follows from this argument that the evaluation of the hyperbolic Nassrallah-Rahman integral is in fact a consequence of fusing trigonometric identities, an approach to hyperbolic beta integrals which was analyzed in detail in [77].

(iii) More generally, specializing (4.6.3) at generic $u\in\mathcal{G}_{2i\omega}$ satisfying $u_1+u_6=ni\omega_1+mi\omega_2$ $(n,m\in\mathbb{Z}_{\geq 0})$, the second term on the right hand side of (4.6.3) still vanishes while the first term reduces to the product of two terminating very-well-poised ${}_{10}\phi_9$ series, one in base q and the other in base $\widetilde{q}$. The terminating ${}_{10}\phi_9$ series is Rahman's [58] biorthogonal rational ${}_{10}\phi_9$ function (cf. Remark 4.5.13**(ii)**), while the resulting expression for S_h is the corresponding two-index hyperbolic analogue of Rahman's biorthogonal rational function, considered by Spiridonov [73, §8.3] (cf. Remark 4.3.1 on the elliptic level).

Corollary 4.6.3. *We have*

$$\begin{aligned}
E_h(u) = {} & \omega_2 e\left(\frac{2\omega^2+\sum_{j=1}^4 u_j^2-u_5^2+u_6^2}{2\omega_1\omega_2}\right)\\
& \times \frac{(\widetilde{q},\widetilde{q};\widetilde{q})_\infty}{2}\frac{\prod_{j=1}^4\theta\big(\widetilde{t}_j\widetilde{t}_6/\widetilde{q};\widetilde{q}\big)}{\theta\big(\widetilde{t}_6/\widetilde{t}_5;\widetilde{q}\big)}V_t\big(\frac{\widetilde{q}^{\frac{3}{2}}}{\widetilde{t}_6},\frac{\widetilde{q}^{\frac{1}{2}}}{\widetilde{t}_1},\ldots,\frac{\widetilde{q}^{\frac{1}{2}}}{\widetilde{t}_5};\widetilde{q}\big)E_t\big(\frac{t_5}{q},t_1,\ldots,t_4,t_6;q\big)\\
& + (u_5\leftrightarrow u_6)
\end{aligned}$$

as meromorphic functions in $u\in\mathbb{C}^6$*, where* $t_j=e\big((iu_j+\omega)/\omega_2\big)$ *and* $\widetilde{t}_j=e\big((iu_j-\omega)/\omega_1\big)$ $(j=1,\ldots,6)$ *as before.*

Proof. For generic $u\in\mathcal{G}_{2i\omega}$ we have

$$E_h(u_2,\ldots,u_7)=\lim_{s\to\infty}S_h(u_1+s,u_2,\ldots,u_7,u_8-s)e\left(\frac{(u_8-u_1-2s)(u_1+u_8)}{2\omega_1\omega_2}\right)$$

under suitable parameter restraints by Proposition 4.4.8. By Theorem 4.6.1, we alternatively have

$$\begin{aligned}
&\lim_{s\to\infty}S_h(u_1+s,u_2,\ldots,u_7,u_8-s)e\left(\frac{(u_8-u_1-2s)(u_1+u_8)}{2\omega_1\omega_2}\right)=\\
&\quad\lim_{s\to\infty}\omega_2 e\left(\frac{2\omega^2+\sum_{j=2}^5 u_j^2-u_6^2+u_7^2}{2\omega_1\omega_2}\right)\frac{(\widetilde{q},\widetilde{q};\widetilde{q})_\infty}{2}\frac{\theta\big(\widetilde{t}_1\widetilde{t}_7\widetilde{u}/\widetilde{q};\widetilde{q}\big)}{\theta\big(\widetilde{t}_7/\widetilde{t}_6;\widetilde{q}\big)}\\
&\qquad\times\prod_{j=2}^5\theta\big(\widetilde{t}_j\widetilde{t}_7/\widetilde{q};\widetilde{q}\big)U_t\big(\widetilde{q}^{\frac{3}{2}}/\widetilde{t}_7,\widetilde{q}^{\frac{1}{2}}/\widetilde{t}_2,\ldots,\widetilde{q}^{\frac{1}{2}}/\widetilde{t}_6,\widetilde{q}^{\frac{1}{2}}/\widetilde{t}_1\widetilde{u},\widetilde{q}^{\frac{3}{2}}\widetilde{u}/\widetilde{t}_8;\widetilde{q}\big)\\
&\qquad\times S_t\big(t_6/q,t_2,\ldots,t_5,t_7,t_1u,t_8/qu;q\big)\\
&\qquad\quad+(u_6\leftrightarrow u_7)
\end{aligned}$$

where $u=e\big(is/\omega_2\big)$ and $\widetilde{u}=e\big(is/\omega_1\big)$. We have $u,\widetilde{u}\to 0$ as $s\to\infty$ since $\Re(\omega_1),\Re(\omega_2)>0$, hence application of Proposition 4.5.18 gives the right hand side of the desired identity with respect to the parameters $(u_2,\ldots,u_7)$. □

Remark 4.6.4. Alternatively Corollary 4.6.3 can be proved by repeating the arguments of Theorem 4.6.1. The argument simplifies, since one now only needs the expression [16, (5.6.1)] of a very-well-poised ${}_8\psi_8$ as a sum of two very-well-poised ${}_8\phi_7$ series, and one does not need to use three term transformations for the trigonometric integrals.

We conclude this section by relating the R-function to the Askey-Wilson function using Corollary 4.6.3. The answer deviates from Ruijsenaars' [66, §6.6] hunch that R is (up to an elliptic prefactor) the product of an Askey-Wilson function in base q and an Askey-Wilson function in base $\widetilde{q}$: it is the appearance below of two such terms which upgrades the $W(D_3)$-symmetry of the Askey-Wilson functions to the $W(D_4)$-symmetry of R (cf. Remark 4.6.2 **i)**). For notational convenience, we write $w_0 = -1 \in W(D_4)$ for the longest Weyl group element, acting as $w_0\gamma = -\gamma$ on the Askey-Wilson parameters γ. We define $\psi(\gamma; x, \lambda) = \psi(\gamma; x, \lambda; \omega_1, \omega_2)$ by

$$\begin{aligned}\psi(\gamma; x, \lambda; \omega_1, \omega_2) &= \frac{h(\hat{\gamma}; \lambda, x; \omega_1, \omega_2)}{h(-\hat{\gamma}; \lambda, x; \omega_1, \omega_2)} \big(w_0 \cdot \phi\big)(\gamma; x, \lambda; \omega_1, \omega_2)\\ &= \frac{h(\gamma; x, \lambda; \omega_1, \omega_2) h(\hat{\gamma}; \lambda, x; \omega_1, \omega_2)}{h(-\gamma; x, \lambda; \omega_1, \omega_2) h(-\hat{\gamma}; \lambda, x; \omega_1, \omega_2)} \phi(-\gamma; x, \lambda; \omega_1, \omega_2),\end{aligned}$$

where the gauge factor h is given by (4.5.28). Note that $\psi(\gamma; x, \lambda; \omega_1, \omega_2)$ is a self-dual solution of the Askey-Wilson difference equation (4.5.25), see Remark 4.5.30. We furthermore define the multiplier

$$M(\gamma; x) = \frac{\theta\big(e((\hat{\gamma}_0 - \gamma_3 - ix)/\omega_2), e((\omega + \gamma_2 + ix)/\omega_2), e((\omega + \gamma_3 - ix)/\omega_2); q\big)}{\theta\big(e((\gamma_3 - \hat{\gamma}_0 - ix)/\omega_2), e((\omega + \gamma_0 - ix)/\omega_2), e((\omega + \gamma_1 - ix)/\omega_2); q\big)},$$

which is elliptic in x with respect to the period lattice $\mathbb{Z}i\omega_1 + \mathbb{Z}i\omega_2$.

Theorem 4.6.5. *We have*

$$\begin{aligned}&R(\gamma; x, \lambda; \omega_1, \omega_2)\\ &\quad = K(\gamma) M(\gamma; x) M(\hat{\gamma}; \lambda) \phi(s_{23}\gamma; x, \lambda; \omega_1, \omega_2) \psi(-\gamma; x, \lambda; -\omega_2, \omega_1) + (\gamma_2 \leftrightarrow \gamma_3),\end{aligned}$$

where $s_{23}\gamma = (\gamma_0, \gamma_1, \gamma_3, \gamma_2)$ *and with*

$$\begin{aligned}K(\gamma; \omega_1, \omega_2) = \sqrt{-i}\, e\big(-\frac{1}{24}(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1})\big) e\Big(-\frac{3\omega(\gamma_0 + \gamma_1 + \gamma_2 - \gamma_3)}{2\omega_1\omega_2}\Big)\\ \times \frac{\prod_{j=1}^{3} G(i\omega + i\gamma_0 + i\gamma_j)}{\theta\big(e((\gamma_2 - \gamma_3)/\omega_2); q\big)}\\ \times e\Big(\frac{-\gamma_0^2 - \gamma_1^2 - \gamma_2^2 + \gamma_3^2 - 2\gamma_0\gamma_1 - 2\gamma_0\gamma_2 + 2\gamma_0\gamma_3}{4\omega_1\omega_2}\Big).\end{aligned}$$

Proof. Using the second hyperbolic Euler integral representation of R from Theorem 4.4.21 and subsequently applying Corollary 4.6.3, we obtain an expression of $R(\gamma; x, \lambda; \omega_1, \omega_2)$ in terms of trigonometric integrals E_t and V_t with parameter specializations which allows us to rewrite them as Askey-Wilson functions by Lemma 4.5.31. This leads to the expression

$$\begin{aligned}R(\gamma; x, \lambda; \omega_1, \omega_2) = C(\gamma; x, \lambda; \omega_1, \omega_2) \phi(s_{23}\gamma; x, \lambda; \omega_1, \omega_2) \psi(-\gamma; x, \lambda; -\omega_2, \omega_1)\\ + (\gamma_2 \leftrightarrow \gamma_3), \quad (4.6.8)\end{aligned}$$

with the explicit prefactor

$$
\begin{aligned}
&C(\gamma;\lambda,x;\omega_1,\omega_2)\\
&=\sqrt{\frac{\omega_2}{\omega_1}}e\biggl(\frac{8\omega^2+8x^2+2(i\omega-i\hat{\gamma}_0-\lambda)^2+(i\omega-2i\gamma_0+i\hat{\gamma}_0+\lambda)^2}{8\omega_1\omega_2}\\
&\qquad+\frac{(i\omega-2i\gamma_1+i\hat{\gamma}_0+\lambda)^2-(i\omega-2i\gamma_2+i\hat{\gamma}_0+\lambda)^2+(i\omega-2i\gamma_3+i\hat{\gamma}_0+\lambda)^2}{8\omega_1\omega_2}\biggr)\\
&\times\frac{\bigl(\widetilde{q};\widetilde{q}\bigr)_\infty}{\bigl(q;q\bigr)_\infty}\frac{\prod_{j=1}^3 G(i\omega+i\gamma_0+i\gamma_j)}{\theta\bigl(e((\gamma_3-\gamma_2)/\omega_1);\widetilde{q}\bigr)}\prod_{j=0}^{2}\frac{\bigl(e((-\omega+\gamma_j+ix)/\omega_1);\widetilde{q}\bigr)_\infty}{\bigl(e((-\omega-\gamma_j+ix)/\omega_1);\widetilde{q}\bigr)_\infty}\\
&\times\frac{\theta\bigl(e((-\omega+\gamma_3\pm ix)/\omega_1);\widetilde{q}\bigr)\bigl(e((\omega-\gamma_2\pm ix)/\omega_2);q\bigr)_\infty\prod_{j=0}^3 G(-i\gamma_j\pm x)}{\bigl(e((\omega+\gamma_0\pm ix)/\omega_2),e((\omega+\gamma_1\pm ix)/\omega_2),e((\omega+\gamma_3\pm ix)/\omega_2);q\bigr)_\infty}\\
&\times\frac{\theta\bigl(e((\gamma_3-\hat{\gamma}_0+ix)/\omega_1);\widetilde{q}\bigr)}{\theta\bigl((\hat{\gamma}_0-\gamma_3+ix)/\omega_1);\widetilde{q}\bigr)}\frac{\bigl(e((-\omega+\gamma_3-ix)/\omega_1);\widetilde{q}\bigr)_\infty}{\bigl(e((-\omega-\gamma_3-ix)/\omega_1);q\bigr)_\infty}\\
&\times\frac{\theta\bigl(e((\hat{\gamma}_3-\gamma_0+i\lambda)/\omega_1),e((-\omega-\hat{\gamma}_2+i\lambda)/\omega_1),e((-\omega+\hat{\gamma}_3+i\lambda)/\omega_1);\widetilde{q}\bigr)}{\theta\bigl(e((\gamma_0-\hat{\gamma}_3+i\lambda)/\omega_1);\widetilde{q}\bigr)}\\
&\times\frac{G(\lambda-i\hat{\gamma}_1,\lambda-i\hat{\gamma}_2,\lambda-i\hat{\gamma}_3)}{G(\lambda+i\hat{\gamma}_0)}\\
&\times\frac{\bigl(e((-\omega+\hat{\gamma}_3-i\lambda)/\omega_1);\widetilde{q}\bigr)_\infty}{\bigl(e((-\omega-\hat{\gamma}_3-i\lambda)/\omega_1);\widetilde{q}\bigr)_\infty}\prod_{j=0}^{2}\frac{\bigl(e((-\omega+\hat{\gamma}_j+i\lambda)/\omega_1);\widetilde{q}\bigr)_\infty}{\bigl(e((-\omega-\hat{\gamma}_j+i\lambda)/\omega_1);\widetilde{q}\bigr)_\infty}\\
&\times\frac{\bigl(e((-\omega-\hat{\gamma}_3+i\lambda)/\omega_1);\widetilde{q}\bigr)_\infty}{\bigl(e((-\omega+\hat{\gamma}_0+i\lambda)/\omega_1),e((-\omega+\hat{\gamma}_1-i\lambda)/\omega_1),e((-\omega+\hat{\gamma}_2-i\lambda)/\omega_1);\widetilde{q}\bigr)_\infty}\\
&\times\frac{\bigl(e((\omega-\hat{\gamma}_2-i\lambda)/\omega_2);q\bigr)_\infty}{\bigl(e((\omega+\hat{\gamma}_0-i\lambda)/\omega_2),e((\omega+\hat{\gamma}_1+i\lambda)/\omega_2),e((\omega+\hat{\gamma}_3+i\lambda)/\omega_2);q\bigr)_\infty}.
\end{aligned}
$$

Elaborate but straightforward computations using (4.6.1), (4.6.4) and (4.6.5) now yields the desired result. □

Chapter 5

Degenerations of multivariate hyperbolic hypergeometric integrals

This chapter is joint work with Eric Rains and consists of an article, which is in preparation [7].

5.1 Introduction

The theory of (classical and basic) hypergeometric functions, which has been developed in the past centuries, has many applications. These applications range from combinatorics and number theory, to representation theory of Lie algebras and quantum groups, to integrable systems. In the last few decades multivariate extensions of many results about univariate series and integrals have been found, see for example the work of Milne, Schlosser and Gustafson (as in [50], [20] and [21] and references therein). These multivariate extensions are naturally connected to root systems since the summand or integrand can be naturally expressed in terms of (classical) root systems. Moreover, Macdonald's [46] conjectured constant term identities for root systems are evaluation formulas for specific multivariate basic hypergeometric integrals associated to root systems.

More recent is the development of the theory of elliptic hypergeometric series and integrals. This theory originated from the work of Frenkel and Turaev [15]. Some results from the basic and classical hypergeometric level have been generalized, but typically on the elliptic level only the most general formulas remain (i.e. there only exist generalizations of summation formulas for ${}_8W_7$ and transformation formulas for ${}_{10}W_9$), see for example [11] and [59].

In this chapter we are interested in another level of hypergeometric functions in which the building block is the hyperbolic gamma function [63] (also known as double sine function). For special values of the parameters, the hyperbolic gamma function can be expressed as the quotient of two q-pochhammer symbols [70], in which case the hyperbolic hypergeometric functions are $(q, \tilde{q})$-bibasic special functions (where $q = \exp(2\pi i\tau)$ and $\tilde{q} = \exp(-2\pi i/\tau)$ are related by modular inversion). An important distinction to the basic hypergeometric functions however is that the hyperbolic gamma function exists for $|q| = 1$ (unlike the basic hypergeometric functions). This is one of the reasons why hyperbolic hypergeometric functions have found applications, one of these being the representation theory of $U_q(\mathfrak{sl}_2(\mathbb{R}))$ in Chapter 3, which was also studied in [32]. Other applications include the harmonic analysis for more general non-compact quantum groups (see for example [55]), knot theory (see [29]), strongly coupled quantum discrete Liouville theory (see [14]) and solutions to the Quantum KZ equations with $|q| = 1$ (see [27]).

Various identities for the hyperbolic hypergeometric functions can be obtained by taking limits from corresponding elliptic hypergeometric functions (as in [61]), although they can also be independently proven in a similar vein as the identities for their elliptic or basic hypergeometric counterparts, which we did with the identities at the univariate level in Chapter 4. In this chapter we consider multivariate hyperbolic hypergeometric integrals. Beginning with the evaluation and transformation formulas on the most general level, we can take limits (of some variables to plus or minus infinity) to obtain identities on lower levels. On the elliptic level itself this is impossible, since the elliptic gamma function has no nice asymptotic behaviour.

The symmetry groups of integrals on the elliptic level are generally larger than the symmetry group of the corresponding identity on the trigonometric level. For example proposition 4.5.10 shows that the basic hypergeometric integral S_t has a smaller symmetry group than its elliptic hypergeometric counterpart S_e. The symmetry group of the corresponding hyperbolic hypergeometric integrals is however identical to the symmetry group of the elliptic hypergeometric counterpart. From now on we will make a distinction between symmetries of an hypergeometric function (which relate a hypergeometric function to another instance of the same hypergeometric function with different variables) and transformations (which relate one hypergeometric function to another kind of hypergeometric function).

In this chapter we start with two integrals, the multivariate hyperbolic hypergeometric BC_n integrals which are the hyperbolic analogues of the type *I* and type *II* elliptic hypergeometric integrals from [59]. The properties of these hyperbolic hypergeometric functions were derived in [61]. Both integrals are n-variate hyperbolic generalizations of the Nassrallah-Rahman integral [16, (6.4.1)] and have a number of free parameters that depends on an extra positive integer m. At the univariate level these two types are identical and correspond to (4.4.6) (for $m = 0$) and (4.4.7) (for $m = 1$). The type *I* integral satisfies transformations between n-variate and m-variate versions of these integrals; if $m = 0$ this leads to evaluation formulas. For $m = 0$ the type *II* integral satisfies evaluation formulas which are closely related to the Macdonald constant term identities for type BC_n. Apart from these evaluation formulas, for $m = 1$ the type *II* integral satisfies symmetries with respect to a Weyl group of type E_7. These symmetries are essentially multivariate hyperbolic versions of Bailey's four term relation [16, (III.39)], and are the multivariate extensions of the $W(E_7)$ symmetry from Theorem 4.4.1.

Given these two top level integrals we consider in what way the parameters can go to infinity such that the integrals have nice limits. Subsequently we can obtain evaluation and transformation formulas for these degenerations by taking the limit in the relevant identities of the top level integrals. This gives us a host of hyperbolic hypergeometric integral identities, some of which are already known in special cases, but most of which are new. In particular we re-obtain the univariate results from Theorem 4.4.11, Proposition 4.4.7 and Corollaries 4.4.6, 4.4.10 and 4.4.14 of the previous chapter. The resulting identities we obtain in this chapter include multivariate hyperbolic analogues of the Askey-Wilson evaluation formula and Barnes' first and second lemma [16, (6.1.1), (4.1.2) and (4.1.3)], and the transformation formulas of very well-poised ${}_8W_7$'s, balanced ${}_3\phi_2$'s and ${}_2\phi_1$'s [16, (III.23), (III.9) and (III.1)]. Here we compare our results with results for series as those are more well-known, but of course our results are about multivariate hyperbolic hypergeometric integrals.

The structure of the kind of degenerations arising from the type *I* and *II* integrals is very similar. However, the identities we prove for the degenerations of the two different top level integrals are quite different, because the top level integrals themselves satisfy completely different kinds of symmetries and transformations. In particular the identities arising from the E_7 symmetry of the type *II* integral are interesting, since its degenerations provide non-trivial transformations between different types of degenerations as well as the symmetries of those degenerations (given by suitable parabolic subgroups of $W(E_7)$).

Apart from the obvious idea of repeating this argument to obtain the degenerations of other top level hyperbolic integrals and their properties (for example starting with an A_n integral, considering contiguous relations or generalizations of families of biorthogonal rational functions), this research raises some interesting other questions. For example, what are the algebraic structures behind these functions? For the univariate case this was partly adressed in Chapter 3. Can we define orthogonal or bi-orthogonal functions with these integrals as kernel? How about Fourier transforms (in the univariate case this was considered in [68])?

Let us finish this introduction by an overview of this chapter. In Section 5.2 we review those properties of the hyperbolic gamma function, which we exploit in the rest of the chapter. In Section 5.3 we give the top level hyperbolic integrals (both type *I* and *II*) and their properties (evaluations, symmetries and transformations). Moreover we discuss a general family of integrals which contains all degenerations of the top level integrals we will consider, and derive the limits between these integrals. Subsequently, in Section 5.4 we heuristically derive the degeneration scheme, i.e. a description of all degenerations of the top level integrals of type I and type II and the relations between them.

In Section 5.5 we then define the degenerations of the type I top level integral according to the degenerations scheme, prove the limit relations between them and obtain the corresponding evaluation and transformation formulas. Section 5.6

has a setup analogous to Section 5.5, but now we consider the degenerations of the type *II* integrals. In Appendix 5.A we give the technical details of the proofs of the limits between the integrals. Finally, in Appendix 5.B we have included the explicit integrals defining the degenerations of the type II integral with Weyl group of type E_7 symmetry, for convenience of the reader.

5.1.1 Notation

Let ω_1 and ω_2 be located in the upper half plane $\{z \in \mathbb{C} \mid \Im(z) > 0\}$. We define

$$\omega = \frac{\omega_1 + \omega_2}{2}. \tag{5.1.1}$$

Throughout this article we will use the branch of the argument which has a branch cut along $-\omega\mathbb{R}_{\geq 0}$ and is zero on the positive real line. In particular (square) roots will also be calculated using this branch. We define

$$\phi_+ = \max(\arg(\omega_1), \arg(\omega_2)), \qquad \phi_- = \min(\arg(\omega_1), \arg(\omega_2)). \tag{5.1.2}$$

Hence we have $0 < \phi_- \leq \phi_+ < \pi$.

Moreover it is convenient to use the following two rescaled exponential functions

$$e(x) = \exp(2\pi i x), \qquad c(x) = \exp\left(\frac{\pi i x}{2\omega_1\omega_2}\right), \tag{5.1.3}$$

where we suppress the ω_1 and ω_2 dependence from the notation of $c(x)$. Finally we will often encounter the complex number

$$\zeta = e\left(\frac{\omega_1^2 + \omega_2^2}{48\omega_1\omega_2}\right) = c\left(\frac{\omega_1^2 + \omega_2^2}{12}\right) = c\left(\frac{\omega^2}{3}\right) e\left(-\frac{1}{24}\right) \tag{5.1.4}$$

We define the left hand cone (LHC) by

$$LHC = \{z \mid \phi_+ < \arg(z) < \phi_- + \pi\}$$

and the right hand cone (RHC) by

$$RHC = \{z \mid \phi_+ - \pi < \arg(z) < \phi_-\}. \tag{5.1.5}$$

For the case $\arg(\omega_1) > \arg(\omega_2)$ we have drawn the left hand cone in Figure 5.1.1 (shaded part). The left and right hand cones are generalizations of the left and right hand plane though note $LHC \cup RHC \neq \mathbb{C}$.

5.2 The hyperbolic gamma function

The integrands of the integrals which are the subject of this article are all expressed in terms of the hyperbolic gamma function. In this section we will introduce this

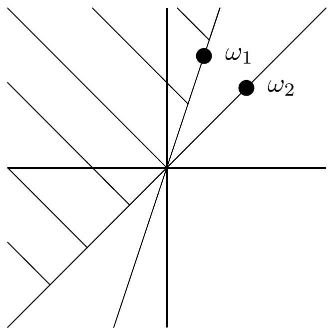


Figure 5.1: The left hand cone

function and mention all results about the hyperbolic gamma function needed in the remainder of the article.

The hyperbolic gamma function [63] is a generalization of Euler's gamma function, like the q-gamma and elliptic gamma functions. The common characteristic of these gamma functions is that they satisfy simple difference equations. For example the regular gamma function satisfies the well-known equation $z\Gamma(z) = \Gamma(z+1)$.

The following definition is due to Ruijsenaars [63].

Definition 5.2.1. *The hyperbolic gamma function* Γ_h *is defined by*

$$\Gamma_h(z;\omega_1,\omega_2) = \exp\left(i\int_0^\infty \left(\frac{z-\omega}{\omega_1\omega_2 x} - \frac{\sin(2x(z-\omega))}{2\sin(\omega_1 x)\sin(\omega_2 x)}\right)\frac{dx}{x}\right),$$

for $z \in \mathbb{C}$ *satisfying* $0 < \Im(z) < \Im(2\omega)$. *There exists ([63, Proposition III.1]) a meromorphic extension of* Γ_h *to* $z \in \mathbb{C}$, *which we also denote by* Γ_h.

Γ_h is related to Ruijsenaars' [63] hyperbolic gamma function G via

$$\Gamma_h(z;\omega_1,\omega_2) = G(-i\omega_1, -i\omega_2; z-\omega),$$

and to the double sine function S_2 (for example [43, (2.2)]) via

$$\Gamma_h(z;\omega_1,\omega_2) = S_2(z \mid \omega_1,\omega_2)^{-1}.$$

If $\Im(\omega_2/\omega_1) > 0$ we can moreover express Γ_h as quotient of two q-pochhammer symbols [70]

$$\Gamma_h(z;\omega_1,\omega_2) = c((z-\omega)^2)\zeta^{-1}\frac{(e(-z/\omega_1)e(\omega_2/\omega_1);e(\omega_2/\omega_1))_\infty}{(e(-z/\omega_2);e(-\omega_1/\omega_2))_\infty},$$

where $(a;q)_\infty = \prod_{r=0}^\infty (1-aq^r)$ for $|q|<1$.

Note that the hyperbolic gamma function is symmetric in ω_1 and ω_2

$$\Gamma_h(z;\omega_1,\omega_2) = \Gamma_h(z;\omega_2,\omega_1).$$

Since the values of ω_1 and ω_2 are usually fixed we will generally omit them from our notation when no confusion can arise. Moreover we use the following short hand notations for products of hyperbolic gamma functions

$$\Gamma_h(a \pm z) = \Gamma_h(a+z)\Gamma_h(a-z), \qquad \Gamma_h(a_1,a_2,\ldots,a_n) = \prod_{j=1}^{n} \Gamma_h(a_j).$$

Proofs of the following propositions can be found in [63, Section III.A].

Proposition 5.2.2. *The hyperbolic gamma function satisfies the following two difference equations*

$$\Gamma_h(z+\omega_1) = 2\sin(\pi z/\omega_2)\Gamma_h(z), \qquad \Gamma_h(z+\omega_2) = 2\sin(\pi z/\omega_1)\Gamma_h(z). \tag{5.2.1}$$

Proposition 5.2.3. *The poles of the hyperbolic gamma function $\Gamma_h(z)$ are located at*

$$\Lambda = -\omega_1\mathbb{Z}_{\geq 0} - \omega_2\mathbb{Z}_{\geq 0}, \tag{5.2.2}$$

and the zeros at $2\omega - \Lambda$.

Proposition 5.2.4. *The hyperbolic gamma function satisfies*

$$\Gamma_h(z;\omega_1,\omega_2) = \overline{\Gamma_h(-\bar{z};-\bar{\omega}_1,-\bar{\omega}_2)}. \tag{5.2.3}$$

Proposition 5.2.5. *The hyperbolic gamma function satisfies the following reflection equation*

$$\Gamma_h(z)\Gamma_h(2\omega - z) = 1. \tag{5.2.4}$$

Now we will consider the asymptotic behaviour of the hyperbolic gamma function. The following generalization of a theorem by Ruijsenaars [63, Proposition III.4] was proven by Rains in [61].

Theorem 5.2.6. *Let a, $\arg(x)$ and ω_j range over the parameter domain $0 < a < \min_j(\Im(-\exp(i\arg(x))/\omega_j))$. Then as $|x| \to \infty$ we have the estimates*

$$\begin{aligned} -2\pi i\left(\frac{(x-\omega)^2}{4\omega_1\omega_2} - \frac{\omega_1^2+\omega_2^2}{48\omega_1\omega_2}\right) + \log(\Gamma_h(x;\omega_1,\omega_2)) &= \mathcal{O}(\exp(-2\pi a|x|)),\\ 2\pi i\left(\frac{(x-\omega)^2}{4\omega_1\omega_2} - \frac{\omega_1^2+\omega_2^2}{48\omega_1\omega_2}\right) + \log(\Gamma_h(-x;\omega_1,\omega_2)) &= \mathcal{O}(\exp(-2\pi a|x|)), \end{aligned} \tag{5.2.5}$$

uniformly on compact subsets of the domain. Specifically, for any compact set $K \subset \mathbb{R}\times[0,2\pi]\times\mathbb{C}^2$, such that any quadruple $(a,\phi,\omega_1,\omega_2) \in K$ satisfies $0 < a < \min_j(\Im(-\exp(i\phi)/\omega_j))$ there exists a constant C_K and an $M \in \mathbb{R}$ such that the left hand sides of 5.2.5 are bounded by $C_K\exp(-2\pi a|x|)$ for $(a,\arg(x),\omega_1,\omega_2) \in K$ and $|x| > M$.

This theorem expresses the asymptotic behaviour of the hyperbolic gamma function in the left and right hand cones. In particular for x in the right hand cone we have $\Gamma_h(x)/(\zeta^{-1}c((x-\omega)^2)) \to 1$ as $|x| \to \infty$ and for x in the left hand cone $\Gamma_h(x)/(\zeta c(-(x-\omega)^2)) \to 1$ as $|x| \to \infty$, where c and ζ are defined in (5.1.3) and (5.1.4). The asymptotic behaviour in the rest of the complex plane, in which the zeros and poles of the hyperbolic gamma function are located is not well behaved.

We will in this article have occasion to use the following corollary to this theorem.

Corollary 5.2.7. *Fix ω_1 and ω_2. For ϕ, ψ satisfying $\phi_- > \phi > \psi > \phi_+ - \pi$, $z \in \mathbb{C}$ and $\epsilon \in \mathbb{R}_{>0}$ we define the set*

$$D_1 = \{x \in \mathbb{C} \mid \phi \geq \arg(x-z) \geq \psi \wedge d(x,\Lambda) \geq \epsilon\},$$

where $d(x,\Lambda) = \min_{k,l\in\mathbb{Z}_{\geq 0}} |x + k\omega_1 + l\omega_2|$ denotes the distance between x and Λ. Then the function

$$\Gamma_h(x)c(-(x-\omega)^2)$$

is bounded on D_1. Moreover the function

$$\Gamma_h(x)c((x-\omega)^2)$$

is bounded on the set

$$D_2 = \{x \mid \phi \geq \arg(z-x) \geq \psi \wedge d(x,\Lambda) \geq \epsilon\}.$$

Proof. The above theorem asserts that $\Gamma_h(x)c(-(x-\omega)^2)$, respectively $\Gamma_h(x)c((x-\omega)^2)$ is uniformly bounded for large $|x|$ in D_1, respectively D_2. Since D_1 and D_2 are closed sets, the corollary follows from continuity of the hyperbolic gamma function on D_1 respectively D_2 (note that we explicitly avoid the poles of the hyperbolic gamma function). □

Basically this corollary bounds the hyperbolic gamma function away from the set of poles, in respectively a shifted right hand cone (for D_1) and a shifted left half cone (D_2).

5.3 The definition of the integrals

We begin this section by discussing two different types of hyperbolic multivariate beta integrals, type *I* and *II* top level hyperbolic integrals. Subsequently we will consider the general family of integrals to which these integrals belongs. All degenerations of the top level integrals we consider are members of this family, as we will see in Sections 5.5 and 5.6. After giving the definitions of this general family of integrals we give evaluation formulas for them in certain special cases and discuss the limits between them.

5.3.1 The type I_{BC} top level integral

The type I_{BC} top level hyperbolic integral is a hyperbolic analogue of a multivariate basic hypergeometric integral by Gustafson [20, (4.2)], which was a multivariate (BC_n-type) analogue of the Nassrallah-Rahman integral [16]. This hyperbolic integral itself was studied in [12] and [61]. The evaluation and transformation formulas it satisfies are very similar to the corresponding formulas for the elliptic hypergeometric analogue, as studied in [11] and [59].

The definition of the type I_{BC} top level integral is given by

Definition 5.3.1. *Let $n, m \in \mathbb{Z}_{\geq 0}$. For parameters $\mu \in \mathbb{C}^{2n+2m+4}$ satisfying the balancing condition*

$$\sum_{r=0}^{2n+2m+3} \mu_r = 2(m+1)\omega \tag{5.3.1}$$

and such that $\mu_r + \mu_s \notin \Lambda$ for $r, s = 0, 1, \ldots, 2n+2m+3$ we define the I_{BC} top level integral by

$$I_{n,T}^m(\mu) = \frac{1}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \frac{\prod_{j=1}^n \prod_{r=0}^{2n+2m+3} \Gamma_h(\mu_r \pm x_j)}{\prod_{1\leq j<k\leq n} \Gamma_h(\pm x_j \pm x_k) \prod_{j=1}^n \Gamma_h(\pm 2x_j)} \prod_{j=1}^n dx_j.$$

Here the contour C is a deformation of the real line $\mathbb{R}$ which separates the poles of the integrand at $x_j = -\mu_r + \Lambda$ ($r = 0, 1, \ldots, 2n+2m+3$) from those at $x_j = \mu_s - \Lambda$ ($s = 0, 1, \ldots, 2n+2m+3$).

We will first show that $I_{n,T}^m$ is well-defined. Note that the symmetries (5.2.1) and (5.2.4) of the hyperbolic gamma function imply

$$\frac{1}{\Gamma_h(\pm z)} = -4 \sin\left(\frac{\pi z}{\omega_1}\right) \sin\left(\frac{\pi z}{\omega_2}\right). \tag{5.3.2}$$

This ensures that the reciprocal of the denominator of the integrand is analytic and hence the poles of the integrand are the poles of the numerator. The condition that $\mu_r + \mu_s \notin \Lambda$ moreover ensures that no two of the poles which should be separated by the contour C collide, hence a contour as desired always exists.

To show convergence of the integral we calculate the asymptotic behaviour of the integrand, using Corollary 5.2.7 for the univariate terms $\prod_{r=0}^{2n+2m+3} \Gamma_h(\mu_r \pm x_j)$ (for $j = 1, \ldots, n$). To deal with the denominator of the integrand we use equation (5.3.2) together with the estimate on the sine given by $|2\sin(x)| \leq \exp(|\Im(x)|)$ and the triangle inequality $|x+y| \leq |x| + |y|$ to show that $|1/\Gamma_h(\pm x_j \pm x_k))| \leq \exp(2|\Im(\pi x_j/\omega_1)| + 2|\Im(\pi x_k/\omega_1)|)$. Using the balancing condition (5.3.1) to simplify leads to the following bound on the integrand

$$\left| \frac{\prod_{j=1}^n \prod_{r=0}^{2n+2m+3} \Gamma_h(\mu_r \pm x_j)}{\prod_{1\leq j<k\leq n} \Gamma_h(\pm x_j \pm x_k) \prod_{j=1}^n \Gamma_h(\pm 2x_j)} \right| \leq K \prod_{j=1}^n |c(-8|x_j|\omega)|,$$

for $x \in C^n$ and some constant $K > 0$, depending on the contour C and the variables μ. We see that the integrand is exponentially decreasing since $\Re(i\omega/\omega_1\omega_2) > 0$, hence the integral converges. Cauchy's theorem now tells us that the value of integral becomes independent of the particular choice of contour, hence $I_{n,T}^m$ is well-defined.

In order to show that $I_{n,T}^m(\mu)$ is a meromorphic function of its variables μ we use the following direct consequence of [61, Theorem 10.2]. Let $d(z)$ be a function with (simple) zeros at $z \in \Lambda$.

Lemma 5.3.2. *Suppose $F(z;t;u;\omega_1,\omega_2)$ is a holomorphic function on $(z,t,u) \in \mathbb{C} \times \mathbb{C}^{s_1} \times \mathbb{C}^{s_2}$. Suppose we can define a function*

$$G(t;u;\omega_1,\omega_2) = \prod_{r=0}^{s_1-1}\prod_{s=0}^{s_2-1} d(t_r+u_s) \int_C F(z;t;u;\omega_1,\omega_2) \times \prod_{r=0}^{s_1-1} \Gamma_h(t_r+z) \prod_{r=0}^{s_2-1} \Gamma_h(u_r-z) dz$$

for t and u in some open domain D, with a fixed contour C (implicating none of the poles $z = -u_r - \Lambda$ and $z = u_r + \Lambda$ meet C for $(t,u) \in D$). Then G extends uniquely to a holomorphic function on $(t,u) \in \mathbb{C}^{s_1} \times \mathbb{C}^{s_2}$.

Corollary 5.3.3. *$I_{n,T}^m(\mu)$ can be uniquely extended to a meromorphic function on $\mu \in \mathbb{C}^{2n+2m+4}$ satisfying the balancing condition* (5.3.1)*.*

Proof. Our integrals are iterated contour integrals of the type described in the lemma. Note that for variables μ such that $\Im(\mu_r) > 0$ we can choose $C = \mathbb{R}$ in Definition 5.3.1, so we can define the function using a fixed contour for some open set of variables μ. □

The following proposition was derived in [61] as a limit from the associated transformation on the elliptic level, derived in [59].

Proposition 5.3.4. *For $\mu \in \mathbb{C}^{2n+2m+4}$ satisfying the balancing condition* (5.3.1) *we have*

$$I_{n,T}^m(\mu) = I_{m,T}^n(\omega - \mu) \prod_{0 \le r < s \le 2n+2m+3} \Gamma_h(\mu_r + \mu_s) \qquad (5.3.3)$$

as meromorphic functions in μ, where $\omega - \mu = (\omega - \mu_0, \omega - \mu_1, \ldots, \omega - \mu_{2n+2m+3})$.

As a special case for $m = 0$ we obtain an evaluation formula. The univariate hyperbolic integral evaluation was proven by Stokman [77], while the general evaluation integral was proven by van Diejen and Spiridonov [12] and in a different manner by Rains [61]. This evaluation formula is in the case $n = 1$ the hyperbolic analogue of the Nasrallah-Rahman integral evaluation [51], and for general n the hyperbolic analogue of an integral by Gustafson [20].

Corollary 5.3.5. *For $\mu \in \mathbb{C}^{2n+4}$ satisfying the balancing condition* (5.3.1) *with $m = 0$ we have*

$$\frac{1}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \frac{\prod_{j=1}^n \prod_{r=0}^{2n+3} \Gamma_h(\mu_r \pm x_j)}{\prod_{1\leq j<k\leq n} \Gamma_h(\pm x_j \pm x_k) \prod_{j=1}^n \Gamma_h(\pm 2x_j)} \prod_{j=1}^n dx_j = \prod_{0\leq r<s\leq 2n+3} \Gamma_h(\mu_r + \mu_s) \quad (5.3.4)$$

as a relation between meromorphic functions, where the contour C is a deformation of $\mathbb{R}$ separating the poles at $x_j = -\mu_r + \Lambda$ from those at $x_j = \mu_s - \Lambda$.

Proof. Note that $I_{0,T}^n = 1$. The result now follows from Proposition 5.3.4 by specializing to $m = 0$. ☐

5.3.2 The type II_{BC} top level integral

The second integral of interest, denoted as type II_{BC}, is another multivariate hyperbolic (BC_n-type) analogue of the Nassrallah-Rahman integral. The univariate ($n = 1$) version of the II_{BC} integral equals the univariate type I_{BC} integral discussed in the previous subsection. Like the I_{BC} integral the general II_{BC} integral was first obtained in [61] as limit of its elliptic counterpart. Again the evaluation and transformation formulas given here are all limits of the corresponding formulas of the elliptic analogue.

The type II_{BC} integral is the hyperbolic analogue of an integral by Gustafson [21, Theorem 2.1]. Gustafson's integral extended the q-Macdonald constant term conjectures [46] for root systems of type B_n, C_n and BC_n. The basic hypergeometric version moreover extends the integral kernel corresponding to the inner-product of the Koornwinder polynomials [41]. The evaluation formula for the elliptic analogue was first conjectured by van Diejen and Spiridonov [11] and proven by Rains [59].

The type II_{BC} hyperbolic top level integral is defined as follows

Definition 5.3.6. *For generic parameters $\mu \in \mathbb{C}^{2m+6}$ and $\tau \in \omega_1\mathbb{R}_{>0} + \omega_2\mathbb{R}_{>0}$ satisfying the balancing condition*

$$2(n-1)\tau + \sum_{r=0}^{2m+5} \mu_r = 2(m+1)\omega \quad (5.3.5)$$

the integral $II_{n,T}^m$ is defined by

$$II_{n,T}^m(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{\prod_{r=0}^{2m+5} \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} dx_j,$$

where the contour C is an even (i.e. $C = -C$) deformation of the real line (i.e. the real line with indentations in some bounded set), separating the poles at $x_j = -\mu_r + \Lambda$ from those at $x_j = \mu_r - \Lambda$ and lying below the contour $\tau - \lambda + C$, for all $\lambda \in \Lambda$. A contour C_2 lies below a contour C_1 if C_2 is contained in the set of points to the right of the oriented contour C_1 (thus being consistent with the intuitive meaning of "lying below" if $C_1 = \mathbb{R}$ traversed from left to right).

It is not trivial to see a desired contour exists, but observe that if $\Im(\mu_r) > 0$ for all r then we can just take $C = \mathbb{R}$. It is convenient to consider adding a circle around some point to the contour, to be a continuous deformation of this contour, as we can obtain the same integral by taking a path from the contour to the edge of the circle, following the circle, and returning to the original contour along the same path (so the integrals along the path cancel each other). The points $x_j = -\mu_r - k\omega_1 - l\omega_2$ (for all j, r, k and l) are poles of the integrand, and if they have positive imaginary part they are located on the wrong side of the contour, so then we have to add (negatively oriented) circles around them to put them on the other side of the contour (and symmetrically put positively oriented circles around their reciprocals). To ensure the contour remains below the shifted contour $\tau - \lambda + C$ we must then put additional circles around the τ shifted circles of the first step. This process then repeats itself and we have to prove that it stops. One way to do this all at once is to add little negatively oriented circles around the points $-\mu_r - j\tau - k\omega_1 - l\omega_2$ for $r = 0, \ldots, 2m+5$ and $j, k, l \in \mathbb{Z}_{\geq 0}$ as long as the imaginary part of such a point is positive (which are only a finite number of points), and symmetrically add positively oriented circles around the reciprocals of these points. The radii of these circles should increase with increasing j to ensure the new circles are larger than the circles they have to enclose. Since we only add a finite amount of circles, this process finishes after some finite time.

We can calculate the asymptotic behaviour of the integrand as

$$\left| \prod_{1 \leq j < k \leq n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \prod_{j=1}^{n} \frac{\prod_{r=0}^{2m+5} \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} \right| \leq K \prod_{j=1}^{n} |c(-8|x_j|\omega)|, \quad (5.3.6)$$

for some constant $K > 0$ for x_j on the contour. This bound clearly shows that the integral converges. In obtaining this bound we use that $\Re(i\tau/\omega_1\omega_2) > 0$, which allows us to estimate the terms $\Gamma_h(\tau \pm (x_j + x_k))/\Gamma_h(\pm(x_j + x_k))$ and $\Gamma_h(\tau \pm (x_j - x_k))/\Gamma_h(\pm(x_j - x_k))$ by $K'|c(4\tau(|x_j| + |x_k|))|$. Together with the bound on the univariate terms the balancing conditions allows us to cancel the μ and τ dependent terms.

The fact that the contour is even and that it lies below the contours $\tau - \Lambda + C$ ensures that if we consider the x_j integral the poles at $x_j = x_k - \tau - k\omega_1 - l\omega_2$ are below the contour, while if we consider the x_k integral the poles at $x_k = -x_j + \tau + k\omega_1 + l\omega_2$ are above the contour. Again we can show $I\!I^m_{n,T}(\mu; \tau)$ has a unique meromorphic extension to $(\mu, \tau) \in \mathbb{C}^{2m+6} \times \mathbb{C}$ satisfying the balancing condition (5.3.5), seeing it as a iterated contour of the type described in Lemma 5.3.2.

The difference with the $I^m_{n,T}$ integral consists of a different number of parameters μ, the additional parameter τ, and related to this, a new balancing condition. These differences lead to quite different behaviour of the integrals themselves.

Let us first consider the evaluation formula for $m = 0$. For $n = 1$ it is of course identical to the evaluation formula (5.3.4) of the $I^0_{1,T}$ integral, since for $n = 1$ the integrals are the same. So the following is another multivariate hyperbolic analogue of the Nassrallah-Rahman integral evaluation. This evaluation formula was also conjectured by van Diejen and Spiridonov [11] and proven by them in [12] while Rains [61] gave a different proof.

Proposition 5.3.7. *For variables $\mu \in \mathbb{C}^6$ and $\tau \in \mathbb{C}$ and a contour C satisfying the conditions of Definition 5.3.6 we have*

$$\frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\leq j<k\leq n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}\prod_{j=1}^n\frac{\prod_{r=0}^5\Gamma_h(\mu_r\pm x_j)}{\Gamma_h(\pm 2x_j)}dx_j$$
$$=\prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\prod_{0\leq r,s\leq 5}\Gamma_h(j\tau+\mu_r+\mu_s) \quad (5.3.7)$$

as a relation between meromorphic functions.

For the $m = 1$ case $I\!I^1_{n,T}$ has symmetries with respect to the Weyl group of type E_7, as in the elliptic case [59]. More precisely this means there is an action of the Weyl group on the parameter space $\mu \in \mathbb{C}^8$ which leaves the integral invariant or multiplies it by a suitable product of hyperbolic gamma functions. The trivial symmetries corresponding to permuting the 8 μ variables are contained in the symmetry structure. The rest of the E_7 Weyl group symmetry is generated by a single symmetry related to one extra reflection in $W(E_7)$, which is described in the next proposition. For the univariate case this corresponds at the basic hypergeometric level to Bailey's 4-term relation for ${}_{10}\phi_9$'s (see [45] and [6], note that the sum of two ${}_{10}\phi_9$'s on one side of Bailey's 4-term relation can be expressed as the basic hypergeometric version of the $I\!I^1_{1,T}$ integral), which leads to a symmetry with respect to a Weyl group of type E_6. Here we see that the integral on the hyperbolic level satisfies more symmetries than its basic hypergeometric analogue.

A proof of this proposition (for general n) was given in [61] by showing it followed from the related symmetry of the elliptic hypergeometric integral [59], and a proof in the univariate case within the framework of hyperbolic hypergeometric integrals was given in Chapter 4.

Proposition 5.3.8. *For variables $\mu \in \mathbb{C}^8$ and $\tau \in \mathbb{C}$ satisfying the conditions of Definition 5.3.6 we have*

$$I\!I^1_{n,T}(\mu;\tau) = I\!I^1_{n,T}(w\mu;\tau)\prod_{j=0}^{n-1}\prod_{0\leq r<s\leq 3}\Gamma_h(j\tau+\mu_r+\mu_s)\prod_{4\leq r<s\leq 7}\Gamma_h(j\tau+\mu_r+\mu_s), \quad (5.3.8)$$

as a relation between meromorphic functions where $w\mu = (\mu_0 + \xi, \mu_1 + \xi, \mu_2 + \xi, \mu_3 + \xi, \mu_4 - \xi, \mu_5 - \xi, \mu_6 - \xi, \mu_7 - \xi)$ *for* $2\xi = \sum_{r=4}^7 \mu_r - 2\omega + (n-1)\tau = -\sum_{r=0}^3 \mu_r + 2\omega - (n-1)\tau$.

This symmetry becomes a Weyl group of type E_7 symmetry after we embed the root system of type E_7 in $\mathbb{R}^8$ by taking the norm $\sqrt{2}$ vectors in $\frac{1}{2}\mathbb{Z}^8$ orthogonal to $(1,1,1,1,1,1,1,1)$. The action of the Weyl group of E_7 on any affine hyperplane $\{\mu \in \mathbb{C}^8 \mid \text{balancing condition}\}$ is then generated by the reflections in the hyperplanes orthogonal to the roots, i.e. $s_\alpha \mu = \mu - \langle \alpha, \mu \rangle \alpha$, for a root α, where $\langle \cdot, \cdot \rangle$ is a complexified version of the inner product on $\mathbb{R}^8$. In particular the w from the above proposition is just reflection in the hyperplane orthogonal to $\frac{1}{2}(1,1,1,1,-1,-1,-1,-1)$. This reflection together with the (trivial) permutation symmetries of the eight variables generate the entire Weyl group of type E_7.

The action of the Weyl group of type E_7 will lead to two different kind of relations for the degenerations of $II_{n,T}^1$. First of all it will provide symmetries of the degenerations of $II_{n,T}^1$. Secondly it will lead to transformation relations between different types of degenerations.

Let us finish our discussion of the top level integral by considering the action of the longest Weyl group element of $W(E_7)$. It leads to a symmetry of $II_{n,T}^1$ integral which reduces for $n=1$ to the symmetry of the $I_{1,T}^1$ integral from Proposition 5.3.4.

Corollary 5.3.9. *For variables* $\mu \in \mathbb{C}^8$ *and* $\tau \in \mathbb{C}$ *satisfying the conditions of Definition 5.3.6 we have*

$$II_{n,T}^1(\mu;\tau) = II_{n,T}^1(\omega - \frac{1}{2}(n-1)\tau - \mu;\tau) \prod_{j=0}^{n-1} \prod_{0 \le r < s \le 7} \Gamma_h(j\tau + \mu_r + \mu_s), \quad (5.3.9)$$

where $\omega - \frac{1}{2}(n-1)\tau - \mu = (\omega - \frac{1}{2}(n-1)\tau - \mu_0, \ldots, \omega - \frac{1}{2}(n-1)\tau - \mu_7)$.

Proof. For each w_{abcd} defined as

$$(w_{abcd}\mu)_j = \begin{cases} \mu_j + \xi & \text{if } j \in \{a,b,c,d\}, \\ \mu_j - \xi & \text{if } j \notin \{a,b,c,d\}, \end{cases}$$

where $2\xi = 2\omega - (n-1)\tau) - \mu_a - \mu_b - \mu_c - \mu_d$ and $|\{a,b,c,d\}| = 4$, we have an associated symmetry of $II_{n,T}^1(\mu;\tau)$ by Proposition 5.3.8 and the permutation symmetry of $II_{n,T}^1(\mu;\tau)$ in its eight variables μ. The symmetry of this corollary is obtained by applying the symmetry from Proposition 5.3.8 three times, first with w_{0123}, then with w_{0145} and finally with w_{0167}, followed by a permutation of the variables. □

5.3.3 Definition of the integrals

We will now define two types of integrals, such that all the degenerations of the $I_{n,T}^m$ and $II_{n,T}^m$ integrals can be expressed as one of these two types.

In order to define these integrals we first have to define the concept of a hook contour. The contours of the integrals we consider are, like the contour of $I_{n,T}^m$ and $II_{n,T}^m$, basically $\mathbb{R}$ traversed from minus infinity to plus infinity. However in order to ensure convergence of the integrals we can not always let our contours approach $-\infty$ or $+\infty$ on the real line, but we must relax our conditions by letting them approach to $q\infty$ for some q in the left, respectively right, hand cone instead.

Definition 5.3.10. *A hook W_{ϕ_1,ϕ_2} is the contour $\exp(i\phi_1)\mathbb{R}_{\leq 0} \cup \exp(i\phi_2)\mathbb{R}_{\geq 0}$ traversed from $-\exp(i\phi_1)\infty$ to 0 to $\exp(i\phi_2)\infty$. An exact parametrization is given by $W_{\phi_1,\phi_2}(t) = t\exp(i\phi_1)$ for $t \in (-\infty, 0]$ and $W_{\phi_1,\phi_2}(t) = t\exp(i\phi_2)$ for $t \in [0,\infty)$.*

We will always insist that $\phi_+ - \pi < \phi_1, \phi_2 < \phi_-$ (recall the definition of $\phi_\pm$ from (5.1.2)), which implies that the contour moves from infinity somewhere in the left hand cone to infinity somewhere in the right hand cone. The contours of the integrals we define will generally be such hook contours with indentations to properly separate the poles of the integrand, where we impose some conditions on ϕ_1 and ϕ_2.

Before we can give the definition of the integrals we also need to define some parameter domains.

Definition 5.3.11. *Let $\phi_\tau = \arg(\tau)$. We define $\mathcal{A}_0 = (\phi_+ - \pi, \phi_-)$. Note that $\frac{\phi_- + \phi_+ - \pi}{2} \in \mathcal{A}_0$, since it is the mean of the endpoints of the interval. Hence $\mathcal{A}_0 \backslash \{\frac{\phi_- + \phi_+ - \pi}{2}\}$ splits in two intervals, the first one will be denoted by $\mathcal{A}_+ = (\phi_+ - \pi, \frac{\phi_- + \phi_+ - \pi}{2})$ and the second one by $\mathcal{A}_- = (\frac{\phi_- + \phi_+ - \pi}{2}, \phi_-)$. Subsequently we define $\mathcal{A}_\epsilon^\tau$ by $\mathcal{A}_\epsilon \cap (\phi_\tau - \pi, \phi_\tau)$ for $\epsilon \in \{+,-\}$ and $\mathcal{A}_0^\tau = \mathcal{A}_0 \cap (\phi_\tau - \pi, \phi_\tau) \cap (\phi_- + \phi_+ - \phi_\tau - \pi, \phi_- + \phi_+ - \phi_\tau)$.*

We will usually consider complex numbers z with $\arg(z) \in \mathcal{A}_\epsilon$. In Figure 5.2 (where we assumed $\arg(\omega_1) > \arg(\omega_2)$) the dashed line is the angle bisector of $\omega_1\mathbb{R}_{\leq 0}$ and $\omega_2\mathbb{R}_{\geq 0}$, hence it depicts $\exp\left(i\frac{\phi_- + \phi_+ - \pi}{2}\right)\mathbb{R}$. The horizontally shaded part are the complex numbers whose argument is in $\mathcal{A}_+$, the vertically shaded part are the complex numbers whose argument is in $\mathcal{A}_-$, while the entire shaded part (including the dashed line) are the complex numbers whose argument are in $\mathcal{A}_0$.

In Figure 5.3 we added τ. The dotted line is $\tau\mathbb{R}$ reflected in the angle bisector of $\omega_1\mathbb{R}_{\leq 0}$ and $\omega_2\mathbb{R}_{\geq 0}$, i.e. $\exp(i(\phi_- + \phi_+ - \phi_\tau))\mathbb{R}$. The horizontally shaded part again denotes the complex numbers with an argument in $\mathcal{A}_+^\tau$. The union of the horizontally and the diagonally shaded part denote the complex numbers with an argument in $\mathcal{A}_0^\tau$, while the union of the diagonally shaded and the vertically shaded part denote those numbers with argument in $\mathcal{A}_-^\tau$. For $\phi_- < \phi_\tau < \phi_+$ we have $\mathcal{A}_{+/0/-}^\tau = \mathcal{A}_{+/0/-}$, and Figure 5.3 reduces to Figure 5.2 in this case.

Now recall the definition (5.1.1) of ω.

Definition 5.3.12. *Let $n, s \in \mathbb{Z}_{\geq 0}$. We define*

$$\mathcal{B}_s^\tau = \{\mu \in \mathbb{C}^s \mid \mu_r + \mu_u \neq -j\tau - k\omega_1 - l\omega_2 \text{ for } j,k,l \in \mathbb{Z}_{\geq 0}\}.$$

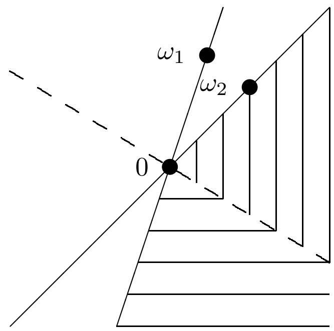


Figure 5.2: The complex numbers z with $\arg(z) \in \mathcal{A}_{+/-/0}$

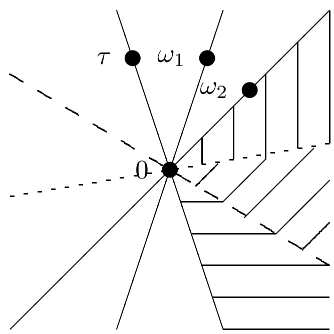


Figure 5.3: The complex numbers z with $\arg(z) \in \mathcal{A}^{\tau}_{+/-/0}$

For $\mu \in \mathbb{C}^s$ we define the angle

$$\alpha^{\tau}(n, s; \mu) = \arg((s-2)\omega - \sum_{r=0}^{s-1} \mu_r - 2(n-1)\tau).$$

Moreover we define the interval $\mathcal{C}^{\alpha} = (\phi_- + \phi_+ - \alpha - \pi, \phi_- + \phi_+ - \alpha)$. Define also the (non-empty) sets

$$\begin{aligned}\mathcal{D}^{\tau}_{n,s} &:= \{\mu \in \mathbb{C}^s \mid \mathcal{C}^{\alpha^{\tau}(n,s;\mu)} \cap \mathcal{A}^{\tau}_0 \neq \emptyset\} \\ &= \{\mu \in \mathbb{C}^s \mid \phi_- + \pi > \alpha > \phi_+ - \pi, |\phi_\tau - \alpha| < \pi, |\phi_\tau + \alpha - \phi_- - \phi_+| < \pi\},\end{aligned}$$

where $\alpha = \alpha^\tau(n, s; \mu, \nu, \lambda)$. *Finally we define*

$$\begin{aligned}
\mathcal{E}_+ &= \{\tau \in \mathbb{C} \mid \frac{1}{2}(\phi_- + \phi_+ - \pi) < \phi_\tau < \phi_- + \pi\},\\
\mathcal{E}_- &= \{\tau \in \mathbb{C} \mid \phi_+ - \pi < \phi_\tau < \frac{1}{2}(\phi_- + \phi_+ + \pi)\},\\
\mathcal{E}_0 &= \{\tau \in \mathbb{C} \mid \frac{1}{2}(\phi_- + \phi_+ - \pi) < \phi_\tau < \frac{1}{2}(\phi_- + \phi_+ + \pi)\}.
\end{aligned}$$

The regions $\mathcal{E}_{+/0/-}$ are depicted in Figure 5.4, where the lines are identical to the lines in Figure 5.2, i.e. the straight lines are $\omega_1\mathbb{R}$, $\omega_2\mathbb{R}$ and the dashed line is $\exp(i(\phi_- + \phi_+ - \pi)/2)\mathbb{R}$. The diagonally shaded area is $\mathcal{E}_0$, while the union of the diagonally shaded area and the horizontally, respectively vertically, shaded area form $\mathcal{E}_-$, respectively $\mathcal{E}_+$.

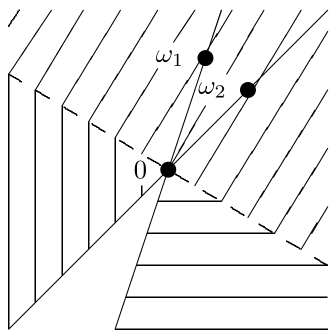


Figure 5.4: The regions $\mathcal{E}_{+/0/-}$

As mentioned before, there are two different kinds of integrals which appear as degenerations of the I_{BC} and $I\!I_{BC}$ integrals. The first kind is characterized by an even integrand and is reminiscent of the $I\!I^m_{n,T}$, respectively $I^m_{n,T}$ integral with the omission of a balancing condition. In particular $I^m_{n,T}$ and $I\!I^m_{n,T}$ are special cases of this kind of integral. The second kind is a Barnes-like integral, as it is reminiscent of the integrals in Barnes' lemma's and Barnes' representation of a ${}_2F_1$ [16, (4.1.1)-(4.1.3)]. Indeed we will prove multivariate hyperbolic analogues of these evaluation formulas in Sections 5.5 and 5.6 as limits of the evaluation formulas of the top level integrals $I^0_{n,T}$ and $I\!I^0_{n,T}$. This dichotomy was already visible in Chapter 4 at the univariate level. Let us now define the first kind of integral.

Definition 5.3.13. *Let $n, s \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$ except for $t = s = 0$. We define*

$$JI_{n,s,t}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^n \frac{\prod_{r=0}^{s-1} \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} c(2tx_j^2) dx_j,$$

for parameters $\mu \in \mathcal{B}_s^\tau$ and $\tau \in \mathcal{E}_\epsilon$, where $\epsilon = -$ for $t < 0$, $\epsilon = 0$ for $t = 0$ and $\epsilon = +$ for $t > 0$. If $t = 0$ we moreover impose the condition $\mu \in \mathcal{D}_{n,s}^\tau$. The even contour $C = -C$ is a deformation of the line $W_{\phi,\phi}$ (for some $\phi \in \mathcal{A}_0$) separating the poles at $x_j = \mu_r - \Lambda$ from those at $x_j = -\mu_r + \Lambda$. Moreover the contour C should be below the contours $\tau - \Lambda + C$. Finally ϕ should satisfy some conditions depending on t, given by

$t < 0$	$t = 0$	$t > 0$
$\phi \in \mathcal{A}_-^\tau$	$\phi \in \mathcal{A}_0^\tau \cap \mathcal{C}^{\alpha^\tau(n,s;\mu)}$	$\phi \in \mathcal{A}_+^\tau$

The conditions $\mu \in \mathcal{B}_s^\tau$ and $\tau \in \mathcal{E}_{+/0/-}$ are imposed to ensure that we can find an appropriate contour. The condition that $\mu \in \mathcal{D}_{n,s}^\tau$ if $t = 0$ ensures we can find a ϕ which satisfy the conditions. If $t = s = 0$ these conditions can not be satisfied, hence the $JI_{n,0,0}$ integral is excluded from this definition.

The condition on ϕ ensures both that the contour moves from the left half cone to the right half cone and that the integral converges. The asymptotic behaviour of the integrand can be estimated in the same way as the asymptotic behaviour of the integrand of the top level integrals. We find that for x_i on a contour C as in the definition we have

$$\left| \frac{\prod_{1\leq j<k\leq n} \Gamma_h(\tau \pm x_j \pm x_k) \prod_{j=1}^n \prod_{r=0}^{s-1} \Gamma_h(\mu_r \pm x_j)}{\prod_{1\leq j<k\leq n} \Gamma_h(\pm x_j \pm x_k) \prod_{j=1}^n \Gamma_h(\pm 2x_j)} c(2t \sum_{j=1}^n x_j^2) \right| \leq K \prod_j f(x_j), \tag{5.3.10}$$

where f is given by

$$f(x) = \begin{cases} \left|c(2tx^2 - 4x((s-2)\omega - \sum_r \mu_r - 2(n-1)\tau))\right| & \text{if } x \in RHC \\ \left|c(2tx^2 + 4x((s-2)\omega - \sum_r \mu_r - 2(n-1)\tau))\right| & \text{if } x \in LHC \\ 1 & \text{else.} \end{cases}$$

and $K > 0$ is some constant (which may depend on C, μ and τ). If $t \neq 0$ the behaviour of the integrand is dominated by the $c(2tx_j^2)$ terms and thus Gaussian. The condition on ϕ then ensures convergence of the integral (by ensuring $\Re(ix^2/\omega_1\omega_2)$ is negative, respectively positive, for all x on the line $W_{\phi,\phi}$, if t is positive, respectively negative). If $t = 0$ the behaviour of the integrand is exponential and again the condition on ϕ ensures the integrand is exponentially decreasing at infinity. Observe that the asymptotic behaviour of the integrand is always exponentially decreasing (given our conditions on the contour) and thus by Cauchy's theorem the value of the integral is independent of the choice of contour.

Being again an iterated integral of the form described in Lemma 5.3.2, we see that $JI_{n,s,t}(\mu;\tau)$ has a unique analytic extension to a meromorphic function on $(\mu,\tau) \in \mathbb{C}^s \times \mathbb{C}$.

Note that the top level integrals are special cases of these integrals. For $\mu \in \mathcal{B}^\tau_{2m+4}$ and $\tau \in \mathcal{E}_0$, satisfying the balancing condition (5.3.5) we have

$$II^m_{n,T}(\mu;\tau) = JI_{n,2m+6,0}(\mu;\tau).$$

In Definition 5.3.6 we previously only defined $II^m_{n,T}(\mu;\tau)$ for τ being a positive linear combinations of ω_1 and ω_2. Note however, that $JI_{n,2m+6,0}(\mu;\tau)$ is defined for more values of τ. We previously restricted the values of τ to simplify the conditions.

Moreover, if $\tau = \omega$ we have $\prod_{\delta,\epsilon\in\{+,-\}} \Gamma_h(\omega+\delta x_j+\epsilon x_k) = 1$ by the reflection equation (5.2.4). Hence those terms disappear from the integrand and we can express $I^m_{n,T}$ for $\mu \in \mathcal{B}^\omega_{2n+2m+2}$ satisfying the balancing condition (5.3.1) as

$$I^m_{n,T}(\mu) = \Gamma_h(\omega)^{-n} JI_{n,2n+2m+4,0}(\mu;\omega) = \Gamma_h(\omega)^{-n} II^{n+m-1}_{n,T}(\mu;\omega). \qquad (5.3.11)$$

Note that $\omega \in \mathcal{E}_0$ so this is well defined. Moreover for $\tau = \omega$ the conditions on the parameters and the integration contour in the definition of $JI_{n,s,t}$ simplify. In particular we do not need to impose the condition that the contour C lies below the contours $\tau - \Lambda + C$ because we don't need to specify the behaviour of the contour around the poles of $\Gamma_h(\omega \pm x_j \pm x_k)$.

For $t \neq 0$ the function $JI_{n,s,t}$ is a meromorphic function on $\mu \in \mathbb{C}^s$. For $t \neq 0$, however, we have defined the function only on $\mu \in \mathcal{D}^\tau_{n,s}$. In this case there exists a unique analytic extension of $JI_{n,s,0}$ to a meromorphic function on $\mu \in \mathbb{C}^s$.

To define the second, Barnes-like integral, we first need to give some new parameter domains, similar to those of Definition 5.3.12

Definition 5.3.14. *Let $n, s_1, s_2 \in \mathbb{Z}_{\geq 0}$. We define*

$$\mathcal{B}^\tau_{s_1,s_2} = \{(\mu,\nu) \in \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \mid \mu_r + \nu_u \neq -j\tau - k\omega_1 - l\omega_2 \text{ for } j,k,l \in \mathbb{Z}_{\geq 0}\}.$$

For $(\mu,\nu,\lambda) \in \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \times \mathbb{C}$ we define the angle

$$\beta^\tau(n,s_1,s_2;\mu,\nu,\lambda) = \arg\Big((s_1+s_2)\omega - \sum_{r=0}^{s_1-1} \mu_r - \sum_{r=0}^{s_2-1} \nu_r - \lambda - 2(n-1)\tau\Big).$$

Define finally the (non-empty) sets

$$\begin{aligned}\mathcal{D}^\tau_{n,(s_1,s_2)} &:= \{(\mu,\nu,\lambda) \in \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \times \mathbb{C} \mid \mathcal{C}^{\beta^\tau(n,s_1,s_2;\mu,\nu,\lambda)} \cap \mathcal{A}^\tau_0 \neq \emptyset\} \\ &= \{(\mu,\nu,\lambda) \in \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \times \mathbb{C} \mid \phi_- + \pi > \beta > \phi_+ - \pi, \\ &\qquad |\phi_\tau - \beta| < \pi, |\phi_\tau + \beta - \phi_- - \phi_+| < \pi\},\end{aligned}$$

for $\beta = \beta^\tau(n,s_1,s_2;\mu;\nu;\lambda)$.

Definition 5.3.15. *For $n, s_1, s_2 \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$ except for $t = s_1 = s_2 = 0$ we define*

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^{n} \prod_{r=0}^{s_1-1} \Gamma_h(\mu_r - x_j) \prod_{r=0}^{s_2-1} \Gamma_h(\nu_r + x_j) c(2\lambda x_j + t x_j^2) dx_j$$

for variables $(\mu,\nu) \in \mathcal{B}^{\tau}_{(s_1,s_2)}$, $\lambda \in \mathbb{C}$ and $\tau \in \mathcal{E}_\epsilon$, where $\epsilon = -$ if $t < -|s_1 - s_2|$, $\epsilon = 0$ if $|t| \leq |s_1 - s_2|$ and $\epsilon = +$ if $t > |s_1 - s_2|$. Moreover if $t + s_1 = s_2$ we impose the condition $(\mu,\nu,-\lambda) \in \mathcal{D}^{\tau}_{n,(s_1,s_2)}$ and if $t + s_2 = s_1$ we have the extra condition $(\mu,\nu,\lambda) \in \mathcal{D}^{\tau}_{n,(s_1,s_2)}$. The contour C is a the hook W_{ϕ_1,ϕ_2} with some indentations separating the poles at $\mu_r - \Lambda$ from the poles at $-\nu_r + \Lambda$, for some (ϕ_1,ϕ_2) which we will specify in a moment. Moreover it should lie below the contours $\tau - \Lambda + C$. The conditions on ϕ_1 and ϕ_2 depend on the sign of $t + s_1 - s_2$ respectively $t - s_1 + s_2$, on τ, and sometimes on μ, ν and λ and are given in the following table

	negative	*zero*	*positive*
$t + s_1 - s_2$	$\phi_1 \in \mathcal{A}^{\tau}_{-}$	$\phi_1 \in \mathcal{A}^{\tau}_{0} \cap \mathcal{C}^{\beta^\tau(n,s_1,s_2;\mu,\nu,-\lambda)}$	$\phi_1 \in \mathcal{A}^{\tau}_{+}$
$t + s_2 - s_1$	$\phi_2 \in \mathcal{A}^{\tau}_{-}$	$\phi_2 \in \mathcal{A}^{\tau}_{0} \cap \mathcal{C}^{\beta^\tau(n,s_1,s_2;\mu,\nu,\lambda)}$	$\phi_2 \in \mathcal{A}^{\tau}_{+}$.

The conditions on μ, ν, λ, τ, ϕ_1 and ϕ_2 again serve the dual purposes of ensuring the integral converges and ensuring that the integration contour can be chosen to separate the poles and moving from infinity in the left hand cone to infinity in the right hand cone. To obtain a contour satisfying the conditions one must perform an algorithm very similar as performed to obtain the contour of the $II^m_{n,T}$ integral of Definition 5.3.6 Using the same techniques as for the calculation of the bound on the integrand of $JI_{n,s,t}$ we can find a bound on the integrand of $JI_{n,(s_1,s_2),t}$ as a product of terms each depending on only one integration variable x_j. This bound is given when all x_j are on the contour by

$$\Bigg| \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=1}^{n} \left(\prod_{r=0}^{s_1-1} \Gamma_h(\mu_r - x_j) \right) \times \prod_{j=1}^{n} \left(\prod_{r=0}^{s_2-1} \Gamma_h(\nu_r + x_j) \right) c(2\lambda x_j + t x_j^2) \Bigg| \leq K \prod_{j=1}^{n} f(x_j),$$

for some $K > 0$ depending only on μ, ν, λ, τ and the contour C, where

$$f(x) = \begin{cases} \left| c\Big((t - s_1 + s_2)x^2 - 2x((s_1+s_2)\omega - \sum_r \mu_r - \sum_r \nu_r - 2(n-1)\tau - \lambda) \Big) \right| & \text{if } x_j \in RHC, \\ \left| c\Big((t + s_1 - s_2)x^2 + 2x((s_1+s_2)\omega - \sum_r \mu_r - \sum_r \nu_r - 2(n-1)\tau + \lambda) \Big) \right| & \text{if } x_j \in LHC, \\ 1 & \text{else.} \end{cases}$$

From this bound we can see that for $x_j \in RHC$, the asymptotic behaviour is in first order determined by the sign of $t - s_1 + s_2$. This provides the conditions on ϕ_2. Similarly the asymptotic behaviour of the integrand in the left hand cone decides the conditions on ϕ_1. The parameter conditions thus ensure that the integrand decays exponentially at infinity. Again, using Lemma 5.3.2 we can extend $JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau)$ uniquely to a meromorphic function on $(\mu,\nu,\lambda,\tau) \in \mathbb{C}^{s_1} \times \mathbb{C}^{s_2} \times \mathbb{C} \times \mathbb{C}$.

Note that the conditions are such that if $|t| < |s_1 - s_2|$ the contour can not be chosen to be a deformed line, i.e. ϕ_1 and ϕ_2 have to be different. This is the principal reason why we are forced to consider hook contours.

We can immediately derive some symmetries of $JI_{n,s,t}$ and $JI_{n,(s_1,s_2),t}$, which allow us later on to reduce the number of different cases we have to consider.

Proposition 5.3.16. *We have for $n, s \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$ and μ and τ as in Definition 5.3.13*

$$JI_{n,s,t}(\mu;\tau;\omega_1,\omega_2) = \overline{JI_{n,s,-t}(-\bar{\mu};-\bar{\tau};-\bar{\omega}_1,-\bar{\omega}_2)}. \tag{5.3.12}$$

For $n, s_1, s_2 \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$ and μ, ν, λ and τ as in Definition 5.3.15 and arbitrary $a \in \mathbb{C}$ we moreover have

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau;\omega_1,\omega_2) = \overline{JI_{n,(s_2,s_1),-t}(-\bar{\nu};-\bar{\mu};-\bar{\lambda};-\bar{\tau};-\bar{\omega}_1,-\bar{\omega}_2)}, \tag{5.3.13}$$

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = JI_{n,(s_1,s_2),t}(\mu - a;\nu + a;\lambda + ta;\tau)c(n(2\lambda a + ta^2)), \tag{5.3.14}$$

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = JI_{n,(s_2,s_1),t}(\nu;\mu;-\lambda;\tau). \tag{5.3.15}$$

Proof. Note that for all identities of this proposition the conditions on the variables μ and τ or μ, ν, λ and τ are such that the integrals on the left hand side of each equation is defined if and only if the integral on the right hand side of the equation is defined.

Equations (5.3.12) and (5.3.13) are special cases of the fact that for any function $f(z)$, integrable over a contour γ, we have

$$\int_\gamma f(z)dz = \overline{\int_{\bar{\gamma}} \overline{f(\bar{z})}dz},$$

where $\bar{\gamma}$ denotes the contour obtained from γ by reflection in the real line (i.e. if $\gamma(t)$ is a parametrisation of γ, the function $\bar{\gamma}(t) = \overline{\gamma(t)}$ is a parametrisation of $\bar{\gamma}$). To calculate the complex conjugate of the integrand, we apply Proposition 5.2.4.

The last two symmetries of the $JI_{n,(s_1,s_2),t}$ integral are obtained by applying simple changes of the integration variables, respectively $x_j \to x_j + a$ and $x_j \to -x_j$, and changing the contour along with this change in integration variables (i.e. shifting it by a or mirroring it in the origin), while avoiding the poles of the integrand on the right side. The new contour does not necessarily follow a hook contour at infinity, so in order for the new integral to satisfy the conditions of Definition 5.3.15 we have to shift the contour away from the origin (so we do not have problems with having to avoid poles) to such a hook contour, which is allowed by Cauchy's theorem and the exponential decay of the integrand. □

Inspired by (5.3.11) showing that $I^m_{n,T}$ is a special case of the $II^m_{n,T}$ integral it is convenient to introduce a new notation for the integrals with $\tau=\omega$.

Definition 5.3.17. *For $n,s,t\in\mathbb{Z}$, excluding the case $s=t=0$ and $\mu\in\mathcal{B}^\omega_s$ and if $t=0$ also $\mu\in\mathcal{D}^\omega_{n,s}$ we define*

$$J_{n,s,t}(\mu)=\Gamma_h(\omega)^{-n}JI_{n,s,t}(\mu;\omega). \tag{5.3.16}$$

For $n,s_1,s_2,t\in\mathbb{Z}$, excluding the case $s_1=s_2=t=0$ and $(\mu,\nu)\in\mathcal{B}^\omega_{(s_1,s_2)}$ and $\lambda\in\mathbb{C}$ and if $s_2=s_1+t$ also $(\mu,\nu,-\lambda)\in\mathcal{D}^\omega_{n,(s_1,s_2)}$ and if $s_1=s_2+t$ also $(\mu,\nu,\lambda)\in\mathcal{D}^\tau_{n,(s_1,s_2)}$ we define

$$J_{n,(s_1,s_2),t}(\mu;\nu;\lambda)=\Gamma_h(\omega)^{-n}JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\omega) \tag{5.3.17}$$

Due to the reflection equation $\Gamma_h(\omega\pm x_j\pm x_k)=1$ and therefore in the integrand of the $J_{n,s,t}$ integral is the integrand of the $JI_{n,s,t}$ integral without the $\Gamma_h(\tau\pm x_j\pm x_k)$ terms. Similarly the integrand of the $J_{n,(s_1,s_2),t}$ integral is the integrand of $JI_{n,(s_1,s_2),t}$ without the $\Gamma_h(\tau\pm(x_j-x_k))$ terms.

Using the newly-defined function $J_{n,s,t}(\mu)$ we can express the $I^m_{n,T}$ integral as

$$I^m_{n,T}(\mu)=J_{n,2n+2m+4,0}(\mu),$$

for $\mu\in\mathcal{B}^\omega_{2n+2m+4}$ satisfying the balancing condition (5.3.1).

5.3.4 Some evaluation formulas

The integrands of the $J_{n,0,t}$ and $J_{n,(0,0),t}$ integrals can be expressed as a product of sines and Gaussians. As such they can be evaluated explicitly. For the $JI_{n,0,t}$ and $JI_{n,(0,0),t}$ integrals there remain some τ-dependent terms which can not in general be expressed in terms of sines. However suitable specializations of τ (for example $\tau=\omega$, which reduces us to the $J_{n,0,t}$ and $J_{n,(0,0),t}$ case) does allow us to write the integrand as a product of sines and Gaussians, after which we can evaluate it.

The significance of the formulas in this subsection is not just that it provides us with these evaluations, but also that it shows when our integrals are non-zero. Showing that the integrals we consider are generically non-zero is non-trivial. In particular it is impossible to find specific values of the variables μ (or μ, ν and λ in the case of $JI_{n,(s_1,s_2),t}$) , τ, ω_1, ω_2 for which the integrand of $JI_{n,s,t}$ (or $II^{(s_1,s_2)}_{n,t}$) is positive real on the real line, unless $t=0$.

First we obtain an evaluation formula for the $J_{n,0,t}$ integral.

Proposition 5.3.18. *We have for $t > 0$*

$$J_{n,0,t}(-) = \frac{1}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \frac{1}{\prod_{1\leq j<k\leq n} \Gamma_h(\pm x_j \pm x_k) \prod_{j=1}^n \Gamma_h(\pm 2x_j)} \times c(2t\sum_{j=1}^n x_j^2) \prod_{j=1}^n dx_j \tag{5.3.18}$$

$$= \frac{e(-\frac{3n}{8})}{\sqrt{t}^n} c\left(-\frac{n(n+1)(2n+1)(\omega_1^2+\omega_2^2)}{3t}\right) \times \prod_{1\leq j<k\leq n} 4\sin\left(\frac{\pi(j+k)}{t}\right)\sin\left(\frac{\pi(j-k)}{t}\right)\prod_{j=1}^n 2\sin\left(\frac{2\pi j}{t}\right),$$

where the contour C is a hook W_{ϕ_1,ϕ_2} with $\phi_1, \phi_2 \in \mathcal{A}_+$.

Proof. Using the Vandermonde-like determinant identity, equivalent to Weyl's character formula for type C_n,

$$\det\{2\sin\left(\frac{2\pi j x_k}{z}\right)\}_{1\leq j,k\leq n} = \prod_{1\leq j<k\leq n} 4\sin\left(\frac{\pi(x_j\pm x_k)}{z}\right)\prod_{j=1}^n 2\sin\left(\frac{2\pi x_j}{z}\right), \tag{5.3.19}$$

and (5.3.2) we can rewrite the integrand of our integral as

$$\frac{1}{\prod_{j<k}\Gamma_h(\pm x_j\pm x_k)\prod_{j=1}^n \Gamma_h(\pm 2x_j)}\prod_{j=1}^n c(2tx_j^2)$$
$$= \prod_{j=1}^n(-c(2tx_j^2)) \prod_{\omega'\in\{\omega_1,\omega_2\}} \left(\prod_{j<k} 4\sin\left(\frac{\pi(x_j+x_k)}{\omega'}\right)\sin\left(\frac{\pi(x_j-x_k)}{\omega'}\right) \times \prod_{j=1}^n 2\sin\left(\frac{2\pi x_j}{\omega'}\right)\right)$$
$$= \det\left\{2\sin\left(\frac{2\pi j x_k}{\omega_1}\right)\right\}_{j,k} \det\left\{2\sin\left(\frac{2\pi j x_k}{\omega_2}\right)\right\}_{j,k} \prod_{j=1}^n\left(-c(2tx_j^2)\right).$$

Now we can calculate the integral using the identity (see e.g. [4])

$$\int_{C^n} \det\{f_j(x_k)\}_{j,k}\det\{g_j(x_k)\}_{j,k}\prod_k h(x_k)dx_k = n!\det\left\{\int_C f_j(x)g_k(x)h(x)dx\right\}_{j,k}, \tag{5.3.20}$$

where in our case $f_j(x) = 2\sin(\frac{2\pi jx}{\omega_1})$, $g_j(x) = 2\sin(\frac{2\pi jx}{\omega_2})$ and $h(x) = -c(2tx^2)$.

Using a simple Gaussian evaluation after expressing the sines as sums of exponentials yields

$$-\int_C 2\sin\left(\frac{2\pi jx}{\omega_1}\right) 2\sin\left(\frac{2\pi kx}{\omega_2}\right) c(2tx^2)dx \\ = -4\sqrt{\frac{\omega_1\omega_2}{it}}c\left(-\frac{2(j^2\omega_1^2+k^2\omega_2^2)}{t}\right)\sin\left(\frac{2\pi jk}{t}\right).$$

Therefore we find that the complete integral equals

$$\begin{aligned} &J_{n,0,t}(-) \\ &= \frac{1}{\sqrt{-\omega_1\omega_2}^n 2^n n!} n! \det\left\{-4\sqrt{\frac{\omega_1\omega_2}{it}}c\left(-\frac{2(j^2\omega_1^2+k^2\omega_2^2)}{t}\right)\sin\left(\frac{2\pi jk}{t}\right)\right\}_{j,k} \\ &= \frac{1}{\sqrt{-\omega_1\omega_2}^n 2^n}\left(-2\sqrt{\frac{\omega_1\omega_2}{it}}\right)^n \\ &\quad \times c\left(-\frac{2(\omega_1^2+\omega_2^2)\sum_{j=1}^n j^2}{t}\right)\det\left\{2\sin\left(\frac{2\pi jk}{t}\right)\right\}_{j,k} \\ &= \frac{e\left(-\frac{3n}{8}\right)}{\sqrt{t}^n}c\left(-\frac{n(n+1)(2n+1)(\omega_1^2+\omega_2^2)}{3t}\right) \\ &\quad \times \prod_{j<k} 4\sin\left(\frac{\pi(j+k)}{t}\right)\sin\left(\frac{\pi(j-k)}{t}\right)\prod_{j=1}^n 2\sin\left(\frac{2\pi j}{t}\right), \end{aligned}$$

where we use (5.3.19) once more in the final equality. □

A similar equation to this one can be obtained for $t < 0$, for example by using the symmetry (5.3.12). Indeed we have for $t < 0$

$$J_{n,0,t}(-) = \frac{e(-\frac{n}{8})}{\sqrt{-t}^n}c\left(-\frac{n(n+1)(2n+1)(\omega_1^2+\omega_2^2)}{3t}\right) \\ \times \prod_{1\leq j<k\leq n} 4\sin\left(\frac{\pi(j+k)}{t}\right)\sin\left(\frac{\pi(j-k)}{t}\right)\prod_{j=1}^n 2\sin\left(\frac{2\pi j}{t}\right).$$

Note that for $0 < t \leq 2n$ one of the sines on the right hand side of (5.3.18) becomes zero (the argument becomes 2π). Hence the integral $J_{n,0,t}(-)$ vanishes if $0 < t \leq 2n$, and likewise if $-2n \leq t < 0$ (recall that the integral does not exist for $s = t = 0$). On the other hand Proposition 5.3.18 shows that $J_{n,0,t}(-)$ does not vanish if $|t| > 2n$.

We also obtain an evaluation formula for the $J_{n,(0,0),t}$ integral.

Proposition 5.3.19. *We have for $t > 0$*

$$\begin{aligned} &J_{n,(0,0),t}(-) \\ &:= \frac{1}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n} \frac{1}{\prod_{1\le j<k\le n}\Gamma_h(\pm(x_j-x_k))} c(2\lambda\sum_{j=1}^n x_j + t\sum_{j=1}^n x_j^2)\prod_{j=1}^n dx_j \\ &= \frac{e(-\frac{n^2}{8})\sqrt{2}^n}{\sqrt{t}^n} c\left(-\frac{n\lambda^2}{t}\right) c\left(-\frac{(n^3-n)(\omega_1^2+\omega_2^2)}{3t}\right)\prod_{j=1}^n \left(2\sin\left(\frac{2\pi j}{t}\right)\right)^{n-j}, \end{aligned}$$

where the contour C is a hook W_{ϕ_1,ϕ_2} with $\phi_1,\phi_2 \in \mathcal{A}_+$.

Proof. The derivation is completely similar to the proof of the previous proposition. This time we apply the Vandermonde determinant formula

$$\det\left\{e\left(\frac{((n+1)/2-j)x_k}{z}\right)\right\}_{1\le j<k\le n} = e\left(\frac{n(n-1)}{8}\right)\prod_{1\le j<k\le n} 2\sin\left(\frac{\pi(x_j-x_k)}{z}\right), \quad (5.3.21)$$

and use (5.3.20) for $f_j(x) = e\left(\frac{((n+1)/2-j)x}{\omega_1}\right)$, $g_j(x) = e\left(\frac{((n+1)/2-j)x}{\omega_2}\right)$ and $h(x) = c(2\lambda x + tx^2)$. In the end we use moreover the simplification

$$\prod_{1\le j<k\le n} 2\sin\left(\frac{\pi(k-j)}{z}\right) = \prod_{j=1}^n \left(2\sin\left(\frac{\pi j}{z}\right)\right)^{n-j}. \qquad \square$$

Again there exists a similar identity for $t < 0$. This expression also allows us to see when $J_{n,(0,0),t}(-)$ vanishes. If t is even, this integral vanishes if $|t| < 2n$, while for odd t the integral vanishes if $|t| < n$.

We would like to have similar evaluation formulas for $JI_{n,0,t}(-;\tau)$, but without specializing τ the integrand can not be written as product of sines and Gaussians, and hence not evaluated with the methods as discussed above for the J integrals. Setting $\tau = \omega$ reduces $JI_{n,0,t}(-;\omega)$ to $J_{n,0,t}(-)$ and thus provides us with an evaluation formula. However this evaluation does not suffice for our purposes, since it turns out to vanish for more values of t than the generic $JI_{n,0,t}(-;\tau)$ integral does.

Specializing τ to $k\omega_1/2 + l\omega_2/2$ ($k, l \in \mathbb{Z}_{>0}$) allows us to write the cross terms $\prod_{j<k}\Gamma_h(\tau \pm x_j \pm x_k)$ as a product of sine functions. Indeed applying the reflection equation (5.2.4) and the elementary difference equations (5.2.1) we obtain

$$\begin{aligned} \Gamma_h(k\frac{\omega_1}{2}+l\frac{\omega_2}{2}+z, k\frac{\omega_1}{2}+l\frac{\omega_2}{2}-z) &= (-1)^{(k-1)(l-1)}\prod_{r=1}^{k-1} 2\sin(\frac{\pi((k-2r)\omega_1+l\omega_2+2z)}{2\omega_2}) \\ &\quad\times\prod_{r=1}^{l-1} 2\sin(\frac{\pi(k\omega_1+(l-2r)\omega_2+2z)}{2\omega_1}). \end{aligned}$$

Note that the special case $k = l = 1$ corresponds to $\tau = \omega$, i.e. the $J_{n,0,t}(-)$ integral. For τ equal to $k\omega_1$ or $k\omega_2$ ($k \in \mathbb{Z}_{>0}$) we also obtain an integrand of a Gaussian times some sine functions using just the difference equations (5.2.1)

$$\prod_{\delta,\epsilon\in\{+,-\}} \frac{\Gamma_h(k\omega_1 + \delta x_j + \epsilon x_k)}{\Gamma_h(\delta x_j + \epsilon x_k)} = \prod_{r=0}^{k-1} \prod_{\delta,\epsilon=\pm 1} 2\sin\left(\frac{\pi(r\omega_1 + \delta x_j + \epsilon x_k)}{\omega_2}\right), \quad (5.3.22)$$

and similarly with $\omega_1 \leftrightarrow \omega_2$. A similar equation exists for $\prod_{\epsilon\in\{+,-\}} \Gamma_h(k\omega_1 + \epsilon(x_j - x_k))/\Gamma_h(\epsilon(x_j - x_k))$, when considering evaluations of $JI_{n,(0,0),t}$. In principle we can calculate the value of the integral for all these values of τ (given n and t) as a sum of products of elementary functions. However the resulting integrals quickly become intangible due to the overabundance of sine terms, and it is not clear if and how this expression factors in terms of a single product of elementary functions.

We therefore restrict to considering to an explicit evaluation formula for the $JI_{n,(0,0),t}(-;-;\lambda;\omega_1)$ integral, which does give a simple expression.

Proposition 5.3.20. *We have for $t > 0$*

$$\begin{aligned} &JI_{n,(0,0),t}(-;-;\lambda;\omega_1) \\ &\quad = \frac{1}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\omega_1 \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=1}^n c(2\lambda x_j + t x_j^2) dx_j \\ &\quad = \sqrt{\frac{2}{t}}^n e\left(-\frac{n^2}{8}\right) c\left(-\frac{n\lambda^2}{t}\right) c\left(-\frac{2(n^3-n)\omega_1^2}{3t}\right) \prod_{j=1}^n \left(2\sin\left(\frac{\pi j \omega_1}{\omega_2 t}\right)\right)^{n-j}. \end{aligned}$$

Proof. The proof is similar to the proofs of Propositions 5.3.18 and 5.3.19. We use the difference equations (5.2.1) of the hyperbolic gamma function and the Vandermonde determinant identity (5.3.21) to express the integrand as

$$\begin{aligned} &\prod_{1\le j<k\le n} \frac{\Gamma_h(\omega_1 \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=1}^n c(2\lambda x_j + t x_j^2) \\ &\qquad = \left(\det\{e\left(\frac{((n+1)/2 - j)x_k}{\omega_2}\right)\}_{j,k}\right)^2 \prod_{j=1}^n c(2\lambda x_j + t x_j^2), \end{aligned}$$

and perform a similar calculation as before, using (5.3.20) with $f_j(x) = g_j(x) = e\left(\frac{((n+1)/2-j)x}{\omega_2}\right)$ and $h(x) = c(2\lambda x + t x^2)$. □

Note that the product of sines on the right hand side of the evaluation formula of Proposition 5.3.20 is now non-zero for all values of $t > 0$ if $\omega_1/\omega_2 \notin \mathbb{Q}$. Again a similar evaluation formula exists for $JI_{n,(0,0),t}(-;-;\lambda;\omega_1)$ for $t < 0$.

For the $JI_{n,0,t}$ integrals we are unable to find a simple evaluation formula (i.e. factoring as a product of elementary functions) for the integrals for any special value of τ other than ω. However, we can still show $II_{n,t}^0$ is non-zero in the special case $\tau = \omega_1$.

Proposition 5.3.21. *The integral $JI_{n,0,t}(-;\tau) \neq 0$ as a meromorphic function of ω_1, ω_2 and τ if $|t| > 2$.*

Proof. It suffices to show that $JI_{n,0,t}(-;\omega_1)$ is non-zero for generic values of ω_1 and ω_2. In this case we perform the calculations as for Propositions 5.3.18, 5.3.19 and 5.3.20. However we get stuck on evaluating the final determinant, but we will show without explicitly evaluating it, that this determinant is generically non-zero.

We only consider the case $t > 0$, for $t < 0$ the results are analogous. We use (5.3.22) (for $k = 1$), the determinental formulas (5.3.19) and the following determinant, which is equivalent to Weyl's characters formula of type D_n,

$$\det\left\{2\cos\left(\frac{2\pi(j-1)x_k}{z}\right)\right\}_{j,k} = 2\prod_{j<k} 4\sin\left(\frac{\pi(x_j \pm x_k)}{z}\right),$$

to write the integrand of $JI_{n,0,t}(-;\omega_1)$ as

$$(-1)^n \det\left\{\cos\left(\frac{2\pi(j-1)x_k}{\omega_2}\right)\right\}_{j,k} \det\left\{\sin\left(\frac{2\pi j x_k}{\omega_2}\right)\right\}_{j,k} \prod_{j=1}^n 2\sin\left(\frac{2\pi x_j}{\omega_1}\right) c(2tx_j^2).$$

Now we apply (5.3.20) and perform a calculation as before to arrive at

$$\begin{aligned} JI_{n,0,t}(-;\omega_1) &= \frac{\Gamma_h(\omega_1)^n 2^{n-1} e(\frac{2n^2-5n+4}{8})}{\sqrt{t}^n} c\left(\frac{-n}{6t}((2n^2+1)\omega_1^2 + 3\omega_2^2)\right) \\ &\times \det\left\{y^{(j-1)k}\sin(\frac{2\pi(j-1-k)}{t}) + y^{-(j-1)k}\sin(\frac{2\pi(j-1+k)}{t})\right\}_{j,k}, \end{aligned}$$

where $y = e(\omega_1/t\omega_2)$. The determinant in this formula is clearly a Laurent polynomial in y. The highest power of y is only obtained by multiplying the positive powers of y along the diagonal (due to the rearrangement inequality [22, Theorem 368]). In particular its coefficient is non-zero if the coefficients of the positive powers of y are non-zero on the diagonal. This holds if $\sin(2\pi(j-1-k)/t)$ is non-zero if $j = k$. For $j = k$ this equals $\sin(-2\pi/t)$ and hence is non-zero if $t > 2$. We conclude that the determinant, and thus also $JI_{n,0,t}$ is generically non-zero. □

5.3.5 Limits of the integrals

In order to calculate the explicit degenerations of the top level integral later on we need to be able to calculate the limits of the $JI_{n,s,t}$ and $JI_{n,(s_1,s_2),t}$ integrals. There are basically three types of limits, those between $JI_{n,s,t}$ integrals themselves, between $JI_{n,s,t}$ integrals and $JI_{n,(s_1,s_2),t}$ integrals and limits between $JI_{n,(s_1,s_2),t}$ integrals themselves.

To accommodate for possible balancing conditions on the variables (such as in the definition of the $I^m_{n,T}$ and $II^m_{n,T}$ integrals) we include not only limits where one variable goes to infinity, but also those where two variables go to infinity in opposite direction (thereby ensuring that the sum of the variables remains constant in the limit and thus a balancing condition can survive the limit). Those limits could also be obtained by consecutively sending one parameter to infinity, but obtaining the limits by such an iterative process does not suffice for our application, since the separate limits would violate the balancing condition.

We first consider the limits of $JI_{n,s,t}$ integrals to $JI_{n,s,t}$ integrals. Recall the definitions of $c(z)$ from (5.1.3) and of ζ from (5.1.4).

Proposition 5.3.22. *We have for $n, s \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$*

$$
\begin{aligned}
JI_{n,s,t}(\mu;\tau) &= \lim_{S\to\infty} JI_{n,s+1,t-1}(\mu,\xi+qS;\tau)\zeta^{2n}c(-2n(\xi+qS-\omega)^2),\\
JI_{n,s,t}(\mu;\tau) &= \lim_{S\to\infty} JI_{n,s+2,t}(\mu,\xi_1+qS,\xi_2-qS;\tau)\\
&\qquad\qquad\qquad \times c(2n((\xi_2-qS-\omega)^2-(\xi_1+qS-\omega)^2),
\end{aligned}
$$

for parameters $q=\exp(i\phi)$, $\mu\in\mathcal{B}_s^\tau$, $\xi\in\mathbb{C}$, $(\xi_1,\xi_2)\in\mathcal{B}_2^\tau$ and $\tau\in\mathcal{E}_\epsilon$ where $\epsilon=-$ if $t<0$, $\epsilon=0$ if $t=0$ and $\epsilon=+$ if $t>0$. In the first limit, if $t=1$, we moreover insist on $\tau\in\mathcal{E}_0$. ϕ has to be contained in some interval depending on the sign of t and the value of τ, μ and $\xi_{1/2}$, which is given in the following table

	$t<0$	$t=0$	$t>0$
first limit:	$\mathcal{A}_-^\tau$	$\mathcal{A}_-^\tau\cap\mathcal{C}^{\alpha^\tau(n,s;\mu)}$	$\mathcal{A}_+^\tau$
second limit:	$\mathcal{A}_-^\tau$	$\mathcal{A}_0^\tau\cap\mathcal{C}^{\alpha^\tau(n,s;\mu)}\cap\mathcal{C}^{\alpha^\tau(n,s+2;\mu,\xi_1,\xi_2)}$	$\mathcal{A}_+^\tau$

Moreover, the first limit also holds if $\phi=\frac{\phi_-+\phi_+-\pi}{2}$, $\mu\in\mathcal{D}_{n,s}^\tau$ and τ and ξ are positive linear combinations of ω_1 and ω_2.

Proof. The proof of this proposition basically consists of showing that limit and integral can be interchanged and then applying the asymptotics of the hyperbolic gamma function from Theorem 5.2.6. It is quite similar to the proof of Proposition 4.4.8, though more technically involved. Indeed, the proof that limit and integral can be interchanged is so technical that we will defer it to Appendix 5.A. Here we will proceed by giving a formal proof of the first limit. The S-dependent part of the integrand of $JI_{n,s+1,t-1}(\mu,\xi+qS)$ is given by $\prod_{j=1}^n\Gamma_h(\xi+qS\pm x_j)$. The pointwise (i.e. for fixed x) asymptotic behaviour of this term is given by

$$
\begin{aligned}
1 &= \lim_{S\to\infty}\prod_{j=1}^n \frac{\Gamma_h(\xi+qS+x_j)}{\zeta^{-1}c((\xi+qS+x_j-\omega)^2)}\frac{\Gamma_h(\xi+qS-x_j)}{\zeta^{-1}c((\xi+qS-x_j-\omega)^2)} \qquad (5.3.23)\\
&= \lim_{S\to\infty}\zeta^{2n}\prod_{j=1}^n c(-(\xi+qS+x_j-\omega)^2-(\xi+qS-x_j-\omega)^2)\Gamma_h(\xi+qS\pm x_j)\\
&= \lim_{S\to\infty}\zeta^{2n}\prod_{j=1}^n c(-2(\xi+qS-\omega)^2-2x_j^2)\Gamma_h(\xi+qS\pm x_j)\\
&= \lim_{S\to\infty}\zeta^{2n}c(-2n(\xi+qS-\omega)^2\prod_{j=1}^n c(-2x_j^2)\Gamma_h(\xi+qS\pm x_j).
\end{aligned}
$$

Apart from giving the correct renormalization term $\zeta^{2n}c(-2n(\xi+qS-\omega)^2)$ with which to multiply $JI_{n,s+1,t-1}(\mu,\xi+qS;\tau)$ in order to obtain a nice limit, this calculation also shows that while s decreases by one, since the Gamma functions containing $\xi+qS$ disappear in the limit, t increases by one due to the appearance of the Gaussian term in the limit. □

For each n, s and t there are values of μ, ξ, ξ_1, ξ_2, τ and ϕ satisfying all the required conditions. For example for the first equation we must have $\alpha^\tau(n,s;\mu) \in (\frac{\phi_-+\phi_++\pi}{2}, \phi_-+\pi)$ to ensure that $\mathcal{A}_-^\tau \cap \mathcal{C}^{\alpha^\tau(n,s;\mu)} \neq \emptyset$.

The parameter domains are basically such that the integral on the left hand side and the integrals within the limit on the right hand side of the equations in the proposition are well-defined. Moreover the angle ϕ is chosen such that the parameters going to infinity and their reciprocals (i.e. $\pm(\xi+qS)$ or $\pm(\xi_1+qS)$ and $\pm(\xi_2-qS)$) stay within a finite (i.e. S-independent) distance of the integration contour. This ensures first of all that the contour is only deformed around an S-independent number of poles in the limit, and secondly that we can apply the asymptotic formula for the hyperbolic gamma function on the integrand over the contour. In particular in both limits ϕ should be such that the integration contours (of the integrals on both sides of the equation) can be chosen as deformations of the straight line $W_{\phi,\phi}$.

By taking the complex conjugate of the first limit in the proposition and applying (5.3.12) we obtain a similar equation in which $JI_{n,s,t}$ is given as a limit of $JI_{n,s+1,t+1}$, in particular

$$JI_{n,s,t}(\mu;\tau) = \lim_{S\to\infty} JI_{n,s+1,t+1}(\mu,\xi-qS;\tau)\zeta^{-2n}c(2n(\xi-qS-\omega)^2),$$

under suitable conditions.

Observe that the second limit of Proposition 5.3.22 (i.e. the limit expressing $JI_{n,s,t}$ as a limit of $JI_{n,s+2,t}$) allows us to remove a balancing condition in the degeneration process. Suppose the variables of the $JI_{n,s+2,t}$ integral have to satisfy $\sum_{r=0}^{s+1}\mu_r = K$ for some constant K. For the variables in the limit this translates into the condition $\sum_{r=0}^{s-1}\mu_r+\xi_1+\xi_2 = K$. Since the limit is independent of ξ_1 and ξ_2 this implies there is no condition anymore on the variables of the $JI_{n,s,t}$ integral as the degeneration, as we can always choose suitable ξ_1 and ξ_2 to satisfy the balancing condition.

Now we consider the limit from a $JI_{n,s,t}$ integral to a $JI_{n,(s_1,s_2),t}$ integral.

Proposition 5.3.23. *For $n,s\in\mathbb{Z}_{\geq 0}$ and $t\in\mathbb{Z}$ such that $s+t$ is even and $|t|\leq s$, let $s_1=\frac{s-t}{2}$ and $s_2=\frac{s+t}{2}$. Then we have*

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = \lim_{S\to\infty} JI_{n,s,t}(\mu+qS,\nu-qS;\tau)G(s),$$

where

$$G(s) = \zeta^{-nt}c(n(2tq^2S^2-4qS(\lambda-(n-1)\tau-2\omega)-\sum_{r=0}^{s_1-1}(\mu_r-\omega)^2+\sum_{r=0}^{s_2-1}(\nu_r-\omega)^2)),$$

and $\lambda = 2(n-1)\tau + \sum_r \mu_r + \sum_r \nu_r + (4-s)\omega$, $\tau \in \mathcal{E}_0$ *and* $(\mu,\nu,\lambda) \in \mathcal{B}^\tau_{s_1,s_2} \cap \mathcal{D}^\tau_{n,s_1,s_2}$. *Here the notation* $\mu + qS$ *denotes* $(\mu_0 + qS, \mu_1 + qS, \ldots, \mu_{s_1-1} + qS)$ *and* $\nu - qS = (\nu_0 - qS, \ldots, \nu_{s_2-1} - qS)$. *Finally* $q = \exp(i\phi)$ *under the following conditions on* ϕ*, depending on the value of* t*,* τ*,* μ *and* ν*.*

$t>0$	$t=0$	$t<0$
$\mathcal{A}^\tau_+$	$\mathcal{A}^\tau_0 \cap \mathcal{C}^{\beta^\tau(n,s;\mu,\nu,-\lambda)}$	$\mathcal{A}^\tau_-$

Proof. Like the proof of Proposition 5.3.22 we will defer the technical details of the proof to Appendix 5.A. The proof (in the univariate case) is rather similar to the proof of Proposition 4.4.5. The idea is that the location of the maximum of the absolute value of the integrand changes with S, in particular the maximum is located at approximately $x = (\epsilon_1 qS, \ldots, \epsilon_n qS)$ for $\epsilon_j = \pm$. Due to evenness of the integrand of $J_{n,s,t}$ we only have to consider the part of the integral in the quadrant where all integration variables are in the right half cone. This provides us with a multiplicative factor of 2^n, which is incorporated in the definition of $JI_{n,(s_1,s_2),t}$. Performing the change of variables $x_j \to x_j + qS$ and interchanging limit and integral, now gives the desired result.

Assuming interchanging limit and integral is allowed and the contour is a straight line, we can then perform the following calculation, which formally shows the correctness of the limit. Let $s_1 = \frac{s-t}{2}$ and $s_2 = \frac{s+t}{2}$. We define

$$F_1(\mu,\nu;x) = \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=1}^n \prod_{r=0}^{s_1-1} \Gamma_h(\mu_r - x_j) \prod_{r=0}^{s_2-1} \Gamma_h(\nu_r + x_j),$$

which is the integrand of $JI_{n,(s_1,s_2),t}$ except for the exponential terms. Observe that $F_1(\mu + qS, \nu - qS; x + qS) = F_1(\mu,\nu;x)$. Let moreover

$$F_2(\mu,\nu;x) = \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j + x_k))}{\Gamma_h(\pm(x_j + x_k))} \times \prod_{j=1}^n \frac{\prod_{r=0}^{s_1-1} \Gamma_h(\mu_r + x_j) \prod_{r=0}^{s_2-1} \Gamma_h(\nu_r - x_j)}{\Gamma_h(\pm 2x_j)} c(2tx_j^2),$$

such that the integrand of $JI_{n,s,t}(\mu,\nu;\tau)$ equals $F_1(\mu,\nu;x)F_2(\mu,\nu;x)$. Now we can calculate the limit by first restricting to a single quadrant, then performing the change in variable $x_j \to x_j + qS$ and finally interchanging limit and integral, i.e.

$$
\begin{aligned}
&\lim_{S\to\infty} \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} F_1(\mu+qS,\nu-qS;x)F_2(\mu+qS,\nu-qS;x)G(S)dx_j \\
&= \lim_{S\to\infty} \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{[0,q\infty)^n} F_1(\mu+qS,\nu-qS;x)F_2(\mu+qS,\nu-qS;x)G(S)dx_j \\
&= \lim_{S\to\infty} \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{[-qS,q\infty)^n} F_1(\mu+qS,\nu-qS;x+qS) \\
&\qquad\qquad \times F_2(\mu+qS,\nu-qS;x+qS)G(S)dx_j \\
&= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{(-q\infty,q\infty)^n} \lim_{S\to\infty} F_1(\mu,\nu;x)F_2(\mu+qS,\nu-qS;x+qS)G(S)dx_j.
\end{aligned}
$$

To find the limit of the integrand we note, using the asymptotics of the hyperbolic gamma function, see Theorem 5.2.5, to substitute $\Gamma_h(z+2qS)$ by $q^{-1}c((z+2qS-\omega)^2)$ and $\Gamma_h(z-2qS)$ by $qc((z-2qS-\omega)^2)$, that

$$
\begin{aligned}
&\lim_{S\to\infty} G(S)F_2(\mu+qS,\nu-qS;x+qS) \\
&= \lim_{S\to\infty} G(S) \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau\pm(x_j+x_k+2qS))}{\Gamma_h(\pm(x_j+x_k+2qS))} \\
&\quad \times \prod_{j=1}^n \frac{\prod_{r=0}^{s_1-1}\Gamma_h(\mu_r+x_j+2qS)\prod_{r=0}^{s_2-1}\Gamma_h(\nu_r-x_j-2qS)}{\Gamma_h(\pm(2x_j+2qS))} c(2t(x_j+qS)^2) \\
&= G(S) \prod_{1\le j<k\le n} \frac{c((\tau+x_j+x_k+2qS-\omega)^2-(\tau-x_j-x_k-2qS-\omega)^2)}{c((x_j+x_k+2qS-\omega)^2-(-x_j-x_k-2qS-\omega)^2)} \\
&\quad \times \prod_{j=1}^n \frac{\prod_{r=0}^{s_1-1}\zeta^{-1}c((\mu_r+x_j+2qS-\omega)^2)\prod_{r=0}^{s_2-1}\zeta c((\nu_r-x_j-2qS-\omega)^2)}{c((2x_j+2qS-\omega)^2-(-2x_j-2qS-\omega)^2)} \\
&\quad \times c(2t(x_j+qS)^2) \\
&= \prod_{j=1}^n c(2\lambda x_j + t x_j^2).
\end{aligned}
$$

The last equality follows by a rather tedious computation (which moreover shows the next to last equation is actually S-independent). Moreover note that we obtain the desired exponential part of the integrand of $JI_{n,(s_1,s_2),t}$ in the limit. □

The condition on ϕ is again necessary to be able to choose an integration contour parallel to the direction in which the parameters go to infinity.

In this limit the value of λ is determined by μ and ν, and we can rewrite this expression as a balancing condition. The condition on λ also ensures that $(\mu,\nu,-\lambda)\in\mathcal{D}^\tau_{n,(s_1,s_2)}$, as required by the definition of the $JI_{n,(\frac{s-t}{2},\frac{s+t}{2}),t}$ integral, since $s_2=\frac{s+t}{2}=\frac{s-t}{2}+t=s_1+t$.

Finally we consider the limits of $JI_{n,(s_1,s_2),t}$ integrals to $JI_{n,(s_1,s_2),t}$ integrals.

Proposition 5.3.24. *Let $s_1, s_2 \in \mathbb{Z}_{\geq 0}$ and $t \in \mathbb{Z}$ and let $q = \exp(i\phi)$. We have*

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = \lim_{S\to\infty} JI_{n,(s_1+1,s_2),t-1}(\mu,\xi+qS;\nu;\lambda+qS+\xi-\omega;\tau) \times \zeta^n e\Big(\frac{-n}{4\omega_1\omega_2}(\xi+qS-\omega)^2\Big),$$

and

$$\begin{aligned} JI_{n,(s_1,s_2),t}&(\mu;\nu;\lambda;\tau) \\ &= \lim_{S\to\infty} JI_{n,(s_1+1,s_2+1),t}(\mu,\xi_1+qS;\nu,\xi_2-qS;\lambda+\xi_1+\xi_2-2\omega;\tau) \\ &\qquad \times e\Big(\frac{n}{4\omega_1\omega_2}((\xi_2-qS-\omega)^2-(\xi_1+qS-\omega)^2)\Big), \end{aligned}$$

for $(\mu,\nu) \in \mathcal{B}^\tau_{s_1,s_2}$, $\xi_1 \in \mathbb{C}$, $(\xi_1,\xi_2) \in \mathcal{B}_{1,1}$ and $\tau \in \mathcal{E}_\epsilon$ with $\epsilon = -$ if $t<0$, $\epsilon = 0$ if $t=0$ and $\epsilon = +$ if $t>0$. In the first limit if $t=1$ we moreover insist $\tau \in \mathcal{E}_0$. The conditions on ϕ are determined by the value of $t+s_2-s_1$, can depend on τ, μ, ν and λ, and are given by

$t+s_2-s_1<0$	$t+s_2-s_1=0$	$t+s_2-s_1>0$
$\mathcal{A}^\tau_-$	$\mathcal{A}^\tau_0 \cap \mathcal{C}^{\beta^\tau(n;s_1+s_2;\mu,\nu,\lambda)} \cap \mathcal{C}^{\beta^\tau(n,s_1+s_2+2;\mu,\xi_1,\nu,\xi_2,\lambda)}$	$\mathcal{A}^\tau_+$

If $t+s_1-s_2=0$ we moreover insist in both equations that $(\mu,\nu,-\lambda) \in \mathcal{D}^\tau_{n,(s_1,s_2)}$.

The first limit also holds if $\phi = \frac{\phi_-+\phi_+-\pi}{2}$, $(\mu,\nu,\lambda) \in \mathcal{D}^\tau_{n,(s_1,s_2)}$ and if τ and ξ are both positive linear combinations of ω_1 and ω_2.

Proof. The proof is similar to that of Proposition 5.3.22 and is again expanded upon in Appendix 5.A. □

Like for the limit between $JI_{n,s,t}$ and $JI_{n,s,t}$ integrals we can find different limits by applying the symmetries (5.3.12) and (5.3.13) to these equations. This allows us to write $JI_{n,(s_1,s_2),t}$ as a limit of $JI_{n,(s_1+1,s_2),t\pm1}$ and of $J_{n,(s_1,s_2+1),t\pm1}$. Moreover we obtain a different limit from $JI_{n,(s_1+1,s_2+1),t}$ to $JI_{n,(s_1,s_2),t}$, in which we take the limit from $JI_{n,(s_1+1,s_2+1),t}(\mu,\xi_1-qS;\nu,\xi_2+qS)$.

5.4 The degeneration scheme

The purpose of this section is to discuss which are possible interesting degenerations of the $I^m_{n,T}$ and $II^m_{n,T}$ hyperbolic hypergeometric integrals. A degeneration will be an integral obtained by taking the limit in a suitable renormalization of one of these top level integrals while letting some, or all, parameters go to infinity. We could analyse what integrals we can obtain by considering which of the limits from Propositions 5.3.22-5.3.24 we can repeatedly apply to the top level integral, but here we consider a different approach. We consider only direct limits from the top level integrals in which the variables μ go to infinity in some general (to be specified later) way and determine the asymptotic behaviour of the integrand. This allows us to heuristically identify the integrals with interesting limits, which gives us our (at this point: heuristic) degeneration scheme. In Sections 5.5 and 5.6 we rigorously establish the limits in this degeneration scheme.

Indeed, in the subsequent sections we show that each element of this classification scheme corresponds in some canonical way to an integral which can be obtained by a series of limits from the top level integral using Propositions 5.3.22-5.3.24 (i.e. in contrast to taking a direct limit from the top level integral). The advantage of considering direct limits from the top level integral in this section is that every degeneration now corresponds to some given way in which the variables of the top level integral go to infinity, which tells us where to place this degeneration in the big picture.

As the main purpose of this section is to give some insight in the structure of the degeneration scheme, while the proof the limits actually hold are reserved for the other sections, the arguments used in this section are mostly heuristic and intuitive.

5.4.1 The conditions

Recall that the integrand of $I\!I_{n,T}^m$ equals

$$I\!I(\mu;\tau;x) = \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{\prod_{r=0}^{2m+5} \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)},$$

while the integrand of $I_{n,T}^m$ equals

$$I(\mu;x) = \prod_{1\leq j<k\leq n} \frac{1}{\Gamma_h(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{\prod_{r=0}^{2n+2m+5} \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)}.$$

In particular the μ-dependent part of both integrands is the same and equals

$$\prod_{j=1}^n \prod_{r=0}^{2v-1} \Gamma_h(\mu_r \pm x_j),$$

where $2v = 2m+6$ (in the case of $I\!I_{n,T}^m$), respectively $2v = 2n+2m+6$ (for $I_{n,T}^m$), denotes the number of μ-parameters. The remaining part of the integrands (the cross-terms and $1/\Gamma_h(\pm 2x_j)$ terms) turn out to have no influence on the kinds of possible degenerations, so the degeneration scheme of both the $I\!I_{n,T}^m$ and $I_{n,T}^m$ integrals are very similar (only differing in the value of v). In this subsection we will therefore only focus on the $I\!I$ integrand.

We assume that the variables μ_r ($r = 0, \ldots, 2v-1$) depend in an affine linear fashion on a parameter $S \in \mathbb{R}$ (i.e. $\mu_r(S) = a_r + b_r S$ for some $a_r \in \mathbb{C}$ and $b_r \in \mathbb{R}$), and we will consider $\lim_{S\to\infty} I\!I_{n,T}^m(\mu(S);\tau)$ if it exists. The asymptotic behaviour of the integrand of $I\!I_{n,T}^m$, and therefore of $I\!I_{n,T}^m$ itself, now depends on the values of the b_r. Hence we can control this asymptotic behaviour by formulating conditions in terms of the b_r. In particular we obtain a first condition by insisting the balancing conditions (5.3.5) (for the $\mathit{II}^m_{n,T}$ integral) hold for all values of S:

$$\sum_r b_r = 0, \tag{5.4.1}$$

which we therefore assume throughout this section. Moreover we assume the a_r satisfy the balancing condition (5.3.5).

We now consider the asymptotic behaviour of the integrand when $S \to \infty$ for the variables $\mu_r = a_r + b_r S$. In particular we consider when the integrand is maximized. Note that for each fixed S, the absolute value of the integrand attains a maximum if the a_r and the b_r satisfy their respective balancing conditions (5.3.5) and (5.4.1), as the asymptotic behaviour for $x \to \infty$ in this case is given by (5.3.6) and thus converges to zero as $x \to \infty$.

The following proposition considers where the location of the absolute value of the integrand can be for large S.

Proposition 5.4.1. *Suppose $a \in \mathbb{C}^{2v}$, $b \in \mathbb{R}^{2v}$ and $\tau \in \mathbb{C}$ are given, such that $\mathit{II}(a+bS;\tau;x)$ has no poles for $x \in \mathbb{R}^n$. Moreover assume a satisfies the balancing condition (5.3.5), b the balancing condition (5.4.1) and that $\omega_1\omega_2 \notin \mathbb{R}$. Let $x^*(S) = (x_1^*(S),\dots,x_n^*(S))$, with $x_j^*(S) \geq 0$, denote for fixed S a location of the maximum of the absolute value of the integrand $\mathit{II}(a+bS;\tau;Sx)$ (where $Sx = (Sx_1,\dots,Sx_n)$) as a function of $x \in \mathbb{R}^n$. Thus if the maximum of the absolute value of the integrand is attained for several different values of x, we just choose one of these points. Due to the evenness of the integrand in the x_j we know the absolute value of the integrand attains its maximum in at least one point in the quadrant $x_j \geq 0$ $(j = 1,\dots,n)$. Then the limit points of $x^*(S)$ as $S \to \infty$ have to be an absolute maximum of*

$$\phi(b;x) = \sum_{j=1}^{n}\sum_{r=0}^{2v-1} |b_r + x_j|(b_r + x_j) + |b_r - x_j|(b_r - x_j), \tag{5.4.2}$$

if $\Im(\omega_1\omega_2) > 0$ and an absolute minimum of $\phi(b;x)$ if $\Im(\omega_1\omega_2) < 0$. Within the set of vectors $x \in \mathbb{R}^n$ which maximize, respectively minimize, $\phi(b;x)$ the limit points of $x^*(S)$ have to maximize the real part of

$$\begin{aligned}\chi(a,b;\tau;x) = \ &\frac{2\pi i}{4\omega_1\omega_2}\Bigg(\sum_{1\leq j<k\leq n} 4\tau|x_j + x_k| + 4\tau|x_j - x_k| \\ &+ \sum_{j=1}^{n} 8\omega|x_j| + 2\sum_{r=0}^{2v-1}(|b_r + x_j| + |b_r - x_j|)(a_r - \omega)\Bigg).\end{aligned}$$

Sketch of proof. According to Theorem 5.2.6 we have for given $u \in \mathbb{C}$, $v \in \mathbb{R}$ and all $S > 0$

$$\frac{1}{S}\left| -\frac{\mathbf{sign}(v)2\pi i}{4\omega_1\omega_2}(v^2S^2 + 2v(u-\omega)S) + \log(\Gamma_h(u+vS))\right| \leq \frac{C}{S},$$

for some constant $C > 0$, which depends on v and u, as long as there are no poles and zeros of Γ_h on the line $\Im(u) + \mathbb{R}$. Here the definition of $\mathbf{sign}(v)$ for $v = 0$ is arbitrary as we multiply $\mathbf{sign}(v)$ by v. Here we use that for $u + vS$ in the right or left hand cones we have an uniform bound, while if $u + vS$ is not in the right or left hand cone, it is on the interval of finite length of the intersection of the line $\Im(u) + \mathbb{R}$ with the compliment of the left and right hand cones, while the difference between $\log(\Gamma_h(u + vS))$ and any continuous function is bounded on any finite fixed interval. In particular we find that the integrand $\mathit{\Pi}(a + bS; Sx)$ can be approximated by

$$\left| \frac{\log(\mathit{\Pi}(a + bS; Sx))}{S} - k\phi(b; x)S - \chi(a, b, \tau; x) \right| \leq \frac{C'}{S},$$

where $k = \frac{2\pi i}{4\omega_1\omega_2}$ and $C' > 0$ is some constant independent of S and x.

Suppose x is not at an absolute maximum of the real part $\Re(k\phi(b; x))$ of the coefficient in front of S (which exists as ϕ is constant if $|x_j| > \max_r(b_r)$), while $\Re(k\phi(b; y))$ is the absolute maximum of $\Re(k\phi(b; \cdot))$. Then for large enough S we see that $\Re(\log(\mathit{\Pi}(a+bS; Sy)) > \Re(\log(\mathit{\Pi}(a+bS; Sx)))$, hence x cannot be the location of the absolute maximum of the absolute value of $\mathit{\Pi}(a + bS; S\cdot)$. Since $\phi(b; \cdot)$ is a continuous function this means that the limit points of locations $x^*(S)$ of the absolute maxima of the absolute value of $\mathit{\Pi}(a+bS; S\cdot)$ maximize $\Re(k\phi(b; x))$. Since $k \notin i\mathbb{R}$ by our assumption that $\omega_1\omega_2 \notin \mathbb{R}$, it follows that the absolute maximum of $\Re(k\phi(b; x))$ is attained at an absolute extremal value of $\phi(b; x)$.

Similarly, within the set of extremal values of $\phi(b; x)$ we obtain that any point outside an absolute extremal value of $\Re(\chi(a, b; \tau; x))$ cannot occur as a limit point of $x^*(S)$. □

Now it is convenient to consider the integers $h_i(b, f)$ $(i = 1, \ldots, 5)$ for $b \in \mathbb{R}^n$ and $f \in \mathbb{R}_{\geq 0}$ defined by

$$\begin{aligned} &h_1(b, f) = |\{r \mid b_r < -f\}|, && h_2(b, f) = |\{r \mid b_r = -f\}|, \\ &h_3(b, f) = |\{r \mid -f < b_r < f\}|, && h_4(b, f) = |\{r \mid b_r = f\}|, \\ &h_5(b, f) = |\{r \mid b_r > f\}|. \end{aligned} \tag{5.4.3}$$

If there can be no confusion we will omit the arguments of $h_i(b, f)$.

Proposition 5.4.2. *If $\phi(b; x)$ (given in (5.4.2)) has for given $b \in \mathbb{R}^{2v}$ satisfying (5.4.1) a local extremal value at $x_j = f$ $(j = 1, \ldots, n)$, with $f \geq 0$ then we have*

$$\sum_{r, |b_r| > f} b_r = (h_5(b, f) - h_1(b, f))f. \tag{5.4.4}$$

Proof. Note that $\phi(b; x)$ is differentiable in the x_j, in particular

$$\frac{\partial}{\partial x_j}\phi(b; x) = 2\sum_{r=0}^{2m+5} |b_r + x_j| - |b_r - x_j| = 4\sum_{r=0}^{2m+5} \min(|b_r|, x_j)\mathbf{sign}(b_r),$$

as $x_j \geq 0$. Note that this expression again does not depend on the sign of $\mathbf{sign}(b_r)$ when $b_r = 0$. An extremal value of ϕ can therefore only occur where this derivative vanishes, hence if

$$\begin{aligned} 0 &= 4\sum_{r=0}^{2m+5} \min(|b_r|, x_j)\mathbf{sign}(b_r) = 4 \sum_{r,|b_r|>x_j} x_j \mathbf{sign}(b_r) + 4 \sum_{r,|b_r|\leq x_j} b_r \\ &= -4 \sum_{r,|b_r|>x_j} b_r + 4x_j(h_5(b,x_j) - h_1(b,x_j)). \end{aligned} \tag{5.4.5}$$

Here we used the balancing condition (5.4.1) to obtain the final expression. □

Propositions 5.4.1 and 5.4.2 show it is not unnatural to assume that the maximum of the absolute value of the integrand is located near the diagonal of $\mathbb{R}^n$. Indeed, for large S the x_j at a location of the maximum of the absolute value of the integrand has to be near to a solution to the same piecewise linear equation (5.4.5). In generic cases a piecewise linear function vanishes at only a finite number of points. Within the set of solutions of this piecewise linear equation the x_j have to be at a maximum of $\Re(\chi(a,b;\tau;x))$, which has a large part $(\sum_{j=1}^n 8\omega|x_j| + 2\sum_{r=0}^{2v-1}(|b_r + x_j| + |b_r - x_j|)(a_r - \omega)$ which is a sum of univariate terms. So especially for small τ (so that the non-univariate part of χ is small) it is not uncommon for the maximum of $\Re(\chi)$ to be attained at the diagonal of $\mathbb{R}^n$.

Note that all $x_j \geq \max_r |b_r|$ are a solution to (5.4.5). For these values of x_j we can simplify χ (using the balancing condition (5.5.5) on the a_r) to

$$\chi(a,b;\tau;x) = \frac{4\pi i}{\omega_1\omega_2}\left(\sum_{j<k} \tau \max(x_j,x_k) - (\omega + (n-1)\tau)\sum_{j=1}^n x_j\right),$$

whose real part is maximized by taking as low values for x_j as possible (the "derivative" to each x_j is negative, as $\Re(i\omega/\omega_1\omega_2), \Re(i\tau/\omega_1\omega_2) > 0$, by the conditions on ω_1, ω_2 and $\tau \in \mathcal{E}_0$), hence it is maximized for $x_j = \max_r |b_r|$ for all j, which is a point on the diagonal of $\mathbb{R}^n$. Another important solution to (5.4.5) is $x_j = 0$.

Observe also that, while it is often true that the maximum of the absolute value of the integrand of $\mathit{\Pi}(a+bS;\tau;x)$ is located at the diagonal this is not always true. In particular if one pairs the b_r such that $b_{2r} = -b_{2r-1}$ $(r = 0,\ldots,v-1)$, the function ϕ (5.4.2) always vanishes, and if we set $b_0 = -b_1 = 1$ and $b_r = 0$ for $r \geq 2$, we can choose a_r such that the maximum of $\Re(\chi(a,b;\tau;x))$ is not attained at the diagonal of $\mathbb{R}^n$. Nevertheless from now on we proceed by assuming that the maximum of the absolute value of the integrand of $\mathit{\Pi}(a+bS;\tau;\cdot)$ is attained for all S near the diagonal of $\mathbb{R}^n$, in particular we assume that the distance between the location of such a maximum and the point $(fS,\ldots,fS)$ is bounded as a function of S.

If we moreover assume the location of the maximum of the absolute value of the integrand of $\mathit{\Pi}_{n,T}^m$ is located near the origin ((i.e. the case $f=0$) we would like to calculate the limit of $\mathit{\Pi}_{n,T}^m(a+bS;\tau)$ by interchanging limit and integral, i.e. by using (in the case $0 < \Im(a_r) < 2\Im(\omega)$ so that we can use $\mathbb{R}^n$ as integration contour)

$$\lim_{S\to\infty} g(S) \int_{\mathbb{R}^n} \mathit{\Pi}(a+bS;\tau;x)dx = \int_{\mathbb{R}^n} \lim_{S\to\infty} g(S)\mathit{\Pi}(a+bS;\tau;x)dx, \tag{5.4.6}$$

where $g(S)$ is some x-independent renormalization factor such that the limit $\lim_{S\to\infty} g(S)\mathit{\Pi}(a+bS;\tau;x)$ exists and does not vanish identically. In explicit cases the renormalization factor $g(S)$ can be calculated using the asymptotic behaviour of the hyperbolic gamma function.

In the case $f>0$ the BC_n symmetry of the integrand ensures that the maximum of the absolute value of $\mathit{\Pi}(a+bS;\tau;\cdot)$ is attained not only near $(fS,\ldots,fS)$ but near all points of the form $(\pm fS,\ldots,\pm fS)$. In this case we can use the evenness of the integrand in all variables to restrict our integral to the quadrant $[0,\infty)^n$ and then interchange limit and integral. In particular we have heuristically

$$\lim_{S\to\infty} g(S) \int_{\mathbb{R}^n} \mathit{\Pi}(a+bS;\tau;x)dx = \lim_{S\to\infty} 2^n \int_{[0,\infty)^n} g(S)\mathit{\Pi}(a+bS;\tau;x)dx \tag{5.4.7}$$

$$= 2^n \lim_{S\to\infty} \int_{[-fS,\infty)^n} g(S)\mathit{\Pi}(a+bS;\tau;x+fS)dx \tag{5.4.8}$$

$$= 2^n \int_{\mathbb{R}^n} \lim_{S\to\infty} g(S)\mathit{\Pi}(a+bS;\tau;x+fS)dx,$$

where $g(S)$ is again a renormalization factor which ensures that $\lim_{S\to\infty} g(S)\mathit{\Pi}(a+bS;\tau;x+fS)$ exists and does not vanish identically.

Using (5.4.6) or (5.4.7) to calculate a degeneration we will obtain the same degeneration if the limits of the integrands for two different sets (a,b) and (a',b') are proportional. Let us define the following equivalence relation which encodes this.

Definition 5.4.3. *For each $v\in\mathbb{N}$ and for each $f\in\mathbb{R}_{\geq 0}$ we define the equivalence relation $\sim_f$ between pairs $(a,b)\in\mathbb{C}^{2v}\times\mathbb{R}^{2v}$ such that b and b' satisfy* (5.4.1) *and a and a' the balancing condition* (5.3.5) *by setting $(a,b)\sim_f(a',b')$ if*

i) $h_i(b,f)=h_i(b',f)$ *for* $i=1,\ldots,5$.

ii) There exists a permutation σ of $\{0,1,\ldots,2v-1\}$ such that $a_r=a'_{\sigma(r)}$ and $b_r=b'_{\sigma(r)}$ for all r with $|b_r|=f$.

iii) $\sum_{r,|b_r|<f} a_r = \sum_{r,|b'_r|<f} a'_r$ *and* $\sum_{r,|b_r|<f} b_r = \sum_{r,|b'_r|<f} b'_r$.

Note that this definition also implies that $\sum_{r,|b_r|>f} a_r = \sum_{r,|b'_r|>f} a'_r$ and $\sum_{r,|b_r|>f} b_r = \sum_{r,|b'_r|>f} b'_r$, due to the balancing conditions (5.3.5) and (5.4.1).

The following lemma shows that the limits of renormalized integrands taken around the point $x+fS=(x_1+fS,x_2+fS,\ldots,x_n+fS)$ are basically the same for two similar sets of parameters $(a,b)\sim_f(a',b')$ (they can only differ by some x and S independent factor).

Lemma 5.4.4. *Let $f \geq 0$. Suppose $a, a' \in \mathbb{C}^{2m+6}$ satisfy the balancing condition* (5.3.5) *and that $b, b' \in \mathbb{R}^{2m+6}$ satisfy the balancing condition* (5.4.1)*. If $(a,b) \sim_f (a',b')$ then there exist a function $g(a,a',b,b')$ independent of x and S such that for the integrands $I\!I(\mu;\tau;x)$ of the $I\!I^m_{n,T}$ integral we have*

$$\lim_{S\to\infty} \frac{I\!I(a+bS;\tau;x+fS)}{I\!I(a'+b'S;\tau;x+fS)} g(a,a',b,b') = 1.$$

pointwise for all $x \in \mathbb{R}^{2n}$ (or indeed $x \in \mathbb{C}^{2n}$). Note the difference here between $a+bS = (a_0+b_0S, \ldots, a_{2v-1}+b_{2v-1}S)$ and $x+fS = (x_1+fS, x_2+fS, \ldots, x_n+fS)$.

Proof. Define

$$g(a,a',b,b') = c(2n(\sum_{r,b_r<-f} (a_r+b_r-\omega)^2 - \sum_{r,b_r>f} (a_r+b_r-\omega)^2 - \sum_{r,b'_r<-f} (a'_r+b'_r-\omega)^2 + \sum_{r,b'_r>f} (a'_r+b'_r-\omega)^2)).$$

Now we can calculate the above limits using the asymptotic formulas from Theorem 5.2.6 for the hyperbolic gamma function, and see that they are indeed equal to 1. □

Recall that if two μ parameters, say μ_0 and μ_1 add up to 2ω the resulting term $\prod_{i=1}^n \Gamma_h(\mu_0 \pm x_i, \mu_1 \pm x_i) = 1$ by the reflection equation (5.2.4), hence the integral $I\!I^m_{n,T}$, reduces to the integral $I\!I^{m-1}_{n,T}$. In particular we would like to consider an integral with such a pair of μ-parameters as an integral with a lower m. Similarly we consider any degeneration obtained from an integral with μ parameters which are specialized like this as a degeneration of an integral with lower m. Moreover any degeneration equivalent to a degeneration in which two μ parameters sum to 2ω can therefore also be obtained as a degeneration from a top level integral with lower m. The following lemma says when this occurs.

Lemma 5.4.5. *For any $f \in \mathbb{R}_{\geq 0}$, given a pair $(a,b) \in \mathbb{C}^{2v} \times \mathbb{R}^{2v}$, such that a and b satisfy the balancing conditions* (5.3.5) *and* (5.4.1)*, there exists a pair (a',b'), such that a' and b' also satisfy these balancing conditions, with $(a,b) \sim_f (a',b')$ and such that $a'_0 + a'_1 = 2\omega$ and $b'_0 + b'_1 = 0$ if*

$$h_1(b,f) \geq 2 \wedge h_5(b,f) \geq 2 \tag{5.4.9}$$

or if

$$\left| \sum_{r,|b_r|<f} b_r \right| < (h_3(b,f)-2)f \tag{5.4.10}$$

holds.

Proof. Suppose that $h_1(b,f) \geq 2$ and $h_5(b,f) \geq 2$. Let $b_r, b_s > f$ and $b_u, b_t < -f$ (for some $r \neq s$ and $u \neq t$). Then the new set of variables (a', b') defined by $a'_t = 2\omega - a_r$, $a'_u = a_t + a_u - a'_t$ and $a'_w = a_w$ for $w \neq t, u$, and $b'_t = -b_r$, $b'_s = b_s + b_r$, $b'_u = b_t + b_u$ and $b_w = b'_w$ for $w \neq s, t, u$ satisfies $(a', b') \sim_f (a, b)$, $a'_r + a'_t = 2\omega$ and $b'_r + b'_t = 0$.

Similarly, if $h_3(b,f) > 2$ and $|b_0|, \dots, |b_{h_3(b,f)-1}| < f$ such that

$$\left| \sum_{r=0}^{h_3(\mu,f)-1} b_r \right| < (h_3(b,f) - 2)f \tag{5.4.11}$$

we can define a new set (a', b') by setting $a'_0 = a'_1 = \omega$, $a'_r = \frac{1}{h_3-2}(\sum_{s=0}^{h_3-1} a_s - 2\omega)$ for $r = 2, \dots, h_3 - 1$ and $a'_r = a_r$ if $r \geq h_3$, and $b'_0 = b'_1 = 0$, $b'_r = \frac{1}{h_3-2} \sum_{s=0}^{h_3-1} b_s$ for $r = 2, \dots, h_3 - 1$ and $b'_r = b_r$ if $r \geq h_3$. Now $(a', b') \sim_f (a, b)$, $a'_0 + a'_1 = 2\omega$ and $b'_0 + b'_1 = 0$ (Observe that $|\frac{1}{h_3-2} \sum_{s=0}^{h_3-1} b_s| < f$, hence $h_3(b', f) = h_3(b, f)$). □

In order to obtain interesting degenerations of $\mathit{II}^m_{n,T}$ we therefore impose the negations of the conditions (5.4.9) and (5.4.10), i.e.

$$h_1(b,f) \leq 1 \vee h_5(b,f) \leq 1 \tag{5.4.12}$$

and

$$\left| \sum_{r, |b_r| < f} b_r \right| \geq (h_3(b,f) - 2)f. \tag{5.4.13}$$

Now we reduce the equivalence relation $\sim_f$ to an equivalence relation $\approx_f$ on the b-vectors in $\mathbb{R}^{2v}$.

Definition 5.4.6. *For $f \in \mathbb{R}_{\geq 0}$ and vectors $b, b' \in \mathbb{R}^{2v}$ satisfying* (5.4.1) *we define $b \approx_f b'$ if there exists vectors $a, a' \in \mathbb{C}^{2v}$, such that $(a,b) \sim_f (a',b')$. This definition is equivalent to setting $b \approx_f b'$ if and only if*

i) $h_i(b,f) = h_i(b',f)$ *for* $i = 1, \dots, 5$,

ii) $\sum_{r, |b_r| < f} b_r = \sum_{r, |b'_r| < f} b'_r$.

While $(a,b) \sim_f (a',b')$ implicated that the two related degenerations would be identical, the new equivalence relation $\approx_f$ only indicates that the degenerations with these b-vectors are similar, i.e. the corresponding degenerations will be given by an $\mathit{JI}_{n,s,t}$ integral with the same n, s and t (but could still have different values of the remaining parameters in the degenerations).

We can reformulate all the conditions obtained thus far to ensure an interesting degeneration in terms of just the h_i. This simplifies classifying all classes of interesting degenerations.

Lemma 5.4.7. *If $f = 0$ there is a bijective correspondence between equivalence classes with respect to $\approx_f$ of $b \in \mathbb{R}^{2v}$ satisfying the conditions* (5.4.1), (5.4.4), (5.4.12) *and* (5.4.13) *and quintuples $(h_1, \dots, h_5) \in \mathbb{Z}^5_{\geq 0}$ with $h_1 + h_2 + h_5 = 2v$ such that*

$$h_1 = 1 \vee h_5 = 1 \vee h_1 = 0 = h_5, \qquad h_2 = h_4, \qquad h_3 = 0. \tag{5.4.14}$$

For a given $f > 0$ there moreover is a bijective correspondence between equivalence classes with respect to $\approx_f$ of $b \in \mathbb{R}^{2v}$ satisfying the conditions (5.4.1), (5.4.4), (5.4.12) *and* (5.4.13) *and quintuples $(h_1, \dots, h_5) \in \mathbb{Z}^5_{\geq 0}$ with $\sum_{i=1}^5 h_i = 2v$ satisfying the conditions*

$$\begin{gathered} h_1 = 1 \vee h_5 = 1 \vee h_1 = 0 = h_5, \\ h_3 = 0 = h_5 + h_4 - h_1 - h_2 \vee |h_5 + h_4 - h_1 - h_2| = h_3 - 2. \end{gathered} \tag{5.4.15}$$

Proof. Observe that if $h_1 = 0$ equation (5.4.4) reduces to $\sum_{r, b_r > f} b_r = h_5 f$, but $\sum_{r, b_r > f} b_r > h_5 f$ if $h_5 > 0$ and hence $h_5 = 0$ as well. Similarly $h_5 = 0$ implies $h_1 = 0$. Together with (5.4.12) this gives the condition $h_1 = 1 \vee h_5 = 1 \vee h_1 = 0 = h_5$ (for both $f = 0$ and $f > 0$) for $b \in \mathbb{R}^{2v}$ satisfying (5.4.4) and (5.4.12).

For $f = 0$ we further remark that $h_2(b, 0) = h_4(b, 0)$ and $h_3(b, 0) = 0$, by definition of h_i. Given a quintuple of h's satisfying the conditions (5.4.14) we can define b by $b_r = -h_5$ for $r = 0, \dots, h_1 - 1$, $b_r = h_1$ for $r = h_1, \dots, h_1 + h_5 - 1$ and $b_r = 0$ for all other r. This vector b satisfies all the conditions (5.4.1), (5.4.4), (5.4.12) and (5.4.13). Since the only condition on two b-vectors to be equivalent in the case $f = 0$ is that the values of the h_i are identical this is the only equivalence class corresponding to these h's.

If $f > 0$ we can moreover calculate for $b \in \mathbb{R}^{2v}$ satisfying (5.4.1) and (5.4.4)

$$\sum_{r, |b_r| < f} b_r = \sum_r b_r - \sum_{r, |b_r| > f} b_r - \sum_{r, |b_r| = f} b_r = (h_1 + h_2 - h_4 - h_5) f.$$

The equation $\left|\sum_{r, |b_r| < f} b_r\right| \leq h_3 f$ (where equality occurs if and only if $h_3 = 0$) together with (5.4.13) now implies

$$h_3 f \geq |h_1 + h_2 - h_4 - h_5| f \geq (h_3 - 2) f.$$

Since $f > 0$ we can divide this equation by f and by a parity argument using that $h_1 + \dots + h_5 = 2v$ is even, we see that either $h_3 = |h_1 + h_2 - h_4 - h_5|$ or $h_3 - 2 = |h_1 + h_2 - h_4 - h_5|$. In order for $h_3 = |h_1 + h_2 - h_4 - h_5|$ to hold we must have equality in $\left|\sum_{r, |b_r| < f} b_r\right| \leq h_3 f$ hence this occurs if and only if $h_3 = 0$.

Reversely, given h's satisfying the conditions (5.4.15) we can define a b-vector by $b_r = -\frac{h_5 + 1}{h_5} f$ for $r = 0, \dots, h_1 - 1$, $b_r = -f$ for $r = h_1, \dots, h_1 + h_2 - 1$, $b_r = \frac{h_1 + h_2 - h_4 - h_5}{h_3} f$ for the next h_3 values of r, $b_r = f$ for the next h_4 values of r and $b_r = \frac{h_1 + 1}{h_1} f$ for the final h_5 values of r. Note that there are no problems with division by zero because if the numerator of one of the above expressions is zero, the number of parameters equal to this expression is zero as well. Moreover there is again only one equivalence class with these values of h, as we have seen that $\sum_{r,|b_r|<f} b_r$ is determined by the values of the h_i. □

5.4.2 Classifying the degenerations

Now we can classify all solutions to (5.4.14) and (5.4.15) and give a parametrization of these solutions. In the next sections we will attach a particular integral to each of these solutions. We will consider these integrals, and also the related solution to (5.4.14) and (5.4.15), to be the degenerations of the top level integral. Therefore already in this subsection we will call the solutions degenerations.

Proposition 5.4.8. *The set of solutions to* (5.4.14) *with* $h_1 + h_2 + h_5 = 2v$ *if* $f = 0$ *are given in the next table. Since* $h_3 = 0$ *and* $h_2 = h_4$ *we omit those values in the table.*

(h_1, h_2, h_5)	*conditions*	*name*
$(0, 2v, 0)$	-	T
$(k-1, 2v-k, 1)$	$2 \le k \le 2v$	ka
$(1, 2v-k, k-1)$	$2 \le k \le 2v$	kb

If $f > 0$ *the solutions to* (5.4.15) *for* $\sum_{i=1}^{5} h_i = 2v$ *are given by*

$(h_1, h_2, h_3, h_4, h_5)$	*conditions*	*name*
$(0, v, 0, v, 0)$	-	$(1, 1)$
$(k-1, v-k+1, 0, v-1, 1)$	$2 \le k \le v+1$	$(k, 1)a$
$(0, v-1, l, v-l+1, 0)$	$2 \le l \le v+1$	$(1, l)a$
$(k-1, v-k, l, v-l, 1)$	$2 \le k, l \le v$	$(k, l)a$
$(k-1, v-k-l+2, l, v-2, 1)$	$3 \le k, l$ *and* $k + l \le v + 2$	$(k, l)^*a$

For all solutions with $f > 0$ *there moreover exists a companion solution obtained by setting* $(h'_1, h'_2, h'_3, h'_4, h'_5) = (h_5, h_4, h_3, h_2, h_1)$*. These extra solutions have the same name, but with* a *replaced by* b*. For* $k \le 2$ *and* $l \le 2$ *the* $(k, l)a$ *and* $(k, l)b$ *solutions coincide and we will sometimes omit the* a *or* b *from these solution types.*

Proof. For $f = 0$ we choose whether h_1 and h_5 are both zero or not and if not, which one is 1. Given the value of h_2 we can then easily determine which values h_1 and h_5 should have.

For $f > 0$ we have to choose whether or not $h_3 = 0$ and whether or not h_1 and h_5 are zero. If $h_3 > 0$ we subsequently have to choose whether or not $h_1 + h_2 - h_4 - h_5$ is positive and, if $h_1, h_5 > 0$, which of the two equals 1. Having made these choices all h_i's can be calculated from the value of h_3 and either h_1 or h_5 (whichever of the pair is greater than 1). □

We will see that the degenerations corresponding to the a and b solutions of the same type are related by complex conjugation (as in equations (5.3.12) and (5.3.13)), and hence define basically the same function.

The names of the degenerations are inspired by their interpretation as degenerations of $II^m_{n,T}$. When $k \geq 2$ the value of k denotes the number of μ parameters going to infinity at a rate which is in absolute value greater than f. Similarly for $l \geq 2$ the value of l denotes the number of parameters staying small (i.e. going to infinity at a rate less than f, or with $b_r = 0$). If $f = 0$ the concept of going to infinity at a rate slower than f is non-existent so we omit the l in the solutions for $f = 0$. For $f > 0$, it is convenient to set k or l equal to 1 if there are no parameters going to infinity at a rate faster, respectively slower, than f to infinity; Note that there are no solutions where exactly one parameter goes to infinity at a rate faster or slower than f. This convention simplifies the following definition

Definition 5.4.9. *For a degeneration ξ we define the level of the degeneration, which is independent of a or b, according to the following table*

Degeneration	T	k	(k,l)	$(k,l)^*$
Level	1	k	$k+l$	$k+l$

for all $k, l \in \mathbb{Z}_{\geq 1}$ such that k, (k,l), respectively $(k,l)^*$ occur in the tables of Proposition 5.4.8.

We will see that the level of a degeneration determines the number of remaining free parameters in the degeneration, namely a degeneration of level m has $2v - m$ free parameters. In particular the top level integral has $2v$ parameters, but we have to subtract one to account for the balancing condition.

If we would have extended the definition of the $(k,l)^*$ degenerations to $k, l \geq 2$, we would be able to express the extra degenerations in regular (k,l) degenerations using $(2,l)^*a/b = (2,l)b/a$ and $(k,2)^*a/b = (k,2)a/b$. This is the reason why we define the * degenerations only for $k, l \geq 3$. When convenient however, we will give formulas for * degenerations also for $k, l = 2$.

In Figures 5.5 and 5.6 we display the degenerations scheme for a fixed top level integral in a graph. Note that now we return from considering degenerations as limits directly from the top-level integral, to looking at degenerations as obtained by a sequence of limit transitions. Each vertex in the scheme represents a degeneration, while the lines connecting them represent limit transitions of the degeneration higher in the scheme downwards. Due to our method of derivation it is at this stage unclear whether all these limits actually exist, but in the next sections we will give the limits explicitly.

To simplify the picture we have restricted ourselves to a-degenerations and the left half of the degeneration scheme, moreover in Figure 5.5 we have omitted the $(k,l)^*$ degenerations (since otherwise the graph would be non-planar). The degeneration scheme for the b-degenerations is identical to the degeneration scheme for the a-degenerations, with the remark that the degeneration schemes overlap at the top (specifically at the degenerations T, 2, $(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$). In Figure 5.6 we have given the degeneration scheme for the $(k,l)^*a$ degenerations, and how it is connected to the degeneration scheme for the a degenerations as given in Figure 5.5. The $(k,l)^*a$ degenerations are moreover also connected to the degeneration scheme for the $(k,l)b$ degenerations, from the right hand side, there

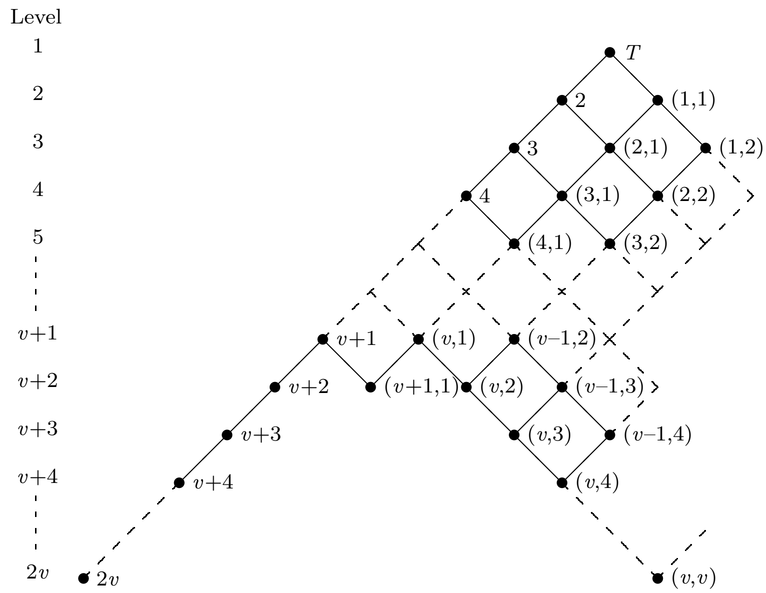


Figure 5.5: The left half of the degeneration scheme of $I^m_{n,T}$ (for $v = n + m + 2$) and $II^m_{n,T}$ (for $v = m + 3$)

are limit transitions from $(k,2)b$ to $(k,3)^*a$. The degenerations scheme for the $(k,l)^*b$ degeneration is an a-b inversion of the scheme for the $(k,l)^*a$ as depicted.

5.4.3 Picture of the classification scheme

Fix $h_1, h_2, \ldots, h_5 \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^5 h_i = 2v$ and such that $(h_1, \ldots, h_5)$ corresponds to a degeneration. For each degeneration we can consider the set of vectors $b = (b_0, \ldots, b_{2v-1})$ which correspond to this degeneration. By restricting to those vectors for which the first h_1 elements correspond to b_r less than $-f$, the next h_2 to b_r equal to $-f$ etc., this set is given by a collection of linear (in)equalities in the b_r and f. Hence if we either set $f = 0$ or allow f to vary over $\mathbb{R}_{>0}$ this set forms a convex cone in $\mathbb{R}^{2v+1}$. If we then look at the projection on the $\mathbb{R}^{2v}$ subspace of only the b_r's we obtain a convex cone of only b-vectors. These cones are open except in the case of the "degeneration" corresponding to the top level integral itself. Indeed, the top level integral corresponds to the case $b_r = 0$ for all r. The cones corresponding to the other degenerations are now given by strictly positive linear combinations of only a few vectors.

For each degeneration we can give the vectors generating this cone. Doing this we see that we have to add one extra vector for each level of degeneration. Let ϵ_j $(j = 0, \ldots, 2v-1)$ denote the j'th standard basis vector of $\mathbb{R}^{2v}$. The vectors occurring in the generating sets of the cones are all of the form

$$a_{(i,j)} = \epsilon_j - \epsilon_i,$$

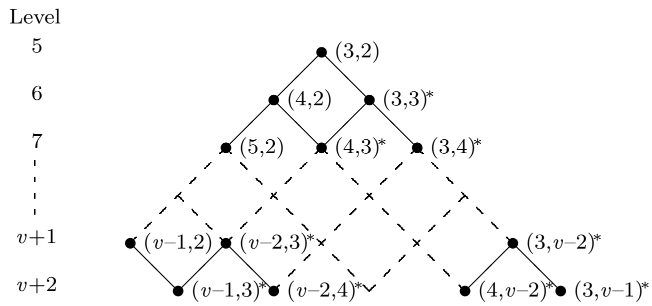


Figure 5.6: The part of the degeneration scheme of $I^m_{n,T}$ (for $v = n + m + 2$) and $II^m_{n,T}$ (for $v = m + 3$) containing the $(k, l)^*$

for $i < j$,

$$b = \frac{1}{2}(-\sum_{k=0}^{v-1} \epsilon_k + \sum_{k=v}^{2v-1} \epsilon_k)$$

and

$$b_{(i,j)} = b + a_{(j,i)},$$

for $v \leq i \leq 2v - 1$ and $0 \leq j \leq v - 1$. In particular the $b_{(i,j)}$ are all of the form $\frac{1}{2}(\pm\epsilon_0 \pm \epsilon_1 \pm \cdots \pm \epsilon_{2v-1})$.

The vectors giving the open convex cones (thus omitting the top level integral) are now tabulated as follows

degeneration	vectors
ka	$a_{(1,2v)}, a_{(2,2v)}, \ldots, a_{(k-1,2v)}$
$(1,1)$	b
$(k,1)a$	$b, a_{(1,2v)}, a_{(2,2v)}, \ldots, a_{(k-1,2v)}$
$(1,l)a$	$b, b_{(v+1,v)}, b_{(v+2,v)}, \ldots, b_{(v+l-1,v)}$
$(k,l)a$	$b, a_{(1,2v)}, a_{(2,2v)}, \ldots, a_{(k-1,2v)}, b_{(v+1,v)}, b_{(v+2,v)}, \ldots, b_{(v+l-1,v)}$
$(k,l)^*a$	$b, a_{(1,2v)}, a_{(2,2v)}, \ldots, a_{(k-1,2v)}, b_{(v+1,v)}, b_{(v+1,v-1)}, \ldots, b_{(v+1,v-l+1)}$.

(5.4.16)

The vectors corresponding to the cones of the b degenerations can be obtained by applying the matrix

$$\begin{pmatrix} 0 & \cdots & 0 & -1 \\ 0 & \cdots & -1 & 0 \\ \vdots & \iddots & \vdots & \vdots \\ -1 & \cdots & 0 & 0 \end{pmatrix}$$

to all vectors defining the cone of the corresponding a degeneration. This matrix basically sends $\epsilon_1 \leftrightarrow -\epsilon_{2v}$, $\epsilon_2 \leftrightarrow -\epsilon_{2v-1}$, $\ldots$.

Finally observe that the level of a degeneration (Definition 5.4.9) corresponds to 1 plus the number of vectors generating the convex cones (if we use the convention that the top level integral, associated to the "cone" consisting of only the origin, is generated by 0 vectors).

5.5 Type I degenerations

In this section we will consider the degenerations of the $I_{n,T}^m$ integral. We will give the explicit integrals corresponding to each vertex in the degeneration scheme, which basically are reparametrizations of the $J_{n,s,t}$ and $J_{n,(s_1,s_2),t}$ integrals from Definition 5.3.17. Subsequently we show how they occur as limits of each other (and hence of the top level integral). Finally we consider the consequences of the type I_{BC} transformation formula (5.3.3) of the $I_{n,T}^m$ integral for these degenerations.

5.5.1 The definition of the degenerations

In order to be able to identify integrals with vertices in the degeneration scheme we introduce the following reparametrization of the $J_{n,s,t}$ and $J_{n,(s_1,s_2),t}$ integrals. Later we obtain limit formulas showing that the integrals in the following definitions are indeed degenerations of the top level integral.

For completeness we include the top level integral, which was already defined in Definition 5.3.1, in the following definition. Recall the definitions of the parameter domains from Definition 5.3.12 and that of $J_{n,s,t}$ (5.3.16).

Definition 5.5.1. *Let $m,n \in \mathbb{Z}_{\geq 0}$. For parameters $\mu \in \mathcal{B}^{\omega}_{2n+2m+4}$ satisfying the balancing condition*

$$\sum_{r=0}^{2n+2m+3} \mu_r = 2(m+1)\omega, \tag{5.3.1}$$

we define

$$I_{n,T}^m(\mu) = J_{n,2n+2m+4,0}(\mu).$$

For parameters $\mu \in \mathcal{B}^{\omega}_{2n+2m+2} \cap \mathcal{D}^{\omega}_{n,2n+2m+2}$ we define

$$I_{n,2}^m(\mu) = J_{n,2n+2m+2,0}(\mu)$$

and for $2 < k \leq 2n+2m+4$ and $\mu \in \mathcal{B}^{\omega}_{2n+2m+4-k}$ we define

$$I_{n,ka}^m(\mu) = J_{n,2n+2m+4-k,2-k}(\mu), \qquad I_{n,kb}^m(\mu) = J_{n,2n+2m+4-k,k-2}(\mu). \tag{5.5.1}$$

As noted before we omitted the a/b demarcation of the $I_{n,2}^m$ integral, since in this case $I_{n,2a}^m$ and $I_{n,2b}^m$ coincide if we would have defined them via (5.5.1). Observe that the $I_{n,T}^{m-1}$ integral equals the $I_{n,2}^m$ integral with a balancing condition. This shows that specializing parameters can reduce m but it can also reduce the level of degeneration.

Not all $J_{n,s,t}$ integrals are reparametrized as $I_{n,ka}^m$ or $I_{n,kb}^m$ integrals. In particular $s+t$ is always even and moreover $s+|t| \geq 2n+2$. We will see that the $J_{n,s,t}$ integrals with $s+|t| \leq 2n$ and $s+t$ even vanish for all values of μ, hence we do not include them. We can express the $J_{n,s,t}$ integrals in terms of the $I_{n,k}^m$ integrals for given for $s+t$ even and $s+|t| \geq 2n+2$ by the following parameter specialization

condition	m	type
$t \leq 0$	$\frac{s-t-2-2n}{2}$	$(2-t)a$
$t \geq 0$	$\frac{s+t-2-2n}{2}$	$(2+t)b$

The symmetry (5.3.12) now interchanges a and b integrals via the formula

$$I^m_{n,ka}(\mu;(\omega_1,\omega_2)) = \overline{I^m_{n,kb}(-\bar{\mu};(-\bar{\omega}_1,-\bar{\omega}_2))}, \tag{5.5.2}$$

for $k \geq 2$.

The other degenerations are expressed in terms of $J_{n,(s_1,s_2),t}$ integrals (5.3.17).

Definition 5.5.2. *For $m, n \in \mathbb{Z}_{\geq 0}$ we define*

$$I^m_{n,\xi}(\mu;\nu;\lambda) = J_{n,(s_1,s_2),t}(\mu;\nu;\lambda),$$

where ξ is either $(k,l)a$ or $(k,l)b$ ($1 \leq k,l \leq n+m+2$ or $k=1$ and $l=n+m+3$ and vice versa) or $(k,l)^*a$ or $(k,l)^*b$ ($k,l \geq 3$ and $k+l \leq n+m+4$). The correspondence between the different parameters is given in the following table

ξ	$s_1(\xi)$	$s_2(\xi)$	$t(\xi)$
$(k,l)a$	$n+m+2-l+\delta_{k,1}$	$n+m+2-k+\delta_{l,1}$	$4-k-\delta_{k,1}-l-\delta_{l,1}$
$(k,l)b$	$n+m+2-k+\delta_{l,1}$	$n+m+2-l+\delta_{k,1}$	$k+\delta_{k,1}+l+\delta_{l,1}-4$
$(k,l)^*a$	$n+m$	$n+m+4-k-l$	$l-k$
$(k,l)^*b$	$n+m+4-k-l$	$n+m$	$k-l$

where $\delta_{a,b}$ denotes the Kronecker delta. The variables μ, ν and λ should satisfy conditions such that the $J_{n,(s_1,s_2),t}$ is defined (see Definition 5.3.15 for $\tau=\omega$), i.e. $(\mu,\nu) \in \mathcal{B}^{\omega}_{(s_1,s_2)}$ and if $k \leq 2$ we impose the extra condition $(\mu,\nu,\lambda) \in \mathcal{D}_{n,(s_1,s_2)}$ while if $l \leq 2$ we impose the condition $(\mu,\nu,-\lambda) \in \mathcal{D}_{n,(s_1,s_2)}$. Moreover we impose the following balancing conditions in some cases. For the $I^m_{n,(k,1)a}$ and $I^m_{n,(k,1)b}$ ($k \geq 1$) degenerations we impose the balancing condition

$$\sum_r \mu_r + \sum_r \nu_r = \lambda + (2m+2-k+\delta_{k,1})\omega, \tag{5.5.3}$$

whereas for the $I^m_{n,(1,l)a}$ and $I^m_{n,(1,l)b}$ ($l \geq 1$) degenerations we impose the balancing condition

$$\sum_r \mu_r + \sum_r \nu_r = -\lambda + (2m+2-l+\delta_{l,1})\omega. \tag{5.5.4}$$

From this definition we see that the degenerations are just a reparametrization of the $J_{n,(s_1,s_2),t}$ integrals with sometimes some balancing conditions imposed on the variables. Note, however, that the degenerations $I^m_{n,\xi}$ again do not reparametrize all possible $J_{n,(s_1,s_2),t}$ integrals. In particular $t+s_1+s_2$ is always even for the degenerations $I^m_{n,\xi}$. Moreover if $s_1, s_2 < n$ the value of t has to satisfy $|t| \geq 2n-s_1-s_2$ for the degenerations. We will see that the remaining integrals with $t+s_1+s_2$ even vanish identically.

For all integer values of s_1, s_2 and t, such that $t+s_1+s_2$ is even and such that if $s_1, s_2 < n$ we have $|t| \geq 2n-s_1-s_2$, the $J_{n,(s_1,s_2),t}$ integral can be expressed as an $I^m_{n,\xi}$ integral with parameters given by

condition	type	m	k	l
$t \leq -\lvert s_1 - s_2\rvert$	$(k,l)a$	$\frac{s_1+s_2-t-2n}{2}$	$\frac{s_1-s_2-t+4}{2}$	$\frac{s_2-s_1-t+4}{2}$
$\lvert t\rvert < \lvert s_1 - s_2\rvert$ and $s_1 > s_2$	$(k,l)^*a$	$s_1 - n$	$\frac{s_1-s_2-t+4}{2}$	$\frac{s_1-s_2+t+4}{2}$
$\lvert t\rvert < \lvert s_1 - s_2\rvert$ and $s_1 < s_2$	$(k,l)^*b$	$s_2 - n$	$\frac{s_2-s_1+t+4}{2}$	$\frac{s_2-s_1-t+4}{2}$
$t \geq \lvert s_1 - s_2\rvert$	$(k,l)b$	$\frac{s_1+s_2+t-2n}{2}$	$\frac{s_2-s_1+t+4}{2}$	$\frac{s_1-s_2+t+4}{2}$

This will lead to $I^m_{n,(k,l)}$ or $I^m_{n,(k,l)^*}$ integrals with $k, l \geq 2$. Note that the $I^m_{n,(k,1)}$ and $I^m_{n,(1,l)}$ integrals are equal to the $I^{m+1}_{n,(k,2)}$ and $I^{m+1}_{n,(2,l)}$ integrals with an extra balancing condition. Also note that in this list m will always be non-negative. Consider for example the case $m = s_1 - n$ for the $(k,l)^*a$ integral; if $s_1 > s_2$ and $|t| < |s_1 - s_2|$ and $s_1 - n < 0$ we have $n > s_1, s_2$, however then $|t| < |s_1 - s_2| = |(n - s_1) - (n - s_2)| \leq (n - s_1) + (n - s_2) = 2n - s_1 - s_2$, which was not allowed.

As we will see more clearly when we write down the limits between the $I^m_{n,\xi}$ integrals, the μ-parameters denote those parameters of the top level integral which go to infinity in the limit as fS (with f and S as in the previous section), i.e. at the same rate as the location of the maximum of the integrand. Hence the number $s_1(\xi)$ of μ parameters is equal to $h_4(\xi)$ (where $h_4(\xi)$ denotes the value of h_4 related to ξ in the table of Proposition 5.4.8). Similarly the ν parameters denote the parameters which go to infinity at the same rate as minus the location of the maximum of the integrand and hence $s_2(\xi) = h_2(\xi)$. Moreover, a calculation, which we will not include here, gives $t(\xi) = 2h_5(\xi) + h_4(\xi) - h_2(\xi) - 2h_1(\xi)$.

In the specific case of the $I^m_{n,(1,1)}$ degeneration, we have two balancing conditions and they can be simplified to $\lambda = 0$ and

$$\sum_{r=0}^{n+m+1} \mu_r + \nu_r = 2(m+1)\omega. \tag{5.5.5}$$

Since λ is fixed for the $I^m_{n,(1,1)}$ degeneration we will usually just write $I^m_{n,(1,1)}(\mu;\nu)$ for $I^m_{n,(1,1)}(\mu;\nu;0)$.

In the case of $I^m_{n,(k,1)}$ and $I^m_{n,(1,l)}$ degenerations the balancing condition can be viewed as a definition of the value of λ, given the values of μ and ν, so again we can remove λ from the set of variables. Moreover the balancing condition in those cases ensures that $(\mu,\nu,-\lambda) \in \mathcal{D}_{n,(s_1,s_2)}$, respectively $(\mu,\nu,\lambda) \in \mathcal{D}_{n,(s_1,s_2)}$, hence the extra condition for $l \leq 2$, respectively $k \leq 2$, is always satisfied of $l = 1$, respectively $k = 1$.

The symmetries (Proposition 5.3.16) of the $J_{n,(s_1,s_2),t}$ integral reduce to symmetries which flip the k and l of a degeneration

$$\begin{aligned} I^m_{n,(k,l)a}(\mu;\nu;\lambda;(\omega_1,\omega_2)) &= I^m_{n,(l,k)a}(\nu;\mu;-\lambda;(\omega_1,\omega_2)) \\ I^m_{n,(k,l)^*a}(\mu;\nu;\lambda;(\omega_1,\omega_2)) &= \overline{I^m_{n,(l,k)^*a}(-\bar{\mu};-\bar{\nu};\bar{\lambda};(-\bar{\omega}_1,-\bar{\omega}_2))} \end{aligned} \tag{5.5.6}$$

(where we have to use both (5.3.13) and (5.3.15) to derive the second equation) and symmetries which turn a-integrals in b-integrals

$$\begin{aligned} I^m_{n,(k,l)a}(\mu;\nu;\lambda;(\omega_1,\omega_2)) &= \overline{I^m_{n,(k,l)b}(-\bar{\nu};-\bar{\mu};-\bar{\lambda};(-\bar{\omega}_1,-\bar{\omega}_2))} \\ I^m_{n,(k,l)^*a}(\mu;\nu;\lambda;(\omega_1,\omega_2)) &= \overline{I^m_{n,(k,l)^*b}(-\bar{\nu};-\bar{\mu};-\bar{\lambda};(-\bar{\omega}_1,-\bar{\omega}_2))}. \end{aligned} \tag{5.5.7}$$

Moreover we obtain a symmetry which allows us to shift the variables by some constant from (5.3.14).

5.5.2 Degeneration formulas

We will now translate the limits between the $J_{n,s,t}$ integrals from Subsection 5.3.5 to limits between the $I^m_{n,\xi}$ integrals. We will first describe those degenerations where m remains fixed. This gives a rather long list in which every line in the degeneration scheme (as seen in Figure 5.5 and 5.6) is materialized by an explicit formula.

All of the limits in the following theorems are special cases of the limits from Propositions 5.3.22-5.3.24. In order to simplify the statement of the theorems we will not explicitly state the parameter domains for which the theorems hold; these domains can be directly obtained from the statements in Propositions 5.3.22-5.3.24 using the parameter correspondence from the previous subsection. Note however that there always exist such appropriate variables for which the limit exists.

We work our way through the degeneration scheme basically from left to right, and top to bottom. Moreover we omit those limits which can be obtained from others by applying the $a \leftrightarrow b$ and $(k,l) \leftrightarrow (l,k)$ interchanging symmetries (5.5.2), (5.5.6) and (5.5.7) to both sides of the equation.

First we obtain the limits between $I^m_{n,ka}$ integrals. Note that the limit from $I^m_{n,T}$ to $I^m_{n,2}$ removes the balancing condition. Recall the definition (5.1.5) of the right hand cone (RHC).

Theorem 5.5.3. *Let $n,m \in \mathbb{Z}_{\geq 0}$. For appropriate $\mu \in \mathbb{C}^{2n+2m+2}$ and $\xi \in \mathbb{C}^2$ satisfying the balancing condition 5.3.1 and $q \in RHC$ we have*

$$I^m_{n,2}(\mu) = \lim_{S\to\infty} I^m_{n,T}(\mu,\xi_1+qS,\xi_2-qS)c(2n((\xi_2-qS-\omega)^2-(\xi_1+qS-\omega)^2). \tag{5.5.8}$$

Let $3 \leq k \leq 2n+2m+4$. For appropriate $\mu \in \mathbb{C}^{2n+2m+4-k}$, $\xi \in \mathbb{C}$ and $q \in RHC$ we have

$$I^m_{n,ka}(\mu) = \lim_{S\to\infty} I^m_{n,(k-1)a}(\mu,\xi-qS)\zeta^{-2n}c(2n(\xi-qS-\omega)^2). \tag{5.5.9}$$

Proof. (5.5.8) is an instance of the second limit of Proposition 5.3.22, for $\tau = \omega$. (5.5.9) can be obtained from the second limit of Proposition 5.3.22 by taking the complex conjugate of both sides of the equation and using (5.3.12). □

Secondly, we obtain the limits $ka \to (k,1)a$.

Theorem 5.5.4. *Let $n, m \in \mathbb{Z}_{\geq 0}$. For appropriate $\mu, \nu \in \mathbb{C}^{n+m+2}$ and $q \in RHC$ satisfying the balancing condition* (5.5.5) *we have*

$$I^m_{n,(1,1)}(\mu;\nu) = \lim_{S\to\infty} I^m_{n,T}(\mu+qS, \nu-qS)c(n(4(n+1)\omega qS-\sum_r(\mu_r-\omega)^2+\sum_r(\nu_r-\omega)^2)), \tag{5.5.10}$$

where $\mu + qS = (\mu_0 + qS, \ldots, \mu_{n+m+1} + qS)$ and similarly for $\nu + qS$. For $2 \leq k \leq n+m+3$ and appropriate $\mu \in \mathbb{C}^{n+m+1}$, $\nu \in \mathbb{C}^{n+m+3-k}$, $\lambda \in \mathbb{C}$ and $q \in RHC$ satisfying (5.5.3) *we have*

$$\begin{aligned} I^m_{n,(k,1)a}(\mu;\nu;\lambda) &= \lim_{S\to\infty} I^m_{n,ka}(\mu+qS, \nu-qS)\zeta^{n(k-2)} \\ &\times c(n(2(2-k)q^2S^2 - 4qS(\lambda-(n+1)\omega) - \sum_{r=0}^{n+m}(\mu_r-\omega)^2 + \sum_{r=0}^{n+m+2-k}(\nu_r-\omega)^2)). \end{aligned} \tag{5.5.11}$$

Proof. These limits are instances of the limit in Proposition 5.3.23 for $\tau = \omega$. □

Thirdly, we consider the limits of $(k,1) \to (k+1,1)$. Using the $k \leftrightarrow l$ interchanging symmetry (5.5.6) these limits also give us the limits $(1,l) \to (1,l+1)$.

Theorem 5.5.5. *Let $n, m \in \mathbb{Z}_{\geq 0}$. For appropriate $\mu, \nu \in \mathbb{C}^{n+m+1}$, $\xi \in \mathbb{C}^2$, $\lambda = 2\omega - \xi_1 - \xi_2 \in \mathbb{C}$ and $q \in RHC$ satisfying the balancing condition* (5.5.3) *we have*

$$\begin{aligned} I^m_{n,(2,1)}(\mu;\nu;\lambda) = \lim_{S\to\infty} & I^m_{n,(1,1)}(\mu, \xi_1+qS; \nu, \xi_2-qS; \lambda+\xi_1+\xi_2-2\omega) \\ &\times c(n((\xi_2-qS-\omega)^2 - (\xi_1+qS-\omega)^2)), \end{aligned} \tag{5.5.12}$$

Let $3 \leq k \leq n+m+3$. For appropriate $\mu \in \mathbb{C}^{n+m+1}$, $\nu \in \mathbb{C}^{n+m+3-k}$, $\lambda, \xi \in \mathbb{C}$ and $q \in RHC$ satisfying the balancing condition (5.5.3) *we have*

$$\begin{aligned} I^m_{n,(k,1)a}(\mu;\nu;\lambda) = \lim_{S\to\infty} & I^m_{n,(k-1,1)a}(\mu; \nu, \xi-qS; \lambda-qS+\xi-\omega) \\ &\times \zeta^{-n} c(n(\xi-qS-\omega)^2). \end{aligned} \tag{5.5.13}$$

Proof. The first limit is an instance of the second limit in Proposition 5.3.24. The second limit can be obtained from the first limit in Proposition 5.3.24 by taking the complex conjugate on both sides of the equation and applying (5.3.13). □

Fourthly, there are the limits $(k,1) \to (k,2)$, which remove one balancing condition.

Theorem 5.5.6. *Let $n, m \in \mathbb{Z}_{\geq 0}$ and $2 \leq k \leq n+m+2$. For appropriate $\mu \in \mathbb{C}^{n+m}$, $\nu \in \mathbb{C}^{n+m+2-k}$, $\xi \in \mathbb{C}^2$, $\lambda \in \mathbb{C}$ and $q \in RHC$ such that $(\mu, \xi_1; \nu, \xi_2; \lambda)$ satisfies the balancing condition* (5.5.3) *we have*

$$\begin{aligned} I^m_{n,(k,2)a}(\mu;\nu;\lambda) = \lim_{S\to\infty} & I^m_{n,(k,1)a}(\mu, \xi_1-qS; \nu, \xi_2+qS; \lambda-\xi_1-\xi_2+2\omega) \\ &\times c(n((\xi_1-qS-\omega)^2 - (\xi_2+qS-\omega)^2)). \end{aligned} \tag{5.5.14}$$

Proof. This limit is an instance of the second limit in Proposition 5.3.24. □

Finally we obtain the limits $(k-1,l) \to (k,l)$, $(2,l) \to (3,l)^*$ and $(k-1,l)^* \to (k,l)^*$. Note that by the $k \leftrightarrow l$ interchanging symmetry (5.5.6) this also gives limits $(k,l-1) \to (k,l)$ and $(k,l-1)^* \to (k,l)^*$.

Theorem 5.5.7. *Let $n,m \in \mathbb{Z}_{\geq 0}$ and for $n+m+2 \geq k \geq 3$ and $n+m+2 \geq l \geq 2$. For appropriate $\mu \in \mathbb{C}^{n+m+2-l}$, $\nu \in \mathbb{C}^{n+m+2-k}$, $\xi, \lambda \in \mathbb{C}$ and $q \in RHC$ we have*

$$I^m_{n,(k,l)a}(\mu;\nu;\lambda) = \lim_{S\to\infty} I^m_{n,(k-1,l)a}(\mu;\nu,\xi-qS;\lambda-qS+\xi-\omega) \times \zeta^{-n} c(n(\xi-qS-\omega)^2), \quad (5.5.15)$$

Let $k \geq 3$, $l \geq 3$ and $k+l \leq n+m+4$. For appropriate $\mu \in \mathbb{C}^{n+m}$, $\nu \in \mathbb{C}^{n+m+4-k-l}$, $\xi, \lambda \in \mathbb{C}$ and $q \in RHC$ we have

$$I^m_{n,(k,l)^*a}(\mu;\nu;\lambda) = \lim_{S\to\infty} I^m_{n,(k-1,l)^*a}(\mu;\nu,\xi-qS;\lambda-qS+\xi-\omega) \times \zeta^{-n} c(n(\xi-qS-\omega)^2), \quad (5.5.16)$$

where we use the convention $I^m_{n,(2,l)^*a} = I^m_{n,(2,l)b}$ for $k=3$.

Proof. These limits are instances of the second limit in Proposition 5.3.24, after taking the complex conjugate on both sides and using (5.3.13). □

It might now seem that the limit transitions between $I^m_{n,(k,l)a}$ integrals and those between $I^m_{n,(k,l)^*a}$ integrals are the same as the k-increasing limits (5.5.15) and (5.5.16) of these integrals are similar. However the l-increasing limits are different, since the (k,l) to (l,k) interchanging symmetry (5.5.6) is different for the $(k,l)a$ and $(k,l)^*a$ integrals. In particular we have (for appropriate values of μ, ν, ξ and q) for $k,l \geq 2$

$$I^m_{n,(k,l)a}(\mu;\nu;-\lambda) = \lim_{S\to\infty} I^m_{n,(k,l-1)a}(\mu,\xi-qS;\nu;-\lambda+qS-\xi+\omega) \times \zeta^{-n} c(n(\xi-qS-\omega)^2), \quad (5.5.17)$$

and for $k,l \geq 3$

$$I^m_{n,(k,l)^*a}(\mu;\nu;\lambda) = \lim_{S\to\infty} I^m_{n,(k,l-1)^*a}(\mu;\nu,\xi+qS;\lambda-qS-\xi+\omega) \times \zeta^{n} c(-n(\xi+qS-\omega)^2)$$

(using the convention $I^m_{n,(3,2)^*a} = I^m_{n,(3,2)a}$).

Note that these limits allow us to express every defined degeneration as a limit (possibly a sequence of limits) of the top level integral $I^m_{n,T}$ with the same parameters n and m. The form of the limits in Theorems 5.5.3-5.5.7 moreover show that the $I^m_{n,\xi}$ integrals are indeed the degenerations corresponding to ξ with sets of parameters going to infinity as described in the previous section. For example

(5.5.9) shows that to increase the k of an ka degeneration we have to send one more variable to minus infinity. As previously noted the transitions (5.5.10) and (5.5.11) from ka to $(k,1)$ integrals show that the μ parameters are those which go to infinity at the same rate as the maximum and the ν parameters those which go to minus infinity at that rate.

Furthermore (5.5.15) shows that increasing k in a (k,l) degeneration (with $k \geq 2$) involves letting a ν variable, one which was already "going to" minus infinity at the same rate as f (from the previous section), go to minus infinity even faster, hence it ends up going to minus infinity at a rate higher than f. Increasing l for such a degeneration as in (5.5.17) involves letting a μ parameter go to minus infinity. This implies that a parameter which went to plus infinity at rate as f (i.e. a μ-parameter) is now going less quickly to (plus) infinity and thus remains in the regime of parameters with an asymptotic behaviour between $-fS$ and fS.

There also exist several m-lowering limits. The interest in these limits is not so that we can learn more about the degenerations with lower m, since their properties can already be obtained from the properties of the top level integral with lower m. However they do allow us to pull up some results about the degenerations with lower m to the degeneration with higher m. Note that these limits show that we can reduce the level of degeneration by reducing m.

Theorem 5.5.8. *Let $n,m \in \mathbb{Z}_{\geq 0}$. Let $k \geq 2$. For appropriate $\mu \in \mathbb{C}^{2n+2m+4-k}$, $\xi \in \mathbb{C}$ and $q \in RHC$ we have*

$$I^m_{n,ka}(\mu) = \lim_{S\to\infty} I^{m+1}_{n,(k+1)a}(\mu, \xi + qS)\zeta^{2n} c(-2n(\xi + qS - \omega)^2). \tag{5.5.18}$$

Let $k \geq 2$ and $l \geq 1$. For appropriate $\mu \in \mathbb{C}^{n+m+2-l}$, $\nu \in \mathbb{C}^{n+m+2-k-\delta_{l,1}}$, $\xi, \lambda \in \mathbb{C}$ and $q \in RHC$ satisfying the balancing condition (5.5.3) *if $l = 1$, we have*

$$\begin{aligned} I^m_{n,(k,l)a}(\mu;\nu;\lambda) = \lim_{S\to\infty} & I^{m+1}_{n,(k+1,l)a}(\mu, \xi + qS; \nu; \lambda + qS + \xi - \omega) \\ & \times \zeta^n c(-n(\xi + qS - \omega)^2). \end{aligned} \tag{5.5.19}$$

Let $k \geq 2$ and $l \geq 3$. For appropriate $\mu \in \mathbb{C}^{n+m}$, $\nu \in \mathbb{C}^{n+m+4-k-l}$, $\xi, \lambda \in \mathbb{C}$ and $q \in RHC$ we have

$$\begin{aligned} I^m_{n,(k,l)^*a}(\mu;\nu;\lambda) = \lim_{S\to\infty} & I^{m+1}_{n,(k+1,l)^*a}(\mu, \xi + qS; \nu; \lambda + qS + \xi - \omega) \\ & \times \zeta^n c(-n(\xi + qS - \omega)^2), \end{aligned} \tag{5.5.20}$$

where we use once again the convention $I^m_{n,(2,l)^*a} = I^m_{n,(2,l)b}$.

Proof. These limits are instances of the first limits from Propositions 5.5.3 (for (5.5.18)) and 5.5.5 (for (5.5.19) and (5.5.20)). □

5.5.3 Transformations

We will now reap the benefits of writing the $I^m_{n,\xi}$ integrals as degenerations of the top level integral, by considering what becomes of the I_{BC} transformation formula from Proposition 5.3.4. Recall that this transformation interchanges n and m of the $I^m_{n,T}$ integral. For the case $m = 0$ it therefore reduces to an evaluation formula of the $I^0_{n,T}$ integral.

Using the limits of the previous section we can now easily calculate transformation formulas for the degenerations. In the univariate case $n = 1$ the transformations of the first few cases have already been obtained in [6]. We will not give the explicit formulations of the evaluation formulas for the $I^0_{n,\xi}$ degenerations, since they can be obtained by specializing m to 0 in the transformation formulas and using that $I^n_{0,\xi} = 1$ for all n and ξ. The evaluations of univariate integrals are identical to the univariate type II_{BC} integrals. Some of these were previously known, references are given when we discuss the type II_{BC} integrals in the next section, as we give the integrals more explicitly there.

First we obtain transformation formulas for the $I^m_{n,ka}$ degenerations. We do not repeat the transformation formula of the $I^m_{n,T}$ integral, which was given in Proposition 5.3.4.

Theorem 5.5.9. *Let $n, m \in \mathbb{N}$. For parameters $\mu \in \mathcal{B}_{2n+2m+2} \cap \mathcal{D}_{n,2n+2m+2}$ such that $\omega - \mu \in \mathcal{B}_{2n+2m+2} \cap \mathcal{D}_{m,2n+2m+2}$ we have*

$$I^m_{n,2}(\mu) = I^n_{m,2}(\omega-\mu)\Gamma_h(2(m+1)\omega - \sum_{r=0}^{2n+2m+1} \mu_r) \prod_{0\le r<s\le 2n+2m+1} \Gamma_h(\mu_r+\mu_s).$$

For $3 \le k \le 2n+2m+4$ and $\mu \in \mathcal{B}_{2n+2m+4-k}$, such that $\omega - \mu \in \mathcal{B}_{2n+2m+4-k}$ we have

$$I^m_{n,ka}(\mu) = I^n_{m,kb}(\omega-\mu) \prod_{1\le r<s\le 2n+2m+4-k} \Gamma_h(\mu_r+\mu_s)\zeta^{-\frac{(k-3)(k-4)}{2}}$$
$$\times c((k-2)(2n - \frac{(k-3)}{2})\omega^2 + (2-k)\sum_r (\mu_r-\omega)^2 + ((2m+1)\omega - \sum_r \mu_r)^2).$$

Proof. The proofs of these transformation formulas consist of taking the transformation formula one level higher and taking the appropriate limit on both sides of the equation using Theorem 5.5.3. For the second formula we moreover have to use induction on k (note that the base case is $k = 3$, and that the $k = 2$ transformation formula is not a special case of the second formula). As an example of the kind of calculations involved we consider the formula for $k = 3$. In the derivation we apply first (5.5.9), then the transformation formula for $I^m_{n,2}$, as given above and finally the complex conjugate of (5.5.9) together with the asymptotics of the hyperbolic gamma function from Theorem 5.2.6.

In order to be able to apply the limit formulas to both $I^m_{n,3a}$ and $I^m_{n,3b}$ we first restrict to parameters μ such that $\alpha(n, 2n+2m+1; \mu), \alpha(n, 2n+2m+1; \omega-\mu) \in$

$(\frac{\phi_-+\phi_+-\pi}{2}, \frac{\phi_-+\phi_++\pi}{2})$. Subsequently we choose ξ such that ξ and $\omega-\xi$ are both positive linear combinations of ω_1 and ω_2 (i.e. $\xi = a\omega_1 + b\omega_2$ for $0 < a, b < 1/2$). Finally we let $q = \exp(i(\phi_- + \phi_+ - \pi)/2)$. Now we can use (5.5.9) in the special case $q = \exp(i\phi)$ with $\phi = (\phi_- + \phi_+ - \pi)/2$ and we calculate

$$
\begin{aligned}
I^m_{n,3a}(\mu) &= \lim_{S\to\infty} I^m_{n,2}(\mu, \xi - qS)\zeta^{-2n} c(2n(\xi - qS - \omega)^2) \\
&= \lim_{S\to\infty} I^n_{m,2}(\omega-\mu, \omega-\xi+qS)\Gamma_h(2(m+1)\omega - \sum_r \mu_r - \xi + qS) \\
&\quad \times \prod_{r<s}\Gamma_h(\mu_r+\mu_s)\prod_r \Gamma_h(\mu+\xi-qS)\zeta^{-2n}c(2n(\xi-qS-\omega)^2) \\
&= I^n_{m,3b}(\omega-\mu)\zeta^{-2m}c(2m(\omega-\xi+qS-\omega)^2) \\
&\quad \times \zeta^{-1}c((2(m+1)\omega - \sum_r \mu_r - \xi + qS - \omega)^2)\prod_{r<s}\Gamma_h(\mu_r+\mu_s) \\
&\quad \times \zeta^{2n+3m+1}c(-\sum_r(\mu_r+\xi-qS-\omega)^2)\zeta^{-2n}c(2n(\xi-qS-\omega)^2) \\
&= I^n_{m,3b}(\omega-\mu)\prod_{r<s}\Gamma_h(\mu_r+\mu_s) \\
&\quad \times c(2n\omega^2 - \sum_r(\mu_r-\omega)^2 + ((2m+1)\omega - \sum_r \mu_r)^2).
\end{aligned}
$$

Finally, we use analytic continuation in μ to obtain the relation for all values of μ.

Note that in the derivation of the transformation formula for the $I^m_{n,2}$ degeneration using (5.5.8) on the transformation of the top level integral (5.3.3) we have to choose the values of the new variables ξ_1 and ξ_2 such that the balancing condition (5.3.1) of $I^m_{n,T}$ is satisfied. □

Now we consider the transformations of $I^m_{n,(k,1)a}$. Using (5.5.6) these transformations also give transformations for $I^m_{n,(1,k)}$ degenerations.

Theorem 5.5.10. *Let $m, n \in \mathbb{N}$. For $(\mu, \nu) \in \mathcal{B}_{(n+m+2,n+m+2)}$ such that $(\omega-\nu, \omega-\mu) \in \mathcal{B}_{(n+m+2,n+m+2)}$ satisfying the balancing condition* (5.5.5) *we have*

$$
I^m_{n,(1,1)}(\mu;\nu) = I^n_{m,(1,1)}(\omega-\nu;\omega-\mu)\prod_{r,s=0}^{n+m+1}\Gamma_h(\mu_r+\nu_s).
$$

For $(\mu,\nu) \in \mathcal{B}_{(n+m+1,n+m+1)}$ and $\lambda \in \mathbb{C}$ such that $(\omega-\nu,\omega-\mu) \in \mathcal{B}_{(n+m+1,n+m+1)}$ and such that the balancing condition (5.5.3) *is satisfied we have*

$$
\begin{aligned}
I^m_{n,(2,1)}(\mu;\nu;\lambda) = I^n_{m,(2,1)}(\omega-\nu;\omega-\mu;2\omega-\lambda)\Gamma_h(2\omega-\lambda)\prod_{r,s=0}^{n+m}\Gamma_h(\mu_r+\nu_s) \\
\times c(\sum_{r=0}^{n+m}(\nu_r^2-\mu_r^2) + \lambda\sum_{r=0}^{n+m}(\mu_r-\nu_r)).
\end{aligned}
$$

For $3 \leq k \leq n+m+3$ *and* $(\mu,\nu) \in \mathcal{B}_{(n+m+1,n+m+3-k)}$ *and* $\lambda \in \mathbb{C}$ *such that* $(\omega-\nu,\omega-\mu) \in \mathcal{B}_{(n+m+3-k,n+m+1)}$ *and such that the balancing condition* (5.5.3) *is satisfied we have*

$$\begin{aligned}
&I^m_{n,(k,1)a}(\mu;\nu;\lambda)\\
&\quad = I^n_{m,(k,1)b}(\omega-\nu;\omega-\mu;k\omega-\lambda)\prod_{r=0}^{n+m}\prod_{s=0}^{n+m+2-k}\Gamma_h(\mu_r+\nu_s)\zeta^{(k-3)}c(\sum_r \nu_r^2)\\
&\qquad \times c((km-2m+k-1)\omega^2+2\lambda(\sum_r \mu_r-(m+1)\omega)-(k-1)\sum_r \mu_r^2).
\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 5.5.9. We have to use Theorem 5.5.4 to prove the transformation of the $I^m_{n,(1,1)}$ degeneration. For the other degenerations we can choose in which higher level degeneration's transformation formula we take the limit. Either we use the limits from Theorem 5.5.4 and the transformations from Theorem 5.5.9, or we use the limits from Theorem 5.5.5 and the transformations from this theorem (and induction on k). Note that the explicit calculations are rather tedious. □

Now we consider the transformations of $I^m_{n,(k,l)}$ integrals with $k,l \geq 2$. The transformation formula of the $I^m_{n,(2,l)a}$ degeneration can again be obtained from the transformation formula of the $I^m_{n,(l,2)a}$ degeneration by applying (5.5.6) to both sides of the equation.

Theorem 5.5.11. *Let* $m,n \in \mathbb{N}$. *For* $(\mu,\nu) \in \mathcal{B}_{(n+m,n+m)}$ *such that* $(\omega-\nu,\omega-\mu) \in \mathcal{B}_{(n+m,n+m)}$, $(\mu,\nu,\lambda),(\mu,\nu,-\lambda) \in \mathcal{D}_{n,(n+m,n+m)}$ *and* $(\omega-\mu,\omega-\nu,\lambda),(\omega-\mu,\omega-\nu,-\lambda) \in \mathcal{D}_{m,(n+m,n+m)}$ *we have*

$$\begin{aligned}
I^m_{n,(2,2)}(\mu;\nu;\lambda) &= I^n_{m,(2,2)}(\omega-\nu;\omega-\mu;-\lambda)\prod_{r,s=0}^{n+m-1}\Gamma_h(\mu_r+\nu_s)\\
&\quad\times \Gamma_h((m+1)\omega-\frac{1}{2}\sum_{r=0}^{n+m-1}(\mu_r+\nu_r)\pm\frac{1}{2}\lambda)c(\lambda\sum_{r=0}^{n+m-1}(\mu_r-\nu_r)). \quad (5.5.21)
\end{aligned}$$

For $3 \leq k \leq n+m+2$ *and* $(\mu,\nu) \in \mathcal{B}_{(n+m,n+m+2-k)}$ *and* $\lambda \in \mathbb{C}$ *such that* $(\omega-\nu,\omega-\mu) \in \mathcal{B}_{(n+m+2-k,n+m)}$, $(\mu,\nu,\lambda) \in \mathcal{D}_{n,(n+m,n+m+2-k)}$ *and* $(\omega-\mu,\omega-$

$\nu, (k-2)\omega - \lambda) \in \mathcal{D}_{m,(n+m,n+m+2-k)}$ *we have*

$$I^m_{n,(k,2)a}(\mu;\nu;\lambda) \tag{5.5.22}$$
$$= I^n_{m,(k,2)b}(\omega-\nu;\omega-\mu;(k-2)\omega-\lambda)\zeta^{-1}\prod_{r=0}^{n+m-1}\prod_{s=0}^{n+m+1-k}\Gamma_h(\mu_r+\nu_s)$$
$$\times \Gamma_h(\frac{1}{2}((2m+4-k)\omega+\lambda-\sum_{r=0}^{n+m-1}\mu_r-\sum_{r=0}^{n+m+1-k}\nu_r))$$
$$\times c((\lambda-(k-2)\omega)(\sum_r \mu_r - \sum_r \nu_r)-(k-2)\sum_{r=0}^{n+m-1}(\mu_r-\omega)^2)$$
$$\times c((k-2)n\omega^2+\frac{1}{4}((2m+k-2)\omega-\sum_r\mu_r-\sum_r\nu_r-\lambda)^2).$$

For $3 \le k, l \le n+m+2$ *and* $(\mu,\nu) \in \mathcal{B}_{(n+m+2-l,n+m+2-k)}$ *and* $\lambda \in \mathbb{C}$, *such that* $(\omega-\nu,\omega-\mu) \in \mathcal{B}_{(n+m+2-k,n+m+2-l)}$ *we have*

$$I^m_{n,(k,l)a}(\mu;\nu;\lambda)$$
$$= I^n_{m,(k,l)b}(\omega-\nu;\omega-\mu;(k-l)\omega-\lambda)\prod_{r,s}\Gamma_h(\mu_r+\nu_s)\zeta^{-6+2k+2l-kl}$$
$$\times c(\frac{1}{2}((4-k-l)(m-n)+(l-k)^2)\omega^2+\frac{1}{2}\lambda^2)$$
$$\times c((2-k)\sum_r\mu_r^2+(2-l)\sum_r\nu_r^2+\frac{1}{2}(2m\omega-\sum_r\mu_r-\sum_r\nu_r)^2)$$
$$\times c(\lambda(\sum_r\mu_r-\sum_r\nu_r+(l-k)\omega)+(k+l-4)(\sum_r\nu_r+\sum_r\mu_r)\omega).$$

Proof. The proof is again similar to the proof of Theorem 5.5.9. Now we have to apply the limits from Theorems 5.5.6 and 5.5.7 to the transformation formulas of the higher level degenerations. We also use a double induction on k and l to obtain the transformation formulas of the $I^m_{n,(k,l)a}$ degenerations. □

And finally we consider the transformations of $I^m_{n,(k,l)^*}$ integrals.

Theorem 5.5.12. *Let* $m,n \in \mathbb{N}$ *and* $k,l \ge 3$ *with* $k+l \le n+m+4$. *For variables* $(\mu,\nu) \in \mathcal{B}_{(n+m,n+m+4-k-l)}$ *and* $\lambda \in \mathbb{C}$ *such that* $(\omega-\nu,\omega-\mu) \in \mathcal{B}_{(n+m+4-k-l,n+m)}$ *we have*

$$I^m_{n,(k,l)^*a}(\mu;\nu;\lambda) = I^n_{m,(k,l)^*b}(\omega-\nu;\omega-\mu;(k-l)\omega-\lambda)\prod_{r,s}\Gamma_h(\mu_r+\nu_s)$$
$$\times c(-2m\lambda\omega+2\lambda\sum_r\mu_r+(l-k)(\sum_r\mu_r^2-m\omega^2)).$$

Proof. Once again the proof is obtained by taking the appropriate limit of the transformation formula for a higher level degeneration, using the limits from Theorem 5.5.7 and a double induction on k and l. □

Note that for the degenerations in which the parameters have to satisfy a balancing condition, if the balancing condition is satisfied on one side of the equation it is automatically also satisfied on the other side. For example for the $I^m_{n,(1,1)}$ degenerations we see that if (μ,ν) satisfies (5.5.5) for m, then $(\omega-\mu,\omega-\nu)$ satisfies (5.5.5) with m replaced by n, since then $\sum_r(\omega-\mu_r)+(\omega-\nu_r) = (2n+2m+4)\omega-\sum_r \mu_r+\nu_r = (2n+2)\omega$.

In contrast, the convergence conditions on the parameters can differ on both sides of the equation, as in the case of the transformation formula (5.5.22) of the $I^m_{n,(k,2)}$ degeneration. For the degeneration on the left hand side to converge we must have $(\mu,\nu,\lambda)\in\mathcal{D}_{n,(n+m,n+m+2-k)}$ and for the degeneration on the right hand side to converge we must impose $(\omega-\mu,\omega-\nu,(k-2)\omega-\lambda)\in\mathcal{D}_{m,(n+m,n+m+2-k)}$. Therefore the transformation formulas define analytic extensions of those integrals where the parameters should satisfy a convergence condition.

In our example we see that for arbitrary $(\mu,\nu,\lambda)\in\mathbb{C}^{2n+2m+3-k}$ always either $(\mu,\nu,\lambda)\in\mathcal{D}_{n,(n+m,n+m+2-k)}$ or $(\omega-\mu,\omega-\nu,(k-2)\omega-\lambda)\in\mathcal{D}_{m,(n+m,n+m+2-k)}$. For if we define

$$\kappa=(2m+1)\omega-\sum_r \mu_r-\sum_r \nu_r+\lambda,$$

we have $(\mu,\nu,\lambda)\in\mathcal{D}_{n,(n+m,n+m+2-k)}$ if and only if $\phi_-+\pi>\arg(\omega+\kappa)>\phi_+-\pi$ and $(\omega-\mu,\omega-\nu,(k-2)\omega-\lambda)\in\mathcal{D}_{m,(n+m,n+m+2-k)}$ holds if and only if $\phi_-+\pi> \arg(\omega-\kappa)>\phi_+-\pi$. Since $(\omega+\kappa)+(\omega-\kappa)=2\omega$ is in the upper half plane, at least one of the two numbers $\omega+\kappa$ and $\omega-\kappa$ is in the upper half plane. Therefore, for all values of μ, ν and λ the degeneration on at least one of the two sides of (5.5.22) is defined.

So far we couldn't tell by the explicit integral definitions of the degenerations whether they were identically zero or not. However the transformations from Theorems 5.5.9-5.5.12 reduce to evaluation formulas for $m=0$ degenerations, which show that those degenerations are at least not identically zero. This result even leads to the following stronger corollary.

Corollary 5.5.13. *The degenerations $I^m_{n,\xi}$ for $n,m\geq 0$ and any degeneration ξ are not identically zero either as function of μ (in the case of $I^m_{n,ka/b}$) or as a function of μ, ν and λ (for the other degenerations).*

Proof. Proposition 5.3.18 gives evaluation formulas for the integrals $I^m_{n,(2n+2m+4)a}$ and $I^m_{n,(2n+2m+4)b}$, thereby showing they are not identically zero (using $m\geq 0$). Similarly Proposition 5.3.19 gives evaluation formulas for the $I^m_{n,(n+m+2,n+m+2)a}$ and $I^m_{n,(n+m+2,n+m+2)b}$ degenerations, thereby showing these are also not identically zero. Finally the transformations from Theorems 5.5.9-5.5.12 show that $I^0_{n,\xi}$ is non-zero for all degenerations ξ.

Any degeneration not of this type can be degenerated into one of these integrals using the limits from Theorem 5.5.3-5.5.8 (thus including the m-lowering limits). In particular the degenerations $I^m_{n,T}$ and $I^m_{n,k}$ $(k\geq 2)$ can be degenerated into $I^m_{n,(2n+2m+4)}$. All degenerations of the form $I^m_{n,(k,l)}$ $(1\leq k,l\leq n+m+2)$ can be degenerated to $I^m_{n,(n+m+2,n+m+2)}$. The degeneration $I^m_{n,(n+m+3,1)}$ can be degenerated to $I^0_{n,(n+3,1)}$. The degenerations $I^m_{n,(k,l)^*}$ with $k \leq m+2$ can be degenerated first to $I^{m+2-k}_{n,(2,l)^*}$ and then to $I^{m+2-k}_{n,(n+m+4-k,n+m+4-k)}$. Finally the degenerations $I^m_{n,(k,l)^*}$ with $k > m+2$ can be degenerated to $I^0_{n,(k-m,l)^*}$.

Since all degenerations have limits which are non-zero functions, these degenerations itself are non-zero as well. □

Note that by specializing two parameters in the $m=0$ integrals to ω (for the (k,l) degenerations one of the two should be a μ parameter and the other a ν parameter), the evaluation formula for the $m=0$ integral shows that the resulting integral is zero. However setting two parameters equal to ω allows us to remove these parameters from the integral (since $\Gamma_h(\omega+x)\Gamma_h(\omega-x)=1$). Therefore we see that some of the general $J_{n,s,t}$ and $J_{n,(s_1,s_2),t}$ integrals are identically zero (in contrast to our degenerations). In particular $J_{n,s,t}$ integrals where $s+|t| \leq 2n$ (while s and t have the same parity) and $J_{n,(s_1,s_2),t}$ integrals with $s_1+s_2+|t| \leq 2n$ (and s_1+s_2+t even) are identically zero.

Combining the transformations of the $I^m_{n,(2n+2m+4)a}$ and $I^m_{n,(n+m+2,n+m+2)a}$ degenerations with the evaluation formulas given in Propositions 5.3.18 and 5.3.19 for the highest level degenerations also leads to some interesting identities.

Corollary 5.5.14. *The following trigonometric equation holds for all $m,n \geq 0$, let $s=n+m+1$ then*

$$\frac{e(\frac{n-n^2}{4})}{\sqrt{2s}^n} \prod_{1\leq j<k\leq n} 4\sin\left(\frac{\pi(j+k)}{2s}\right)\sin\left(\frac{\pi(j-k)}{2s}\right)\prod_{j=1}^n 2\sin\left(\frac{\pi j}{s}\right)$$
$$= \frac{e(\frac{m^2-m}{4})}{\sqrt{2s}^m} \prod_{1\leq j<k\leq m} 4\sin\left(\frac{\pi(j+k)}{2s}\right)\sin\left(\frac{\pi(j-k)}{2s}\right)\prod_{j=1}^m 2\sin\left(\frac{\pi j}{s}\right),$$

and as a special case we have

$$2^{n-1}\prod_{1\leq j\leq n-1}\cos\left(\frac{\pi j}{2n}\right)=\sqrt{n}.$$

Proof. The first identity is obtained by using the transformation formula expressing $I^m_{n,(2n+2m+4)a}(-)$ in $I^n_{m,(2n+2m+4)b}(-)$ and inserting the explicit evaluation we found in Proposition 5.3.18. The second identity is obtained from the first by inserting $m=n-1$ and simplifying using the identity $2\sin(x+y)\sin(x-y) = \cos(2y)-\cos(2x)$. Note that this identity was previously known, (see for example [43, Remark 2.3]), and can also be derived by taking the limit $z \to 0$ of the quotient of the two sides of (1.3.16) using l'Hôpitals rule. □

One would suppose that the evaluation of the $I^m_{n,(n+m+2,n+m+2)a}$ integral would lead to another interesting trigonometric identity. However, this only leads to an equation which is quickly seen to be equivalent to the second equality of the corollary.

5.6 Type $I\!I$ degenerations

In this section we will consider the degenerations of the $I\!I^m_{n,T}$ integral. As mentioned the degeneration scheme of the $I\!I^m_{n,T}$ integral is identical to that of the $I^m_{n,T}$ integral, and hence the first part of this section will be very similar to that of the previous section. However, since the evaluation formulas and symmetries satisfied by $I\!I^m_{n,T}$ differ from those of $I^m_{n,T}$ we obtain different evaluation formulas, symmetries and transformations for the degenerations. The evaluation formula of the $I\!I^0_{n,T}$ integral reduces to evaluation formulas for its degenerations. The Weyl group of E_7-type transformation symmetry of the $I\!I^1_{n,T}$ integral reduces to both symmetries for its degenerations and transformations relating different degenerations. Note that we reserve the term symmetry for an equation of some integral equating it to the same type of integral with different values of the variables, while transformations are equations relating one type of integral to another type of integral.

5.6.1 The definition of the degenerations

We begin by giving a reparametrization of the $JI_{n,s,t}$ and $I\!I^{(s_1,s_2)}_{n,t}$ integrals suitable for identification of integrals to vertices in the degeneration scheme of $I\!I^m_{n,T}$. As this reparametrization tends to be rather convoluted we have written down the explicit defining integrals for the special case $m=1$ (which is the most interesting case, due to the Weyl group of type E_7 symmetry of the top level integral for $m=1$) in Appendix 5.B.

First we give the integrals related to the vertices k on the left of the degeneration scheme. For completeness we include the $I\!I^m_{n,T}$ integral. This definition is very similar to Definition 5.5.1.

Definition 5.6.1. *For variables $\mu \in \mathcal{B}_{2m+6}$ and $\tau \in \mathcal{E}_0$ satisfying the balancing condition*

$$2(n-1)\tau + \sum_{r=0}^{2m+5} \mu_r = 2(m+1)\omega, \tag{5.6.1}$$

we define

$$I\!I^m_{n,T}(\mu;\tau) = JI_{n,2m+6,0}(\mu;\tau).$$

For variables $\tau \in \mathcal{E}_0$ and $\mu \in \mathcal{B}_{2m+4} \cap \mathcal{D}^\tau_{n,2m+4}$ we define

$$I\!I^m_{n,2}(\mu;\tau) = JI_{n,2m+4,0}(\mu;\tau).$$

Finally for $2 < k \leq 2m+6$, $\mu \in \mathcal{B}_{2m+6-k}$ and $\tau \in \mathcal{E}_-$ respectively $\tau \in \mathcal{E}_+$ we define

$$I\!I^m_{n,ka}(\mu;\tau) = JI_{n,2m+6-k,2-k}(\mu;\tau), \qquad I\!I^m_{n,kb}(\mu;\tau) = JI_{n,2m+6-k,k-2}(\mu;\tau).$$

This definition implies that the $I\!I^m_{n,ka}$ and $I\!I^m_{n,kb}$ integrals nearly reparametrize all $JI_{n,s,t}$ integrals with $s+t$ even. Indeed only the $JI_{n,2,0}$, $JI_{n,1,\pm1}$ and $JI_{n,0,\pm2}$

integrals can not be expressed as one of the $II^m_{n,ka/b}$ integrals (while $JI_{n,0,0}$ was not defined). The other $JI_{n,s,t}$ integrals with $s+t$ even are expressed as a $II^m_{n,\xi}$ integral with parameters

condition	m	type
$t \leq 0$	$\frac{s-t-4}{2}$	$(2-t)a$
$t \geq 0$	$\frac{s+t-4}{2}$	$(2+t)b$

Definition 5.6.2. *We define*

$$II^m_{n,\xi}(\mu;\nu;\lambda;\tau) = JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau),$$

where ξ is either $(k,l)a$ or $(k,l)b$ ($1 \leq k,l \leq m+3$, or $k=1$ and $l=m+4$, or $k=m+4$ and $l=1$) or $(k,l)^*a$ or $(k,l)^*b$ ($k,l \geq 3$ and $k+l \leq m+5$). The correspondence between the different parameters is given in the following table

ξ	$s_1(\xi)$	$s_2(\xi)$	$t(\xi)$
$(k,l)a$	$m+3-l+\delta_{k,1}$	$m+3-k+\delta_{l,1}$	$4-k-\delta_{k,1}-l-\delta_{l,1}$
$(k,l)b$	$m+3-k+\delta_{l,1}$	$m+3-l+\delta_{k,1}$	$k+\delta_{k,1}+l+\delta_{l,1}-4$
$(k,l)^*a$	$m+1$	$m+5-k-l$	$l-k$
$(k,l)^*b$	$m+5-k-l$	$m+1$	$k-l$

We assume $(\mu,\nu) \in \mathcal{B}_{(s_1,s_2)}$ and $\tau \in \mathcal{E}_\epsilon$ with $\epsilon = -$ if $\xi = (k,l)a$ for $k,l > 2$, with $\epsilon = +$ if $\xi = (k,l)b$ for $k,l > 2$, and with $\epsilon = 0$ otherwise. Moreover, if $k \leq 2$ we moreover have the condition $(\mu,\nu,-\lambda) \in \mathcal{D}^\tau_{n,(s_1,s_2)}$ and if $l \leq 2$ we have the extra condition $(\mu,\nu,\lambda) \in \mathcal{D}^\tau_{n,(s_1,s_2)}$. In the cases of $\xi = (k,1)$ or $\xi = (1,l)$ we moreover impose some balancing conditions. For $II^m_{n,(k,1)}$ ($k \geq 1$) we impose the balancing condition

$$2(n-1)\tau + \sum_r \mu_r + \sum_r \nu_r = \lambda + (2m+2-k+\delta_{k,1})\omega, \tag{5.6.2}$$

whereas for $II^m_{n,(1,l)}$ ($l \geq 1$) we have the balancing condition

$$2(n-1)\tau + \sum_r \mu_r + \sum_r \nu_r = -\lambda + (2m+2-l+\delta_{l,1})\omega. \tag{5.6.3}$$

Most of the remarks after the definition of the type I integrals are still valid and will not be repeated here. However we do want to note that the two balancing conditions for the $II^m_{n,(1,1)}$ integral (both (5.6.2) and (5.6.3)) now reduce to $\lambda = 0$ and

$$2(n-1)\tau + \sum_r \mu_r + \sum_r \nu_r = 2(m+1)\omega. \tag{5.6.4}$$

In this case, all $JI_{n,(s_1,s_2),t}$ integrals can be expressed as a $JI_{n,m,\xi}$ integral as long as s_1+s_2+t is even. The reverse transformation is nearly identical to the one for the type I integrals, the only difference is that now m is always $n-1$ higher. For example if $|t| < |s_1-s_2|$ and $s_1 > s_2$ we have $JI_{n,(s_1,s_2),t} = II^m_{n,\xi}$ with $m = s_1 - n + (n-1) = s_1 - 1$ and $\xi = (\frac{s_1-s_2-t+4}{2}, \frac{s_1-s_2+t+4}{2})^*a$.

We remark that the $k \leftrightarrow l$ and $a \leftrightarrow b$ transformations (5.5.2), (5.5.6) and (5.5.7) still hold with I replaced by II, in particular we have

$$\begin{aligned} II^m_{n,ka}(\mu;\tau;(\omega_1,\omega_2)) &= \overline{II^m_{n,kb}(-\bar{\mu};-\bar{\tau};(-\bar{\omega}_1,-\bar{\omega}_2))}, \qquad (5.6.5)\\ II^m_{n,(k,l)a}(\mu;\nu;\lambda;\tau;(\omega_1,\omega_2)) &= \overline{II^m_{n,(k,l)b}(-\bar{\nu};-\bar{\mu};-\bar{\lambda};-\bar{\tau};(-\bar{\omega}_1,-\bar{\omega}_2))}\\ II^m_{n,(k,l)^*a}(\mu;\nu;\lambda;\tau;(\omega_1,\omega_2)) &= \overline{II^m_{n,(k,l)^*b}(-\bar{\nu};-\bar{\mu};-\bar{\lambda};-\bar{\tau};(-\bar{\omega}_1,-\bar{\omega}_2))}. \end{aligned}$$

and

$$\begin{aligned} II^m_{n,(k,l)a}(\mu;\nu;\lambda;\tau;(\omega_1,\omega_2)) &= II^m_{n,(l,k)a}(\nu;\mu;-\lambda;\tau;(\omega_1,\omega_2)) \qquad (5.6.6)\\ II^m_{n,(k,l)^*a}(\mu;\nu;\lambda;\tau;(\omega_1,\omega_2)) &= \overline{II^m_{n,(l,k)^*a}(-\bar{\mu};-\bar{\nu};\bar{\lambda};-\bar{\tau};(-\bar{\omega}_1,-\bar{\omega}_2))}. \end{aligned}$$

5.6.2 Degeneration formulas

Recall that the limit formulas for the JI integrals from Propositions 5.3.22 and 5.3.24 were the same as the limit formulas of the J integral. Since these formulas are identical the translations of these formulas into limits between $II^m_{n,\xi}$ integrals will be identical as well.

Hence all degeneration formulas which do not give a limit between a $II^m_{n,k}$ and a $II^m_{n,(k,1)}$ integral are given by the same formulas as before.

Theorem 5.6.3. *Let $n,m \in \mathbb{Z}_{\geq 0}$. For appropriate (in this case ensuring both that the limits are allowed and that the variables satisfy the right balancing conditions) values of μ, ξ, τ and appropriate $q \in RHC$, the formulas from Theorem 5.5.3 hold with $I^m_{n,\psi}(\mu)$ replaced by $II^m_{n,\psi}(\mu;\tau)$, for $\psi = 2$ or $\psi = ka$.*

For appropriate μ, ν, λ, ξ and τ and appropriate $q \in RHC$ the formulas from Theorems 5.5.5-5.5.7 hold with $I^m_{n,\psi}(\mu;\nu;\lambda)$ replaced by $II^m_{n,\psi}(\mu;\nu;\lambda;\tau)$, for $\psi = (k,l)a$ or $\psi = (k,l)^*a$.

Proof. These limits are all instances of the limits in Propositions 5.3.22 and 5.3.24 and limits obtained from those by simple symmetries. The exact parameter domains for μ, ν, λ, ξ and τ can all be found in the statements of those propositions. □

The limits of $II^m_{n,ka}$ integrals to $II^m_{n,(k,1)a}$ integrals are slightly different from the limits in Theorem 5.5.4 and are given by

Theorem 5.6.4. *Let $n,m \in \mathbb{Z}_{\geq 0}$. For appropriate $\mu,\nu \in \mathbb{C}^{m+3}$ and $\tau \in \mathbb{C}$ satisfying the balancing condition* (5.3.5) *(in which the pair (μ,ν) plays the role of μ) and appropriate $q \in RHC$ we have*

$$\begin{aligned} II^m_{n,(1,1)}(\mu;\nu;\tau) &= \lim_{S\to\infty} II^m_{n,T}(\mu+qS,\nu-qS;\tau)\\ &\quad \times c(4n(n-1)qS\tau + 8nqS\omega - n\sum_r(\mu_r-\omega)^2 + n\sum_r(\nu_r-\omega)^2), \qquad (5.6.7) \end{aligned}$$

using the notation $\mu + qS = (\mu_0 + qS, \ldots, \mu_{m+2} + qS)$ *and similarly for* $\nu + qS$. *Let* $2 \leq k \leq m+4$. *For appropriate* $\mu \in \mathbb{C}^{m+2}$, $\nu \in \mathbb{C}^{m+4-k}$, $\tau \in \mathbb{C}$ *and* $q \in RHC$ *we have*

$$\mathit{II}^m_{n,(k,1)a}(\mu;\nu;\lambda;\tau) = \lim_{S\to\infty} \mathit{II}^m_{n,ka}(\mu + qS, \nu - qS; \tau)\zeta^{n(k-2)} c(2n(2-k)q^2S^2) \\ \times c(n(-4qS(\lambda - (n-1)\tau - 2\omega) - \sum_r (\mu_r - \omega)^2 + \sum_r (\nu_r - \omega)^2)), \quad (5.6.8)$$

where $\lambda = 2(n-1)\tau + \sum_r \mu_r + \sum_r \nu_r + (k-2-2m)\omega$.

Proof. These limits are instances of the limits from Proposition 5.3.23. The parameter domains for this theorem can be translated from the statement of that proposition. □

The m-lowering limits for the $\mathit{II}^m_{n,\xi}$ degenerations are also identical to the m-lowering limits of the $I^m_{n,\xi}$ degenerations, being deduced from Propositions 5.3.22 and 5.3.24.Therefore we have

Theorem 5.6.5. *Let* $n, m \in \mathbb{Z}_{\geq 0}$. *For appropriate values of* μ, ν, ξ *and* τ *and* $q \in RHC$ *the limits from Theorem 5.5.8 hold with* $I^m_{n,\psi}(\cdot)$ *replaced by* $\mathit{II}^m_{n,\psi}(\cdot;\tau)$, *for an arbitrary degeneration* ψ. *Note that if* $\psi = (k,1)a$ *for some* $k \geq 1$ *we have to impose the balancing condition* (5.5.3) *on the argument of the* $\mathit{II}^m_{n,\psi}$ *integral.*

Proof. These limits are instances of the limits in Propositions 5.3.22 and 5.3.24. □

5.6.3 Evaluation formulas

Recall that in the $m = 0$ case the $\mathit{II}^m_{n,T}$ integral can be evaluated (Proposition 5.3.7). Using the limits between the degenerations this leads to evaluations for all degenerations with $m = 0$. In the univariate case these evaluations are of course the same as the evaluations for the type I integral. Note that the degeneration scheme (see figure 5.6.3) for $m = 0$ simplifies a lot, in particular there exist no $(k,l)^*$ integrals.

Recall that the evaluation formula for the top level integral was given by an multivariate hyperbolic analogue of the Nasrallah-Rahman integral evaluation formula (5.3.7)

$$\mathit{II}^0_{n,T}(\mu;\tau) = \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau) \prod_{0\leq r,s\leq 5} \Gamma_h(j\tau + \mu_r + \mu_s), \quad (5.6.9)$$

for μ satisfying the balancing condition (5.3.5).

We will now give the corresponding evaluation formulas for all other degenerations. Note that by the elementary symmetries (equations (5.6.5) and (5.6.6)) satisfied by the degenerations we only have to consider type a degenerations and

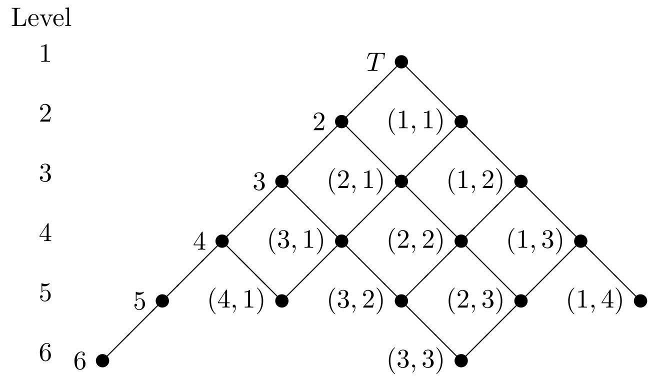


Figure 5.7: The degeneration scheme for $m = 0$ and type II integrals, only showing the a degenerations

for (k, l) degenerations only those with $k \geq l$ (i.e. those on the left hand side of the degeneration scheme).

The evaluation formulas corresponding to the $II^m_{n,k}$ integrals correspond to a multivariate hyperbolic analogue of the Askey-Wilson integral evaluation [16, (6.1.1)] and instances thereof in which some parameters are set to zero. In the univariate case $II^0_{1,2}$ we reobtain the integral evaluation (4.4.19), which was also given in [68] and in [77].

Theorem 5.6.6. *Let $n \geq 0$. In all integrals below C denotes an even contour separating the poles at $\mu_r - \Lambda$ from those at $-\mu_r + \Lambda$, lying below the contours $\tau - \Lambda + C$ such that the integral converges.*

For $\tau \in \mathcal{E}_0$ and $\mu \in \mathcal{B}_4 \cap \mathcal{D}^\tau_{n,4}$, we have

$$
\begin{aligned}
&II^0_{n,2}(\mu; \tau) \\
&\quad = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{\prod_{r=0}^3 \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} dx_j \\
&\quad = \prod_{j=0}^{n-1} \frac{\Gamma_h((j+1)\tau)}{\Gamma_h((2n-2-j)\tau + \sum_{r=0}^3 \mu_r)} \prod_{0\leq r<s\leq 3} \Gamma_h(j\tau + \mu_r + \mu_s).
\end{aligned}
$$

For $\tau \in \mathcal{E}_-$ *and* $\mu \in \mathcal{B}_3$ *we have*

$$\begin{aligned}
&I\!I^0_{n,3a}(\mu;\tau)\\
&= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}\prod_{j=1}^n\frac{\prod_{r=0}^2\Gamma_h(\mu_r\pm x_j)}{\Gamma_h(\pm 2x_j)}c(-2x_j^2)dx_j\\
&= \prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\prod_{0\le r<s\le 2}\Gamma_h(j\tau+\mu_r+\mu_s)\\
&\quad\times c(n(2(\mu_0\mu_1+\mu_1\mu_2+\mu_2\mu_0)+2(n-1)\tau\sum_r\mu_r+\frac{1}{3}(n-1)(4n-5)\tau^2)).
\end{aligned}$$

For $\tau \in \mathcal{E}_-$ *and* $\mu \in \mathcal{B}_2$ *we have*

$$\begin{aligned}
&I\!I^0_{n,4a}(\mu;\tau)\\
&\quad= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}\\
&\qquad\qquad\times\prod_{j=1}^n\frac{\Gamma_h(\mu_0\pm x_j,\mu_1\pm x_j)}{\Gamma_h(\pm 2x_j)}c(-4x_j^2)dx_j\\
&\quad= \prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\Gamma_h(j\tau+\mu_0+\mu_1)c(n(-(\mu_0-\mu_1)^2+2\omega(\mu_0+\mu_1)))\\
&\qquad\times c(n((n-1)\tau(\mu_0+\mu_1+2\omega)+\frac{2}{3}(n-1)(n-2)\tau^2)).
\end{aligned}$$

For $\tau \in \mathcal{E}_-$ *and* $\mu \in \mathbb{C}$ *we have*

$$\begin{aligned}
&I\!I^0_{n,5a}(\mu;\tau)\\
&\quad= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}\prod_{j=1}^n\frac{\Gamma_h(\mu\pm x_j)}{\Gamma_h(\pm 2x_j)}c(-6x_j^2)dx_j\\
&\quad= \prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\zeta^{-n}\\
&\qquad\times c(n(-2\mu^2+4\mu\omega+\omega^2+3(n-1)\tau\omega+\frac{1}{6}(n-1)(2n-7)\tau^2)).
\end{aligned}$$

For $\tau \in \mathcal{E}_-$ *we have*

$$\begin{aligned}
&I\!I^0_{n,6a}(-;\tau)\\
&\quad= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}\prod_{j=1}^n\frac{1}{\Gamma_h(\pm 2x_j)}c(-8x_j^2)dx_j\\
&\quad= \prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\zeta^{-3n}c(n(3\omega^2+3(n-1)\tau\omega+\frac{1}{6}(n-1)(2n-7)\tau^2)).
\end{aligned}$$

Proof. The proofs of these evaluation formulas consist of taking the appropriate limit in the evaluation formula one level higher using Theorem 5.6.3 to obtain the limit of the degeneration on the left hand side and Theorem 5.2.6 to calculate the limits of the hyperbolic gamma functions on the right hand side.

As an example we calculate the evaluation formula for the $I\!I^0_{n,2}$ degeneration. We start with the evaluation formula for the top level integral (5.3.7). Choose ξ_1 and ξ_2 such that (μ, ξ) satisfies the balancing condition (5.3.5). And let $q = \exp(i\phi)$, with $\phi \in \mathcal{A}_0 \cap \mathcal{C}^{\alpha^\tau(n,4;\mu)}$. We calculate

$$
\begin{aligned}
&I\!I^0_{n,2}(\mu;\tau) \\
&\quad = \lim_{S\to\infty} I\!I^0_{n,T}(\mu, \xi_1 + qS, \xi_2 - qS) c(2n((\xi_2 - qS - \omega)^2 - (\xi_1 + qS - \omega)^2)) \\
&\quad = \lim_{S\to\infty} \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau, j\tau + \xi_1 + \xi_2) \prod_{0\le r,s\le 3} \Gamma_h(j\tau + \mu_r + \mu_s) \\
&\qquad\quad \times \prod_{r=0}^{3} \Gamma_h(j\tau + \mu_r + \xi_1 + qS, j\tau + \mu_r + \xi_2 - qS) \\
&\qquad\quad \times c(2n((\xi_2 - qS - \omega)^2 - (\xi_1 + qS - \omega)^2)) \\
&\quad = \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau, j\tau + \xi_1 + \xi_2) \prod_{0\le r,s\le 3} \Gamma_h(j\tau + \mu_r + \mu_s) \\
&\qquad\quad \times \prod_{j=0}^{n-1} \prod_{r=0}^{3} c((j\tau + \mu_r + \xi_1 + qS - \omega)^2 - (j\tau + \mu_r + \xi_2 - qS - \omega)^2) \\
&\qquad\quad \times c(2n((\xi_2 - qS - \omega)^2 - (\xi_1 + qS - \omega)^2)) \\
&\quad = \prod_{j=0}^{n-1} \frac{\Gamma_h((j+1)\tau)}{\Gamma_h((2n-2-j)\tau + \sum_{r=0}^{3} \mu_r)} \prod_{0\le r<s\le 3} \Gamma_h(j\tau + \mu_r + \mu_s) \\
&\qquad\quad \times c(2n(2qS + \xi_1 - \xi_2)(2(n-1)\tau + \sum_r \mu_r + \xi_1 + \xi_2 - 2\omega)).
\end{aligned}
$$

In the first equation we applied Theorem 5.6.3, for the case related to the first equation of Theorem 5.5.3. In the second equality we applied the evaluation formula (5.3.7) for the top level integral. In the third equation we used Theorem 5.2.6 to calculate the limits of the hyperbolic gamma function, and in the last equation we simplified the resulting expression and removed the ξ_1 and ξ_2 dependence by using the balancing condition (5.3.5). This balancing condition also ensures that the remaining exponential term equals one.

For lower degenerations the calculations become even more tedious and were partly performed by the computer algebra program Mathematica. □

Subsequently we consider the degenerations $I\!I^0_{n,(k,1)}$. These evaluation formulas are multivariate hyperbolic analogues of Barnes' first and second lemma [16, (4.1.2) and (4.1.3)] (for the $I\!I^0_{n,(2,1)a}$ and $I\!I^0_{n,(1,1)}$ degenerations respectively). The univariate evaluation for $I\!I^0_{1,(1,1)}$ was previously obtained in (4.4.14), also it was given in [80] and [8]. The evaluation for $I\!I^0_{1,(2,1)}$ was already given in [30], [80] and [8].

Theorem 5.6.7. *Let $n \geq 0$. In all integrals below C denotes a contour separating the poles at $\mu_r - \Lambda$ from those at $-\nu_r + \Lambda$, which lies below the contours $\tau - \Lambda + C$, such that the integrals converge.*

For $\tau \in \mathcal{E}_0$ and $(\mu, \nu) \in \mathcal{B}_{(3,3)}$, satisfying the balancing condition (5.6.4) *we have*

$$
\begin{aligned}
&I\!I^0_{n,(1,1)}(\mu;\nu;\tau)\\
&\quad= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\leq j<k\leq n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}\prod_{j=1}^{n}\prod_{r=0}^{2}\Gamma_h(\mu_r-x_j,\nu_r+x_j)dx_j\\
&\quad= \prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\prod_{r,s=0}^{2}\Gamma_h(j\tau+\mu_r+\nu_s).
\end{aligned}
$$

For $\tau \in \mathcal{E}_0$, $(\mu, \nu) \in \mathcal{B}_{(2,2)}$ and $\lambda \in \mathbb{C}$ satisfying the balancing condition (5.6.2)*, such that $(\mu, \nu, -\lambda) \in \mathcal{D}^{\tau}_{n,(2,2)}$ we have*

$$
\begin{aligned}
&I\!I^0_{n,(2,1)}(\mu;\nu;\lambda;\tau)\\
&= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\leq j<k\leq n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}\prod_{j=1}^{n}\prod_{r=0}^{1}\Gamma_h(\mu_r-x_j,\nu_r+x_j)c(2\lambda x_j)dx_j\\
&= \prod_{j=0}^{n-1}\frac{\Gamma_h((j+1)\tau)}{\Gamma_h(\lambda-j\tau)}\prod_{r,s=0}^{1}\Gamma_h(j\tau+\mu_r+\nu_s)\\
&\qquad\times c(n(2\mu_0\mu_1-2\nu_0\nu_1+(n-1)\tau(\mu_0+\mu_1-\nu_0-\nu_1))).
\end{aligned}
$$

For $\tau \in \mathcal{E}_-$, $(\mu, \nu) \in \mathcal{B}_{(2,1)}$ and $\lambda \in \mathbb{C}$ satisfying the balancing condition (5.6.2) *we have*

$$
\begin{aligned}
&I\!I^0_{n,(3,1)a}(\mu;\nu;\lambda;\tau)\\
&\quad= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\leq j<k\leq n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}\\
&\qquad\qquad\times\prod_{j=1}^{n}\Gamma_h(\mu_0-x_j,\mu_1-x_j,\nu+x_j)c(2\lambda x_j-x_j^2)dx_j\\
&\quad= \prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\Gamma_h(j\tau+\mu_0+\nu)\Gamma_h(j\tau+\mu_1+\nu)\\
&\qquad\times c(n(\nu^2+2\nu(n-1)\tau+2\nu(\mu_0+\mu_1)+4\mu_0\mu_1-2\nu\omega))\\
&\qquad\times c(n(3(\mu_0+\mu_1)(n-1)\tau-(n-1)\tau\omega+\frac{1}{6}(n-1)(10n-11)\tau^2)),
\end{aligned}
$$

For $\tau \in \mathcal{E}_-$, $(\mu,\nu) \in \mathcal{B}_{(2,0)}$ and $\lambda \in \mathbb{C}$ satisfying the balancing condition (5.6.2) *we have*

$$\begin{aligned}
&\mathit{II}^0_{n,(4,1)a}(\mu;-;\lambda;\tau)\\
&\quad= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}\\
&\qquad\qquad\times\prod_{j=1}^n \Gamma_h(\mu_0-x_j,\mu_1-x_j)c(2\lambda x_j-2x_j^2)dx_j\\
&\quad=\prod_{j=0}^{n-1}\Gamma_h((j+1)\tau)\zeta^n c(n(-\mu_0^2-\mu_1^2-\omega^2+2(\mu_0+\mu_1)\omega+4\mu_0\mu_1))\\
&\qquad\times c(n((n-1)\tau(\omega+2\mu_0+2\mu_1)+\frac{1}{2}(n-1)(2n-3)\tau^2)).
\end{aligned}$$

Proof. The proof is very similar to the proof of Theorem 5.6.6. However for the evaluation formula of the $\mathit{II}^0_{n,(1,1)}$ degeneration we have to use the limits from Theorem 5.6.4 to obtain it as a limit of the evaluation formula for the top level integral $\mathit{II}^0_{n,T}$. The evaluation formulas for the $\mathit{II}^0_{n,(k,1)a}$ degenerations with $k \ge 2$ can be obtained either from the evaluation formulas of $\mathit{II}^0_{n,(k-1,1)a}$ or $\mathit{II}^0_{n,ka}$, but the first derivation requires less calculational effort. Again most calculations were performed using Mathematica. □

Finally we retrieve some Barnes' like (they are similar to Barnes' lemmas, but not direct analogues) integral evaluations for the (k,l) degenerations with $k,l \ge 2$. The univariate versions of these evaluations were previously known. The evaluation formula for $\mathit{II}^0_{1,(2,2)}$ was already given in [56], [31], [77], [8] and [80]. The evaluation for $\mathit{II}^0_{1,(3,2)a}$ (or, equivalently, $\mathit{II}^0_{1,(3,2)b}$) can be found in [82], [30], [31], [32] and [80]. Finally the integrand of $\mathit{II}^0_{1,(3,3)a}$ is just a Gaussian so its evaluation is well-known (though observe that the integrands of $\mathit{II}^0_{n,(3,3)a}$ for $n \ge 2$ are not Gaussian).

Theorem 5.6.8. *Let $n \ge 0$. In all integrals below C denotes a contour separating the poles at $\mu-\Lambda$ from those at $-\nu+\Lambda$ (if they exist), which lies below the contours $\tau-\Lambda+C$, such that the integrals converge.*

For $\tau \in \mathcal{E}_0$, $(\mu,\nu) \in \mathcal{B}^\tau_{(1,1)}$ and $\lambda \in \mathbb{C}$ such that $(\mu,\nu,\pm\lambda) \in \mathcal{D}^\tau_{n,(1,1)}$ we have

$$\begin{aligned}
&\mathit{II}^0_{n,(2,2)}(\mu;\nu;\lambda;\tau)\\
&= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}\prod_{j=1}^n\Gamma_h(\mu-x_j,\nu+x_j)c(2\lambda x_j)dx_j\\
&=\prod_{j=0}^{n-1}\Gamma_h((j+1)\tau, j\tau+\mu+\nu, -\frac{\mu+\nu}{2}+\omega-j\tau\pm\frac{\lambda}{2})c(n\lambda(\mu-\nu))
\end{aligned}$$

For $\tau \in \mathcal{E}_-$ and $\mu, \lambda \in \mathbb{C}$ such that $(\mu, -, \lambda) \in \mathcal{D}^{\tau}_{n,(1,0)}$ we have

$$\begin{aligned}
&I\!I^0_{n,(3,2)a}(\mu; -; \lambda; \tau) \\
&= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=1}^n \Gamma_h(\mu - x_j) c(2\lambda x_j - x_j^2) dx_j \\
&= \zeta^{-n} \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau, \frac{\omega + \lambda - \mu}{2} - j\tau) \\
&\quad \times c(\frac{n}{4}(-3\mu^2 + (\lambda - \omega)^2 + 2\mu(3\lambda + \omega) + 2(n-1)\tau(\lambda + \omega - \mu))),
\end{aligned}$$

For $\tau \in \mathcal{E}_-$ and $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
&I\!I^0_{n,(3,3)a}(-; -; \lambda; \tau) \\
&= \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \prod_{j=1}^n c(2\lambda x_j - 2x_j^2) dx_j \\
&= \zeta^{-3n} \prod_{j=0}^{n-1} \Gamma_h((j+1)\tau) c(\frac{n}{2}(2\omega^2 + \lambda^2 + 2(n-1)\tau\omega + \frac{1}{3}(n-1)(2n-1)\tau^2)).
\end{aligned}$$

The evaluation of the $I\!I^0_{n,(3,3)a}$ degeneration for $\tau = \omega_1$ is of course a special case of the evaluation in Proposition 5.3.20, and a simple check shows the expressions indeed correspond. In the univariate case the evaluations of Theorem 5.6.8 give Fourier transforms of certain products of hyperbolic Gamma functions multiplied by a given Gaussian. In some sense we can see them as generalizations of the invariance of the Gaussian function under Fourier transforms, which is the evaluation formula for $I\!I^0_{1,(3,3)a}$.

Knowing these evaluation formulas gives us as a corollary that all degenerations do not identically vanish as in the type I case.

Corollary 5.6.9. *All degenerations $I\!I^m_{n,\xi}$ for $n, m \geq 0$ and any degeneration ξ are not identically zero either as function of μ and τ or as a function of μ, ν, λ and τ.*

Proof. The proof is identical to the proof of Corollary 5.5.13. However now we use the evaluation formulas for $I\!I^0_{n,6a}$ (from Theorem 5.6.6) and $I\!I^0_{n,(3,3)a}$ (from Theorem 5.6.8), together with Propositions 5.3.20 and 5.3.21, to show that the base integrals are non-zero. □

5.6.4 General remarks about symmetries and transformation

In the case $m = 1$ the top level type $I\!I^1_{n,T}$ integral satisfies an E_7 symmetry Proposition 5.3.8. In the degenerations this symmetry will lead to symmetries of the degenerations, where the symmetry group of the integrals will typically be a parabolic subgroup of the symmetry group $W(E_7)$ of the top level integral. Moreover the symmetries of the top level integral lead to transformations between the different degenerations.

Recall from Subsection 5.4.3 that each degeneration is obtained by letting the variables μ of the top level integral go to infinity in the direction of a certain b-vector. The possible vectors giving one type of degeneration are those located within an open cone, of which we gave the generating vectors (5.4.16). Also recall that the norm $\sqrt{2}$ vectors in $\left(\frac{1}{2}\mathbb{Z}\right)^8$ orthogonal to the vector $(1,1,\ldots,1)$ form the root system of type E_7. In the case of the $II^1_{n,T}$ integral we have 8 variables and by inspection we see that the generating vectors as given in (5.4.16) of the open cones corresponding to the different degenerations are all elements of this choice of presentation of the root system of type E_7. Moreover the representation of the Weyl group $W(E_7)$ of type E_7 acting on 8 parameters by reflections in the hyperplanes orthogonal to the roots is exactly the representation giving the E_7 type symmetries of the top level integral. Also observe that the angle between two generating vectors of the same cone always equals $60°$, hence all cones related to degenerations of the same level are congruent. We will want to use the following lemma about these cones

Lemma 5.6.10. *Let C and C' be two open cones generated by roots of some simply laced root system (for example E_7), such that the generators of each cone make angles of $60°$ with each other. Then either $C = C'$ or $C \cap C' = \emptyset$.*

Proof. Suppose without loss of generality that the root system is normalized such that $\langle \alpha, \alpha \rangle = 2$ for all roots α.

Suppose $v \in C$ and C is generated by the vector α_j $(j = 1, \ldots, r_1)$, so $v = \sum_{j=1}^{r_1} a_j \alpha_j$. For any k $(k = 1, \ldots, r_1)$ we then calculate

$$\langle v, \alpha_k \rangle = \sum_{j=1}^{r_1} a_j \langle \alpha_j, \alpha_k \rangle = a_k + \sum_{j=1}^{r_1} a_j,$$

where we use $\langle \alpha_j, \alpha_k \rangle = 1 + \delta_{j,k}$ by assumption. For any other root β not in the generating set we have $\langle \beta, \alpha_j \rangle \leq 1$ $(j = 1, \ldots, r_1)$, hence

$$\langle v, \beta \rangle = \sum_{j=1}^{r_1} a_j \langle \alpha_j, \beta \rangle \leq \sum_{j=1}^{r_1} a_j < \langle v, \alpha_k \rangle,$$

for any $k \in \{1, 2, \ldots, r_1\}$.

Let $C \neq C'$. If $C \cap C' \neq \emptyset$, then there exists a $v \in C \cap C'$. If there is a generator α of C which is not a generator of C' and similarly a generator β of C' which is not a generator of C, then by the above argument (for C) we have $\langle v, \alpha \rangle > \langle v, \beta \rangle$, but by the same argument for C' we also have $\langle v, \beta \rangle > \langle v, \alpha \rangle$, which is impossible. Hence the generators of C must be contained in the generators of C', or vice versa. But then there must be a linear relation in the larger set of generators (namely

$\sum_{j=1}^{r_1} a_j\alpha_j = v = \sum_{j=1}^{r_2} b_j\alpha_j$ for some positive constants a_j and b_j), which is impossible since for vectors at angles of 60° we have

$$\langle \sum c_j\alpha_j, \sum c_j\alpha_j\rangle = 2\sum_j c_j^2 + 2\sum_{j<k} c_jc_k = \sum_j c_j^2 + \left(\sum_j c_j\right)^2 \geq 0$$

with equality only when all $c_j = 0$.

Therefore there exists no such v and $C \cap C' = \emptyset$ if $C \neq C'$. □

A symmetry of a degeneration can be found as a degeneration of the symmetry of the top level integral related to $w \in W(E_7)$ if w preserves the cone C of b-vectors generating this degeneration. For in this case the limits of both $I\!I_{n,T}^1(a + bS;\tau)$ and $I\!I_{n,T}^1(wa + (wb)S;\tau)$ are given by the same type of degeneration. In fact, if for some $w \in W(E_7)$ we have $C \cap wC \neq \emptyset$ the w-symmetry of $I\!I_{n,T}^1$ leads to a symmetry of its degeneration, but the lemma showed that two cones in the root system of E_7 which are generated by a set of roots at angles of 60° to each other are either disjoint or identical. We have $wC = C$ if and only if w permutes the roots generating the open cone, and thus preserves the central vector (the sum of all the roots generating the cone) of the cone. Therefore the symmetries of a degeneration are given by the isotropy group of the central vector in the cone associated to this degeneration in $W(E_7)$. In the univariate case, for the first degenerations ($I\!I_{n,2}^1$ and $I\!I_{n,(1,1)}^1$), this was already observed in Chapter 4.

These considerations allow us to determine the symmetry groups of the degenerations, by calculating the isotropy groups of certain vectors in the root lattice, namely the central vectors of the cones. These isotropy groups are parabolic subgroups of the Weyl group $W(E_7)$ generated by the reflections in roots orthogonal to these central vectors [24, Theorem 1.12c]. These symmetry groups are now easily calculated. For example, since the roots orthogonal to $v = (1,0,0,0,0,0,0,-1)$ form a root system of type D_6, with basis $\frac{1}{2}(1,-1,-1,-1,-1,1,1,1)$ and $\epsilon_j - \epsilon_{j+1}$ ($j = 2,\ldots,6$) we obtain that the symmetry group of $I\!I_{n,2a}^1$, the integral connected to the cone generated by v is the Weyl group $W(D_6)$.

Similarly we find that transformations between different degenerations are obtained by considering elements $w \in W(E_7)$ sending the central vector of the cone generating one degeneration to the central vector of the cone generating another degeneration. Therefore we can tell which degenerations are related by transformation formulas by considering the $W(E_7)$ orbits of the central vectors. We have calculated these orbits using the computer algebra program GAP, which told us which degenerations were connected by transformation formulas.

In this and the next subsection we will explicitly give the symmetries and transformation obtained in this way. We have annotated the symmetry groups of the degenerations and the transformation formulas between them in the degeneration scheme for the $I\!I_{n,T}^1$ integral depicted in figure 5.6.4.

There exist transformations between all degenerations on the same level except on level 6, where there are two orbits of degenerations, which are differentiated

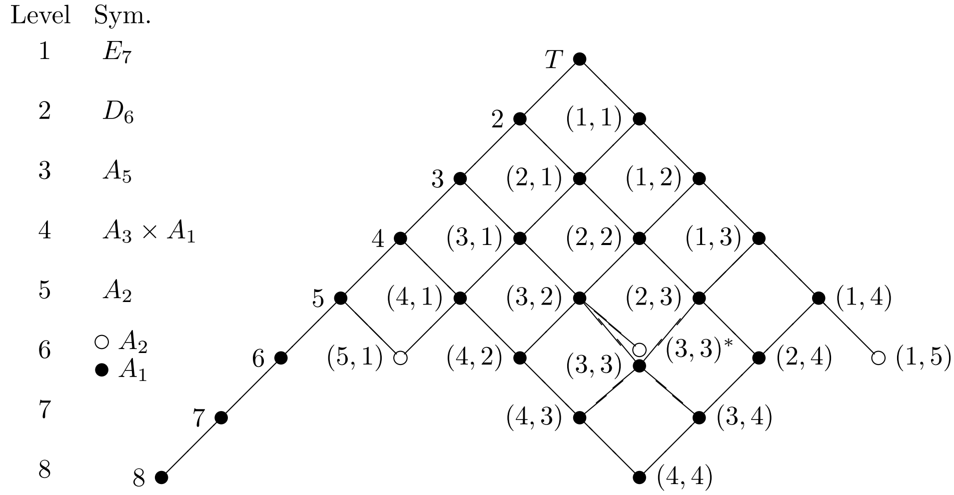


Figure 5.8: The degeneration scheme for $m = 1$ and type II integrals, together with the corresponding symmetry groups

by open and closed dots. In particular we have an orbit containing $II^1_{n,6}$, $II^1_{n,(4,2)}$, $II^1_{n,(3,3)}$ and $II^1_{n,(2,4)}$, and an orbit containing $II^1_{n,(5,1)}$, $II^1_{n,(3,3)^*}$ and $II^1_{n,(1,5)}$. Note that the a and b integrals of the same type (such as $(k,l)a$ and $(k,l)b$) are always located in the same orbit (indeed, we will see that they are connected by the longest Weyl group element of the $W(E_7)$ symmetry of the top level integral). On the left hand side the symmetry groups related to the degenerations of that level are given, by the type of Weyl group they form (degenerations related by some transformation will naturally have the same symmetry group). Note that the two different orbits at level 6 satisfy different symmetry groups. In the case of the $II^1_{n,7}$ and $II^1_{n,8}$ degenerations the symmetry group consists of just the identity element, and is therefore omitted from the figure.

On the classical and q-hypergeometric level transformation formulas are usually presented as transformation formulas between series. However most of these relations can be also expressed as transformations between integrals, since the series can generally be expressed as an integral. In particular the connection between Barnes' type integrals and series is very close, as one can calculate the integral by picking up the residues corresponding to the poles of the integrand, and the corresponding series is exactly this sum of residues (as for example in [16, (4.5.1)]).

This means that we consider our hyperbolic hypergeometric integrals as multivariate generalizations not only of (basic) hypergeometric integrals, but also of the series which they equal. Therefore we will relate the symmetries of our degenerations to transformation formulas of the related series. For example the symmetry of the $II^1_{n,T}$ is the hyperbolic multivariate analogue of Bailey's four term relation, as the related univariate basic hypergeometric integral equals the sum of two ${}_{10}\phi_9$'s.

5.6.5 Symmetries

We now present the explicit symmetries. To reduce the amount of formulas to be given, we restrict ourselves to showing the symmetry group for only one representative degeneration in each orbit of degenerations connected with transformations. In the next subsection we give the transformations expressing each degeneration in an orbit in this representative. The symmetries of each degeneration, and the transformations between two arbitrary degenerations (in the same degeneration orbit) can subsequently be calculated from these formulas. We have chosen the $I\!I^1_{n,ka}$ degenerations as the representative of the degenerations of each orbit containing such a degeneration, because they exist on each level and have many trivial permutation symmetries. As representative of the orbit on level 6 which does not contain $I\!I^1_{n,6a}$ we choose $I\!I^1_{n,(5,1)a}$.

The symmetries of the $I\!I^1_{n,2}$ integral are a multivariate generalization of the transformation (in our terminology: symmetry) formula for very-well poised ${}_8\phi_7$'s, since the basic hypergeometric analogue of the integral is Nassrallah-Rahman's integral representation of very well poised ${}_8\phi_7$'s as [16, (III.23)], generating a symmetry group of type $W(D_6)$ (see [45]). In the univariate case this situation was more extensively studied in Section 4.4.

Theorem 5.6.11. *The $I\!I^1_{n,2}$ degeneration has a symmetry group $W(D_6)$ generated by permutations of the six variables and the following transformation. Let $w\mu = (\mu_0+\xi, \ldots, \mu_3+\xi, \mu_4-\xi, \mu_5-\xi)$ where $2\xi = 2\omega-(n-1)\tau-\sum_{r=0}^{3}\mu_r$. For $\tau \in \mathcal{E}_0$ and $\mu, w\mu \in \mathcal{B}_6 \cap \mathcal{D}^\tau_{n,6}$, we have*

$$I\!I^1_{n,2}(\mu;\tau) = I\!I^1_{n,2}(w\mu;\tau) \times \prod_{j=0}^{n-1} \frac{\Gamma_h(j\tau+\mu_4+\mu_5)}{\Gamma_h(\sum_r \mu_r + (2n-j-2)\tau - 2\omega)} \prod_{0\le r<s\le 3} \Gamma_h(j\tau+\mu_r+\mu_s). \quad (5.6.10)$$

Proof. The fact that $I\!I^1_{n,2}$ is invariant under permutations of the variables is clear from the definition. To prove the final relation for the $I\!I^1_{n,2}$ integral we have to take the appropriate limit on both sides of the symmetry (5.3.8) of $I\!I^1_{n,T}$. In particular we have to choose two extra parameters ζ_1 and ζ_2 such that (μ,ζ_1,ζ_2) satisfies the balancing condition (5.3.5). Moreover we choose $q=\exp(i\phi)$ such that $\phi \in \mathcal{A}^\tau_0 \cap \mathcal{C}^{\alpha^\tau(n;6;\mu)} \cap \mathcal{C}^{\alpha^\tau(n;6;w\mu)}$. This set is at least non-empty for variables μ and τ such that $|(n-1)\tau+\mu_4+\mu_5|, |(n-1)\tau+\sum_{r=0}^{3}\mu_r| < |\omega|$, for then $0 < \alpha^\tau(n;6;\mu), \alpha^\tau(n;6;w\mu) < \pi$. After the calculation we can remove this restriction on the variables by analytic continuation.

In particular we have (with w the w from this theorem, not the w from (5.3.8)),

$$\begin{aligned}
II^1_{n,2}(\mu;\tau) &= \lim_{S\to\infty} II^1_{n,T}(\mu,\zeta_1+qS,\zeta_2-qS)c(2n((\zeta_2-qS-\omega)^2-(\zeta_1+qS-\omega)^2)\\
&= \lim_{S\to\infty} II^1_{n,T}(w\mu,\zeta_1+qS-\xi,\zeta_2-qS-\xi;\tau)\\
&\quad \times c(2n((\zeta_2-qS-\omega)^2-(\zeta_1+qS-\omega)^2)\\
&\quad \times \prod_{j=0}^{n-1}\Gamma_h(j\tau+\mu_4+\mu_5)\Gamma_h(j\tau+\zeta_1+\zeta_2)\prod_{0\leq r<s\leq 3}\Gamma_h(j\tau+\mu_r+\mu_s)\\
&\qquad \times \prod_{r=4}^{5}\Gamma_h(j\tau+\mu_r+\zeta_1+qS, j\tau+\mu_r+\zeta_2-qS)\\
&= II^1_{n,2}(w\mu;\tau)c(-2n((\zeta_2-qS-\xi-\omega)^2-(\zeta_1+qS-\xi-\omega)^2))\\
&\quad \times c(2n((\zeta_2-qS-\omega)^2-(\zeta_1+qS-\omega)^2)\\
&\quad \times \prod_{j=0}^{n-1}\Gamma_h(j\tau+\mu_4+\mu_5)\Gamma_h(j\tau+\zeta_1+\zeta_2)\prod_{0\leq r<s\leq 3}\Gamma_h(j\tau+\mu_r+\mu_s)\\
&\quad \times c(\sum_{r=4}^{5}((j\tau+\mu_r+\zeta_1+qS-\omega)^2-(j\tau+\mu_r+\zeta_2-qS-\omega)^2)).
\end{aligned}$$

We obtain that the desired expression after expanding the exponential term and using the balancing condition to see it vanishes, and after replacing the $\zeta_1+\zeta_2$ in the hyperbolic gamma function using the balancing condition once more. ☐

The following transformation is a multivariate hyperbolic generalization of the symmetry for balanced ${}_3\phi_2$'s or Kummer-Thomae-Whipple's formula for ${}_3F_2$'s. as in [16, (3.2.7) and (3.2.8)] (being the non-trivial symmetry of a $W(A_4)$ symmetry group).

Theorem 5.6.12. *Let $n\geq 0$. The symmetry group of the $II^1_{n,3a}$ degeneration is a Weyl group of type $W(A_5)$ generated by permutations of the 5 variables and the w where $w\mu=(\mu_0+\xi,\ldots,\mu_3+\xi,\mu_4-\xi)$ with $2\xi=2\omega-(n-1)\tau-\sum_{r=0}^3\mu_r$. The symmetry of $II^1_{n,3a}$ corresponding to w is given as follows. For $\tau\in\mathcal{E}_-$ and $\mu\in\mathcal{B}_5^\tau$ such that $w\mu\in\mathcal{B}_5^\tau$ we have*

$$II^1_{n,3a}(\mu;\tau)=II^1_{n,3a}(w\mu;\tau)\prod_{j=0}^{n-1}\prod_{0\leq r<s\leq 3}\Gamma_h(j\tau+\mu_r+\mu_s)c(2n\xi(\xi-2\mu_4-(n-1)\tau)).$$

Proof. That $II^1_{n,3a}$ satisfies the permutation symmetries is again trivial. The symmetry corresponding to w can be obtained by taking the limit $S\to\infty$ of (5.6.10) with $\mu_S=(\mu_0,\mu_1,\mu_2,\mu_3,\mu_4,\mu_5-qS)$, for $q=\exp(i\phi)$ and $\phi\in\mathcal{A}_-$.

In order to take this limit we first restrict τ to positive linear combinations of ω_1 and ω_2. Then for large S we have $\arg(4\omega-\sum_r\mu_r+qS-2(n-1)\tau)\to\arg(q)=\phi$, hence for the restricted τ we have $\mu_S \in \mathcal{D}^\tau_{n,6}$. Likewise we have for large S and restricted τ that $w\mu_S \in \mathcal{D}^\tau_{n,6}$.

Thus we see that in this case (5.6.10) holds for all large S, and we can take the limit on both sides using the limit of a $II^1_{n,2}$ degeneration to a $II^1_{n,3a}$ degeneration from Theorem 5.6.3 and the asymptotic behaviour of the hyperbolic gamma function from Theorem 5.2.6. These calculations, similar to those in the proof of Theorem 5.6.11, were again partly performed using Mathematica.

Finally we extend the result analytically in τ so that it holds for all $\tau \in \mathcal{E}_-$. □

Note that the $W(A_5)$ symmetry of the $II^1_{n,3a}$ degeneration is not given by permutations of 6 variables. Finally we consider the symmetries of the $II^1_{n,4a}$ degeneration, which is a multivariate hyperbolic analogue of Heine's transformation of ${}_2\phi_1$ series [16, (1.4.1)], which generates a symmetry group of type $W(A_2 \times A_1)$.

Theorem 5.6.13. *Let $n \geq 0$. The symmetries of the $II^1_{n,4a}$ degeneration form a Weyl group of type $A_3 \times A_1$. It is generated by permutations of the four variables and the element w where $w\mu = (\mu_0 + \xi, \ldots, \mu_3 + \xi)$ with $2\xi = 2\omega - (n-1)\tau - \sum_{r=0}^3 \mu_r$. The symmetry related to w is given for $\tau \in \mathcal{E}_-$ and $\mu \in \mathcal{B}^\tau_4$ such that $w\mu \in \mathcal{B}^\tau_4$ by*

$$II^1_{n,4a}(\mu;\tau) = II^1_{n,4a}(w\mu;\tau) \prod_{j=0}^{n-1} \prod_{0\leq r<s\leq 3} \Gamma_h(j\tau + \mu_r + \mu_s) c(-2n\xi(2\omega + (n-1)\tau)).$$

Proof. The proof is similar to the proof of Theorem 5.6.12, except now we don't have to restrict the possible values of τ. □

The symmetry groups of the remaining representatives of the orbits of degenerations ($II^1_{n,ka}$ for $5 \leq k \leq 8$ and $II^1_{n,(5,1)a}$) are simply the permutation groups of the remaining variables. Note, however, that this does not imply that the symmetries of all degenerations on the same level are trivial as well. For example the $II^1_{n,(3,2)a}$ degeneration has $2+1$ (counting μ and ν variables separately) variables and thus it is not trivial that its symmetry group is isomorphic the permutation group of 3 variables (and indeed the symmetries are not given by permuting the μ and ν variables arbitrarily).

5.6.6 Transformations

First we will obtain the transformation formulas relating the a and b integrals of the same type to each other. After that we consider the other transformations.

Recall that the longest element of $W(E_7)$ induced the symmetry (5.3.9) for the type II integral. The degenerations of this symmetry provide transformations between $II^1_{n,ka}$ and $II^1_{n,kb}$ ($k \geq 2$) and also between $II^1_{n,(k,l)a}$ and $II^1_{n,(k,l)b}$ and between $II^1_{n,(3,3)^*a}$ and $II^1_{n,(3,3)^*b}$. It is sufficient to obtain these transformations only for the representatives of the orbits of degenerations, which we do below. Note that these are different relations than the $a \leftrightarrow b$ interchanging relations from (5.6.5), in particular they leave ω_1 and ω_2 invariant.

For the $k = 2$ level $II^1_{n,2a} = II^1_{n,2b}$, hence in this case the "transformation" reduces to a symmetry of the $II^1_{n,2}$ degeneration, and in fact it reduces to one of the symmetries of Theorem 5.6.11. Indeed it is the symmetry corresponding to the longest Weyl group element of its symmetry group of type D_6.

Theorem 5.6.14. *Let v act on vectors $\mu = (\mu_0, \mu_1, \ldots)$ of arbitrary length by $v\mu = \omega - \frac{1}{2}(n-1)\tau - \mu := (\omega - \frac{1}{2}(n-1)\tau - \mu_0, \omega - \frac{1}{2}(n-1)\tau - \mu_1, \ldots)$.*

For $\tau \in \mathcal{E}_0$ and $\mu, v\mu \in \mathcal{B}_6 \cap \mathcal{D}^\tau_{n,6}$ we have

$$II^1_{n,2}(\mu;\tau) = II^1_{n,2}(v\mu;\tau) \prod_{j=0}^{n-1} \frac{\prod_{0\leq r<s\leq 5} \Gamma_h(j\tau + \mu_r + \mu_s)}{\Gamma_h(-j\tau - 2\omega + 2(n-1)\tau + \sum_r \mu_r)}.$$

Let $3 \leq k \leq 8$. For $\tau \in \mathcal{E}_0$ and $\mu, v\mu \in \mathcal{B}_{8-k}$ we have

$$\begin{aligned} II^1_{n,ka}(\mu;\tau) = II^1_{n,kb}(v\mu;\tau)\zeta^{-\frac{n(k-3)(k-4)}{2}} \prod_{j=0}^{n-1} \prod_{0\leq r<s\leq 7-k} \Gamma_h(j\tau + \mu_r + \mu_s) \\ \times c(n(-(\sum_r \mu_r)(\sum_r (v\mu)_r) - (2-k)\sum_r \mu_r (v\mu)_r) \\ \times c(n(\frac{1}{2}(k^2 - 11k + 36)\omega^2 - \frac{1}{2}(k^2 - 15k + 44)(n-1)\tau\omega)) \\ \times c\left(n(n-1)\tau^2 \left(\frac{n}{6}(k^2 - 12k + 34) - \frac{1}{12}(k^2 - 9k + 40)\right)\right). \end{aligned}$$

For $\tau \in \mathcal{E}_0$, $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}^3$, satisfying the balancing condition (5.6.2) we have

$$\begin{aligned} II^1_{n,(5,1)a}(\mu; -; \lambda; \tau) = II^1_{n,(5,1)b}(-; v\mu; 5\omega + \frac{5}{2}(n-1)\tau - \lambda; \tau)\zeta^{2n} c(4n \sum_r \mu_r (v\mu)_r) \\ \times c(n(-2(\sum_r \mu_r)(\sum_r (v\mu)_r) + 3\omega^2 + \frac{1}{12}(n-1)(n-17)\tau^2)). \end{aligned}$$

Proof. The symmetry for the $II^1_{n,2}$ degeneration can be obtained by iterating the symmetries from Theorem 5.6.11. Indeed $v = ws_{15}s_{26}ws_{35}s_{46}w$, where s_{ij} denotes the permutation of the i'th and j'th variable. It can also be obtained by taking the appropriate limit from (5.3.9) using a calculation similar to that of the proof of Theorem 5.6.11.

The relations for $k \geq 3$ are obtained by taking the limit $S \to \infty$ of the relation for $k-1$ with variables $\mu_S = (\mu_0, \mu_1, \ldots, \mu_{7-k}, \omega - qS)$, where $q = \exp(i\phi)$ and $\phi = \frac{\phi_- + \phi_+ - \pi}{2}$. Moreover we first assume $\mu, v\mu \in \mathcal{D}_{n,8-k}$ and that τ is a positive linear combination of ω_1 and ω_2. To calculate this limit we again use Theorems 5.6.3 and 5.2.6. Subsequently we use analytic extension to extend the result for all values of $\mu, v\mu \in \mathcal{B}_{8-k}$ and all values of $\tau \in \mathcal{E}_0$.

Finally the relation for the $II^1_{n,(5,1)a}$ degeneration is obtained as the limit $S \to \infty$ of the relation for the $II^1_{n,5a}$ degeneration, with variables $\mu + qS := (\mu_0 + qS, \mu_1 + qS, \mu_2 + qS)$.

The explicit and rather tedious calculations were again partly performed using Mathematica. □

Note that applying complex conjugation to these equations and using the $a \leftrightarrow b$ interchanging symmetry (5.6.5) the relations from this theorem transform to equations relating $I\!I^1_{n,ka}(v\mu;\tau)$ to $I\!I^1_{n,kb}(\mu;\tau)$. Since v^2 is the identity element, this relation should be identical to the relation given in the theorem. Indeed this can be shown using the reflection equation (5.2.4) for the hyperbolic gamma function and the fact that the exponential terms are $\mu \leftrightarrow v\mu$ invariant.

Now we consider the transformation formulas between different degenerations. As mentioned before we relate each degeneration to the representative of the orbit of degenerations it is in. Except for $k = 6$ this means giving a transformation formula of all degenerations on level k to $I\!I^1_{n,ka}$. On level 6 we either relate a degeneration to $I\!I^1_{n,6a}$ or to $I\!I^1_{n,(5,1)a}$.

We only display transformations between a degenerations, as the transformation (5.6.5) between $a \leftrightarrow b$ degenerations allows us to compute transformations between any b degenerations and $I\!I^1_{n,kb}$ (for some k) or $I\!I^1_{n,(5,1)b}$. The transformation between $I\!I^1_{n,ka}$ and $I\!I^1_{n,kb}$ degenerations and between $I\!I^1_{n,(5,1)a}$ and $I\!I^1_{n,(5,1)b}$ from Theorem 5.6.14 subsequently allows us to express those b degenerations in terms of $I\!I^1_{n,ka}$ or $I\!I^1_{n,(5,1)a}$. Moreover we only give relations of $I\!I^1_{n,(k,l)a}$ degenerations with $k \geq l$, since those for $l > k$ can be derived using the $k \leftrightarrow l$ interchanging symmetry (5.6.6).

Level 2

The transformation is a multivariate generalization of Theorem 4.4.11. It connects two hyperbolic multivariate integral representations of ${}_8W_7$ series (as the $I\!I^1_{n,2}$ integral is a hyperbolic multivariate generalization of [16, (6.3.2)]).

Theorem 5.6.15. *For $\tau \in \mathcal{E}_0$ and $(\mu,\nu) \in \mathcal{B}^\tau_{(4,4)}$ satisfying the balancing condition (5.6.4), such that $\mu_\sigma = (\mu_0+\sigma, \mu_1+\sigma, \mu_2+\sigma, \nu_0-\sigma, \nu_1-\sigma, \nu_2-\sigma) \in \mathcal{B}_6 \cap \mathcal{D}^\tau_{n,6}$ for $2\sigma = \mu_3+\nu_0+\nu_1+\nu_2-2\omega+(n-1)\tau = 2\omega-(n-1)\tau-\nu_3-\mu_0-\mu_1-\mu_2$ we have*

$$I\!I^1_{n,(1,1)}(\mu;\nu;\tau) = I\!I^1_{n,2}(\mu_\sigma;\tau) \prod_{j=0}^{n-1}\prod_{r=0}^{2} \Gamma_h(j\tau+\mu_r+\nu_3, j\tau+\mu_3+\nu_r).$$

Proof. The proof is similar to the proof of Theorem 5.6.11.

Let $q = \exp(i\phi)$ with $\phi \in \mathcal{A}^\tau_0 \cap \mathcal{C}^{\alpha^\tau(n,6;\mu_\sigma)}$, where the last set is non-empty since $\mu_\sigma \in \mathcal{D}^\tau_{n,6}$. Subsequently we substitute $\mu = (\mu_0+qS, \mu_1+qS, \mu_2+qS, \nu_3-qS, \nu_0-qS, \nu_1-qS, \nu_2-qS, \mu_3+qS)$ in the symmetry (5.3.8) of $I\!I^1_{n,T}$ and take the limit $S \to \infty$ using Theorems 5.6.3, 5.6.4 and 5.2.6. □

It might seem the $I\!I^1_{n,(1,1)}$ degeneration depends on two more variables than the $I\!I^1_{n,2}$ degeneration (more precisely 8 versus 6 variables, excluding τ). However the balancing condition (5.6.4) and the fact that $I\!I^1_{n,(1,1)}$ has a shifting freedom (as (5.3.14)) in the variables reduce the true number of variables for the $II^1_{n,(1,1)}$ degeneration by 2. The shifting freedom also forces us to make a choice if we want to obtain a reverse transformation. In particular we can find such a reverse transformation by choosing $\sigma = 0$, and solving for μ_3 and ν_3 in the definition of σ. Similar considerations hold for all of the following transformations as well.

Level 3

The integrals on level 3 are multivariate hyperbolic generalizations of ${}_3\phi_2$ series, the $II^1_{n,(2,1)}$ integral is a generalization of the integral expression [16, (4.5.1)] of ${}_3\phi_2$. Note that we don't have to consider the transformation formula with $II^1_{n,(1,2)}$ as we can obtain this transformation by applying (5.6.6) to the equation below.

Theorem 5.6.16. *Let $\tau \in \mathcal{E}_0$, $\lambda \in \mathbb{C}$ and $(\mu,\nu) \in \mathcal{B}^\tau_{(3,3)}$ such that $(\mu,\nu,\lambda) \in \mathcal{D}^\tau_{n,(3,3)}$ and such that the balancing condition* (5.6.2) *is satisfied. For $2\sigma = 2\omega - (n-1)\tau - \mu_0 - \mu_1 - \mu_2 - \nu_2$ define $\mu_\sigma = (\mu_0+\sigma, \mu_1+\sigma, \mu_2+\sigma, \nu_0-\sigma, \nu_1-\sigma)$. If $\mu_\sigma \in \mathcal{B}^\tau_5$ then we have*

$$II^1_{n,(2,1)}(\mu;\nu;\lambda;\tau) = II^1_{n,3a}(\mu_\sigma;\tau) \prod_{j=0}^{n-1} \frac{\prod_{r=0}^2 \Gamma_h(j\tau+\mu_r+\nu_2)}{\Gamma_h(\lambda - j\tau)}$$
$$\times c(n(2\sigma^2 - 2\nu_0\nu_1 - (n-1)\tau(\sum_{r=0}^3 \mu_r+\nu_r) - \frac{1}{3}(n-1)(n-2)\tau^2)). \quad (5.6.11)$$

Proof. Choose μ_3 and ν_3 such that (μ,μ_3,ν,ν_3) satisfies the balancing condition (5.6.4). Subsequently take the limit of $S \to \infty$ in the equation of Theorem 5.6.15 with (μ,ν) replaced by $(\mu_0,\mu_1,\mu_2,\mu_3+qS,\nu_3-qS,\nu_0,\nu_1,\nu_2)$, where $q=\exp(i\phi)$ and $\phi \in \mathcal{A}^\tau_-$. This calculation can certainly be performed if the variables are restricted by τ and $3\omega - \sum_r \mu_r - \sum_r \nu_r - 2(n-1)\tau$ being positive linear combinations of ω_1 and ω_2. Finally we can lift this restriction on the variables by analytic continuation. □

Level 4

The $II^1_{n,(2,2)a}$ degeneration is a multivariate hyperbolic analogue of Barnes' representations of a ${}_2F_1$ [16, (4.1.1)], by applying a shift in the variables to write it as $II^1_{n,(2,2)a}(\mu_0,0;\nu_0,\nu_1;\lambda;\tau)$.

Theorem 5.6.17. *Let $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ and $(\mu,\nu) \in \mathcal{B}^\tau_{(3,2)}$ such that (μ,ν,λ) satisfy the balancing condition* (5.6.2). *Let $2\sigma = 2\omega - (n-1)\tau - \mu_0 - \mu_1 - \mu_2 - \nu_1$ and define $\mu_\sigma = (\mu_0+\sigma,\mu_1+\sigma,\mu_2+\sigma,\nu_0-\sigma)$. Then we have if $\mu_\sigma \in \mathcal{B}^\tau_4$*

$$II^1_{n,(3,1)a}(\mu;\nu;\lambda;\tau) = II^1_{n,4a}(\mu_\sigma;\tau) \prod_{j=0}^{n-1}\prod_{r=0}^{2} \Gamma_h(j\tau+\mu_r+\nu_1) \quad (5.6.12)$$
$$\times c(n(\lambda^2 - (\lambda+\nu_0)((n-1)\tau+2\omega) + \omega^2 + \frac{1}{2}(n-1)\tau^2)).$$

Moreover, for $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ *and* $(\mu,\nu) \in \mathcal{B}^\tau_{(2,2)}$ *such that* $(\mu,\nu,\pm\lambda) \in \mathcal{D}^\tau_{n,(2,2)}$, *let* $4\sigma' = \nu_0+\nu_1-\mu_0-\mu_1-\lambda$ *and* $\mu_{\sigma'} = (\mu_0+\sigma',\mu_1+\sigma',\nu_0-\sigma',\nu_1-\sigma')$. *Then if* $\mu_{\sigma'} \in \mathcal{B}^\tau_4$ *we have*

$$\begin{aligned} I\!I^1_{n,(2,2)}(\mu;\nu;\lambda;\tau) = I\!I^1_{n,4a}(\mu_{\sigma'};\tau) \prod_{j=0}^{n-1} \Gamma_h(j\tau + \frac{1}{2}(\pm\lambda-\mu_0-\mu_1-\nu_0-\nu_1)+2\omega) \\ \times c(n(4\sigma'^2 - 2\mu_0\mu_1 - 2\nu_0\nu_1 - (n-1)\tau(\mu_0+\mu_1+\nu_0+\nu_1))) \\ \times c(n(-\frac{2}{3}(n-1)(n-2)\tau^2)). \end{aligned} \tag{5.6.13}$$

Proof. The first equation is obtained by taking the limit $S \to \infty$ in (5.6.11) with (μ,ν) replaced by $(\mu_0,\mu_1,\mu_2,\nu_0,\nu_2-qS,\nu_1)$. The second equation is obtained by taking the limit $S \to \infty$ in (5.6.11) with (μ,ν) replaced by $(\mu_0,\mu_1,\mu_2-qS,\nu_0,\nu_1,\nu_2+qS)$. □

Level 5

Theorem 5.6.18. *Let* $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ *and* $(\mu,\nu) \in \mathcal{B}(3,1)^\tau$ *such that* (μ,ν,λ) *satisfy the balancing condition* (5.6.2). *For* $2\sigma = 2\omega-(n-1)\tau-\mu_0-\mu_1-\mu_2-\nu$ *we define* $\mu_\sigma = (\mu_0+\sigma,\mu_1+\sigma,\mu_2+\sigma)$. *If* $\mu_\sigma \in \mathcal{B}^\tau_3$ *we have*

$$\begin{aligned} I\!I^1_{n,(4,1)a}(\mu;\nu;\lambda;\tau) = I\!I^1_{n,5a}(\mu_\sigma;\tau)\zeta^n \prod_{j=0}^{n-1}\prod_{r=0}^{2} \Gamma_h(j\tau+\mu_r+\nu) \\ \times c(\frac{n}{2}(\lambda^2-6\omega^2-6(n-1)\tau\omega-(n-1)(n-2)\tau^2)). \end{aligned} \tag{5.6.14}$$

Moreover, let $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ *and* $(\mu,\nu) \in \mathcal{B}^\tau_{(2,1)}$ *such that* $(\mu,\nu,\lambda) \in \mathcal{D}^\tau_{n,(2,1)}$ *and define* $4\sigma' = \omega+\nu-\mu_0-\mu_1-\lambda$ *and* $\mu_{\sigma'} = (\mu_0+\sigma',\mu_1+\sigma',\nu-\sigma')$. *If* $\mu_{\sigma'} \in \mathcal{B}^\tau_3$ *we have*

$$\begin{aligned} &I\!I^1_{n,(3,2)a}(\mu;\nu;\lambda;\tau) \\ &= I\!I^1_{n,5a}(\mu_{\sigma'};\tau) \prod_{j=0}^{n-1} \Gamma_h(-j\tau+\frac{1}{2}(\lambda-\mu_0-\mu_1-\nu+3\omega)) \\ &\quad \times c(n(6\sigma'^2-2\mu_0\mu_1+\nu(\lambda+\mu_0+\mu_1-3\omega))) \\ &\quad \times c(n(\frac{1}{2}(n-1)\tau(\lambda-\mu_0-\mu_1-\nu-3\omega)-\frac{1}{6}(n-1)(2n-7)\tau^2)). \end{aligned} \tag{5.6.15}$$

Proof. Equation (5.6.14) is obtained by taking the limit $S \to \infty$ in (5.6.12) with $(\mu,\nu) = (\mu_0,\mu_1,\mu_2,\nu_1-qS,\nu_0)$. Equation (5.6.15) is obtained by taking the limit $S \to \infty$ in (5.6.13) with $(\mu,\nu,\lambda) = (\mu_0,\mu_1,\nu_0,\nu_1-qS,\lambda-qS+\nu_1-\omega)$. □

Level 6

The transformations in the orbit containing $II^1_{n,6a}$ are given by

Theorem 5.6.19. *For $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}^2$ such that $(\mu, \lambda) \in \mathcal{D}^\tau_{(2,0)}$ we define $4\sigma = 2\omega - \mu_0 - \mu_1 - \lambda$. If $(\mu_0 + \sigma, \mu_1 + \sigma) \in \mathcal{B}^\tau_2$ we have*

$$\begin{aligned} &II^1_{n,(4,2)a}(\mu; -; \lambda; \tau) \qquad (5.6.16)\\ &\quad = II^1_{n,6a}(\mu_0 + \sigma, \mu_1 + \sigma; \tau)\zeta^n \prod_{j=0}^{n-1} \Gamma_h(-j\tau + \frac{1}{2}(\lambda - \mu_0 - \mu_1) + \omega))\\ &\quad \times c(\frac{n}{4}((\lambda + \mu_0 + \mu_1)^2 - 8\mu_0\mu_1 - 8\omega^2))\\ &\quad \times c(\frac{n}{2}(-(n-1)\tau(\mu_0 + \mu_1 - \lambda + 4\omega) - \frac{1}{3}(n-1)(2n-7)\tau^2)) \end{aligned}$$

Moreover, for $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ and $(\mu, \nu) \in \mathcal{B}^\tau_{(1,1)}$ we define $4\sigma' = \nu - \mu - \lambda$. We have

$$\begin{aligned} II^1_{n,(3,3)a}(\mu; \nu; \lambda; \tau) &= II^1_{n,6a}(\mu + \sigma', \nu - \sigma'; \tau)\\ &\times c(\frac{n}{2}(\lambda^2 + (\mu + \nu)^2 - 4(\mu + \nu)\omega - 4(n-1)\tau\omega + 2(n-1)\tau^2)). \qquad (5.6.17) \end{aligned}$$

Proof. Letting $S \to \infty$ in (5.6.15) with $(\mu, \nu, \lambda) = (\mu_0, \mu_1, \nu_0 - qS, \lambda + \nu_1 - qS - \omega)$ gives (5.6.16). If we substitute $(\mu, \nu, \lambda) = (\mu_0, \mu_1 - qS, \nu_0, \lambda - \mu_1 + qS + \omega)$ in the same equation (5.6.15) we obtain (5.6.17). ☐

Moreover we have the orbit containing $II^1_{n,(5,1)a}$. Since in for the $(5,1)a \to (3,3)^*a$ transformation both sides of the equation satisfy a shifting freedom in the variables we have to make a choice on how we shift the parameters on both sides of the equation. We resolved this choice by trying to find the simplest formula.

Theorem 5.6.20. *Let $n \geq 1$. For $\tau \in \mathcal{E}_-$, $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}^3$ satisfying the balancing condition* (5.6.2)*, we have*

$$\begin{aligned} &II^1_{n,(5,1)a}(\mu; -; \lambda; \tau) \qquad (5.6.18)\\ &\quad = II^1_{n,(3,3)^*a}(\mu_0, \mu_1; -; 2\mu_2 - \mu_0 - \mu_1; \tau)\zeta^n c(n(4\mu_0\mu_1 - \mu_2^2 + 2\omega\mu_2))\\ &\quad \times c(n(2(n-1)\tau(\mu_0 + \mu_1) - \omega^2 + \frac{2}{3}(n-1)(n-2)\tau^2)). \end{aligned}$$

Proof. Using (5.6.15) with variables $(\mu_0, \mu_1; qS; \lambda - qS + \omega)$ and (5.6.14) with variables $(\mu_0 + \sigma - qS/2, \mu_1 + \sigma - qS/2, qS/2 - \sigma; 2\omega - (n-1)\tau - \mu_0 - \mu_1 - qS/2 - \sigma; (n-1)\tau + 2\omega - qS)$, where $4\sigma = -\mu_0 - \mu_1 - \lambda$ we obtain a transformation between the $II^1_{n,(3,2)a}$ and $II^1_{n,(4,1)a}$ degenerations. The desired equation is obtained by taking the limit $S \to \infty$. Subsequently simplify by shifting the variables of the right hand side using (5.3.14). ☐

Level 7

Theorem 5.6.21. *For $\tau \in \mathcal{E}_-$ and $\lambda, \mu \in \mathbb{C}$ we have*

$$I\!I^1_{n,(4,3)a}(\mu; -; \lambda; \tau) = I\!I^1_{n,7a}(\frac{3\mu-\lambda+\omega}{4}; \tau)\zeta^n c(\frac{n}{8}(3\lambda^2-2\lambda\mu+3\mu^2+8(n-1)\tau^2)) \times c(\frac{n}{8}(2\lambda\omega-6\mu\omega-16(n-1)\tau\omega-13\omega^2)) \quad (5.6.19)$$

Proof. Take the limit $S \to \infty$ in (5.6.17) with (μ, ν, λ) replaced by $(\mu, \nu - qS, \lambda + \nu - qS - \omega)$. Alternatively one can take the limit $S \to \infty$ in (5.6.16) with $(\mu, \lambda) = (\mu, \mu_1 + qS, \lambda - \mu_1 + qS + \omega)$. □

Level 8

Theorem 5.6.22. *For $\tau \in \mathcal{E}_-$ and $\lambda \in \mathbb{C}$ we have*

$$I\!I^1_{n,(4,4)a}(-; -; \lambda; \tau) = I\!I^1_{n,8a}(-; \tau)\zeta^{2n} c(\frac{n}{4}(\lambda^2 + 4(n-1)\tau^2 - 8(n-1)\tau\omega - 8\omega^2)). \quad (5.6.20)$$

Proof. Take the limit $S \to \infty$ in (5.6.19) with $(\mu, \lambda) = (\mu+qS, \lambda-\mu+qS+\omega)$. □

As a corollary we find that the transformation (5.6.20) between $I\!I^1_{n,(4,4)a}$ and $I\!I^1_{n,8a}$, combined with the evaluation formula for the $I\!I^1_{n,(4,4)a}(-; -; \lambda; \omega_1)$ degeneration from Proposition 5.3.20, allows us to calculate an evaluation formula for the $I\!I^1_{n,8a}(-; \omega_1)$ degeneration.

Corollary 5.6.23. *We have*

$$\begin{aligned} &I\!I^1_{n,8a}(-; \omega_1) \\ &= \frac{\Gamma_h(\omega_1)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\omega_1 \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \prod_{j=1}^n \frac{1}{\Gamma_h(\pm 2x_j)} c(-12x_j^2) dx_j \\ &= \frac{\Gamma_h(\omega_1)^n}{\sqrt{2}^n} e(\frac{3n^2}{8}) c(n(\frac{1}{6}(n^2+1)\omega_1^2 + \frac{1}{3}\omega_2^2)) \prod_{j=1}^n (2\sin(\frac{j\pi\omega_1}{2\omega_2}))^{n-j}. \end{aligned}$$

Proof. Note that $I\!I^1_{n,(4,4)a}(-; -; \lambda; \omega_1) = JI_{n,(0,0),-4}(-; -; \lambda; \omega_1)$ and apply complex conjugation to the evaluation in Proposition 5.3.20 to obtain the evaluation formula for $I\!I^1_{n,(4,4)a}(-; -; \lambda; \omega)$. Subsequently use (5.6.20) to obtain the desired relation. □

5.A Proofs of the limits

Before giving the proofs we first define the concept of semi-constancy of a family of integration contours depending on some real limiting parameter S. In order to take limits of integrals it is important to have an integration contour which does not depend too much on the limiting parameter. Otherwise bounds on the integrand will not directly lead to bounds of the integral, since we cannot exclude that the growth of the length of the contour offsets the decrease of the integrand. However, since the contours of the integrals in our case have to separate some poles whose locations depend on S the contour can not always be chosen S-independently. For example the contour of the integral in the first limit in Proposition 5.3.22 should separate the poles at $\xi + qS + \omega_1\mathbb{Z}_{\geq 0} + \omega_2\mathbb{Z}_{\geq 0}$ from the poles at $-\xi - qS - \omega_1\mathbb{Z}_{\geq 0} - \omega_2\mathbb{Z}_{\geq 0}$. The definition of semi-constancy provides us with a tool that allows us to choose contours which are allowed to depend on S in such a way that they can separate these S-dependent poles of our integrands, but whose S-dependence is still so restricted that bounds on the integrand immediately lead to bounds on the integral itself.

Since the parameters in the limit are affine linear in S, the corresponding poles go to infinity linearly in S. Therefore the poles of the integrand split in some, let's say p, groups which move at the same rate as S goes to infinity. In practice p will be two or three. The poles of the k'th group are all of the form $\nu_{r,k} + a_k S$ for some fixed $a_k \in \mathbb{C}$. For example in the first limit of Proposition 5.3.22 we have 3 groups of poles; one group of poles going to infinity like $-qS$, one group of poles independent of S remaining fixed near 0 and a final group of poles going to infinity like qS. Hence $a_1 = -q$, $a_2 = 0$ and $a_3 = q$, and for the first group the values of $\nu_{r,1}$ are $-\xi - \omega_1\mathbb{Z}_{\geq 0} - \omega_2\mathbb{Z}_{\geq 0}$.

For large S we can now split the contour such that these different groups of poles are located near different parts of the contour. In the setting of the first limit of Proposition 5.3.22 we would cut the contour, which is a deformation of $W_{\phi,\phi}$ with $q = \exp(i\phi)$, at roughly $-qS/2$ and $qS/2$ to divide it in three parts, each corresponding to one group of poles. Note that $\pm qS/2$ are on the contour $W_{\phi,\phi}$ and for large enough S we can choose the contour such that there are no deformations around $\pm qS/2$, so that these points are on the contour of the integral. For each part we want to choose a fixed contour, such that the (S-dependent) contour on that part is given by a part of this fixed contour shifted along $-a_k qS$ (like the poles of the integrand in this part) so that it always separate the correct poles from each other and such that we can glue the contours thus obtained together at the endpoints. We accomplish this by first shifting the poles in the k'th group back to the origin by adding $-a_k S$ to all of them. Subsequently we choose an S-independent contour C_k separating these shifted poles (which are now S-independent) from each other in the appropriate way and finally moving the resulting contour back by adding $a_k S$ to it. For the first group of poles in our example we would therefore consider a contour C_1 which lies below the points $-\xi - \omega_1\mathbb{Z}_{\geq 0} - \omega_2\mathbb{Z}_{\geq 0}$. Subsequently we would shift it by $-qS$ to obtain a contour which lies below the S-dependent poles of the integrand in the first group.

Having obtained p of these contours, we need to glue them together to obtain one single S-dependent contour C_S for the integral with S-dependent parameters. In order for this gluing to be possible we insist that far away from the (shifted, S-independent) poles the contours C_k follow exactly the original hook contour, so they connect to each other. In particular this implies that the contour C_k for

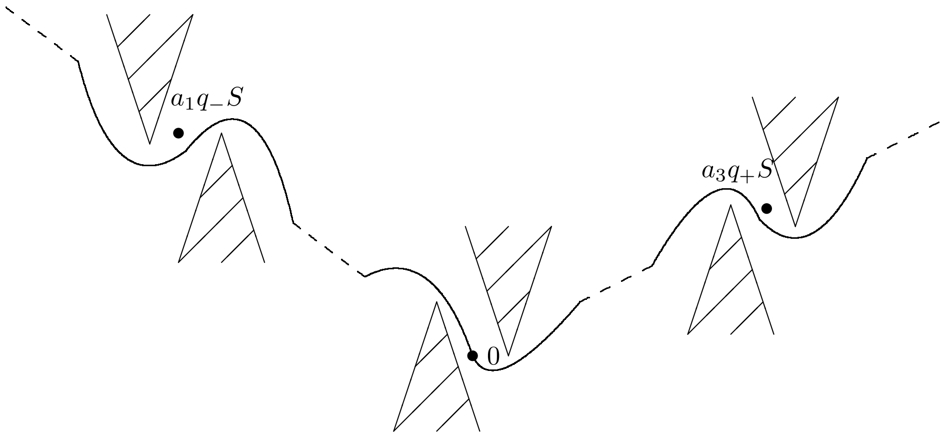


Figure 5.9: An example of a contour from a semi-constant family, with $a_1 < a_2 = 0 < a_3$ and $q_- = \exp(-i\pi/4)$ and $q_+ = \exp(i\pi/6)$.

groups of poles with $a_k \neq 0$ should always be a deformation of a straight line.

Having obtained such a family of contours we can take limits of the integral over each part of the contour, knowing that the lengths of each part of the contour increases only linearly in S and behaves rather constant. This construction leads to the definition of a semi-constant family of contours. In this definition we adopt the convention that $\mathbf{sign}(0) = +$.

Definition 5.A.1. *A family of contours C_S ($S \in \mathbb{R}_{\geq 0}$) in $\mathbb{C}$ is called semi-constant if there exist $q_\pm \in \mathbb{C}$, a finite family of contours C_k parametrized by $\gamma_k : \mathbb{R} \to \mathbb{C}$ ($k = 1, \ldots, p$) and a strictly increasing sequence of real numbers $(a_k)_{k=1}^p$, satisfying the following conditions:*

i) If $q_+ \neq q_-$ then $a_j = 0$ for some j;

ii) There exists M_1 such that for $|x| > M_1$ the continuous contour C_k is parametrized by $\gamma_k(x) = q_{\boldsymbol{sign}(a_k,x)}x$, where $\boldsymbol{sign}(a_k, x) = \boldsymbol{sign}(a_k)$ if $a_k \neq 0$ and $\boldsymbol{sign}(0, x) = \boldsymbol{sign}(x)$;

iii) There exists M_2 such that for $S > M_2$ the contour C_S is parametrized by

$$\gamma_S(t) = \gamma_k(t - a_kS) + a_k q_{\boldsymbol{sign}(a_k)}S \quad \text{if} \quad \frac{a_{k-1} + a_k}{2}S < t \leq \frac{a_k + a_{k+1}}{2}S,$$

where we set $a_0 = -\infty$ and $a_{n+1} = \infty$.

We write $C_{k,S}$ for the partial contours parametrized by $\gamma_S\big|_{(\frac{a_{k-1}+a_k}{2}S, \frac{a_k+a_{k+1}}{2}S)}$ for $k = 1, \ldots, n$.

Note that in this definition the contour C_S is connected for S large enough (i.e. $S > \max(M_2, M_1/\max_j\{|a_j - a_{j+1}|\})$), because the equation $\gamma_k(x) = q_{\mathbf{sign}(a_k,x)}x$

for large $|x| > M_1$ and all k implies there are no jumps in the contour γ_S at $t = \frac{a_k + a_{k+1}}{2} S$ $(k = 1, \ldots, n-1)$ for large S. Indeed, for t with $|t - a_k S| > M_1$ for all k we have $\gamma_S(t) = q_{\mathbf{sign}(t)} t$. Checking this is rather tedious and involves using the inclusion of 0 in the list (a_k) to account for the transition of the behaviour of the contour from $\gamma_S(t) \approx q_- t$ (for $t < 0$) to $\gamma_S(t) \approx q_+ t$ (for $t > 0$). If $q_- = q_+$ there is no such transition and thus 0 does not need to be included in the list (a_k).

In Figure 5.9 we have depicted an example of a contour from a semi-constant family. Here we have three parameters $a_1 < a_2 = 0 < a_3$ and two different q_- and q_+. The shaded areas denote places where poles of the integrand occur, which have to be separated by the contour. The dashed lines denote parts of the contour (of an S-dependent length) where it is a straight line. The only places where the contour is allowed to deviate from this line is near $a_1 q_- S$, $0 = a_2$ and $a_3 q_+ S$ (and it therefore does not necessarily run through these three points).

Proof of the first limit of Proposition 5.3.22. Recall that we want to show that

$$JI_{n,s,t}(\mu;\tau) = \lim_{S \to \infty} JI_{n,s+1,t-1}(\mu, \xi + qS; \tau) \zeta^{2n} c(-2n(\xi + qS - \omega)^2),$$

under certain conditions on $q = \exp(i\phi)$, μ and ξ. First we prove it for $\phi \in \mathcal{A}_-^\tau$ if $t < 0$, $\phi \in \mathcal{A}_-^\tau \cap \mathcal{C}^{\alpha^\tau(n,s;\mu)}$ if $t = 0$, and $\phi \in \mathcal{A}_+^\tau$ if $t > 0$.

As contours for the $JI_{n,s+1,t-1}(\mu, \xi + qS; \tau)$ integrals we choose a semi-constant family of contours $C_S = C_{1,S} \cup C_{2,S} \cup C_{3,S}$ with $a_1 = -1$, $a_2 = 0$ and $a_3 = 1$ and $q_\pm = q$. This relates to the pole sequences $-\xi - qS + \Lambda$ for $C_{1,S}$, $\pm(\mu_r - \Lambda)$ $(r = 0, \ldots, s-1)$ for $C_{2,S}$, and $\xi + qS - \Lambda$ for $C_{3,S}$. Note that such a semi-constant family of contours exists.

Using the decomposition of C_S we can decompose C_S^n as a union of contours of the form $C_{(n_1,n_2,n_3),S} = C_{1,S}^{n_1} \times C_{2,S}^{n_2} \times C_{3,S}^{n_3}$, where $n_1 + n_2 + n_3 = n$ (using the permutation symmetry of the integrand to order the $C_{j,S}$'s). We now bound the integrand on the contours $C_{(n_1,n_2,n_3),S}$ for $n_2 < n$ (i.e. either $n_1 > 0$ or $n_3 > 0$) to show the integrals over these contours converge to zero. This implies that the limit of the complete integral equals the limit of the integral over just the piece $C_{(0,n,0),S} = C_{2,S}^n$ of the total contour.

If all x_i lie on the contour C_S with $x_i = \gamma_S(t_i)$ for a decreasing sequence $t_1 > t_2 > \cdots > t_n$ (such that if the contour were just a wedge we would have

$x_i \in RHC$ and $x_j - x_k \in RHC$ if $j < k$) we have using Corollary 5.2.7

$$\left| \prod_{j<k} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \right| \tag{5.A.1}$$
$$\begin{aligned} &\leq K \prod_{j<k} |c((\tau + x_j + x_k - \omega)^2 - (\tau - x_j - x_k - \omega)^2 - (x_j + x_k - \omega)^2 \\ &\qquad + (-x_j - x_k - \omega)^2 + (\tau + x_j - x_k - \omega)^2 - (\tau - x_j + x_k - \omega)^2 \\ &\qquad - (x_j - x_k - \omega)^2 + (x_k - x_j - \omega)^2)| \\ &= K \prod_{j<k} |c(8\tau x_j)| \leq K \exp(4\pi(n-1) \sum_j |\Re(\frac{i\tau x_j}{\omega_1\omega_2})|). \end{aligned}$$

Using the Weyl group of B_n symmetry of the integrand this bound actually holds for all x on the contour C_s^n. Moreover we have

$$\left| \frac{1}{\prod_{j=1}^n \Gamma_j(\pm 2x_j)} \right| = \left| \prod_{\delta \in \{1,2\}} \prod_{j=1}^n 2\sin(\frac{2\pi x_j}{\omega_\delta}) \right| \leq K_2 \prod_{j=1}^n \exp(4\pi \sum_{j=1}^n |\Re(ix_j\omega/\omega_1\omega_2)|)$$

Since we can bound the cross-terms by a product of univariate functions (see (5.A.1)) we can bound the integral over $C_{(n_1,n_2,n_3),S}$ by a product of univariate integrals as

$$\begin{aligned} &\Bigg| c(-2n(\xi + qS - \omega)^2) \int_{C_{(n_1,n_2,n_3),S}} \prod_{1 \leq i < j \leq n} \frac{\Gamma_h(\tau \pm x_i \pm x_j)}{\Gamma_h(\pm x_i \pm x_j)} \\ &\qquad \times \prod_{i=1}^n \frac{\Gamma_h(\xi + qS \pm x_i) \prod_{r=0}^{s-1} \Gamma_h(\mu_r \pm x_i)}{\Gamma_h(\pm 2x_i)} c(2(t-1)x_i^2) dx_i \Bigg| \\ &\leq \prod_{j=1}^3 \Bigg(\int_{C_{j,S}} K \exp(4\pi|\Re(\frac{ix\omega}{\omega_1\omega_2})| + 4\pi(n-1)|\Re(\frac{ix\tau}{\omega_1\omega_2})) \, |\Gamma_h(\xi + qS \pm x)| \\ &\qquad \times \prod_{r=0}^{s-1} \Gamma_h(\mu_r \pm x)|c(2(t-1)x^2 - 2(\xi + qS - \omega)^2)|dx \Bigg)^{n_j}. \end{aligned}$$

To show that the integrals over the contours $C_{(n_1,n_2,n_3),S}$ converge to zero as $S \to \infty$ it is sufficient to show that the above univariate integral over $C_{1,S}$ and the integral over $C_{3,S}$ converge to zero as $S \to \infty$, while the integral over $C_{2,S}$ stays bounded. By evenness of the original integrand the integral over $C_{3,S}$ is the same as the integral over $C_{1,S}$ and hence we will restrict ourselves to the latter case.

To bound the integrals over $C_{1,S}$ we choose a point z on the contour C_1 and split the contour $C_{1,S}$ in two parts, $D_{1,S}$ and $D_{2,S}$, where $D_{1,S}$ is the part of the contour from $-q\infty$ to $z - qS$ and $D_{2,S}$ the part of the contour from $z - qS$ to $-qS/2$. Note that $z - qS$ is on the contour $C_{1,S}$ for any z on C_1. So we now want to bound the integrals

$$\int_{D_{1,S}} \exp(4\pi|\Re(\frac{ix\omega}{\omega_1\omega_2})| + 4\pi(n-1)|\Re(\frac{ix\tau}{\omega_1\omega_2})) \left|\Gamma_h(\xi+qS\pm x)\prod_{r=0}^{s-1}\Gamma_h(\mu_r\pm x)\right| \times |c(2(t-1)x^2 - 2(\xi+qS-\omega)^2)|dx,$$

$$\int_{D_{2,S}} \exp(4\pi|\Re(\frac{ix\omega}{\omega_1\omega_2})| + 4\pi(n-1)|\Re(\frac{ix\tau}{\omega_1\omega_2})) \left|\Gamma_h(\xi+qS\pm x)\prod_{r=0}^{s-1}\Gamma_h(\mu_r\pm x)\right| \times |c(2(t-1)x^2 - 2(\xi+qS-\omega)^2)|dx$$

On $D_{1,S}$ the functions $\Gamma_h(\mu_r+x)c((\mu_r+x-\omega)^2)$, $\Gamma_h(\mu_r-x)c(-(\mu_r-x-\omega)^2)$ $(r=0,\ldots,s-1)$ and $\Gamma_h(\xi+qS+x)c((\xi+qS+x-\omega)^2)$ and $\Gamma_h(\xi+qS-x)c(-(\xi+qS-x-\omega)^2)$ are all bounded uniformly in S and x due to Corollary 5.2.7. Hence we obtain the following bound on the integrand

$$\begin{aligned}\exp(4\pi|\Re(\frac{ix\omega}{\omega_1\omega_2})| + 4\pi(n-1)|\Re(\frac{ix\tau}{\omega_1\omega_2})|) \left|\Gamma_h(\xi+qS\pm x)\prod_{r=0}^{s-1}\Gamma_h(\mu_r\pm x)\right| \\ \times |c(2(t-1)x^2 - 2(\xi+qS-\omega)^2)| \\ \leq K_1|c(2(t-1)(x+qS)^2 - 4(x+qS)(tqS+2\omega+2(n-1)\tau+\sum_r\mu_r+\xi-(s+1)\omega)) \\ \times c(2tq^2S^2 + 4qS((2-s)\omega+\sum_r\mu_r+2(n-1)\tau))|\end{aligned} \tag{5.A.2}$$

for some constant K_1 and for $x \in D_{1,S}$. We will now show this bound converges uniformly exponentially to zero as $S\to\infty$. To do this it is convenient to perform a shift in integration variable as follows

$$\begin{aligned}&\int_{-q\infty}^{z-qS} |c(2(t-1)(x+qS)^2 - 4(x+qS)(tqS+2\omega+2(n-1)\tau+\sum_r\mu_r+\xi-(s+1)\omega)) \\ &\qquad \times c(2tq^2S^2+4qS((2-s)\omega+\sum_r\mu_r+2(n-1)\tau))|dx \\ &= \int_{-q\infty}^{z} |c(2(t-1)(x^2 - 4x(tqS+2\omega+2(n-1)\tau+\sum_r\mu_r+\xi-(s+1)\omega)) \\ &\qquad \times c(2tq^2S^2+4qS((2-s)\omega+\sum_r\mu_r+2(n-1)\tau))|dx,\end{aligned}$$

where the contour of the first integral from $-q\infty$ to $z-qS$ is $D_{1,S}$ and the contour of the second integral is $D_{1,S}$ shifted by qS, or C_1 restricted to $-q\infty$ to z. It is now a rather tedious check to see that the second integral converges exponentially to zero as $S\to\infty$.

Consider the case $t > 0$. The fact that (in this case) $\phi \in \mathcal{A}_+^\tau$ (as in the statement of the proposition) implies that $c(2tq^2S^2) \to 0$ (as a Gaussian term, as $S \to \infty$) and the rest of the S-dependent part of the integrand increases at most as $\exp(cS)$ for some constant c (independent of x, even for the $c(-4txqS)$ term, because the contour only extends a finite amount to the right). So all we have to check is whether the integral converges for some value of S. If $t > 1$ then the $c(2(t-1)x^2)$ term converges like a Gaussian to zero as $x \to -q\infty$, so then the integral converges. If $t = 1$ the main x-dependent part of the integral is $c(-4xtqS)$, which converges exponentially to zero as $x \to -q\infty$, so in that case the integral converges as well.

For $t < 0$ the analysis is very similar so for the rest we only consider the case $t = 0$. If $t = 0$ we see that $c(2(t-1)x^2)$ converges Gaussian to zero as $x \to -q\infty$ since $\phi \in \mathcal{A}_-^\tau$. Moreover since $\phi \in \mathcal{C}^{\alpha^\tau(n,s;\mu)}$ we have $c(4qS((2-s)\omega + \sum_r \mu_r + 2(n-1)\tau)) \to 0$ exponentially as $S \to \infty$. Together this shows the integral converges to zero as $S \to \infty$.

For $x \in D_{2,S}$ we can bound the integrand similarly by using the same estimates as before, except now $\Gamma_h(\xi + qS + x)c(-(\xi + qS + x - \omega)^2)$ is bounded uniformly in S and x (note the change of sign in the exponential). We find that the integrand is bounded on $D_{2,S}$ by

$$K_2|c(2tx^2 - 4x((2-s)\omega + \sum_r \mu_r + 2(n-1)\tau))|. \tag{5.A.3}$$

For $x \in D_{2,S}$ (and S large enough) this is moreover bounded by its value in $x = -qS/2$, i.e. $K_2|c(\frac{1}{2}tq^2S^2 + 2qS((2-s)\omega + \sum_r \mu_r + 2(n-1)\tau)|$, as for $t \neq 0$ (5.A.3) decreases like a Gaussian for x on the line $(-q\infty, 0)$ so the maximum of this bound on $D_{2,S}$ (for large enough S) is attained at $x = -qS/2$ (for $|x - qS| > M$ (with M as in Definition 5.A.1) the contour follows the line $(-q\infty, 0)$ exactly), while for $t = 0$ it decreases exponentially in x on the line $(-q\infty, 0)$. Since the length of the contour only increases linearly in S we conclude that the integral over $D_{2,S}$ converges to zero as $S \to \infty$.

To bound the integral over $C_{2,S}$ we use a similar method and first choose a point w on the contour C_2. Then on the part of the contour from $-qS/2$ to w we can bound the integrand by the same bound as for $x \in D_{2,S}$, with just a different constant, i.e.

$$K_3|c(2tx^2 - 4x((2-s)\omega + \sum_r \mu_r + 2(n-1)\tau))| \tag{5.A.4}$$

which is independent of S and has a finite integral from $-q\infty$ to w. Hence the integral over this part of the contour stays bounded. Similarly we find that the integral over the other part of $C_{2,S}$ (from w to $qS/2$) stays bounded as well. Combining these bounds we obtain that the integral over the contour $C_{(n_1,n_2,n_3),S}$ converges to zero as $S \to \infty$ if either $n_1 > 0$ or $n_3 > 0$.

The above argument moreover gives us a uniform (in S), integrable bound on the integrand for integration variables in $C_{(0,n,0),S}$ (namely the product of n

copies of (5.A.4), one for each integration variable, if all x_j are on the part of the contour from $-qS/2$ to w, and a similar bound if some or all x_j are on the part of the contour from w to $qS/2$). Lebesgue's theorem of dominated convergence now asserts that

$$\begin{aligned}
&\lim_{S\to\infty} JI_{n,s+1,t-1}(\mu,\xi+qS;\tau)\zeta^{2n}c(-2n(\xi+qS-\omega)^2) \\
&\quad= \lim_{S\to\infty}\int_{C_{(0,n,0),S}} \prod_{1\le i<j\le n}\frac{\Gamma_h(\tau\pm x_i\pm x_j)}{\Gamma_h(\pm x_i\pm x_j)}\prod_{i=1}^n \frac{\Gamma_h(\xi+qS\pm x_i)\prod_{r=0}^{s-1}\Gamma_h(\mu_r\pm x_i)}{\Gamma_h(\pm 2x_i)} \\
&\qquad\qquad \times c(2(t-1)x_i^2-2(\xi+qS-\omega)^2)dx_i \\
&\quad= \int_{C_2^n}\lim_{S\to\infty}\frac{\prod_{i=1}^n\Gamma_h(\xi+qS\pm x_i)\prod_{r=0}^{s-1}\Gamma_h(\mu_r\pm x_i)}{\prod_{1\le i<j\le n}\Gamma_h(\pm x_i\pm x_j)\prod_{i=1}^n\Gamma_h(\pm 2x_i)}c(2(t-1)x_i^2) \\
&\qquad \times\prod_i c(-2n(\xi+qS-\omega)^2)dx_i \\
&= J_{n,s,t}(\mu),
\end{aligned}$$

where we use Theorem 5.2.6 to calculate the limit of the integrand (see the calculation (5.3.23)).

To extend the validity of the proposition to $\phi=\frac{\phi_-+\phi_+-\pi}{2}$ (i.e. the point in between the intervals $\mathcal{A}_-$ and $\mathcal{A}_+$) we insist first of all that ξ is a positive linear combination of ω_1 and ω_2. This ensures that the contours C_1 and C_3 do not need to be deformed to be below, respectively above, the poles at $\pm(qS+\xi+\omega_1\mathbb{Z}_{\ge 0}+\omega_2\mathbb{Z}_{\ge 0})$. Hence we choose them to be just straight lines. Subsequently we use the same proof as before, but we have to reconsider whether the bounds we found are indeed still good enough, because for example now $|c(q^2S^2)|=1$ for all S, instead of being exponentially decreasing. Note that also $|c(tx^2)|=1$ on C_1 and C_3 and outside a bounded region on C_2 since there are no deformations in the contour (if x would deviate slightly from the line $q\mathbb{R}$, it could immediately become huge, as $\exp(iz^2)$ does for $z\notin\mathbb{R}$).

The first bound (5.A.2) still converges since by our extra condition $\mu\in\mathcal{D}_{n,s}^{\tau}$ we have $\alpha^\tau(n,s;\mu)\in(\frac{\phi_-+\phi_+-\pi}{2},\frac{\phi_-+\phi_++\pi}{2})$ (so that the $c(4qS((2-s)\omega+\sum_r\mu_r+2(n-1)\tau))$ part converges exponentially to zero), and as ξ is a positive linear combination of ω_1 and ω_2 we moreover have $\alpha^\tau(n,s+1;(\mu,\xi))\in(\frac{\phi_-+\phi_+-\pi}{2},\frac{\phi_-+\phi_++\pi}{2})$ (so that $c(-4(x+qS)(2\omega+\sum_r\mu_r+\xi-(s+1)\omega+2(n-1)\tau))$ converges exponentially if $x+qS\to -q\infty$). The second bound (5.A.3) remains exponentially convergent due again to $\alpha^\tau(n,s+1;(\mu,\xi))\in(\frac{\phi_-+\phi_+-\pi}{2},\frac{\phi_-+\phi_++\pi}{2})$. And finally the third bound (5.A.4) again converges exponentially for large x, since C_2 only has deformations in some bounded region. □

The proofs of the other limits are all in the same vein as this proof. We therefore only stress the differences to this proof, instead of writing them all down explicitly. In particular we omit the proof of the second limit of Proposition 5.3.22, since the only difference is that the bounds are different.

Proof of Proposition 5.3.23. We want to show that

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = \lim JI_{n,s,t}(\mu+qS,\nu-qS;\tau)\zeta^{-nt}c(n(2tq^2S^2)) \\ \times c(n(-4qS(\lambda-2\omega-(n-1)\tau) - \sum_r (\mu_r-\omega)^2 + \sum_r (\nu_r-\omega)^2)),$$

where $s_1 = \frac{s-t}{2}$, $s_2 = \frac{s+t}{2}$, $\lambda = 2(n-1)\tau + \sum_r \mu_r + \sum_r \nu_r + (4-s)\omega$ and $q = \exp(i\phi)$, such that $\phi \in \mathcal{A}_-$ if $t<0$, $\phi \in \mathcal{A}_0 \cap \mathcal{C}^{\alpha^\tau(n,s;\mu,\nu)}$ if $t=0$ and $\phi \in \mathcal{A}_+$ if $t>0$.

As contours for the $JI_{n,s,t}(\mu+qS,\nu-qS;\tau)$ integrals we take a semi-constant family of contours C_S with $a_1=-1$ and $a_2=1$, while $q_\pm = q$. The poles of the integrand are at $x = \pm(\mu+qS-\Lambda)$ and $\pm(\nu-qS-\Lambda)$ and hence there exists such a family of contours. Using the decomposition $C_S = C_{1,S}\cup C_{2,S}$ we obtain a decomposition of C_S^n as a union of contours of the form $C_{1,S}^{n_1}\times C_{2,S}^{n_2}$ (using the permutation symmetries of the integration variables). Due to evenness of the integrand the integrals over all these 2^n parts are identical and hence the total integral is 2^n times the integral over just $C_{2,S}^n$.

We now bound the integrand on $C_{2,S}^n$, to be able to apply Lebesgue's theorem. In contrast to the proof of Proposition 5.3.22 we now need a bound on the integrand which is a good approximation near $x_j = qS$ (for all j). Using a similar derivation as for (5.A.1) but using $|\Re(\pi i(x_i-x_j)/\omega_\delta)| \leq |\Re(\pi i(x_i-qS)/\omega_\delta)| + |\Re(\pi i(x_j - qS)/\omega_\delta)|$ for $\delta=1,2$ we obtain

$$\left|\frac{\prod_{1\leq i<j\leq n}\Gamma_h(\tau\pm x_i\pm x_j)}{\prod_{1\leq i<j\leq n}\Gamma_h(\pm x_i\pm x_j)\prod_{i=1}^n\Gamma_h(\pm 2x_i)}\right| \\ \leq K\exp(2\pi\sum_{j=1}^n |\Re(\frac{i((n-1)\tau+2\omega)x_j}{\omega_1\omega_2})| + |\Re(\frac{(n-1)i\tau(x_j-qS)}{\omega_1\omega_2})|)$$

for some constant K if all x_i are on the contour $C_{2,S}$.

Using this bound the integrand becomes a product of univariate terms, and we only have to give an integrable uniform bound of each univariate integral. In particular on the part of $C_{2,S}$ from 0 to $z+qS$ for some fixed z we find the bound

$$H_1(x) = K|c(8(x-qS)\omega)|. \tag{5.A.5}$$

This bound is clearly integrable in $x-qS$ over the relevant part of the contour, since from 0 to $z+qS$ this term is exponentially decreasing in $x-qS$. For the part of the contour from $z+qS$ to $q\infty$ we find the bound

$$H_2(x) = K|c(2t(x-qS)^2 + 4(x-qS)(\lambda-2\omega))|, \tag{5.A.6}$$

which again is integrable on this part of the contour, due to the conditions on ϕ.

So we find that the shifted integrand is bounded (uniformly in S) by

$$\Bigg| \frac{\prod_{i=1}^n \prod_{r=0}^{s_1-1} \Gamma_h(\mu_r + 2qS + x_i, \mu_r - x_i) \prod_{r=0}^{s_2-1} \Gamma_h(\nu_r + x_i, \nu_r - 2qS - x_i)}{\prod_{1\leq i<j\leq n} \Gamma_h(\pm(2qS + x_i + x_j), \pm(x_i - x_j)) \prod_{i=1}^n \Gamma_h(\pm(2qS + 2x_i))}$$

$$\times \prod_i \zeta^{-t} c(2t(x_i+qS)^2 + 2tq^2S^2 - 4qS(\lambda-(n+1)\omega) - \sum_r (\mu_r-\omega)^2 - \sum_r (\nu_r-\omega)^2) \Bigg|$$

$$\leq \prod_{j=1}^n H(x_j),$$

where

$$H(x) = \begin{cases} H_1(x+qS) = K|c(8x\omega)| & \text{on the part of } C_2 \text{ from } -qS \text{ to } w \\ H_2(x+qS) = K|c(2tx^2+4x(|\lambda-2\omega))| & \text{on the part of } C_2 \text{ from } w \text{ to } q\infty. \end{cases}$$

The conditions on ϕ ensure that H is integrable, hence we can calculate the desired limit using Lebesgue's theorem of dominated convergence:

$$\lim_{S\to\infty} J_{n,s,t}(\mu + qS; \nu - qS)\zeta^{-nt}$$
$$\times c(n(2tq^2S^2 - 4qS(\lambda - (n+1)\omega) - \sum_r (\mu_r - \omega)^2 + \sum_r (\nu_r - \omega)^2))$$
$$= \lim_{S\to\infty} \int_{C_{2,S}^n} \frac{\prod_{i=1}^n \prod_{r=0}^{s_1-1} \Gamma_h(\mu_r + qS \pm x_i) \prod_{r=0}^{s_2-1} \Gamma_h(\nu_r - qS \pm x_i)}{\prod_{1\leq i<j\leq n} \Gamma_h(\pm x_i \pm x_j) \prod_{i=1}^n \Gamma_h(\pm 2x_i)}$$
$$\times \prod_i \zeta^{-t} c(2tx_i^2 + 2tq^2S^2 - 4qS(\lambda-(n+1)\omega) - \sum_r (\mu_r-\omega)^2 - \sum_r (\nu_r-\omega)^2) dx_i$$
$$= \int_{C_2^n} \lim_{S\to\infty} \frac{\prod_{i=1}^n \prod_{r=0}^{s_1-1} \Gamma_h(\mu_r+2qS+x_i, \mu_r-x_i) \prod_{r=0}^{s_2-1} \Gamma_h(\nu_r+x_i, \nu_r-2qS-x_i)}{\prod_{1\leq i<j\leq n} \Gamma_h(\pm(2qS+x_i+x_j), \pm(x_i-x_j)) \prod_{i=1}^n \Gamma_h(\pm(2qS+2x_i))}$$
$$\times \prod_i \zeta^{-t} c(2t(x_i+qS)^2 + 2tq^2S^2 - 4qS(\lambda-(n+1)\omega) - \sum_r (\mu_r-\omega)^2 - \sum_r (\nu_r-\omega)^2) dx_i$$
$$= J_{n,(\frac{s-t}{2},\frac{s+t}{2}),t}(\mu;\nu;\lambda).$$

In the second equality we shifted the integration variable by qS, to obtain an integral over C_2^n, and interchanged limit and integral, which is allowed by Lebesgue's theorem since (5.A.5) and (5.A.6) provide uniform integrable bounds for the shifted integrand. The limit of the hyperbolic gamma functions was again calculated using Theorem 5.2.6. □

Proof of Proposition 5.3.24. The difference of this proof to the previous ones, is that we have to give a different bound on the cross terms, and that there's only a sequence of poles going to infinity in one direction (either plus or minus infinity).

We will only consider the first limit, i.e.

$$JI_{n,(s_1,s_2),t}(\mu;\nu;\lambda;\tau) = \lim_{S\to\infty} JI_{n,(s_1+1,s_2),t-1}(\mu, \xi + qS; \nu; \lambda + \xi - \omega; \tau)$$
$$\times \zeta^n c(-n(\xi + qS - \omega)^2),$$

for $q = \exp(i\phi)$, where $(\mu,\nu) \in \mathcal{B}_{s_1,s_2}$, $\xi_1 \in \mathbb{C}$, and $\tau \in \mathcal{E}_\epsilon$ with $\epsilon = -$ if $t < 0$, $\epsilon = 0$ if $t = 0, 1$ and $\epsilon = +$ if $t > 1$. The conditions on ϕ are determined by the value of $t + s_2 - s_1$, can depend on τ, μ, ν and λ and are given by

$t+s_2-s_1<0$	$t+s_2-s_1=0$	$t+s_2-s_1>0$
$\mathcal{A}_-^\tau$	$\mathcal{A}_0^\tau \cap \mathcal{C}^{\beta^\tau(n;s_1+s_2;\mu,\nu,\lambda)} \cap \mathcal{C}^{\beta^\tau(n,s_1+s_2+2;\mu,\xi_1,\nu,\xi_2,\lambda)}$	$\mathcal{A}_+^\tau$

If $t + s_1 - s_2 = 0$ we moreover insist that $(\mu,\nu,-\lambda) \in \mathcal{D}^\tau_{n,(s_1,s_2)}$.

Note that the poles of the integrand of $JI_{n,(s_1+1,s_2),t-1}(\mu,\xi+qS;\nu;\lambda+\xi-\omega;\tau)$ are located at $\mu_r-\Lambda$, $-\nu_r+\Lambda$ and $\xi+qS-\Lambda$. We assume the integration contours are given by a semi-constant family of integration contours C_S with $a_1 = 0$ and $a_2 = 1$. Moreover $q_+ = q$ and $q_- = \exp(i\phi)$ for some $\phi \in \mathcal{A}_{\mathbf{sign}(2t-1)}$ (this ensures that the integrals on both sides of the equation converge). Note that sometimes we cannot choose the integration contours as a deformation of a straight line (i.e. with $q_- = q_+$), which is why we had to add this extra flexibility in the definition of a semi-constant family of contours.

For the cross terms we use a similar calculation as for (5.A.1) to obtain for all x_i on the contour C_S

$$\left|\frac{1}{\prod_{i<j}\Gamma_h(\pm(x_i-x_j))}\right| \leq K\exp(2(n-1)\pi\sum_i\left|\Re(\frac{i\omega x_i}{\omega_1\omega_2})\right|).$$

Using this bound we can once again split the integration contour in $C_S^n = C_{1,S}^{n_1} \times C_{2,S}^{n_2}$ and consider univariate integrals. Subsequently we can find a bound as before to show that the integral over $C_{2,S}$ vanishes as $S \to \infty$, so we only have to consider the integral over $C_{1,S}^n$. We obtain a uniform integrable bound for this and use Lebesgue's lemma of dominated converge as before to obtain the desired limit relation. □

5.B Explicit definitions of the $II_{n,\xi}^1$ degenerations

In this appendix we give the explicit integrands and balancing conditions of the degenerations $II_{n,\xi}^1$, for the a-versions and only the (k,l) degenerations with $k \geq l$. We omit the parameter conditions and just mention the sets the variables are chosen from. Moreover we ignore the conditions on the contours.

Level 1 • For $\mu \in \mathbb{C}^8$ and $\tau \in \mathbb{C}$ satisfying the balancing condition

$$2(n-1)\tau + \sum_{r=0}^{7}\mu_r = 4\omega$$

we define

$$I\!I^1_{n,T}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^n \frac{\prod_{r=0}^7 \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} dx_j.$$

Level 2

- For $\mu \in \mathbb{C}^6$ and $\tau \in \mathbb{C}$ we define

$$I\!I^1_{n,2}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^n \frac{\prod_{r=0}^5 \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} dx_j.$$

- For $\mu,\nu \in \mathbb{C}^4$ and $\tau \in \mathbb{C}$ satisfying the balancing condition

$$2(n-1)\tau + \sum_{r=0}^3 \mu_r + \sum_{r=0}^3 \nu_r = 4\omega$$

we define

$$I\!I^1_{n,(1,1)}(\mu;\nu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^n \prod_{r=0}^3 \Gamma_h(\mu_r - x_j, \nu_r + x_j) dx_j.$$

Level 3

- For $\mu \in \mathbb{C}^5$ and $\tau \in \mathbb{C}$ the $I\!I^1_{n,3a}$ integral is given by

$$I\!I^1_{n,3a}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^n \frac{\prod_{r=0}^4 \Gamma_h(\mu_r \pm x_j)}{\Gamma_h(\pm 2x_j)} c(-2x_j^2) dx_j.$$

- For $\mu,\nu \in \mathbb{C}^3$ and $\lambda,\tau \in \mathbb{C}$ satisfying the balancing condition

$$2(n-1)\tau + \sum_{r=0}^2 \mu_r + \sum_{r=0}^2 \nu_r = \lambda + 2\omega$$

we define

$$II^1_{n,(2,1)}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}$$
$$\times\prod_{j=1}^n\prod_{r=0}^2\Gamma_h(\mu_r-x_j,\nu_r+x_j)c(2\lambda x_j)dx_j.$$

Level 4

- For $\mu\in\mathbb{C}^4$ and $\tau\in\mathbb{C}$ the $II^1_{n,4a}$ integral is given by

$$II^1_{n,4a}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}$$
$$\times\prod_{j=1}^n\frac{\prod_{r=0}^3\Gamma_h(\mu_r\pm x_j)}{\Gamma_h(\pm 2x_j)}c(-4x_j^2)dx_j.$$

- For $\mu\in\mathbb{C}^3$, $\nu\in\mathbb{C}^2$ and $\lambda,\tau\in\mathbb{C}$ satisfying the balancing condition

$$2(n-1)\tau+\sum_{r=0}^2\mu_r+\sum_{r=0}^1\nu_r=\lambda+\omega$$

we define

$$II^1_{n,(3,1)a}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}$$
$$\times\prod_{j=1}^n\prod_{r=0}^2\Gamma_h(\mu_r-x_j)\prod_{r=0}^1\Gamma_h(\nu_r+x_j)c(2\lambda x_j-x_j^2)dx_j.$$

- For $\mu,\nu\in\mathbb{C}^2$ and $\lambda,\tau\in\mathbb{C}$ we define

$$II^1_{n,(2,2)}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm(x_j-x_k))}{\Gamma_h(\pm(x_j-x_k))}$$
$$\times\prod_{j=1}^n\prod_{r=0}^1\Gamma_h(\mu_r-x_j,\nu_r+x_j)c(2\lambda x_j)dx_j.$$

Level 5

- For $\mu\in\mathbb{C}^3$ and $\tau\in\mathbb{C}$ the $II^1_{n,5a}$ integral is given by

$$II^1_{n,5a}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!}\int_{C^n}\prod_{1\le j<k\le n}\frac{\Gamma_h(\tau\pm x_j\pm x_k)}{\Gamma_h(\pm x_j\pm x_k)}$$
$$\times\prod_{j=1}^n\frac{\prod_{r=0}^2\Gamma_h(\mu_r\pm x_j)}{\Gamma_h(\pm 2x_j)}c(-6x_j^2)dx_j.$$

- For $\mu \in \mathbb{C}^3$ and $\nu, \lambda, \tau \in \mathbb{C}$ satisfying the balancing condition

$$2(n-1)\tau + \sum_{r=0}^{2} \mu_r + \nu_0 = \lambda$$

we define

$$I\!I^1_{n,(4,1)a}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^{n} \prod_{r=0}^{2} \Gamma_h(\mu_r - x_j)\Gamma_h(\nu_0 + x_j) c(2\lambda x_j - 2x_j^2) dx_j.$$

- For $\mu \in \mathbb{C}^2$ and $\nu, \lambda, \tau \in \mathbb{C}$ we define

$$I\!I^1_{n,(3,2)a}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^{n} \Gamma_h(\mu_0 - x_j, \mu_1 - x_j, \nu_0 + x_j) c(2\lambda x_j - x_j^2) dx_j.$$

Level 6

- For $\mu \in \mathbb{C}^2$ and $\tau \in \mathbb{C}$ the $I\!I^1_{n,6a}$ integral is given by

$$I\!I^1_{n,6a}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^{n} \frac{\Gamma_h(\mu_0 \pm x_j, \mu_1 \pm x_j)}{\Gamma_h(\pm 2x_j)} c(-8x_j^2) dx_j.$$

- For $\mu \in \mathbb{C}^3$ and $\lambda, \tau \in \mathbb{C}$ satisfying the balancing condition

$$2(n-1)\tau + \sum_{r=0}^{2} \mu_r = \lambda - \omega$$

we define

$$I\!I^1_{n,(5,1)a}(\mu;-;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^{n} \prod_{r=0}^{2} \Gamma_h(\mu_r - x_j) c(2\lambda x_j - 3x_j^2) dx_j.$$

- For $\mu \in \mathbb{C}^2$ and $\lambda, \tau \in \mathbb{C}$ we define

$$II^1_{n,(4,2)a}(\mu;-;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^n \Gamma_h(\mu_0 - x_j, \mu_1 - x_j) c(2\lambda x_j - 2x_j^2) dx_j.$$

- For $\mu, \nu, \lambda, \tau \in \mathbb{C}$ we define

$$II^1_{n,(3,3)a}(\mu;\nu;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^n \Gamma_h(\mu_0 - x_j, \nu_0 + x_j) c(2\lambda x_j - 2x_j^2) dx_j.$$

- For $\mu \in \mathbb{C}^2$ and $\lambda, \tau \in \mathbb{C}$ we define

$$II^1_{n,(3,3)^*a}(\mu;-;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^n \Gamma_h(\mu_0 - x_j, \mu_1 - x_j) c(2\lambda x_j) dx_j.$$

Level 7

- For $\mu, \tau \in \mathbb{C}$ the $II^1_{n,7a}$ integral is given by

$$II^1_{n,7a}(\mu;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^n \frac{\Gamma_h(\mu_0 \pm x_j)}{\Gamma_h(\pm 2x_j)} c(-10x_j^2) dx_j.$$

- For $\mu, \lambda, \tau \in \mathbb{C}$ we define

$$II^1_{n,(4,3)a}(\mu;-;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^n \Gamma_h(\mu_0 - x_j) c(2\lambda x_j - 3x_j^2) dx_j.$$

Level 8

- For $\tau \in \mathbb{C}$ the $II^1_{n,8a}$ integral is given by

$$II^1_{n,8a}(-;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n 2^n n!} \int_{C^n} \prod_{1\le j<k\le n} \frac{\Gamma_h(\tau \pm x_j \pm x_k)}{\Gamma_h(\pm x_j \pm x_k)} \times \prod_{j=1}^n \frac{1}{\Gamma_h(\pm 2x_j)} c(-12x_j^2) dx_j.$$

- For $\lambda, \tau \in \mathbb{C}$ we define

$$I\!I^1_{n,(4,4)a}(-;-;\lambda;\tau) = \frac{\Gamma_h(\tau)^n}{\sqrt{-\omega_1\omega_2}^n n!} \int_{C^n} \prod_{1\leq j<k\leq n} \frac{\Gamma_h(\tau \pm (x_j - x_k))}{\Gamma_h(\pm(x_j - x_k))} \times \prod_{j=1}^{n} c(2\lambda x_j - 4x_j^2) dx_j.$$

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Samenvatting: Hyperbolisch Hypergeometrische Functies

Deze samenvatting poogt een beeld te geven van de inhoud van dit proefschrift dat begrijpelijk is voor niet-wiskundigen.

Zoals de titel van het proefschrift zegt gaat dit proefschrift over hyperbolisch hypergeometrische functies. Dit zijn een bepaald soort speciale functies. Functies worden speciaal genoemd als ze zo vaak voorkomen in bijvoorbeeld de getaltheorie, de combinatoriek, de analyse, de natuurkunde of andere gebieden waar wiskunde wordt toegepast, dat ze een eigen naam en een aparte studie verdienen. De bekendste speciale functies zijn e^x en de sinus en cosinus. Hypergeometrische functies vormen een belangrijke deelklasse van speciale functies, waartoe de net genoemde exponentiële en trigonometrische functies behoren. Aangezien hypergeometrische functies op een uniforme manier gedefinieerd zijn, krijgt de theorie ervan meer structuur en is die minder een "postzegelverzameling" van functies en hun eigenschappen.

Er zijn verschillende soorten hypergeometrische functies. De oorspronkelijke "klassieke" hypergeometrische functies bestaan al enige eeuwen en zijn onder andere bestudeerd door wiskundige grootheden als Euler en Gauß. In toepassingen in de getaltheorie, de combinatoriek en later ook de mathematische fysica is een natuurlijke deformatie gevonden van de hypergeometrische functies, de q-hypergeometrische functies. Deze functies hangen af van een parameter q en voor $q = 1$ reduceren ze tot de klassieke hypergeometrische functies. De q-hypergeometrische functies bestaan echter niet voor alle waarden van q, alhoewel de waarden van q waarvoor de functies niet bestaan wel van belang zijn in sommige toepassingen.

Hyperbolisch hypergeometrische functies kun je zien als een soort intrinsieke vervlechting van twee q-hypergeometrische functies. Voor toepassingen die een q gebruiken waarvoor q-hypergeometrische functies niet bestaan biedt de hyperbolisch hypergeometrische theorie uitkomst. In deze gevallen kunnen we de problemen soms symmetrischer maken door aan de echte wereld een parallelle wereld toe te voegen die intrinsiek met de echte wereld vervlochten is (een beetje zoals in Lewis Carrol's Through the Looking Glass). De oplossingen van dit "verdubbelde" probleem zijn dan hyperbolisch hypergeometrische functies. De oorsprong van de hyperbolisch hypergeometrische theorie is terug te voeren tot het begin van de twintigste eeuw (het werk van Barnes), maar het onderzoek hierin is pas de laatste tien jaar tot bloei gekomen, onder andere door toepassingen in de mathematische fysica. Toepassingen van hyperbolisch hypergeometrische functies zijn onder andere te vinden in de studie van relativistische quantummechanische systemen en in de knopentheorie.

In hoofdstuk drie wordt beschreven hoe hyperbolisch hypergeometrische functies kunnen ontstaan in de representatietheorie van de modulaire dubbel van een quantumgroep. Laten we proberen uit te leggen wat dit betekent.

In de wiskunde kun je abstract nadenken over bijvoorbeeld rotaties, zonder een concrete ruimte in gedachte te nemen die je roteert. Als je vervolgens daadwerkelijk de ruimte (en de dingen erin) gaat roteren spreek je over een representatie van die rotaties. Doordat het denken aan alleen de rotatie, en niet direct hetgeen dat je roteert, een hoop informatie weglaat zorgt dit dat je veel makkelijker over bepaalde zaken (bijvoorbeeld functies met rotatiesymmetrie) kunt nadenken. Dit stelt ons in staat om te werken met veel ingewikkelder symmetrieën dan alleen rotaties. In deze context treden hypergeometrische functies op natuurlijke wijze op. Onder andere worden functies die zich netjes gedragen onder de representatie (i.e. symmetrisch zijn) vaak geschreven in termen van hypergeometrische functies. De sinus en de cosinus kom je bijvoorbeeld op verschillende plaatsen tegen bij het bestuderen van rotaties.

Als we nu bijvoorbeeld de rotaties van net op geschikte wijze vervormen krijgen we een zogeheten quantumgroep. Deze deformatie wordt weer gegeven door een extra parameter q, in analogie met de situatie dat de q-hypergeometrische functies een deformatie zijn van de klassieke hypergeometrische functies. Bij het bestuderen van deze quantumgroepen treden q-hypergeometrische functies op natuurlijke wijze op. De modulaire dubbel van een quantumgroep krijg je nu door twee quantumgroepen te combineren, zoals we eerder hyperbolisch hypergeometrische functies verkregen door twee q-hypergeometrische functies met elkaar te vervlechten. Bij de beschrijving van representaties van de modulaire dubbel vinden we dan hyperbolisch hypergeometrische functies.

In de hierop volgende hoofdstukken bestuderen we bepaalde belangrijke hyperbolisch hypergeometrische functies door ze te beschouwen als degeneraties van een heel algemene functie met een rijke symmetriestructuur (de “stammoeder” van al deze functies). Door de hyperbolisch hypergeometrische functies op deze manier te beschrijven kunnen we veel, bekende en nog niet bekende, resultaten voor ze afleiden. Bovendien vinden we op deze manier een aantal nog niet eerder bestudeerde hyperbolisch hypergeometrische functies en een hoop van hun eigenschappen. Een van de interessantere type resultaten is dat symmetrieën van de “stammoeder” in degeneraties leiden tot identiteiten die verschillende degeneraties op niet triviale wijze aan elkaar relateren.

In hoofdstuk 4 ligt de focus op de verschillen in de structuur die je krijgt als je zo’n analyse voor hyperbolisch hypergeometrische functies uitvoert in vergelijking met de structuur die je krijgt als je zo’n analyse voor andere klassen hypergeometrische functies uitvoert. In het laatste hoofdstuk bekijken we tenslotte meer-dimensionale hyperbolisch hypergeometrische functies.

Dankwoord

Ten eerste wil ik Jasper bedanken voor zijn steun tijdens het hele promotietraject. Weliswaar was het natuurlijk ook zijn baan, dus zouden sommigen redeneren dat een blijk van dank daarvoor overbodig is, maar ik heb het idee dat hij regelmatig meer deed dan strikt noodzakelijk was. Verder wil ik ieder ander op het KdV instituut bedanken voor het altijd open hebben staan van de deur (ten minste figuurlijk) en de bereidheid om vragen over ieder wiskundig of administratief onderwerp te beantwoorden.

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Ik wil ook nog de deelnemers aan de wiskunde olympiade training en de IMC-trainingsgroep noemen omdat zij ervoor zorgden dat ik, wat er verder ook die week te doen was, altijd even de gelegenheid had om weer gewoon even met wiskunde te spelen.

Mijn beide paranimfen, Dion en Iris, verdienen een aparte vermelding. Dion als inspiratiebron voor hoe je met wiskunde kunt blijven spelen ook als je ouder wordt. Iris omdat ze altijd in is voor elk idioot plan en daardoor de meest gekke dingen doet en natuurlijk voor al het andere.

Mijn ouders legden al wiskunde uit toen ik nog in de buik van mijn moeder zat (al weet ik niet hoeveel ik heb opgestoken van de lessen die mijn moeder toen gaf) en voor deze onaflatende steun in het stillen van mijn honger naar wiskunde wil ik hun danken. Tenslotte wil ik mijn zus bedanken voor het feit dat ze het al die jaren met mij heeft uitgehouden terwijl ik over wiskunde sprak.

Curriculum Vitae

Fokko van de Bult is op 25 november 1980 geboren in Hoogmade, gemeente Woubrugge. In Leiden ging hij naar het Stedelijk Gymnasium waar hij in 1998 met succes zijn eindexamen afrondde. Vervolgens is hij aan de Universiteit van Amsterdam begonnen met de studies wiskunde en econometrie. Na een half jaar ruilde hij econometrie in voor natuurkunde, zodat hij aan het eind van het eerste jaar cum laude zijn propedeuse haalde in wis- en natuurkunde. Bovendien kreeg hij de aanmoedigingsprijs voor de beste eerstejaars student wiskunde. In 2003 rondde hij zijn studie wiskunde cum laude af, met als specialisatie richting statistiek, waarna hij begonnen is aan zijn promotie. Dit mondde uit in het voor u liggende proefschrift, wat bijna precies 4 jaar na het beginnen van het promotietraject zal worden verdedigd.

Naast zijn studie is Fokko actief geweest in verscheidene olympiades en olympiade-achtige wedstrijden. Als scholier won hij onder andere de wiskunde olympiade in 1997 en de natuurkunde olympiade in 1998. Bovendien haalde hij twee zilveren medailles op de internationale wiskunde olympiades van 1997 en 1998. Als student was hij nog lid van het winnende team van de PION in 1999. Vanaf 1999 tot 2006 heeft hij bovendien geholpen bij de training van het Nederlandse team voor de internationale wiskunde olympiade, waar hij driemaal (van 2004 tot en met 2006) is heengegaan als deputy-leader. Bovendien heeft hij in 2006 een training opgezet voor deelname aan de internationale wiskunde competitie voor studenten waar hij vervolgens tweemaal (2006 en 2007) als leider van een Nederlandse delegatie naartoe is gegaan.