Affine Weyl groups and integrable systems with delta-potentials

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. mr. P. F. van der Heijden

ten overstaan van een door het college voor promoties ingestelde

commissie, in het openbaar te verdedigen in de Aula der Universiteit

op dinsdag 17 oktober 2006, te 10.00 uur

door

Erdal Emsiz

geboren te Kiğı, Turkije

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Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Dit proefschrift werd mede mogelijk gemaakt met financiële steun van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). Het onderzoek dat in dit proefschrift beschreven is, maakt deel uit van het project *Symmetrie en het breken van symmetrie in de wiskunde en de mathematische fysica* van NWO.

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

Omslag: Houtgravures uit 1861 door Gustave Doré, geïnspireerd door Gezang 1 (voorkant) en Gezang 34 (achterkant) uit *De Hel* van Dante Alighieri
Ontwerp omslag: L. Zisme

Druk: PrintPartners Ipskamp, Enschede

2000 Mathematics Subject Classification: 20C08, 81R12 (37J35, 82C10)
NUR 921
ISBN-10 90 5776 159 9
ISBN-13 978 90 5776 159 1

Affine Weyl groups and integrable systems with delta-potentials

Mephistopheles: Wo bin ich denn? wo wills hinaus?
(Goethe, *Faust*, V. 7801)

Aan E.

Contents

CHAPTER 1

General introduction

1.1. Introduction

In this introductory chapter we give some historical background and discuss integrable systems that are closely connected to the integrable systems treated in this thesis. In the last section a brief summary of the contents of the thesis is given.

In this thesis we study mathematical generalizations of a quantum mechanical many-body system introduced by Lieb and Liniger [**56**]. The system describes n spinless quantum particles on a circle with pair-wise repulsive contact interaction. The Lieb-Liniger system is related to the quantum system of n spinless quantum particles on a line with pair-wise repulsive contact interaction. This system is described by the following formal quantum[1] Hamiltonian

$$H = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + k \sum_{1\leq i\neq j\leq n} \delta(x_i - x_j). \tag{1.1.1}$$

Here δ denotes Dirac's delta-function and k is a nonnegative finite coupling constant that determines the strength of the interaction between two particles.

It was early on realized that (1.1.1) is formally equivalent to the bosonic n-particle sector of a quantum field theory. The time evolution equation of the corresponding infinite dimensional quantum Hamiltonian system is given by a quantum version of the nonlinear integrable partial differential equation in 2-dimension known as the classical nonlinear Schrödinger equation[2],

$$i\frac{\partial \phi}{\partial t} = -\frac{\partial^2 \phi}{\partial x^2} + 2k\,|\phi|^2\,\phi. \tag{1.1.2}$$

In the next section we give a sketch of this equivalence. This observation allowed the study of (1.1.1) (and the Lieb-Liniger model) by quantum inverse scattering methods (cf. Section 1.4)

[1]We work in units where Planck's constant $\hbar$ equals 1 and the mass of the particles equals $1/2$.

[2]This equation arises for example in the theory of two-dimensional selffocusing of a strong light beam in a nonlinear medium [**52**] and the theory of weakly nonideal Bose gas at zero Kelvin [**30**].

1.2. The nonlinear Schrödinger equation and a quantum Bose-gas

Before considering the quantum field theory itself, we briefly recall some facts from classical Hamiltonian systems of finitely many degrees of freedom. A state of such a system is described by n position variables $q_1, \dots q_n$ and n momentum variables $p_1, \dots p_n$. The evolution of the dynamical system is determined by a single real-valued function $H(q_1, \dots, q_n; p_1, \dots, p_n)$ (the Hamiltonian) on the phase space $\mathbb{R}^{2n}$. In complex coordinates

$$z_j = (q_j + ip_j)/\sqrt{2}, \quad \overline{z}_j = (q_j - ip_j)/\sqrt{2}$$

the time-evolution of the system is described by Hamilton's equations:

$$dz_j/dt = \{H, z_j\} = -i\frac{dH}{d\overline{z}_j}, \tag{1.2.1}$$

with the $\{,\}$ the standard Poisson bracket on the phase space which is completely determined by the fundamental commutation relations

$$\{z_j, z_k\} = 0 = \{\overline{z}_j, \overline{z}_k\}, \quad \{z_j, \overline{z}_k\} = i\delta_{jk}. \tag{1.2.2}$$

We now consider a classical Hamiltonian system of infinitely many degrees of freedom i.e. a classical field theory (for a detailed account we refer the reader to the book [**21**] by Faddeev and Takhtajan) with phase space $\mathcal{S}(\mathbb{R})$ the space of the Schwartz space of rapidly decreasing functions on the line,

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) | \lim_{x\to\pm\infty} \left| x^k \frac{d^l f}{dx^l}(x) \right| = 0 \quad \forall k, l \geq 0\}.$$

The observables $\mathcal{A}$ are certain so-called *polynomial* real-valued *smooth* functionals F on $\mathcal{S}(\mathbb{R})$. They are polynomials in the function $\psi(x)$ and $\overline{\psi}(x)$ ($x \in \mathbb{R}$), and their derivatives, where $\psi(x)$ and $\overline{\psi}(x)$ are given by,

$$(\psi(x))(\phi) = \phi(x), \quad (\overline{\psi}(x))(\phi) = \overline{\phi(x)}.$$

There is a Poisson bracket $\{,\}$ on $\mathcal{A}$, completely determined by the fundamental commutation relations (cf. (1.2.2))

$$\{\psi(x), \psi(y)\} = 0 = \{\overline{\psi}(x), \overline{\psi}(y)\}, \quad \{\psi(x), \overline{\psi}(y)\} = i\delta(x-y) \tag{1.2.3}$$

(these relations have to be interpreted [**21**, Part 1, I.1] in the sense of distributions).

A smooth functional H (the Hamiltonian) defines an infinite dimensional dynamical system, with the time evolution equation given by (cf. (1.2.1))

$$\partial_t \psi(x) = \{H, \psi(x)\} = -i\frac{\delta H}{\delta\overline{\psi}(x)} \tag{1.2.4}$$

The time evolution of an observable $F \in \mathcal{A}$ is given by $\partial_t F = \{H, F\}$.

We now introduce the Hamiltonian for the nonlinear Schrödinger model:

$$H(\phi) = \int_{\mathbb{R}} \left(|\partial_x \phi(x)|^2 + k\,|\phi(x)|^4 \right) dx. \tag{1.2.5}$$

A calculation shows that

$$(\delta H/\delta\overline{\psi}(x))(\phi) = -\partial_x^2\phi(x) + 2k\,|\phi(x)|^2\,\phi(x).$$

Therefore the time evolution equation (1.2.4) of the dynamical system corresponding to the Hamiltonian (1.2.5) is given by the classical nonlinear Schrödinger equation (1.1.2).

It is easily checked that the observable

$$N(\phi) = \int_{\mathbb{R}} |\phi(x)|^2\,dx \tag{1.2.6}$$

is an integral of motion and is called the *charge* observable.

We now give a summary of the quantized version of this classical field theory (see also [**65**, Formal Preliminaries and Part III, Chapter 1], [**35**] and [**67**, Section X.7]).

Quantization of a classical Hamiltonian systems of finitely many degrees of freedom is in its simplest form the following procedure. Attach to the (complex) canonical variables z_j and $\overline{z}_j$ (necessarily unbounded) linear operators $\widehat{z}_j$ and $\widehat{z}_j^\dagger$ on a separable Hilbert space $\mathcal{H}$ such that they satisfy the canonical commutation relations (cf. (1.2.2))

$$[\widehat{z}_j,\widehat{z}_k] = 0 = [\widehat{z}_j^\dagger,\widehat{z}_k^\dagger],\quad [\widehat{z}_j,\widehat{z}_k^\dagger] = \delta_{jk}\,\mathrm{Id}_{\mathcal{H}}\quad (j,k = 1,2,\ldots,n). \tag{1.2.7}$$

Let A be a suitable classical observable (for example, A is a polynomial in the z_j and $\overline{z}_j$). One quantizes A (i.e. attaches a linear operator $\widehat{A}$ on $\mathcal{H}$ to A) as follows. Bring all the $\overline{z_j}$'s to the left of the z_i's and replace in this expression any z_j by $\widehat{z}_j$ and any $\overline{z}_j$ by $\widehat{z}_j^\dagger$. The operator one gets is denoted by $\widehat{A}$ and is called the *normal ordering* of A.

In analogy with the quantization of classical Hamiltonian system of finite many degrees of freedom, a quantization of the classical fields $\psi(x),\overline{\psi}(x)$ is given by the following prescription:

- **(i)** For any $x\in\mathbb{R}$ we attach *quantum fields*, i.e. "operators" $\Psi(x)$ and $\overline{\Psi(x)}$ on a Hilbert space $\mathcal{H}$ satisfying the *canonical commutation relations*:

$$[\Psi(x),\Psi(y)] = [\Psi^\dagger(x),\Psi^\dagger(y)] = 0,\quad [\Psi(x),\Psi^\dagger(y)] = \delta(x-y)\,\mathrm{Id}_{\mathcal{H}}, \tag{1.2.8}$$

- **(ii)** Given a suitable classical observable $A\in\mathcal{A}$, replace the fields in the normal ordered form, i.e. put the $\Psi^\dagger$'s in front of the Ψ's, to get a quantum observable $\widehat{A}$.

It is clear that $\Psi(x)$, $\Psi^\dagger(x)$ can not be bone-fide operators (not even unbounded) because of the delta-function in (1.2.8). To make sense of (1.2.8) the quantum fields have to be interpreted as operator-valued distributions.

Normal ordering of the classical Hamiltonian (1.2.5) gives formally

$$\widehat{H} = \int_{\mathbb{R}} \left[(\partial_x\Psi^\dagger(x))(\partial_x\Psi(x)) + k\Psi^\dagger(x)^2\Psi(x)^2\right]dx.$$

A formal calculation shows then that we have

$$-[\widehat{H},\Psi(x)] = -\partial_x^2\Psi(x) + 2k\Psi^\dagger(x)\Psi(x)^2.$$

Whence the associated equation of motion (cf. (1.2.4)) is given by the quantum nonlinear Schrödinger equation

$$i\partial_t \Psi(x,t) = -\partial_x^2 \Psi(x,t) + 2k\Psi^\dagger(x,t)\Psi(x,t)^2, \tag{1.2.9}$$

with the time-dependent quantum fields $\Psi(x,t)$ and $\Psi^\dagger(x,t)$ given formally by

$$\Psi(x,t) = \exp(i\widehat{H}t)\Psi(x)\exp(-i\widehat{H}t), \quad \Psi^\dagger(x,t) = \exp(i\widehat{H}t)\Psi^\dagger(x)\exp(-i\widehat{H}t).$$

We now construct the time-independent fields $\Psi(x)$ and $\overline{\Psi}(x)$. The most commonly used representation of (1.2.8) is the so-called Fock representation, defined on a bosonic Fock space. This space has a distinguished cyclic vector called the vacuum vector, which is killed by the annihilation operators $\Psi(x)$ and which is cyclic with respect to creation operators $\Psi^\dagger(x)$.

The Hilbert space we work with is the *bosonic Fock space*

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^n, dx)^{S_n},$$

with $\mathcal{H}_n$ the *bosonic n-particle sector* of $\mathcal{H}$ (convention: $\mathcal{H}_0 := \mathbb{C}$).

Here S_n is the symmetry group on n symbols, acting on functions $\mathbb{R}^n \to \mathbb{C}$ by permutation of the coordinates. As vacuum vector any nonzero element of $\mathcal{H}_0 = \mathbb{C}$ can be taken. Denote the inner product on $\mathcal{H}_n$ by $\langle \cdot,\cdot \rangle_n$ and the inner product on $\mathcal{H}$ by $\langle \cdot,\cdot\rangle$. Let $\mathcal{D}_n = C_c^\infty(\mathbb{R}^n)^{S_n}$ be the space of symmetric compactly supported smooth functions on $\mathbb{R}^n$ (convention: $\mathcal{D}_0 := \mathbb{C}$).

It is a dense subspace of $\mathcal{H}_n$, and

$$\mathcal{D} = \bigoplus_{n\geq 0} \mathcal{D}_n \subsetneq \mathcal{H},$$

is a dense subset of $\mathcal{H}$.

If $D : \mathcal{D}\times\mathcal{D} \to \mathbb{C}$ is a sesqui-linear (convention: linear in the first factor) form , we usually write $\langle Df, g\rangle$ for $D(f,g)$. Any linear operator $L : \mathcal{D} \to \mathcal{D}$ defines uniquely a sesqui-linear form $\mathcal{D}\times\mathcal{D} \to \mathbb{C}$ by $(f,g) \mapsto \langle Lf, g\rangle_{\mathcal{H}}$.

DEFINITION 1.2.1. *Let $x \in \mathbb{R}$. The linear operator $\Psi(x) : \mathcal{D} \to \mathcal{D}$ is uniquely defined by: $\Psi(x) : \mathcal{D}_n \to \mathcal{D}_{n-1}$ ($n \geq 0$), and*

$$(\Psi(x)f)(x_1, x_2, \ldots, x_{n-1}) = \sqrt{n} f(x, x_1, x_2, \ldots, x_n) \quad (f \in \mathcal{D}_n)$$

(convention: $\Psi(x) : \mathcal{D}_0 \to \{0\}$).

The adjoint of the operator $\Psi(x)$ (considered as an unbounded operator with domain of definition $\mathcal{D}$) is given formally by

$$(\Psi^\dagger(x)g)(x_1,\ldots,x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} \delta(x-x_j) g(x_1,\ldots,\widehat{x_j},\ldots,x_{n+1}) \quad (g \in \mathcal{D}_n). \tag{1.2.10}$$

As forms it makes sense:

DEFINITION 1.2.2. *Let $x_1, \dots x_m \in \mathbb{R}$. The following defines uniquely a form $\mathcal{D} \times \mathcal{D} \to \mathbb{C}$,*

$$\begin{aligned}\langle \Psi^\dagger(x_1)\dots\Psi^\dagger(x_m)f, g\rangle &= \langle f, \Psi(x_1)\dots\Psi(x_m)g\rangle_n \\ &= \sqrt{\frac{(m+n)!}{n!}} \int_{\mathbb{R}^n} f(y)\overline{g(x_1,\dots,x_m,y_1,\dots,y_n)}dy\end{aligned}$$

for $f \in \mathcal{D}_n$ and $g \in \mathcal{D}_{n+m}$, and zero otherwise.

In other words: $\Psi^\dagger(x_1)\dots\Psi^\dagger(x_n)$ is by definition the adjoint of $\Psi(x_1)\dots\Psi(x_n)$ in the sense of forms. The fact that formally $\Psi^\dagger(x)$ is given by (1.2.10) suggests that we should to see the $\Psi^\dagger(x_1)\dots\Psi^\dagger(x_n)$ as operator-valued distributions. In the mathematically rigorous treatments of quantum field theory (for example in axiomatic quantum field theory [**37**], [**76**]) one therefore uses "smeared" fields.

DEFINITION 1.2.3. *Let $f \in \mathcal{S}(\mathbb{R})$. The linear operator $\Phi^\dagger(f) : \mathcal{D} \to \mathcal{D}$ is uniquely defined by: $\Phi^\dagger(f) : \mathcal{D}_n \to \mathcal{D}_{n+1}$ and,*

$$((\Phi^\dagger(f))g)(x_1,\dots,x_{n+1}) = \frac{1}{\sqrt{n+1}}\sum_{j=1}^{n+1} f(x_j)g(x_1,\dots,\widehat{x_j},\dots,x_{n+1}) \quad (g \in \mathcal{D}_n).$$

For $f, g \in \mathcal{S}(\mathbb{R})$ one checks easily the commutation relations

$$\big[\Psi(x),\Psi(y)\big] = 0, \ \ \big[\Phi^\dagger(f),\Phi^\dagger(g)\big] = 0, \ \ \big[\Psi(x),\Phi^\dagger(f)\big] = f(x)\,\mathrm{Id}_{\mathcal{H}},$$

which are the rigorous form of (1.2.8).

REMARK 1.2.4. When $x_1, \dots x_m, y_1, \dots y_n \in \mathbb{R}$, we can always define

$$\Psi^\dagger(x_1)\dots\Psi^\dagger(x_m)\Psi(y_1)\dots\Psi(y_n) : \mathcal{D}\times\mathcal{D} \to \mathbb{C},$$

a sesqui-linear form uniquely defined by,

$$\langle \Psi^\dagger(x_1)\dots\Psi^\dagger(x_m)\Psi(y_1)\dots\Psi(y_n)f, g\rangle = \langle \Psi(y_1)\dots\Psi(y_n)f, \Psi(x_1)\dots\Psi(x_n)g\rangle.$$

However if in an expression, $\Psi(x)$ comes earlier then a $\Psi^\dagger(y)$ for some x, y, then such an expression does not make sense, not even as a sesqui-linear form. Thus for example $\Psi(x)\Psi^\dagger(y)$ does not make sense.

Note that the quantization $\widehat{N}$ of the charge observable (1.2.6) has the following property,

$$\langle \widehat{N} f, g\rangle = n\langle f, g\rangle \quad \text{for } f, g \in \mathcal{D}_n \text{ (and zero otherwise)},$$

and is called the *number of particles* observable. Because $\widehat{N}$ is a quantum integral of motion, it natural to consider the Hamiltonian $\widehat{H}$ on the bosonic n-particle sector $\mathcal{D}_n$ (of $\mathcal{H}_n$), which is precisely the eigenvalue space of $\widehat{N}$ with eigenvalue n .

To describe the action of $\widehat{H}$ on $\mathcal{D}_n$ more concretely, we introduce the following notation.

DEFINITION 1.2.5. *For* $1 \leq i < j \leq n$ *consider the hyperplane* $L_{ij} = \{x \in \mathbb{R}^n | x_i - x_j = 0\}$. *We denote by* $\mathcal{L}$ *the collection of hyperplanes* $\{L_{ij} | 1 \leq i < j \leq n\}$ *in* $\mathbb{R}^n$. *For* $L \in \mathcal{L}$ *we denote by* $d'y$ *the volume measure on* L *induced from Lebesgue measure on* $\mathbb{R}^n$. *For example*

$$\int_{L_{12}} g(y) d'y = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-2}} g(t, t, y_1, y_2, \ldots, y_{n-2}) dy_1 \ldots dy_{n-2} \right) dt.$$

Write $\Delta = \sum_{j=1}^{n} (\partial / \partial_{x_j})^2$ for the Laplacian on $\mathbb{R}^n$, acting on smooth functions. The quantum Hamiltonian $\widehat{H}$ takes the following form on $\mathcal{D}_n$.

PROPOSITION 1.2.6. *Let* $n \geq 1$. *For* $f, g \in \mathcal{D}_n$ *the following holds:*

$$\langle \widehat{H} f, g \rangle = \int_{\mathbb{R}^n} \langle -\Delta f, g \rangle_n + \sqrt{2} k \sum_{1 \leq i \neq j \leq n} \int_{L_{ij}} f(y) \overline{g(y)} d'y. \tag{1.2.11}$$

Also: $\langle \widehat{H} f, g \rangle = 0$ *for* $f \in \mathcal{D}_n$, $g \in \mathcal{D}_m$ *and* $n \neq m$.

REMARK 1.2.7. Proposition 1.2.6 tells us, in physics language, that the restriction of the Hamiltonian $\widehat{H}$ to the bosonic n-particle sector Hilbert space $\mathcal{H}_n$ of $\mathcal{H}$, is the Hamiltonian for the quantum system of n indistinguishable spinless quantum bosonic particles on the line with pair-wise delta-function potential of strength k. It seems that this was realized for the first time by Kaup [**50**].

PROOF. Because of symmetry the Laplacian Δ we have $\langle \Delta f, g \rangle_n = n \langle \partial_{y_1}^2 f, g \rangle_n$ for $f, g \in \mathcal{D}_n$. Because of symmetry again, the right hand side of (1.2.11) can be written as

$$\begin{aligned} &- n \int_{\mathbb{R}^n} (\partial_{y_1}^2 f(y)) \overline{g(y)} dy + \\ &kn(n-1) \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-2}} f(t, t, y_1, \ldots, y_{n-2}) \overline{g(t, t, y_1, \ldots, y_{n-2})} dy_1 \ldots dy_{n-2} \right) dt. \end{aligned} \tag{1.2.12}$$

In the same way the left hand side $\langle \widehat{H} f, g \rangle$ of (1.2.11) can be written as

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\langle \partial_x (\Psi(x) f), \partial_x (\Psi(x)) g \rangle + k \langle \Psi(x)^2 f, \Psi(x)^2 g \rangle \right) dx \\ &= n \langle \partial_{y_1} f, \partial_{y_1} g \rangle \\ &\quad + kn(n-1) \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-2}} f(t, t, y_1, \ldots, y_{n-2}) \overline{g(t, t, y_1, \ldots, y_{n-2})} dy_1 \ldots dy_{n-2} \right) dt. \end{aligned}$$

Partial integrating once gives $\langle \partial_{y_1} f, \partial_{y_1} g \rangle = -\langle \partial_{y_1}^2 f, g \rangle$, and therefore (1.2.11) follows, proving the proposition. □

Denote by $C_c^\infty(\mathbb{R}^n)$ the space of smooth functions on $\mathbb{R}$ with compact support and by $C_c^\infty(\mathbb{R}^n)'$ the space of distributions on $\mathbb{R}^n$.

DEFINITION 1.2.8. *The quantum Hamiltonian $H_k^n : C(\mathbb{R}^n) \to C_c^\infty(\mathbb{R}^n)'$ is the map defined formally by the formula*

$$H_k^n = -\Delta + \sqrt{2}k \sum_{1\leq i\neq j\leq n} \delta(x_i - x_j), \tag{1.2.13}$$

which we interpret as

$$(H_k^n f)(\phi) = -\int_{\mathbb{R}^n} f(x)(\Delta\phi)(x)dx + \sqrt{2}k \sum_{1\leq i\neq j\leq n} \int_{L_{ij}} f(x)\phi(x)d'x$$

for all $f \in C(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\mathbb{R}^n)$.

With this definition, Proposition 1.2.6 is the following statement:

$$\langle \widehat{H} f, g\rangle = (H_k^n f)(g) \quad \text{for } f, g \in \mathcal{D}_n.$$

1.3. Generalizations to finite reflection groups

A fundamental insight by Gaudin [**26**], Gutkin and Sutherland [**36**], [**32**] was that the quantum system on the line described by the quantum Hamiltonian (1.2.13) has a natural generalization in the context of reflection groups.

Let $\mathcal{L}$ be an arbitrary collection of affine hyperplanes in a n-dimensional Euclidean space V with inner product $\langle\cdot,\cdot\rangle$. For a $L \in \mathcal{L}$ denote by s_L the orthogonal reflection in V with respect to L. Consider the group G generated by $\{s_L | L \in \mathcal{L}\}$. We may assume with out loss of generality that $\mathcal{L}$ is permuted by G. The group G is called a *reflection group* if $\mathcal{L}$ is *locally finite*, meaning that any compact set meets only finitely many $L \in \mathcal{L}$.

Any reflection group can be decomposed into the product of irreducible reflection groups. These groups have been classified by É. Cartan and Coxeter. The most important finite irreducible reflection groups are the so-called crystallographic ones. G is said to be crystallographic if it stabilizes a full lattice L in V, i.e. $gL \subseteq L$ for all $g \in G$. The crystallographic reflection groups fall into families, the *classical* types A_n ($n \geq 1$), B_n (≥ 2) , C_n ($n \geq 3$), D_n ($n \geq 4$) and the *exceptional* types[3] E_6, E_7, E_8, F_4 and G_2. The crystallographic finite reflection groups arise as the Weyl groups of complex Lie algebras [**44**].

For any reflection group G and $k \in \mathbb{R}$ (called a multiplicity function) consider the formal quantum Hamiltonian on V given by

$$H_k = -\Delta + 2k \sum_{L\in\mathcal{L}} \delta_L, \tag{1.3.1}$$

with δ_L denoting a delta-function on V supported on L and Δ denoting the Laplace operator on V. Gaudin [**26**] started the study of quantum Hamiltonian systems (1.3.1) on V for general finite reflection groups G.

[3]The subscript has the meaning of the dimension of the corresponding Euclidean space V.

EXAMPLE 1.3.1 (Type A_{n-1}: symmetric group S_n). The symmetric group S_n is an irreducible reflection group of type A_{n-1}. Consider the following (finite) collection hyperplanes of hyperplanes $\mathcal{L} = \{L_{ij}|1 \le i < j \le n\}$ (cf. Definition 1.2.5) in the subspace

$$V = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n |\, x_1 + \cdots + x_n = 0\} \tag{1.3.2}$$

of $\mathbb{R}^n$, where $L_{ij} = \{x \in V | x_i - x_j = 0\}$. Note that $s_{ij} := s_{L_{ij}}$ is the transposition that exchanges the i-the position and j-the position of $x \in V$. Then (1.3.1) takes the following form,

$$H_k = -\Delta + k \sum_{1 \le i \ne j \le n} \delta(x_i - x_j),$$

i.e. it is the Hamiltonian (cf. (1.1.1)) of a system of n quantum particles on the line with a delta-function potential with strength k between two particles. By going from $\mathbb{R}^n$ to V corresponds to describing the system in a center of mass. Wave functions invariant under the action of S_n describe n spinless quantum particles on the line $\mathbb{R}$.

For a Weyl group G of classical type, G-invariant wave functions of the quantum system described by the Hamiltonian (1.3.1) have a reasonable physical interpretation.

EXAMPLE 1.3.2 (Type B: hyper-octahedral group). Consider the following finite collection of hyperplanes in $\mathbb{R}^n$,

$$\mathcal{L} = \{\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n | x_i = 0\} | i = 1, 2, \ldots, n\}$$
$$\cup \{\{x \in \mathbb{R}^n | x_i - x_j = 0\} |\, 1 \le i < j \le n\} \cup \{\{x \in \mathbb{R}^n |\, x_i + x_j = 0\} | 1 \le i < j \le n\}.$$

The corresponding finite reflection group is crystallographic, irreducible and acts on $\mathbb{R}^n$ by permutations and arbitrary changes of signs of coordinates. It is called the *hyper-octahedral* group, is of type B_n and has order $n!2^n$. The G-invariant wave functions of the quantum Hamiltonian (1.3.1) describes in this case $2n + 1$ spinless bosons on the line with a delta-function potential that is constrained by the symmetry $x \mapsto -x$ of the line. The middle particle is located at the origin because of this constraint. Alternatively, it is the problem of n quantum spinless bosons on the half-line $[0, \infty)$ with a delta-function potential and with the particles interacting with the boundary 0 of the half-line with a delta-potential.

For finite reflection group G, Gaudin [**26**] described the general form of the wave functions invariant under G. For G of type A_{n-1} and repulsive $k > 0$ there is only purely continuous spectrum and Gaudin [**24**], [**25**] was able to proof a Plancherel formula for the wave functions. In the attracting case multi-particle binding may occur and this may give rise to lower-dimensional contributions to the spectrum, what was shown rigorously by Oxford in his thesis [**65**].

The Plancherel formula for general finite reflection group and $k \in \mathbb{R}$ was found by Heckman and Opdam [**40**]. Furthermore they realized that the underlying symmetry structures of (1.3.1) are governed by the graded Hecke algebra, an infinitesimal version of the affine Hecke algebra introduced by Lusztig [**57**] and Drinfeld [**17**]. Using Kazhdan and Lusztig's [**51**] work, they were able to relate the spectrum of the system to the Langlands parametrization of the irreducible representations of simple p-adic groups (and more general to representations of affine Hecke algebras [**39**]).

1.4. Particles on a circle with pair-wise delta-potential

The dynamical system on the circle introduced by Lieb and Liniger has been studied by physicists by different methods. Bethe [**7**] presented in 1931 a method for obtaining the exact eigenvalues and eigenvectors of the one-dimensional spin-1/2 Heisenberg model, a linear array of electrons with uniform exchange interaction between nearest neighbors. By adapting Bethe's idea, Lieb and Liniger were able to show that the spectral problem for the interacting many-particle system governed by (1.1.1) is ruled by certain system of transcendental equations, nowadays called Bethe ansatz equations. The corresponding eigenfunctions nowadays are called Bethe ansatz eigenfunctions. Fundamental progress was made by C. N. Yang and C. P. Yang in [**81**], where they evaluated the thermodynamics at finite temperature. In [**81**] they introduced the Yang-Yang action (nowadays called the master function), which enabled them to give a convenient parametrization of the solutions of the Bethe ansatz equations and the corresponding eigenfunctions via a variational problem.

Parallel to these investigation were important development in the theory of nonlinear partial differential equations (PDE's). Nonlinear PDE's which at first sight seemed intractable, could be solved explicitly by a nonlinear Fourier transformation (depending on the nonlinear PDE under question), making the nonlinear equation linear and explicitly solvable. The methods are collected under the name *Classical Inverse Scattering Method* [4] [**1**], [**21**]. One of the earlier examples of integrable nonlinear PDE's treated were the Korteweg-de Vries equation by Gardner *et. al.* in their famous paper [**22**] and the nonlinear Schrödinger equation (1.1.2) by Zakharov and Shabat [**82**].

Further developments showed that the Classical Inverse Scattering Method could be quantized to give methods to solve quantum field theories. Fundamental progress was made by the Leningrad school of Faddeev, Sklyanin [**71**], [**73**], [**74**], [**78**]. Together with the work of Baxter [**5**] on exactly solvable vertex models, this led to *Quantum Inverse Scattering Method*. The work on Quantum Inverse Scattering Method was actually a major inspiration in the development of quantum groups (this is described nicely by one of the founders of the theory in [**18**, Section 11]).

The Lieb-Liniger system can be considered as given by the formal quantum Hamiltonian (1.1.1), with the wave functions subjected to periodic boundary conditions (see also Example 1.5.1). Because the Hamiltonian (1.1.1) is equivalent to the bosonic n-particle sector of the quantum field theory governed by (1.2.9) (see Remark 1.2.7), the Lieb-Liniger system on the circle can be treated by Quantum Inverse Scattering Methods. This method yields for example the orthogonality and the norms of the Bethe ansatz eigenfunctions (see the introduction to Chapter 6 for a more detailed description).

[4]Other names used are *Inverse Scattering Transform* and *Inverse Spectral Method*.

1.5. Present work

The quantum system on an Euclidean space V defined by the formal Hamiltonian (1.3.1) is for general finite reflection groups well understood since the work of Heckman and Opdam [**40**]. One of the aims of this thesis is to understand the quantum system defined by the formal Hamiltonian (1.3.1) for affine Weyl groups. These are certain infinite reflection groups that are intimately related to affine Lie algebras [**48**]. The fundamental property of an affine Weyl group W is that it admits a presentation of the form $W = W_0 \ltimes Q^\vee$. Here W_0 is a finite Weyl group acting on the same Euclidean space V and $Q^\vee$ a W_0-stable full lattice in V acting by translations.

Gutkin and Sutherland [**36**],[**77**] started the systematic non-symmetric study of quantum system with Hamiltonian (1.3.1) for general reflection groups G. By *symmetric* we mean the study of G-invariant wave functions. In this thesis we mostly concentrate on the symmetric theory. Now for an affine Weyl group W the study of the symmetric theory is equivalent to period conditions on the system because of the presentation $W = W_0 \ltimes Q^\vee$. Equivalently, it is the study of W_0-invariant wave functions of (1.3.1) on the torus $V/Q^\vee$.

EXAMPLE 1.5.1 (The affine Weyl group $\widehat{S}_n$). Let $V \subset \mathbb{R}^n$ and $L_{ij} \subset V$ be as in Example 1.3.1. Consider the following set of affine hyperplanes

$$\mathcal{L} = \{L_{ij} |\, 1 \leq i < j \leq n\} \cup \{L_{ij,m} |\, 1 \leq i \neq j \leq n,\ \ m \in \mathbb{Z}_{>0}\},$$

where $L_{ij,m} = \{x = (x_1, \ldots, x_n) \in V | x_i - x_j + m = 0\}$. The corresponding reflection group acting on V is an affine Weyl group and is denoted by $\widehat{S}_n$ (and is sometimes called the affine symmetric group). It contains the symmetric group S_n as a normal subgroup. Moreover, $\widehat{S}_n$ admits the presentation $S_n \ltimes Q^\vee$ with $Q^\vee$ the $\mathbb{Z}$-span of the $e_i - e_j$ ($1 \leq i < j \leq n$), with $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$. The Hamiltonians (1.3.1) takes the form

$$-\Delta + k \sum_{m\in\mathbb{Z}} \sum_{1\leq i\neq j\leq n} \delta(x_i - x_j + m). \tag{1.5.1}$$

The $\widehat{S}_n$-invariant theory is essentially the study of the Lieb-Liniger system of n spinless quantum particles on the *circle* (we will say more about this in Chapter 6).

We give now a summary of the thesis chapter by chapter. Detailed information can be found in the introduction section of the chapters.

In the next chapter the main algebraic structure underlying the quantum system governed by the Hamiltonian (1.3.1) H_k for an affine Weyl group W will be introduced, an associative unital algebra H depending on W and k. It is the associated graded algebra of Cherednik's (suitably filtered) degenerate double affine Hecke algebra and contains the group algebra of W as a subalgebra. The main results is the construction of a representation Q of H on the space of U-valued smooth functions on V, with the group algebra of W acting by reflection-integral operators.

In Chapter 3, the heart of the thesis, we start with the systematic study of vector-valued quantum integrable systems with delta-potentials associated to affine Weyl groups. By vector-valued is meant that the wave functions of the quantum systems takes values in a finite-dimensional representation U of W. For $W = \widehat{S}_n$ and special choices of U, these systems describe quantum particles with spin. The formal quantum Hamiltonian (1.3.1) corresponds to the case that U is the trivial representation (we reserve the adjective *scalar* for this case). The associated spectral problem is formulated as a boundary value problem in a space of U-valued functions on V satisfying simple normal derivative jump conditions on the hyperplanes corresponding to W.

After defining U-valued commuting differential-reflection operators, we show that these Dunkl-type operators together with the the standard reflection action of the affine Weyl group define a faithful representation π of the algebra H. Gutkin's generalization of the equivalence between the impenetrable Bose-gas and the free Fermi-gas for the scalar case is generalized to the vector-valued case by defining a U-valued propagation operator T. The formal quantum Hamiltonian is interpreted as an operator on a space of U-valued function on V satisfying higher order normal derivative jump conditions on the hyperplanes corresponding to W. The algebraic complete integrability of the quantum-integrable systems is reflected by the commutativity of the Dunkl-type operators.

The propagation operator intertwines the representations π and Q. This allows us to reformulate the boundary value problem in terms of a space of U-valued *smooth* functions on V. Using this reformulation we show that, under the assumption that the spectral parameters are generic, the symmetric vector-valued theory is governed by U-valued transcendental equations (called Bethe ansatz equations). After describing the general form of the corresponding U-valued eigenfunctions, we show that for unitary representations U and positive k the spectrum of the boundary value problem is, at least generically, purely imaginary.

In Chapter 4 we show that in the scalar case and positive k the results of Chapter 3 holds without the assumption that the spectral parameters have to be generic: the spectrum of the boundary value problem is purely imaginary, regular and completely determined by the Bethe ansatz equations. The regularity conditions is called the Pauli principle since for $W = \widehat{S}_n$ (see Example 1.5.1) it implies that the momenta of the quantum bosons are pairwise different. The main ingredient in the proof of these statements is a generalization of the master function of Yang and Yang to all affine Weyl groups. The master function allows us for example to give a convenient parametrization of the solutions of the Bethe ansatz equations.

Chapter 5 deals with the functional analysis aspects of the scalar system for positive k. The relevant Hilbert space is the space of square-integrable functions on a fundamental domain for the reflection-action of W (with respect to Lebesgue measure). Using perturbation theory of unbounded self-adjoint operators, quadratic forms and Sobolev space theory, positive self-adjoint operators H_k on this Hilbert space are constructed that are associated to the formal quantum Hamiltonians (1.3.1). The completeness of the Bethe ansatz equations for general positive multiplicity functions k is reduced to the completeness at zero coupling $k \equiv 0$. The precise conditions under which such an argument works was given by Dorlas [**16**] in his proof of the completeness of the Bethe ansatz eigenfunctions of particles on the circle with repulsive pair-wise delta-function potential. The arguments of this chapter also give an independent proof of the Pauli principle.

In Chapter 6 we conjecture that the Bethe ansatz eigenfunctions are orthogonal in the Hilbert space of square-integrable functions on a fundamental domain. We also conjecture that the quadratic norms are expressible in terms of the determinant of the Hessian of the master function at the associated spectral point. Similar conjectures about the Lieb-Liniger system of quantum particles on the circle with repulsive pair-wise delta-function potential were proved by Korepin (norms) and Dorlas (orthogonality). By relating the formal quantum Hamiltonian (1.5.1) to the Lieb-Liniger system on the circle we show that these two systems are essentially equivalent. This implies that our conjectures are true for the quantum system of Example 1.5.1.

Publication details

Chapter 4 and the parts of Chapter 2 and 3 relating to the scalar case have been (for the major part) published in

E. Emsiz, E. M. Opdam and J. V. Stokman, *Periodic integrable systems with delta-potentials*, Comm. Math. Phys. **264** (2006), 191-225.

Chapter 2 and 3 are joint work with Eric Opdam and Jasper Stokman and is research in progress.

Chapter 5 will also be offered for publication.

CHAPTER 2

Fundamental representations of degenerate Hecke algebras

2.1. Introduction

In [**10**] Cherednik associates to an affine root system and a multiplicity function an associative unital algebra which he called the double affine Hecke algebra (DAHA, in short). With this algebra at hand he was able to prove the general case of the constant term conjecture of Macdonald [**58**]. This algebra extends the concept of an affine Hecke algebra. Like affine Hecke algebras, double affine Hecke algebras also admit degenerations: a trigonometric one and a rational one. The trigonometric one [**9**] is called the degenerate DAHA.

Degenerate DAHA admits a natural filtration. In our studies of integrable systems with delta potentials, certain representations of the associated graded algebra will be of fundamental importance. In this chapter we explicitly construct these representations.

Let Σ be an affine root system, considered as a subset of affine linear functions on an Euclidean vector space V. Let X be a lattice in V lying between the co-root lattice and the co-weight lattice of Σ, W_X the corresponding Weyl group, $k : \Sigma \longrightarrow \mathbb{C}$ a W_X-invariant function and U a finite dimensional representation of W_X. The associated graded of the (suitably filtered) degenerate DAHA corresponding to the Weyl group W_X will be denoted by H_k^X. The main result of this chapter (Theorem 2.5.13), states that certain integral-reflection operators together with constant coefficient partial differential operators on the space of smooth functions from V to U, defines a representation of H_k^X.

We now give a summary of every section separately. The second section is introductory and will be used throughout this chapter (and thesis). It is used mainly to fix notations. After some well-known facts on affine root systems and affine Weyl groups, the associated graded of the (suitably filtered) degenerate affine Hecke algebras H_k^X is introduced.

Section 3 is the heart of the chapter. We start by attaching to any finite dimensional representation U of W_X a representation of H_k^X on $S(V)_{\mathbb{C}} \otimes U$, with $S(V)_{\mathbb{C}}$ the symmetric algebra of the complexification $V_{\mathbb{C}}$ of V. The group algebra $\mathbb{C}[W_X]$ of W_X, considered as a subalgebra of H_k^X, acts via certain reflection-divided-difference operators (the divided-difference part is also known as BGG-operators). By dualizing, we construct a family of representations of H_k^X on $P(V)_{\mathbb{C}} \otimes U$ with $\mathbb{C}[W_X]$ acting by reflection-integral operators. Here $P(V)_{\mathbb{C}}$ is the space of complex-valued polynomial functions on V. We then show that this family contains, up to scaling isomorphisms, only two representations. Our focus will be on the representation (denoted by Q_k) which is relevant for our studies of integrable systems with delta-potentials.

In Section 4 we define another representation of H_k^X in terms of certain divided-difference operators on $\mathcal{O}(V_{\mathbb{C}}^*) \otimes U$, with $\mathcal{O}(V_{\mathbb{C}}^*)$ the space of analytic functions on $V_{\mathbb{C}}^*$. This representations has the feature that it is the dual of Q_k with respect to a natural non-degenerate pairing.

Section 5 deals with extending the representation space of the representation Q_k of H_k^X from $P(V)_{\mathbb{C}} \otimes U$ to $C^\infty(V) \otimes U$, with $C^\infty(V)$ the space of smooth functions on V. When this representations is restricted to the subalgebra $\mathbb{C}[W_X]$, we show that is possible to extend Q_k to a representation on $C(V) \otimes U$, with $C(V)$ the space of continuous functions on V and with $\mathbb{C}[W_X]$ still acting by reflection-integral operators.

2.2. Weyl groups and degenerate Hecke algebras

2.2.1. Affine root systems and affine Weyl groups. Let V be an Euclidean space of dimension n. Let Σ_0 be a finite, irreducible crystallographic root system in the dual Euclidean space V^*. We denote $\langle \cdot, \cdot \rangle$ for the inner product on V^* and $\| \cdot \|$ for the corresponding norm. The co-root of $\alpha \in \Sigma_0$ is the unique vector $\alpha^\vee \in V$ satisfying

$$\xi(\alpha^\vee) = \frac{2\langle \xi, \alpha \rangle}{\|\alpha\|^2}, \quad \forall\, \xi \in V^*.$$

We write $\Sigma_0^\vee = \{\alpha^\vee\}_{\alpha \in \Sigma_0}$ for the resulting co-root system in V. We fix a basis $I_0 = \{a_1, \ldots, a_n\}$ for the root system Σ_0. Let $\Sigma_0 = \Sigma_0^+ \cup \Sigma_0^-$ be the corresponding decomposition in positive and negative roots. We denote $\rho \in V^*$ for the half sum of positive roots and $\varphi \in \Sigma_0^+$ for the highest root with respect to the basis I_0. The highest root φ is a long root in Σ_0. We define the fundamental Weyl chamber in V^* by

$$V_+^* = \{\xi \in V^* \,|\, \xi(\alpha^\vee) > 0 \quad \forall\ \alpha \in \Sigma_0^+\}. \tag{2.2.1}$$

Let $\widehat{V}$ be the vector space of affine linear functionals on V. Then $\widehat{V} = V^* \oplus \mathbb{R}\delta$ as vector spaces, with δ the constant function $\delta(v) = 1$ ($v \in V$). The gradient map $D : \widehat{V} \to V^*$ is the projection onto V^* along this decomposition.

The subset $\Sigma = \Sigma_0 + \mathbb{Z} \subset \widehat{V}$ is the affine root system associated to Σ_0. We extend the basis I_0 of Σ_0 to a basis $I = \{a_0 = -\varphi + \delta, a_1, \ldots, a_n\}$ of the affine root system Σ. Let $\Sigma = \Sigma^+ \cup \Sigma^-$ be the corresponding decomposition in positive and negative affine roots. Observe that D maps Σ onto Σ_0. The group of affine linear isomorphisms of V is denoted by $\mathrm{Aff}(V)$.

For an affine linear functional $a \in \widehat{V}$ with $Da \neq 0$,

$$s_a(v) = v - a(v) Da^\vee, \qquad v \in V$$

defines the orthogonal reflection in the affine hyperplane $V_a := a^{-1}(0)$. The affine Weyl group W associated to Σ is the subgroup of $\mathrm{Aff}(V)$ generated by the orthogonal reflections s_a ($a \in \Sigma$). The subgroup $W_0 \subset W$ generated by the orthogonal reflections s_α ($\alpha \in \Sigma_0$) is the Weyl group associated to Σ_0. We denote w_0 for the longest Weyl group element in W_0. It is well known that W (respectively W_0) is a Coxeter group with Coxeter generators the simple reflections $s_j = s_{a_j}$ for $j = 0, \ldots, n$ (respectively s_j for $j = 1, \ldots, n$).

Let $Q^\vee = \mathbb{Z}\Sigma_0^\vee \subset V$ be the co-root lattice of Σ_0. For a $\xi \in V$ let $\tau_\xi \in \mathrm{Aff}(V)$ be the translation by ξ on V, i.e. $\tau_\xi(v) = v + \xi$ for all $v \in V$. For an $a \in \Sigma$ we have the simple but useful relation

$$s_a = s_{Da}\tau_{a(0)Da^\vee} \tag{2.2.2}$$

in W, and in particular $s_0 = \tau_{\varphi^\vee}s_\varphi = s_\varphi\tau_{-\varphi^\vee}$. Using (2.2.2) one sees that W admits a second important presentation given by

$$W \simeq W_0 \ltimes Q^\vee, \tag{2.2.3}$$

with the co-root lattice $Q^\vee$ acting by translations on V. The gradient map D induces a surjective group homomorphism $D : W \to W_0$ by $D(s_a) = s_{Da}$ for $a \in \Sigma$. Alternatively, $Dw = v$ if $v \in W_0$ is the W_0-component of w in the semi-direct product decomposition (2.2.3).

The space $\widehat{V}$ of affine linear functionals on V is a W-module by $(wf)(v) = f(w^{-1}v)$ ($w \in W, f \in \widehat{V}, v \in V$). Observe that V^* is W_0-invariant, and

$$s_\alpha(\xi) = \xi - \xi(\alpha^\vee)\alpha, \qquad \xi \in V^*$$

for roots $\alpha \in \Sigma_0$. Furthermore,

$$s_\alpha(\Sigma_0) = \Sigma_0, \qquad s_a(\Sigma) = \Sigma$$

for $\alpha \in \Sigma_0$ and $a \in \Sigma$. The length of $w \in W$ is defined by $l(w) = \#\big(\Sigma^+ \cap w^{-1}\Sigma^-\big)$. Alternatively, $l(w)$ is the minimal positive integer r such that $w \in W$ can be written as product of r simple reflections. Such an expression $w = s_{j_1}s_{j_2}\cdots s_{j_{l(w)}}$ ($j_k \in \{0, \ldots, n\}$) is called reduced.

The root lattice Q is the lattice in V^* generated by all the roots $\alpha \in \Sigma_0$; It is a full lattice because V^* is spanned by the roots in Σ_0.

The weight lattice of Σ_0 is defined by

$$P = \{\lambda \in V^* \mid \lambda(\alpha^\vee) \in \mathbb{Z} \quad \forall\, \alpha \in \Sigma_0\}.$$

Another convenient description is

$$P = \{\lambda \in V^* \mid w\lambda(\varphi^\vee) \in \mathbb{Z} \quad \forall\, w \in W_0\}, \tag{2.2.4}$$

which follows from the fact that $Q^\vee$ is already spanned over $\mathbb{Z}$ by the short co-roots in $\Sigma_0^\vee$. We denote P^+ (respectively P^{++}) for the cone of dominant (respectively strictly dominant) weights with respect to the choice Σ_0^+ of positive roots in Σ_0. Recall that $P^{++} = \rho + P^+$.

We write $V_{irreg} = \bigcup_{a\in\Sigma^+} V_a$ for the irregular vectors in V with respect to the affine root hyperplane arrangement $\{V_a \mid a \in \Sigma^+\}$. Its open, dense complement $V_{reg} := V \setminus V_{irreg}$ is called the set of regular vectors in V.

We denote $\mathcal{C}$ for the collection of connected components of V_{reg}. An element $C \in \mathcal{C}$ is called an alcove. The affine Weyl group W acts simply transitively on $\mathcal{C}$. Explicitly, $V_{reg} = \dot{\bigcup}_{w\in W} w(C_+)$ (disjoint union) with the fundamental alcove C_+ defined by

$$C_+ = \{\, v \in V \,|\, a_j(v) > 0 \ \ (j = 0, \ldots, n)\,\}.$$

We call a vector $v \in V_a$ ($a \in \Sigma^+$) sub-regular if it does not lie on any other root hyperplane V_b ($a \neq b \in \Sigma^+$).

The symmetric algebra $S(V)$ is canonically a W_0-module algebra. Using the standard identification $S(V) \simeq P(V^*)$ where $P(V^*)$ is the algebra of real-valued polynomial functions on V^*, the W_0-module structure takes the form

$$(wp)(\xi) = p(w^{-1}\xi), \qquad w \in W_0,\ \xi \in V^*,$$

on $P(V^*)$. We denote $S(V)^{W_0}$ and $P(V^*)^{W_0}$ for the subalgebra of W_0-invariants in $S(V)$ and $P(V^*)$, respectively.

For $m \in \mathbb{Z}_{\geq 0}$ we denote by $S^{(\leq m)}(V)_{\mathbb{C}}$ for the elements $q \in S(V)_{\mathbb{C}}$ of degree less or equal than m.

Let ∂_v ($v \in V$) be the derivative in direction v,

$$(\partial_v f)(u) = \frac{d}{dt}\Big|_{t=0} f(u + tv)$$

for f continuously differentiable at $u \in V$. The assignment $v \mapsto \partial_v$ uniquely extends to an algebra isomorphism of $S(V)$ onto the algebra of constant coefficient differential operators on V (say acting on $C^\infty(V)$ or $P(V)_{\mathbb{C}}$). We denote $q(\partial)$ for the constant coefficient differential operator corresponding to $q \in S(V)_{\mathbb{C}} \simeq P(V^*)_{\mathbb{C}}$.

2.2.2. Extensions of affine Weyl groups. In this subsection we consider some nontrivial group extensions of W by finite abelian groups. These extension will be parameterized by certain full lattices in V.

The lattice

$$P^\vee := \{\xi \in V \,|\, \alpha(\xi) \in \mathbb{Z}\, \forall \alpha \in \Sigma_0\}.$$

is called the *co-weight lattice* (and the elements in it are called *co-weights*). It contains $Q^\vee$ as a full sublattice. Let X be a lattice in V that lies between $Q^\vee$ and $P^\vee$. We have

$$s_\alpha(x) = x - \alpha(x)\alpha^\vee \in X \tag{2.2.5}$$

for all $\alpha \in \Sigma_0$ and $x \in X$ because $\alpha(x) \in \mathbb{Z}$. Formula (2.2.5) shows that X is W_0-invariant. The group $X/Q^\vee$ is finite since X and $Q^\vee$ are both full lattices in V. Since X and $Q^\vee$ are W_0-invariant we can define an action of the finite Weyl group W_0 on $X/Q^\vee$ in the obvious way. By (2.2.5) it follows actually that W_0 acts trivially on $X/Q^\vee$.

We attach to X its dual lattice Y in V^*:

$$Y := \{\mu \in V^* \,|\, \mu(X) \subseteq \mathbb{Z}\}.$$

Note that $Q \subseteq Y \subseteq P$. Let W_X be the $W_0 \ltimes X$, i.e. the set $W_0 \times X$ with the group operation given by

$$(w', x') \cdot (w, x) = (w'w, w^{-1}(x') + x) \quad (w, w' \in W_0, x, x' \in X).$$

The group W_X is called the *affine Weyl group associated to* X and acts faithfully on V by affine linear isomorphisms via

$$(w,x)v=(w\tau_x)(v)\quad ((w,x)\in W_X, v\in V).$$

By the presentation (2.2.3) we have a canonical isomorphism $W\simeq W_{Q^\vee}$. The group $W_{P^\vee}$ is also denoted by W^e. Some authors (e.g. [**59**]) call W^e also *the extended affine Weyl group*.

The group W_X is an extension of W by a finite abelian group Ω_X:

$$W_X=W\rtimes\Omega_X,$$

where

$$\Omega_X:=\{w\in W_X\,|\,w(C_+)=C_+\}.$$

We skip the subscript X when X equals $P^\vee$ and write Ω instead of $\Omega_{P^\vee}$. Note that Ω_X is a subgroup of Ω. The group Ω_X depends on the alcove C_+. Another choice of a fundamental alcove leads to a W-conjugate of Ω_X.

Observe that a $\omega\in\Omega$ permutes the simple affine roots I, because $\omega(C_+)=C_+$.

2.2.3. Representations of affine Weyl groups. If A and B are two sets, then the set of all maps from A to B is denoted by $\mathrm{Fun}(A,B)$. [If B is a complex vector space, then $\mathrm{Fun}(A,B)$ becomes a complex vector space in the obvious way]. We also write $\mathrm{Fun}(A)$ for $\mathrm{Fun}(A,\mathbb{C})$.

Fix a lattice X between $Q^\vee$ and $P^\vee$. Let U be a finite dimensional representation U of the affine Weyl group W_X.

The natural isomorphism $\mathrm{Fun}(V,U)\simeq\mathrm{Fun}(V)\otimes U$ will be used implicitly to identify these two vector spaces. Unless stated otherwise, we will use the convention that

$$Z(V,U)=Z(V)\otimes U \tag{2.2.6}$$

for any function space $Z(V)$ on V, and $Z(V,U)$ will also be considered as a subspace of $\mathrm{Fun}(V,U)\simeq\mathrm{Fun}(V)\otimes U$ in the obvious way.

For a $f\in\mathrm{Fun}(V,U)$, $w\in W_X$ define $f^w,\pi(w)f\in\mathrm{Fun}(V,U)$ by

$$(f^w)(v)=f(w^{-1}v)\quad (w\in W_X, v\in V), \tag{2.2.7}$$

and

$$(\pi(w)f)(v)=w(f(w^{-1}v))=w(f^w(v))\quad (v\in V). \tag{2.2.8}$$

Both (2.2.7) and (2.2.8) define representation of W_X on $\mathrm{Fun}(V,U)$, with W_X-submodules $P(V,U)$, $C^\infty(V,U)$ and $C(V,U)$. We will use the notation M_π to indicate that the subspace $M\subset\mathrm{Fun}(V,U)$ is a W_X-submodule with respect to the π-action.

Since W^e permutes the affine roots, the set of regular vectors V_{reg} is invariant under W^e, i.e. $w(V_{reg})\subseteq V_{reg}$ for all $w\in W^e$. Hence the usual action (2.2.7) and the π-action (2.2.8) also define an action on $\mathrm{Fun}(V_{reg},U)$. We have

$$\pi(w)(f\otimes u)=f^w\otimes wu\quad (f\otimes u\in\mathrm{Fun}(V)\otimes U).$$

Consider also the $W_X\times W_X$-action defined by

$$(w\otimes w')(f\otimes u)=f^w\otimes w'u\quad (w,w'\in W_X, f\in\mathrm{Fun}(V)),$$

or equivalently

$$((w \otimes w')f)(v) = w'(f(w^{-1}v)) \quad (w_1, w_2 \in W_X, f \in \mathrm{Fun}(V) \otimes U).$$

The restriction to the diagonal $W_X \subset W_X \times W_X$ (i.e. $w \mapsto (w, w)$) is then precisely π.

2.2.4. Associated graded of degenerated double affine Hecke algebras. The algebras we consider are parametrized by what is called a multiplicity function. From the viewpoint of physics, these will play the role of coupling constants between particles in the integrable systems we consider.

DEFINITION 2.2.1. *A multiplicity function k is a W^e-invariant function $k : \Sigma \longrightarrow \mathbb{C}$. The space of all multiplicity functions is denoted by $K_{\mathbb{C}}$. We write also k_a for the value of $k \in K_{\mathbb{C}}$ at the root $a \in \Sigma$.*

If k is a multiplicity function, then it satisfies $k(a) = k(Da)$ for all $a \in \Sigma$.

In this chapter we always write k for a multiplicity function, unless stated otherwise.

We are now able to introduce the fundamental algebraic structure of this thesis.

THEOREM 2.2.2. *There exists a unique complex unital associative algebra $H_k^e = H_k^e(\Sigma)$ satisfying*

- **(a)** *$H_k^e = S(V)_{\mathbb{C}} \otimes \mathbb{C}[W^e]$ as vector space, with $\mathbb{C}[W^e]$ the group algebra of W^e.*
- **(b)** *The maps $q \mapsto q \otimes e$ and $w \mapsto 1 \otimes w$, with $e \in W^e$ the unit element of W^e, are algebra embeddings of $S(V)_{\mathbb{C}}$ and $\mathbb{C}[W^e]$ into H_k^e.*
- **(c)** *The cross relations*
$$s_a \cdot v - (s_{Da}v) \cdot s_a = k_a Da(v)$$
holds in H_k^e for $a \in I$ and $v \in V \subset S(V)_{\mathbb{C}}$. Here we have identified $S(V)_{\mathbb{C}}$ and $\mathbb{C}[W^e]$ with their images in H_k^e through the algebra embeddings of **(b)**.
- **(d)** *The cross relations*
$$\omega \cdot v = (D\omega(v)) \cdot \omega$$
hold in H_k^e for all $\omega \in \Omega$ and $v \in V$.

REMARK 2.2.3. If the values k_a of the multiplicity function k are considered to be independent central variables in the definition of H_k^e (we then use the adjective *generic*), then H_k^e is graded by imposing the degree of $w \in W^e$ to be zero and the degrees of $v \in V$ and k_a to be one. As graded algebra, the generic H_k^e is the associated graded of Cherednik's [**12**] degenerate extended double affine Hecke algebra $\mathbb{H}_k^e$, considered as filtered algebra by the same degree function (the only difference in the definition of $\mathbb{H}_k^e$ are the cross relations (see Theorem 2.2.2**(c)** and **(d)**), which are now of the form

$$s_a \cdot v - v^{s_a} \cdot s_a = k_a Da(v) \qquad (a \in I, v \in V)$$

and

$$\omega \cdot v = v^{\omega} \cdot \omega \qquad (\omega \in \Omega, v \in V). \tag{2.2.9}$$

The W^e-module structure on $S(V)_{\mathbb{C}}$ in this relation comes from the action (2.2.7) of W^e on $P(V)_{\mathbb{C}}$ as functions on V and the identification $P(V)_{\mathbb{C}} \simeq S(V)_{\mathbb{C}}$ via the inner product $\langle \cdot, \cdot \rangle$. Then $v^{s_0} = s_{\varphi}(v) + 2\|\varphi\|^{-2}\varphi(v)\delta$.

In Section 3.4 a proof of Theorem 2.2.2 will be given. The proof is based on a realization of H_k^e as differential-reflection (Dunkl-type) operators on V_{reg} and makes no use of $\mathbb{H}_k^e$.

DEFINITION 2.2.4. *We denote by H_k the subalgebra of H_k^e generated by W and V. For a lattice $X \subset V$ between $Q^\vee$ and $P^\vee$, we denote by H_k^X the subalgebra of H_k^e generated by W_X and V. By $H_k^{(0)}$ we denote the subalgebra of H_k^e generated by W_0 and V. All these subalgebras admit similar descriptions as H_k^e in Theorem 2.2.2.*

REMARK 2.2.5. **(1)** The algebra $H_k^{(0)}$ is isomorphic to the graded Hecke algebra (also called the degenerate affine Hecke algebra) and was introduced independently by Lusztig [**57**] and Drinfeld [**17**].
(2) We have the following filtration

$$H_k^{e,\leq 0} \subset H_k^{e,\leq 1} \subset \dots$$

on H_k^e, where

$$H^{e,\leq i} := S^{(\leq i)}(V)_{\mathbb{C}} \otimes \mathbb{C}[W^e].$$

The associated graded algebra with respect to this filtration is the cross-product $S(V)_{\mathbb{C}} \rtimes \mathbb{C}[W^e]$, and hence independent of k. Note that we are not in the setting of **(i)**, for the k_a are complex values and not formal parameters, and hence of degree zero.

For an $\alpha \in \Sigma_0$ consider the BGG-operators

$$\Delta_\alpha : S(V)_{\mathbb{C}} \longrightarrow S(V)_{\mathbb{C}}, \text{ defined by } \Delta_\alpha(q) = \frac{q - q^{s_\alpha}}{\alpha^\vee}. \tag{2.2.10}$$

Here BGG stands for Bernstein-Bernstein-Gelfand. Some authors (e.g. [**55**]) call Δ_α ($\alpha \in \Sigma_0$) *divided-difference operators*, others call them (e.g. [**13**]) *Lusztig-Demazure operators*. These operators were first introduced in [**6**], and independently in [**14**].

LEMMA 2.2.6. *In H_k^e we have*

$$s_a \cdot q - q^{s_{Da}} \cdot s_a = k_a \Delta_{Da}(q) \tag{2.2.11}$$

for $a \in I$ and $q \in S(V)_{\mathbb{C}}$, and

$$\omega \cdot q = q^{D\omega} \cdot \omega \tag{2.2.12}$$

for $\omega \in \Omega$ and $q \in S(V)_{\mathbb{C}}$.

PROOF. This follows from induction on the degree of q and the cross relations (**(b)**, **(c)** and **(d)** in Theorem 2.2.2) in H_k^X. □

It is known (for a proof see [**4**, Proposition 1.3.6(i)]) that the generic degenerate extended double affine Hecke algebra $\mathbb{H}_k^e$ has trivial center $\mathbb{C}[k]$, the polynomial subring in $\mathbb{H}_k^e$ generated by the central independent variables k_a. But going over to the associated graded algebra this changes dramatically:

PROPOSITION 2.2.7. *The center $\mathcal{Z}(H_k^e)$ of H_k^e contains $S(V)_{\mathbb{C}}^{W_0}$. In particular $\mathcal{Z}(H_k)$ and $\mathcal{Z}(H_k^{(0)})$ contain $S(V)_{\mathbb{C}}^{W_0}$.*

PROOF. This follows from the cross relations (2.2.11) and (2.2.12) in H_k^e. □

REMARK 2.2.8. In the case of the subalgebra $H_k^{(0)}$, we can say more about the center, namely:

$$\mathcal{Z}(H_k^{(0)}) = S(V)_{\mathbb{C}}^{W_0}.$$

For a proof see [**57**, Prop. 4.5].

2.3. The fundamental representations

We fix a lattice X between $Q^\vee$ and $P^\vee$. In this section we attach to any finite dimensional W_X-module U a H_k^X-module. These H_k^X-modules are defined in terms of vector-valued reflection-integral operators. The main result of this section is Theorem 2.3.7.

We fix a finite dimensional representation U of W_X (or equivalently, a $\mathbb{C}[W_X]$-module). By seeing $\mathbb{C}[W_X]$ as a subalgebra of H_k^X, we may form the induced module

$$M_U = \mathrm{Ind}_{\mathbb{C}[W_X]}^{H_k^X} U = H_k^X \otimes_{\mathbb{C}[W_X]} U.$$

By definition the H_k^X-module structure on M_U is giving by

$$b(a \otimes_{\mathbb{C}[W_X]} u) := (ba) \otimes_{\mathbb{C}[W_X]} u \qquad (a, b \in H_k^X, u \in U).$$

The assignment

$$q \otimes u \mapsto q \otimes_{\mathbb{C}[W_X]} u \quad (q \in S(V)_{\mathbb{C}}, u \in U)$$

defines a vector space isomorphism

$$S(V)_{\mathbb{C}} \otimes U \xrightarrow{\sim} H_k^X \otimes_{\mathbb{C}[W_X]} U. \tag{2.3.1}$$

Transporting the H_k^X-module structure from M_U to $S(V)_{\mathbb{C}} \otimes U$ under this isomorphism gives:

PROPOSITION 2.3.1. *The following uniquely defines an H_k^X-module structure (denoted by δ_k) on $S(V)_{\mathbb{C}} \otimes U$:*

$$\begin{aligned}
\delta_k(v)(q \otimes u) &= vq \otimes u \quad (v \in V),\\
\delta_k(s_a)(q \otimes u) &= q^{s_{Da}} \otimes s_a u + k_a \Delta_{Da}(q) \otimes u \quad (a \in I),\\
\delta_k(\omega)(q \otimes u) &= q^{D\omega} \otimes \omega u \quad (\omega \in \Omega_X)
\end{aligned}$$

for $q \otimes u \in S(V)_{\mathbb{C}} \otimes U$. In particular $S(V)_{\mathbb{C}}$ acts on $S(V)_{\mathbb{C}} \otimes U$ by multiplication on the first factor.

We call the H_k^X-representation δ_k of Proposition 2.3.1 the *vector-valued BGG-representation* (because of the BGG-operators Δ_{Da} entering in the description).

PROOF. Let $v \in V$, $a \in I$, $\omega \in \Omega_X$, $q \in S(V)_{\mathbb{C}}$ and $u \in U$. Consider $q \otimes_{\mathbb{C}[W_X]} u \in H_k^X \otimes_{\mathbb{C}[W_X]} U$ with $q \in S(V)_{\mathbb{C}}$ and $u \in U$. Then we have

$$\begin{aligned}
s_a(q \otimes_{\mathbb{C}[W_X]} u) &= (s_a q) \otimes_{\mathbb{C}[W_X]} u = (q^{s_{Da}} s_a + k_a \Delta_{Da}(q)) \otimes_{\mathbb{C}[W_X]} u\\
&= q^{s_{Da}} \otimes_{\mathbb{C}[W_X]} s_a u + k_a \Delta_{Da}(q) \otimes_{\mathbb{C}[W_X]} u
\end{aligned}$$

as identities in $H_k^X \otimes_{\mathbb{C}[W_X]} U$. The first and third equality are a consequence of the definition of an induced module, and the second equality follows from the cross relations (2.2.11) in H_k^X. In a similar way we get, $v(q \otimes_{\mathbb{C}[W_X]} u) = (vq) \otimes_{\mathbb{C}[W_X]} u$, and, using the cross relations (2.2.12), that $\omega(q \otimes_{\mathbb{C}[W_X]} u) = q^{D\omega} \otimes_{\mathbb{C}[W_X]} \omega u$, as identities in $H_k^X \otimes_{\mathbb{C}[W_X]} U$. The results now follows immediately if one uses the isomorphism (2.3.1). □

From now on we denote by M_U the H_k^X-module $(S(V)_{\mathbb{C}} \otimes U, \delta_k)$.

REMARK 2.3.2. Restricting the H_k^X-representation δ_k to the subalgebra $\mathbb{C}[W_0] \subset H_k^X$ gives a representation of $\mathbb{C}[W_0]$. For U the trivial representation this representation was observed by Gutkin (see [**34**, Theorem 4.2]). The proof given in [**34**] is different from the proof given here (the concept of a degenerate affine Hecke algebra is not considered in that article).

LEMMA 2.3.3. *The assignment $w \mapsto w^\dagger := w^{-1}$ ($w \in W_X$) and $v \mapsto v^\dagger := v$ ($v \in V$) uniquely defines a unit preserving, linear anti-involution $^\dagger : H_k^X \to H_k^X$.*

PROOF. One easily checks that the defining relations of H_k^X are preserved (see Theorem 2.2.2). □

The anti-involution † allows use to define a H_k^X-module structure on the full dual $M_U^* = (S(V)_{\mathbb{C}} \otimes U)^* \simeq S(V)_{\mathbb{C}}^* \otimes U^*$ by

$$(hm^*)(m) = m^*(h^\dagger m) \quad (h \in H_k^X, m \in M_U)$$

for $m^* \in M_U^*$.

For an affine linear $a \in \widehat{V}$ with $Da \neq 0$, we now consider an integral operator $I(a) : C(V) \longrightarrow C(V)$. The operator $I(a)$ ($a \in \widehat{V}$, $Da \neq 0$) was introduced by Gutkin in [**32**] to study integrable systems with delta potentials, and is defined by the formula

$$(I(a)(f))(v) := \int_0^{a(v)} f(v - tDa^\vee)dt \tag{2.3.2}$$

for $f \in C(V)$ and $v \in V$.

Consider the following non-degenerate pairing $\langle \cdot, \cdot \rangle : S(V)_{\mathbb{C}} \times P(V)_{\mathbb{C}} \longrightarrow \mathbb{C}$, defined by

$$\langle q, p \rangle = (q(\partial)p)(0). \tag{2.3.3}$$

We collect in the following lemma some properties of the pairing (2.3.3) for later use. Note in particular that Lemma 2.3.4.**(iii)** states that $I(\alpha)$ ($\alpha \in \Sigma_0$) is the adjoint to Δ_α with respect to the pairing $\langle \cdot, \cdot \rangle$.

LEMMA 2.3.4. **(i)** *The operator $I(a)$ ($a \in \widehat{V}$, $Da \neq 0$) is a linear endomorphism of $P(V)_{\mathbb{C}}$.*
(ii) *We have $\langle qq', p \rangle = \langle q, q'(\partial)p \rangle$ for $q, q' \in S(V)_{\mathbb{C}}$ and $p \in P(V)_{\mathbb{C}}$.*
(iii) *We have $\langle \Delta_\alpha(q), p \rangle = \langle q, I(\alpha)(p) \rangle$ for $\alpha \in \Sigma_0$, $q \in S(V)_{\mathbb{C}}$ and $p \in P(V)_{\mathbb{C}}$.*
(iv) *We have $\langle q^w, p \rangle = \langle q, p^{w^{-1}} \rangle$ for $w \in W_0$, $q \in S(V)_{\mathbb{C}}$ and $p \in P(V)_{\mathbb{C}}$.*

PROOF. **(i)** This follows from

$$I(a)(\partial_{Da^\vee}(f)) = f - f^{s_a} \qquad (f \in C^\infty(V)), \tag{2.3.4}$$

(recall that $\partial_{Da^\vee}$ is the derivative in direction $Da^\vee$) and (2.3.4) follows from an application of the fundamental theorem of calculus.
(ii) This follows from

$$\langle qq', p\rangle = ((qq')(\partial))(p)(0) = (q(\partial)(q'(\partial)p))(0) = \langle q, q'(\partial)p\rangle.$$

(iii) Choose a $p' \in P(V)_{\mathbb{C}}$ with $p = \partial_{\alpha^\vee}(p')$. Then

$$\begin{aligned}\langle \Delta_\alpha(q), p\rangle &= \langle \Delta_\alpha(q), \partial_{\alpha^\vee}(p')\rangle = \langle \alpha^\vee \Delta_\alpha(q), p'\rangle \\ &= \langle q - q^{s_\alpha}, p'\rangle = \langle q, p' - (p')^{s_\alpha}\rangle.\end{aligned}$$

Also

$$\langle q, I(\alpha)(p)\rangle = \langle q, I(\alpha)(\partial_{\alpha^\vee} p')\rangle = \langle q, p' - (p')^{s_\alpha}\rangle,$$

and therefore $\langle \Delta_\alpha(q), p\rangle = \langle q, I(\alpha)(p)\rangle$.
(iv) The proof is with induction on $\deg(q)$. It is certainly true for $\deg(q) = 0$, i.e. for q equal to a constant. The induction step follows from

$$\langle (vq)^w, p\rangle = \langle q^w, \partial_{wv} p\rangle = \langle q, (\partial_{wv} p)^{w^{-1}}\rangle = \langle q, \partial_v p^{w^{-1}}\rangle = \langle vq, p^{w^{-1}}\rangle,$$

with the third equality following from $w\partial_v = \partial_{wv} w$ (as operators on $P(V)_{\mathbb{C}}$). □

We can now state and prove one of the main results of this section.

PROPOSITION 2.3.5. *The following defines uniquely a representation L_k of H_k^X on $P(V)_{\mathbb{C}} \otimes U^*$:*

$$\begin{aligned} L_k(v)(p \otimes \psi) &= (\partial_v p) \otimes \psi \quad (v \in V), \\ L_k(s_a)(p \otimes \psi) &= p^{s_{Da}} \otimes s_a \psi + k_a I(Da)(p) \otimes \psi \quad (a \in I), \\ L_k(\omega)(p \otimes \psi) &= p^{D\omega} \otimes \omega\psi \quad (\omega \in \Omega_X) \end{aligned}$$

for $p \otimes \psi \in P(V)_{\mathbb{C}} \otimes U^*$.

PROOF. Consider the embedding

$$\varrho : P(V)_{\mathbb{C}} \hookrightarrow S(V)^*_{\mathbb{C}},$$

defined by sending an p to $\langle \cdot, p\rangle$. From Lemma 2.3.4 follows immediately

$$\begin{aligned} \varrho(p) \circ w^{-1} &= \varrho(p^w) \qquad (w \in W_0), \\ \varrho(p) \circ \Delta_\alpha &= \varrho(I(\alpha)(p)) \qquad (\alpha \in \Sigma_0) \\ \varrho(p) \circ (v\cdot) &= \varrho(\partial_v(p)) \qquad (v \in V), \end{aligned} \tag{2.3.5}$$

for all $p \in P(V)_{\mathbb{C}}$. Here $v\cdot$ denotes the multiplication operator by v on $S(V)_{\mathbb{C}}$. Recall the H_k^X-module $M_U = S(V)_{\mathbb{C}} \otimes U$ (see Proposition 2.3.1). Proposition 2.3.1 together with (2.3.5) shows that

$$\varrho(P(V)_{\mathbb{C}}) \otimes U^* \subseteq M_U^*$$

is a H_k^X-submodule. Furthermore, (2.3.5) also shows that the pull back of this H_k^X-module structure on $\varrho(P(V)_{\mathbb{C}}) \otimes U^*$ to $P(V)_{\mathbb{C}} \otimes U^*$ via the linear isomorphism

$$\varrho \otimes \mathrm{id}_{U^*} : P(V)_{\mathbb{C}} \otimes U^* \xrightarrow{\sim} \varrho(P(V)_{\mathbb{C}}) \otimes U^*$$

yields the representation L_k. □

REMARK 2.3.6. Every representation U of W_0 can be extended to a representation of W_X in such a way that the lattice part X in $W_X = W_0 \ltimes X$ acts trivial on U. Restricting the corresponding representation δ_k of Proposition 2.3.1 to the subalgebra $H_k^{(0)}$ of H_k^X gives the finite type Hecke algebra $H_k^{(0)}$ version of this proposition.

We extend the pairing $\langle \cdot, \cdot \rangle : S(V)_{\mathbb{C}} \times P(V)_{\mathbb{C}} \longrightarrow \mathbb{C}$ to a pairing

$$\langle \cdot, \cdot \rangle : (S(V)_{\mathbb{C}} \otimes U) \times (P(V)_{\mathbb{C}} \otimes U^*) \longrightarrow \mathbb{C}, \tag{2.3.6}$$

by $\langle q \otimes u, p \otimes \psi \rangle = (q(\partial)p)(0)\psi(u)$. Although the following duality is by construction, it is useful to state it explicitly for later reference:

$$\langle \delta_k(h^\dagger)(q \otimes u), p \otimes \psi \rangle = \langle q \otimes u, L_k(h)(p \otimes \psi) \rangle \quad (h \in H_k^X) \tag{2.3.7}$$

for $q \otimes u \in S(V)_{\mathbb{C}} \otimes U$ and $p \otimes \psi \in P(V)_{\mathbb{C}} \otimes U^*$.

In Proposition 2.3.5 we defined a representation L_k of H_k^X on $P(V)_{\mathbb{C}} \otimes U^*$. In the following theorem we define a different representation of H_k^X on the same representation space $P(V) \otimes U^*$.

THEOREM 2.3.7. *The following defines uniquely a representation Q_k of H_k^X on $P(V)_{\mathbb{C}} \otimes U^*$:*

$$\begin{aligned} Q_k(v)(p \otimes \psi) &= (\partial_v p) \otimes \psi \quad (v \in V), \\ Q_k(s_a)(p \otimes \psi) &= p^{s_a} \otimes s_a\psi + k_a I(a)(p) \otimes \psi \quad (a \in I), \\ Q_k(\omega)(p \otimes \psi) &= p^\omega \otimes \omega\psi \quad (\omega \in \Omega_X) \end{aligned}$$

for $p \otimes \psi \in P(V)_{\mathbb{C}} \otimes U^*$.

REMARK 2.3.8. Restricting the H_k^X-representation Q_k to the subalgebra $\mathbb{C}[W]$ gives a representation of $\mathbb{C}[W]$. For U the trivial representation this was shown by Gutkin [**32**].

The topological versions (see Theorems 2.5.12 and 2.5.13) of the above theorem will be of great importance in the next chapters, when studying integrable systems with delta-potentials.

The representation Q_k of H_k^X of Theorem 2.3.7 will be part of a general construction. This will be discussed now.

We restrict ourself for the moment to the special case $X = Q^\vee$. We construct a family of representations L_k^g of H_k on $P(V) \otimes U^*$, parametrized by $g \in \mathrm{Fun}(I, V)$. For $0 \in \mathrm{Fun}(I, V)$, the map that sends an $a \in I$ to $0 \in V$, we will recover L_k, i.e. $L_k^0 = L_k$.

The group $\mathrm{Aff}(V)$ of affine linear isomorphisms of V acts on $P(V)_{\mathbb{C}}$ via

$$p^A(v) = p(A^{-1}v) \quad (p \in P(V)_{\mathbb{C}}, A \in \mathrm{Aff}(V), v \in V)$$

(compare with (2.2.7)). Note that under this action $\widehat{V} \subset P(V)_{\mathbb{C}}$ is an invariant subspace.

For $v \in V$ consider the linear isomorphism $\tau_v \otimes \mathrm{id}_{U^*}$ of $P(V)_{\mathbb{C}} \otimes U^*$, acting as translation on the first factor:

$$(\tau_v \otimes \mathrm{id}_{U^*})(p \otimes \psi) = p^{\tau_v} \otimes \psi = p(\cdot - v) \otimes \psi$$

for $p \otimes \psi \in P(V)_{\mathbb{C}} \otimes U^*$. For a $v \in V$ and $a \in I$ consider the conjugation

$$L_k^v(s_a) := (\tau_v \otimes \mathrm{id}_{U^*}) L_k(s_a)(\tau_{-v} \otimes \mathrm{id}_{U^*})$$

of $L_k(s_a)$ by $\tau_v \otimes \mathrm{id}_{U^*}$.

LEMMA 2.3.9. *The following holds*

$$L_k^v(s_a)(p \otimes \psi) = p^{s_{Da - Da(v)\delta}} \otimes s_a\psi + k_a I(Da - Da(v)\delta)(p) \otimes \psi \quad (a \in I)$$

for $p \in P(V)_{\mathbb{C}}$ and $\psi \in U^*$. Hence $L_k^v(s_a)$ only depends on a and on $Da(v)$.

PROOF. Let $A \in \mathrm{Aff}(V)$ and $b \in \widehat{V}$, $Db \neq 0$. Then $As_bA^{-1} = s_{b^A}$ (as operators on $P(V)_{\mathbb{C}}$). Furthermore, we have the intertwining property $AI(b)A^{-1} = I(b^A)$ as operators on $P(V)_{\mathbb{C}}$ (see [**32**, Lemma 2.1(i)]). The lemma follows from these two results applied to $A = \tau_v$ and $b = Da$, and using $(Da)^{\tau_v} = Da - Da(v)\delta$. □

COROLLARY 2.3.10. *Let $v \in V$ with $\varphi(v) = 1$. Then*

$$L_k^v(s_0)(p \otimes \psi) = p^{s_0} \otimes s_0\psi + k_0 I(a_0)(p) \otimes \psi$$

for $p \otimes \psi \in P(V)_{\mathbb{C}} \otimes U^*$.

PROPOSITION 2.3.11. *Let $g \in \mathrm{Fun}(I, V)$. Then the assignment $s_a \mapsto L_k^{g(a)}(s_a)$ ($a \in I$) and $v \mapsto \partial_v \otimes \mathrm{id}_{U^*}$ ($v \in V$) defines uniquely a representation L_k^g of H_k on $P(V)_{\mathbb{C}} \otimes U^*$.*

PROOF. We divide the proof in four steps.
(a) The restriction to the subalgebra $S(V)_{\mathbb{C}}$ is a representation. This is trivial.
(b) The cross relation

$$L_k^{g(a)}(s_a)(\partial_v \otimes \mathrm{id}_{U^*}) - (\partial_{s_{Da}v} \otimes \mathrm{id}_{U^*}) L_k^{g(a)}(s_a) = k_a Da(v) \quad (a \in I, v \in V)$$

follows from the cross relation between $L_k(s_a)$ and $\partial_v \otimes \mathrm{id}_{U^*}$, and the fact that differentiation commutes with translation.
(c) The quadratic relation is obvious:

$$\left(L_k^{g(a)}(s_a)\right)^2 = (\tau_{g(a)} \otimes \mathrm{id}_{U^*}) L_k(s_a)^2 (\tau_{-g(a)} \otimes \mathrm{id}_{U^*}) = 1.$$

(d) The braid relations. We may assume that the rank of Σ_0 ($= \dim V^*$) is greater then one, since there are no braid relations to check when the rank of Σ_0 is one. Choose $a, b \in I$, $a \neq b$ and denote by $m_{ab} \in \mathbb{N}$ the order of s_as_b in W. Choose a $x \in V$ such that

$Da(x) = Da(g(a))$ and $Db(x) = Db(g(b))$ (possible since the rank of Σ_0 is greater than one). By Lemma 2.3.9 we have $L_k^x(s_a) = L_k^{g(a)}(s_a)$ and $L_k^x(s_b) = L_k^{g(b)}(s_b)$. Hence

$$\begin{aligned}\left(L_k^{g(a)}(s_a)L_k^{g(b)}(s_b)\right)^{m_{ab}} &= \left(L_k^x(s_a)L_k^x(s_b)\right)^{m_{ab}} \\ &= (\tau_x \otimes \mathrm{id}_{U^*})\left(L_k(s_a)L_k(s_b)\right)^{m_{ab}}(\tau_{-x} \otimes \mathrm{id}_{U^*}) \\ &= 1,\end{aligned}$$

the last equality following from the braid relation between $L_k(s_a)$ and $L_k(s_b)$. □

Let $g_0 \in \mathrm{Fun}(I, V)$ be the map defined by

$$g_0(a) = \begin{cases} \frac{\varphi^\vee}{2} & \text{if } a = a_0, \\ 0 & \text{if } a \in I_0. \end{cases}$$

COROLLARY 2.3.12. *The following defines uniquely a representation Q_k of H_k on $P(V)_{\mathbb{C}} \otimes U^*$:*

$$\begin{aligned} Q_k(v)(p \otimes \psi) &= (\partial_v p) \otimes \psi \quad (v \in V), \\ Q_k(s_a)(p \otimes \psi) &= p^{s_a} \otimes s_a\psi + k_a I(a)(p) \otimes \psi \quad (a \in I) \end{aligned}$$

for $p \otimes \psi \in P(V)_{\mathbb{C}} \otimes U^*$.

PROOF. Applying Proposition 2.3.11 with $g = g_0$ and use Corollary 2.3.10. □

For $v \in V$ and $g \in \mathrm{Fun}(I, V)$ we denote by $g + v \in \mathrm{Fun}(I, V)$ the function $(g + v)(a) = g(a) + v$.

PROPOSITION 2.3.13. *The representations $(P(V)_{\mathbb{C}} \otimes U^*, L_k^g)$ and $(P(V)_{\mathbb{C}} \otimes U^*, L_k^{g+v})$ of H_k are equivalent for $g \in \mathrm{Fun}(I, V)$ and $v \in V$.*

PROOF. First observe that

$$(\tau_v \otimes \mathrm{id}_{U^*})L_k^{v'-v}(s_a) = L_k^{v'}(s_a)(\tau_v \otimes \mathrm{id}_{U^*}) \quad (v, v' \in V \text{ and } a \in I).$$

Since $\tau_v \otimes \mathrm{id}_{U^*} : P(V) \otimes U^* \to P(V) \otimes U^*$ is also a linear isomorphism, the proposition follows. □

For a $t \in \mathbb{R}$ and $g \in \mathrm{Fun}(I, V)$ denote by $tg \in \mathrm{Fun}(I, V)$ the map $(tg)(a) = tg(a)$. Proposition 2.3.13 tells us that every representation L_k^g is equivalent to a representation $L_k^{tg_0}$ for some $t \in \mathbb{R}$. This reduces the $(n + 1)$-parameter family of representations $\{L_k^g \mid g \in \mathrm{Fun}(I, V)\}$ to a 1-parameter family of representations $\{L_k^{tg_0} \mid t \in \mathbb{R}\}$ of H_k.

Although the 1-parameter family of representations $\{L_k^{tg_0} | t \in \mathbb{R}^*\}$ do not have to be equivalent in the strict sense, they are *scaling equivalent*. To make this statement precise we introduce for a $t \in \mathbb{R}^*$ the scaling map $S_t : P(V)_{\mathbb{C}} \longrightarrow P(V)_{\mathbb{C}}$, defined by

$$(S_t(p))(v) = p(tv) \qquad (p \in P(V)_{\mathbb{C}})$$

for $v \in V$.

LEMMA 2.3.14. *Let $t \in \mathbb{R}^*$. The assignments $v \mapsto v/t$ ($v \in V$) and $s_a \mapsto s_a$ ($a \in I$) uniquely defines a unit preserving algebra morphism Λ_t from H_k to H_{tk}.*

PROPOSITION 2.3.15. *Let $t \in \mathbb{R}^*$ and $g \in \mathrm{Fun}(I, V)$. Then we have*

$$L_{tk}^{g/t}(\Lambda_t(h))(S_t \otimes \mathrm{id}_{U^*}) = (S_t \otimes \mathrm{id}_{U^*})L_k^g(h) \quad (h \in H_k). \tag{2.3.8}$$

In particular the representations $(P(V)_{\mathbb{C}} \otimes U^*, L_{tk}^{g/t} \circ \Lambda_t)$ and $(P(V)_{\mathbb{C}} \otimes U^*, L_k^g)$ of H_k are equivalent.

PROOF. Note that it suffices to show (2.3.8) for $h = v$ ($v \in V$) and $h = s_a$ ($a \in I$). For $h = v$ this follows without difficulty because $L_k^g(v) = \partial_v \otimes \mathrm{id}_{U^*}$. For $h = s_a$ ($a \in I$) equation (2.3.8) is equivalent with:

$$s_{Da - Da(g(a)/t)\delta} S_t = S_t s_{Da - Da(g(a))\delta}, \text{ and} \tag{2.3.9}$$

$$tI(Da - Da(g(a)/t)\delta)S_t = S_t I(Da - Da(g(a))\delta) \tag{2.3.10}$$

as operators on $P(V)_{\mathbb{C}}$. Identity (2.3.9) follows from:

$$ts_{Da - Da(\xi/t)\delta} = s_{Da - Da(\xi)\delta}(t\cdot) \quad (\xi \in V)$$

as elements in $\mathrm{Aff}(V)$. Identity (2.3.10) is the statement for $\xi = g(a)$,

$$t\int_0^{Da(v) - Da(\xi/t)} p(tv - trDa^\vee)dr = \int_0^{Da(tv) - Da(\xi)} p(tv - rDa^\vee)dr \quad \forall p \in P(V)_{\mathbb{C}},$$

which follows after the change of coordinate $r \mapsto r/t$. The second statement of the proposition follows from (2.3.8) and the fact that $S_t \otimes \mathrm{id}_{U^*}$ is a linear automorphism of $P(V)_{\mathbb{C}} \otimes U^*$. □

Propositions 2.3.13 and 2.3.15 together imply that the family of representations $\{L_k^g | g \in \mathrm{Fun}(I, V)\}$ of H_k contains essentially only two different representations: the Q_k representation from Corollary 2.3.12 (i.e. $L_k^{g_0}$) and the representation L_k from Proposition 2.3.5, while we keep in mind that we have chosen $X = Q^\vee$.

Now let X be any lattice between $Q^\vee$ and $P^\vee$, and U a finite dimensional representation of W_X. By Proposition 2.3.5 we know that the representation L_k of H_k on $P(V)_{\mathbb{C}} \otimes U^*$ admits a natural extension to the larger algebra H_k^X. It is also possible to extend the representation Q_k of H_k from Corollary 2.3.12 to a representation of H_k^X as stated in Theorem 2.3.7. Before proving this we observe that

$$wq(\partial) = q^{Dw}(\partial)w \quad (w \in W_X, q \in S(V)_{\mathbb{C}}) \tag{2.3.11}$$

holds as operators on $C^\infty(V)$ (and hence also on the subspace $P(V)_{\mathbb{C}} \subset C^\infty(V)$). It is not difficult to see that the identity (2.3.11) also holds on $C^\infty(V_{reg})$.

PROOF OF THEOREM 2.3.7. Because of the defining relations of H_k^X (see Theorem 2.2.2) it suffices to prove:

(a) **(i)** and **(ii)** define a representation of the subalgebra H_k of H_k^X on $P(V)_{\mathbb{C}} \otimes U^*$,

(b) the cross relations between $Q_k(\omega)$ ($\omega \in \Omega_X$) and $Q_k(v)$ ($v \in V$) holds (see Theorem 2.2.2**(d)**, but with Ω replaced by Ω_X),

(c) $s_a \mapsto Q_k(s_a)$ ($a \in I$) and $\omega \mapsto Q_k(\omega)$ ($\omega \in \Omega_X$) uniquely define a representation of the subalgebra $\mathbb{C}[W_X]$ of H_k^X on $P(V)_{\mathbb{C}} \otimes U^*$.

Use Corollary 2.3.12 to conclude that (a) is true. The cross relations of (b) follows directly from (2.3.11). Recall from Section 2.2 that a $\omega \in \Omega_X$ permutes I. Condition (c) follows because $s_a \mapsto Q_k(s_a)$ defines a representation of W (see (a)) and

$$Q_k(\omega)Q_k(s_a)Q_k(\omega^{-1}) = Q_k(s_{\omega a}) \quad (a \in I, \omega \in \Omega_X). \tag{2.3.12}$$

The idenity (2.3.12) follows from

$$\omega s_a \omega^{-1} = s_{\omega a} \text{ and } \omega I(a) \omega^{-1} = I(\omega a) \quad (a \in I, \omega \in \Omega_X)$$

as operators on $P(V)_{\mathbb{C}}$ (see also the proof of Lemma 2.3.9). □

2.4. Vector-valued modified BGG-representations

We know that the vector-valued BGG-representation δ_k (see Proposition 2.3.1) and the L_k-representation (see Proposition 2.3.5) of H_k^X are dual representations with respect to the pairing $\langle\cdot,\cdot\rangle$. In this section we construct a representation of H_k^X that is dual to the Q_k-representation of H_k^X with respect to a similar pairing as $\langle\cdot,\cdot\rangle$. In the construction so-called *modified BGG-operators* are used.

Denote by $\mathcal{O}(V_{\mathbb{C}}^*)$ the ring of analytical functions on $V_{\mathbb{C}}^*$. The symmetric algebra $S(V)_{\mathbb{C}}$ is embedded naturally in $\mathcal{O}(V_{\mathbb{C}}^*)$ via the identification $S(V)_{\mathbb{C}} \simeq P(V^*)_{\mathbb{C}}$. We extend the non-degenerate pairing $\langle\cdot,\cdot\rangle : S(V)_{\mathbb{C}} \times P(V)_{\mathbb{C}} \longrightarrow \mathbb{C}$ to a non-degenerate pairing $\mathcal{O}(V_{\mathbb{C}}^*) \times P(V)_{\mathbb{C}} \to \mathbb{C}$ via the same formula: $\langle f, p\rangle = (f(\partial)p)(0)$ ($f \in \mathcal{O}(V_{\mathbb{C}}^*)$, $p \in P(V)_{\mathbb{C}}$), and similarly we consider the non-degenerate pairing

$$\langle\cdot,\cdot\rangle : (\mathcal{O}(V_{\mathbb{C}}^*) \otimes U) \times (P(V)_{\mathbb{C}} \otimes U^*) \longrightarrow \mathbb{C},$$

defined uniquely by $\langle f \otimes u, p \otimes \psi\rangle = (f(\partial)p)(0)\psi(u)$.

For a $v \in V$, denote by $e^v \in \mathcal{O}(V_{\mathbb{C}}^*)$ the analytic function $\lambda \mapsto e^{\lambda(v)}$ ($\lambda \in V_{\mathbb{C}}^*$). Then a Taylor expansion shows that $e^v(\partial) = \tau_{-v}$ (acting on $P(V)_{\mathbb{C}}$), or equivalently,

$$\langle e^v, p\rangle = p(v) \quad (p \in P(V)_{\mathbb{C}}, v \in V). \tag{2.4.1}$$

The finite Weyl group W_0 acts on $\mathcal{O}(V_{\mathbb{C}}^*)$ via algebra automorphisms in the usual way:

$$(wf)(\lambda) = f(w^{-1}\lambda) \quad (w \in W_0, f \in \mathcal{O}(V_{\mathbb{C}}^*), \lambda \in V_{\mathbb{C}}^*).$$

We extend this to an action of the affine Weyl group $W_X = W_0 \ltimes X$ on $\mathcal{O}(V_{\mathbb{C}}^*)$ by $\overline{\tau_x} f := e^x f$ ($x \in X, f \in \mathcal{O}(V_{\mathbb{C}}^*)$). The corresponding W_X-action is denoted by $\overline{w} f$ ($w \in W_X$, $f \in \mathcal{O}(V_{\mathbb{C}}^*)$). Note that this W_X-actions does not act as algebra automorphisms, but

$$\overline{w}(ff') = (\overline{w}f)(Dw)f' \quad (w \in W_X, f, f' \in \mathcal{O}(V_{\mathbb{C}}^*)).$$

Using (2.2.2) we get

$$\overline{s_a} f = e^{-a(0)Da^\vee}(s_{Da} f) \quad (a \in \Sigma, f \in \mathcal{O}(V_{\mathbb{C}}^*)).$$

om (2.4.1)) it follows

$$\langle \overline{w} f, p\rangle = \langle f, p^{w^{-1}}\rangle \quad (w \in W_X, f \in \mathcal{O}(V_{\mathbb{C}}^*), p \in P(V)_{\mathbb{C}}). \tag{2.4.2}$$

LEMMA 2.4.1. *Let $a \in \Sigma$. Then*

$$\overline{\Delta}_a f = \frac{f - \overline{s_a} f}{Da^\vee} \quad (f \in \mathcal{O}(V_{\mathbb{C}}^*)) \tag{2.4.3}$$

defines a linear endomorphism of $\mathcal{O}(V_{\mathbb{C}}^*)$.

The operator $\overline{\Delta}_a$ ($a \in \Sigma$) is called a *modified BGG-operator*. Let $\alpha \in \Sigma_0$. Then $\Delta_\alpha = \overline{\Delta}_\alpha$ on $S(V)_{\mathbb{C}} \simeq P(V^*)_{\mathbb{C}} \subset \mathcal{O}(V_{\mathbb{C}}^*)$, (with Δ_α is the BGG-operator defined in (2.2.10)). It is therefore natural to use the notation Δ_α for $\overline{\Delta}_\alpha$.

PROOF. The operators $\overline{\Delta}_a$ is a priori a map from $\mathcal{O}(V_{\mathbb{C}}^*)$ to the space of meromorphic functions $\mathcal{M}(V_{\mathbb{C}}^*)$ on $V_{\mathbb{C}}^*$. We want to show that $\overline{\Delta}_a$ actually leaves the subspace $\mathcal{O}(V_{\mathbb{C}}^*) \subset \mathcal{M}(V_{\mathbb{C}}^*)$ invariant. Choose complex-linear coordinates $z_1, \ldots, z_n$ on $V_{\mathbb{C}}^*$ such that we have

$$(\overline{s_a} f)(z_1, \ldots, z_n) = e^{-mz_1} f(-z_1, z_2, z_3, \ldots, z_n) \quad \forall f \in \mathcal{O}(V_{\mathbb{C}}^*)$$

for a $m \in \mathbb{R}$. Writing out the power series expansion of $f - \overline{s_a} f$ one sees easily that $f - \overline{s_a} f = z_1 g(z)$ for a $g \in \mathcal{O}(V_{\mathbb{C}}^*)$, whence $\overline{\Delta}_a f = g$, and therefore also $\overline{\Delta}_a f \in \mathcal{O}(V_{\mathbb{C}}^*)$, what had to be shown. □

LEMMA 2.4.2. **(i)** *The* modified Leibniz rule

$$\overline{\Delta}_a(ff') = \overline{\Delta}_a(f)f' + (s_{Da}f)\overline{s_a s_{Da}}\Delta_{Da}(f') \quad (a \in \Sigma, f, f' \in \mathcal{O}(V_{\mathbb{C}}^*)) \tag{2.4.4}$$

holds.
(ii) *We have $\overline{\Delta}_a(ff') - (s_{Da}f)\overline{\Delta}_a(f') = \Delta_{Da}(f)f'$ for $a \in \Sigma$, $f, f' \in \mathcal{O}(V_{\mathbb{C}}^*)$.*

PROOF. **(i)** We start with the case $\alpha \in \Sigma_0$:

$$\Delta_\alpha(ff') = \Delta_\alpha(f)f' + (s_\alpha f)\Delta_\alpha(f') \quad \forall f, f' \in \mathcal{O}(V_{\mathbb{C}}^*). \tag{2.4.5}$$

This follows immediately from $ff' - s_\alpha(ff') = (f - (s_\alpha f))f' + (s_\alpha f)(f' - s_\alpha f')$. The general case (2.4.4) follows directly from (2.4.5),$\tau_{-a(0)Da^\vee} = s_a s_{Da}$ (cf. (2.2.2)) and

$$\overline{\Delta}_a(f) = \frac{1 - e^{-a(0)Da^\vee}}{Da^\vee} f + e^{-a(0)Da^\vee}\Delta_{Da}(f) \quad (a \in \Sigma, f \in \mathcal{O}(V_{\mathbb{C}}^*)). \tag{2.4.6}$$

(ii) Using (2.4.6) we observe that $\overline{s_a s_{Da}}\Delta_{Da} - \overline{\Delta}_a$ is multiplication by the function

$$(1/Da^\vee)(e^{-a(0)Da^\vee} - 1).$$

Therefore

$$\begin{aligned}(s_{Da}f)(\overline{s_a s_{Da}}\Delta_{Da} - \overline{\Delta}_a)(f') &= (\overline{s_a s_{Da}}\Delta_{Da} - \overline{\Delta}_a)(s_{Da}f)f' \\ &= ((1/Da^\vee)(\overline{s_a} - s_{Da}))(f)f',\end{aligned}$$

and where we used (2.4.3) for the last equality. Note that also $\overline{\Delta}_a = \Delta_{Da} - (1/Da^\vee)(\overline{s_a} - s_{Da})$, which easily follows from (2.4.6). Together with (2.4.4) these shows

$$\begin{aligned}\overline{\Delta}_a(ff') - (s_{Da}f)\overline{\Delta}_a(f') &= \overline{\Delta}_a(f)f' + (s_{Da}f)(\overline{s_a s_{Da}}\Delta_{Da} - \overline{\Delta}_a)(f') \\ &= (\overline{\Delta}_a + \frac{1}{Da^\vee}(\overline{s_a} - \overline{s_{Da}}))(f)f' \\ &= \Delta_{Da}(f)f'.\end{aligned}$$

□

We can reformulate Lemma 2.4.2**(ii)** by: the operators $\overline{\Delta}_a$ ($a \in I$) together with $S(V)_\mathbb{C} \simeq P(V^*)_\mathbb{C}$, seen as multiplication operators on $\mathcal{O}(V_\mathbb{C}^*)$, satisfy the cross relations (see (2.2.11)) of H_1, corresponding to multiplicity function $k \equiv 1$. This observation will follow also from the following theorem.

THEOREM 2.4.3. *Let U be a finite dimensional representation of W_X. The following uniquely defines a representation $\overline{\delta_k}$ of H_k^X on $\mathcal{O}(V_\mathbb{C}^*) \otimes U$:*

$$\begin{aligned}\overline{\delta_k}(v)(f \otimes u) &= (vf) \otimes u \quad (v \in V),\\ \overline{\delta_k}(s_a)(f \otimes u) &= (\overline{s_a}f) \otimes s_a u + k_a \overline{\Delta_a}(f) \otimes u \quad (a \in I),\\ \overline{\delta_k}(\omega)(f \otimes u) &= \overline{\omega} f \otimes \omega u \quad (\omega \in \Omega_X)\end{aligned}$$

for $f \in \mathcal{O}(V_\mathbb{C}^*)$, $u \in U$, and with $vf \in \mathcal{O}(V_\mathbb{C}^*)$ the map $\lambda \mapsto \lambda(v)f(\lambda)$.

PROOF. Consider the injective linear map $\xi : \mathcal{O}(V_\mathbb{C}^*) \longrightarrow P(V)_\mathbb{C}^*$ defined by

$$\xi(f)(p) = \langle f, p\rangle = (f(\partial)p)(0) \quad (f \in \mathcal{O}(V_\mathbb{C}^*), p \in P(V)_\mathbb{C}).$$

Then we have

$$\begin{aligned}&\text{a) } \xi(e^v)(p) = p(v) \quad (v \in V, p \in P(V)_\mathbb{C})\\ &\text{b) } \xi(\overline{w}f) = \xi(f) \circ w^{-1} \quad (w \in W_X)\\ &\text{c) } \xi(vf) = \xi(f) \circ \partial_v \quad (v \in V)\\ &\text{d) } \xi(\overline{\Delta_a}f) = \xi(f) \circ I(a) \quad (a \in I)\end{aligned} \tag{2.4.7}$$

Formula a) is a reformulation of $\langle e^v, p\rangle = p(v)$, (b) of (2.4.2), c) of $\langle vf, p\rangle = \langle f, \partial_v p\rangle$ and d) of

$$\langle \overline{\Delta}_a(f), p\rangle = \langle f, I(a)(p)\rangle \quad (a \in \Sigma, f \in \mathcal{O}(V_\mathbb{C}^*), p \in P(V)_\mathbb{C}). \tag{2.4.8}$$

Note that (2.4.8) is a generalization of Lemma 2.3.4**(iii)** and follows from (2.4.2).

Recall the representation Q_k of H_k^X on $P(V)_\mathbb{C} \otimes U^*$ from Proposition 2.3.7. The linear anti-involution † allows us to put a left H_k^X-module structure on $(P(V)_\mathbb{C} \otimes U^*)^* \simeq P(V)_\mathbb{C}^* \otimes U$, and (2.4.7) shows that

$$\xi(\mathcal{O}(V_\mathbb{C}^*)) \otimes U \subset P(V)_\mathbb{C}^* \otimes U$$

is a H_k^X-submodule. Furthermore, (2.4.7) also shows that the pull back of this H_k^X-module structure on $\xi(\mathcal{O}(V_{\mathbb{C}}^*))\otimes U$ to $\mathcal{O}(V_{\mathbb{C}}^*)\otimes U$ under the linear isomorphism

$$\xi\otimes \mathrm{id}_U : \mathcal{O}(V_{\mathbb{C}}^*)\otimes U \xrightarrow{\sim} \xi(\mathcal{O}(V_{\mathbb{C}}^*))\otimes U$$

gives rise to the action $\overline{\delta_k}$.

□

We can now state the duality mentioned at the beginning of the section: the following

$$\langle\overline{\delta_k}(h^\dagger)(f\otimes u), p\otimes\psi\rangle = \langle f\otimes u, Q_k(h)(p\otimes\psi)\rangle \quad (h\in H_k^X) \tag{2.4.9}$$

holds for $f\otimes u\in\mathcal{O}(V_{\mathbb{C}}^*)$ and $p\otimes\psi\in P(V)_{\mathbb{C}}\otimes U^*$. It holds by construction. Compare (2.4.9) with (2.3.7).

2.5. The fundamental representations: topological versions

Recall from Section 2.2.1 that we denote by $\mathcal{C}$ the collection of open alcoves in V_{reg}. For an alcove $C\in\mathcal{C}$ we denote by $C(\overline{C})$ the vector space of continuous functions f on C having a continuous extension to some open $C_f\supset\overline{C}$.

DEFINITION 2.5.1. *Let*

$$B(V)=\prod_{C\in\mathcal{C}} C(\overline{C})$$

as vector space.

REMARK 2.5.2. We have a natural embedding $B(V)\hookrightarrow C(V_{reg})$ of vector spaces, defined by

$$\prod_{C\in\mathcal{C}} f_C\mapsto f=\sum_{C\in\mathcal{C}}\chi_C f_C, \tag{2.5.1}$$

where χ_C is the characteristic function of $C\in\mathcal{C}$, and where we view f_C as continuous function on V by arbitrary choice of continuous extension of $f_C\in C(\overline{C})$.

We also have the injection $C(V)\hookrightarrow B(V)$ of vector spaces defined by

$$f\mapsto\prod_{C\in\mathcal{C}} f_{|\overline{C}}.$$

We use these identifications repeatedly.

Note that $f\in B(V)\subset C(V_{reg})$ has an interpretation as multivalued function on V, where for $v\in V$,

$$f(v)=\big\{\lim_{C\ni v'\to v} f_C(v')|C\in\mathcal{C}_v\big\},$$

where $\mathcal{C}_v$ is the finite collection of alcoves $C\in\mathcal{C}$ such that $v\in\overline{C}$.

In the previous section we defined integral-operators $I(b):C(V)\longrightarrow C(V)$ (see (2.3.2)) for any $b\in\Sigma$. These operators will now be extended to linear endomorphisms of $B(V)$.

DEFINITION 2.5.3 (The integral operator [**32**]). *Let $b \in \Sigma$, $f = \prod_{C \in \mathcal{C}} f_C \in B(V)$, and $D \in \mathcal{C}$. Let $(I(b)f)_D \in \mathrm{Fun}(\overline{D}, \mathbb{C})$ be the function defined by*

$$(I(b)f)_D(v) = \sum_{C \in \mathcal{C}} \int_0^{b(v)} (\chi_C f_C)(v - tDb^\vee)dt \quad \forall v \in \overline{D}. \tag{2.5.2}$$

Let $D \in \mathcal{C}$ and $b \in \Sigma$. Let $A_{D,b}$ be the smallest convex set in V containing $\overline{D} \cup s_b\overline{D}$. Then $A_{D,b}$ is a compact subset of V. Furthermore

$$\mathcal{C}_{D,b} := \{C \in \mathcal{C} |\, \overline{C} \cap A_{D,b} \neq \emptyset\}$$

is finite and $D \in \mathcal{C}_{D,b}$, hence $\mathcal{C}_{D,b}$ is nonempty. Then (2.5.2) indeed defines an element in $\mathrm{Fun}(\overline{D}, \mathbb{C})$, for the summation in (2.5.2) is actually over the subset $\mathcal{C}_{D,b}$ of $\mathcal{C}$, since $\chi_C(v - tDb^\vee) = 0$ for all $v \in D$, $t \in [0, b(v)]$ and $C \notin \mathcal{C}_{D,b}$. Thus

$$(I(b)f)_D(v) = \sum_{C \in \mathcal{C}_{D,b}} \int_0^{b(v)} (\chi_C f_C)(v - tDb^\vee)dt \quad (v \in \overline{D}). \tag{2.5.3}$$

With identification of $B(V)$ as multivalued functions on V (see Remark 2.5.2), we have

$$(I(b)f)(v) = \int_0^{b(v)} f(v - tDb^\vee)dt \quad (v \in V). \tag{2.5.4}$$

LEMMA 2.5.4 (Gutkin [**32**]). *Let $b \in \Sigma$. The assignment*

$$f \mapsto I(b)f = \prod_{C \in \mathcal{C}} (I(b)f)_C \quad (f \in B(V))$$

defines a linear endomorphism of $B(V)$. Furthermore, the restriction of $I(b)$ to $C(V)$ (resp. $C^\infty(V)$) yields linear endomorphisms of $C(V)$ (resp. $C^\infty(V)$).

PROOF. The first statement follows from (2.5.3), while the second statement follows from (2.5.4). □

REMARK 2.5.5. In the proof of [**32**, Theorem 2.7] it is claimed that the integral-operators $I(b)$ ($b \in \Sigma$), in Gutkin's notation, preserve $CB^\infty(V)$. This is not correct however.

Consider the following measure μ_E (E standing for Euclidean) on V. Choose an orthonormal basis $\eta_1, \ldots, \eta_n$ for $(V, \langle, \rangle)$. Then $(V, \langle, \rangle)$ is isomorphic to $\mathbb{R}^n$ (with the standard inner product on $\mathbb{R}^n$) as Euclidean spaces via the map Ψ defined by $e_i \mapsto \eta_i$ ($i = 1, \ldots, n$), and with $e_1, \ldots, e_n$ the standard basis for $\mathbb{R}^n$. The measure μ_E is by definition the forward of the standard Lebesgue measure μ_{Leb} on $\mathbb{R}^n$, i.e. μ_E is the unique measure on V such that

$$\mu_{Leb}(A) = \mu_E(\Psi(A))$$

holds for every Lebesgue measurable set A in $\mathbb{R}^n$. It is independent of the choice of orthonormal basis. The L^1-norm of a function $g \in L^1(V, d\mu_E)$ is denoted by

$$\|g\|_1 = \int_V |g(v)|\, d\mu_E(v).$$

Note that μ_E is a Haar measure on the locally compact abelian group V (the group structure being addition).

We now topologize the vector space $B(V)$.

DEFINITION 2.5.6. *For any $C \in \mathcal{C}$ we put the L^1-topology with respect to the measure μ_E on $C(\overline{C}) \subset L^1(\overline{C}, d\mu_E)$. The topology on $B(V) = \prod_{C\in\mathcal{C}} C(\overline{C})$ is taken to be the product topology.*

REMARK 2.5.7. The topological space $C(\overline{C})$ ($C \in \mathcal{C}$) is a metric space, and therefore first-countable and Hausdorff. Since the collection $\mathcal{C}$ of alcoves is countable, and the countable product of first-countable (resp. Hausdorff) spaces is first-countable (resp. Hausdorff), the topological space $B(V)$ is first-countable and Hausdorff. In particular a map $\Lambda : B(V) \longrightarrow Y$, with Y a topological space, is continuous if and only if Λ is sequential continuous. (The reader is referred to [**61**], in particular paragraph 30 therein).

LEMMA 2.5.8. *Let $b \in \Sigma$. The linear endomorphism $I(b) : B(V) \longrightarrow B(V)$ is continuous.*

PROOF. Denote by pr_D ($D \in \mathcal{C}$) the projection of $B(V) = \prod_{C\in\mathcal{C}} C(\overline{C})$ on the component $C(\overline{D})$, i.e. $\prod_{C\in\mathcal{C}} f_C \mapsto f_D$. For a topological space Y, a map $\Lambda : Y \longrightarrow B(V)$ is continuous if and only if $pr_D \circ \Lambda$ is a continuous map from Y to $C(\overline{D})$ for all $D \in \mathcal{C}$. Taking for $Y = B(V)$ and using (2.5.3) we get the desired result. □

LEMMA 2.5.9. *$P(V)_\mathbb{C} \subset B(V)$ is dense.*

PROOF. Let $f \in B(V)$. Fix a finite number of alcoves $C_1, \ldots, C_m$. We have to show that for all $\varepsilon > 0$ there exists $p \in P(V)_\mathbb{C}$ such that

$$\|\chi_{\overline{C_j}} f_{C_j} - p\chi_{\overline{C_j}}\|_1 < \varepsilon \quad \forall j = 1, 2, \cdots m \text{ in } L^1(\overline{C_j}).$$

Let $\xi > 0$ and $M = \max\left(\left|f_{\overline{C_j}}(v)\right| |v \in \overline{C_j},\, j = 1, 2, \ldots m\right)$.

Set also

$$H = \sum_{j=1}^m \int_{\overline{C_j}} d\mu_E(v) = \mu_E(C_1 \cup \cdots \cup C_m).$$

Let $C_j^\xi \subset C_j$ ($j = 1, \ldots, m$) be compact and such that

$$\sum_{j=1}^m \mu_E(C_j \backslash C_j^\xi) < \xi.$$

Let f_ξ be a continuous function on $\cup_{j=1}^m \overline{C_j}$, uniformly bounded by M, such that

$$f_{\xi_{|\cup_{j=1}^m C_j^\xi}} = f_{|\cup_{j=1}^m C_j^\xi} \quad (\text{so } f_{\xi_{|C_j^\xi}} = f_{C_j}{}_{|C_j^\xi}\ \forall j).$$

By Stone-Weierstrass theorem on approximation of continuous functions by polynomials on compacta there is a $p_\xi \in P(V)_\mathbb{C}$ such that

$$\max_{v \in \cup_{j=1}^m \overline{C}_j} |p_\xi(v) - f_\xi(v)| < \xi.$$

Then

$$\begin{aligned}
\sum_{j=1}^m \|\chi_{\overline{C_j}} f_{C_j} - p_\xi \chi_{\overline{C_j}}\|_1 &= \sum_{j=1}^m \int_{C_j} |f(v) - p_\xi(v)|\, d\mu_E(v) \\
&\le \int_{\cup_{j=1}^m C_j} |f(v) - f_\xi(v)|\, d\mu_E(v) \\
&\quad + \int_{\cup_{j=1}^m C_j} |f_\xi(v) - p_\xi(v)|\, d\mu_E(v) \\
&\le \int_{\cup_{j=1}^m (C_j \setminus C_j^\xi)} |f(v) - f_\xi(v)|\, d\mu_E(v) + H\xi \\
&\le 2M\mu_E(\cup_{j=1}^m C_j \setminus C_j^\xi) + H\xi \\
&\le (2M + H)\xi,
\end{aligned}$$

and this goes to zero as ξ goes to zero. □

REMARK 2.5.10. Another natural topology on $B(V)$ is the product topology $\prod_{C \in \mathcal{C}} C(\overline{C})$, with the topology of uniform convergence on the components $C(\overline{C})$ ($C \in \mathcal{C}$). However, with respect to this topology the subspace $P(V)_\mathbb{C}$ is not dense in $B(V)$.

Let X be a lattice in V between $Q^\vee$ and $P^\vee$, and U a finite dimensional representation of the affine Weyl group W_X.

The topology we put on $B(V, U) = B(V) \otimes U$ is the tensor product topology (which simplifies enormously due to the finite dimensionality of U).

The space $B(V, U)$ admits a natural embedding as multi-valued U-valued functions on V_{reg} in an obvious way (see Remark 2.5.2 and (2.5.1)).

The group W_X acts on $B(V)$:

$$(f^w)_D = f_{w^{-1}D}(w^{-1}\cdot) \quad \forall w \in W_X, f = \prod_{C \in \mathcal{C}} f_C \in B(V), D \in \mathcal{C}. \tag{2.5.5}$$

When $B(V)$ is interpreted as multivalued functions on V, the action (2.5.5) of W_X can be written as

$$(f^w)(v) = \left\{ \lim_{C \ni v' \to v} f(w^{-1}v') | C \in \mathcal{C}_v \right\} \quad \forall v \in V.$$

Note that W_X acts on $B(V)$ by continuous operators. Hence the integral-reflection operators

$$Q_k(s_a) = s_a \otimes s_a + k_a I(a) \otimes \mathrm{id}_U \quad (a \in I)$$

and the operators

$$Q_k(\omega) = \omega \otimes \omega \quad (\omega \in \Omega_X)$$

define continuous linear endomorphisms of $B(V,U)$, with invariant subspaces $C(V,U)$ and $P(V,U)$.

LEMMA 2.5.11. *$P(V,U)$ is dense in $B(V,U)$.*

PROOF. Since U is finite dimensional, this follows directly from Lemma 2.5.9. □

The Q_k-representation (see Theorem 2.3.7) of W_X on $P(V,U)$ can be extended to a representation of W_X on $B(V,U)$.

THEOREM 2.5.12. *The following defines uniquely a representation Q_k of W_X on $B(V,U)$:*

$$\begin{aligned} s_a &\mapsto Q_k(s_a) \quad (a \in I), \\ \omega &\mapsto Q_k(\omega) \quad (\omega \in \Omega_X). \end{aligned}$$

Furthermore, $C(V,U)$ and $C^\infty(V,U)$ are invariant subspaces.

PROOF. The operators $Q_k(s_a)$ ($a \in I$), $Q_k(\omega)$ ($\omega \in \Omega_X$) (and hence also compositions of these operators) are continuous linear endomorphisms of $B(V,U)$ by Lemma 2.5.8. We know (Theorem 2.3.7) that the assignments (i) and (ii) define a W_X-action on the subspace $P(V,U)$ of $B(V,U)$. Use now Lemma 2.5.11 and the first-countability of the topology of $B(V)$ (see also Remark 2.5.7) to conclude that the defining relations of W_X for the operators extend continuously from $P(V,U)$ to $B(V,U)$. □

A W_X-submodule $M \subset C(V,U)$ with respect to the Q_k-action will be denoted by M_{Q_k}.

THEOREM 2.5.13. *The following defines uniquely a representation Q_k of H_k^X on $C^\infty(V,U)$:*

$$\begin{aligned} v &\mapsto \partial_v \otimes \mathrm{id}_U \quad (v \in V), \\ s_a &\mapsto Q_k(s_a) \quad (a \in I), \\ \omega &\mapsto Q_k(\omega) \quad (\omega \in \Omega_X). \end{aligned}$$

PROOF. It is immediate that $v \mapsto \partial_v \otimes \mathrm{id}_U$ ($v \in V$) uniquely defines a representation of $S(V)_{\mathbb{C}}$ on $C^\infty(V) \otimes U$. Because of Theorem 2.5.12 and the defining relations of H_k^X (see Theorem 2.2.2) it therefore suffices to check the cross relations

$$Q_k(\omega)(\partial_v \otimes \mathrm{id}_U) = (\partial_{D\omega v} \otimes \mathrm{id}_U) Q_k(\omega) \quad (\omega \in \Omega_X, v \in V) \tag{2.5.6}$$

and

$$Q_k(s_a) b(\partial_v \otimes \mathrm{id}_U) - (\partial_{s_{Da} v} \otimes \mathrm{id}_U) Q_k(s_a) = k_a Da(v) \quad (a \in I, v \in V) \tag{2.5.7}$$

as operators on $C^\infty(V) \otimes U$. Relation (2.5.6) follows directly from (2.3.11), and (2.5.7) follows from (2.3.11) and

$$I(a)\partial_v - \partial_{s_{Da}v} I(a) = Da(v) \quad (a \in I, v \in V) \tag{2.5.8}$$

as operators on $C^\infty(V)$, which follows from a direct calculation using (2.3.4) (or see [**32**, Lemma 2.1]). □

We will also use the notation M_{Q_k} to indicate that a subspace $M \subset C^\infty(V, U)$ is a H_k^X-submodule with respect to the Q_k-action.

REMARK 2.5.14. Proceeding similar as in Remark 2.3.6 we get a finite type Hecke algebra $H_k^{(0)}$ version of Theorem 2.5.13. The corresponding representation of $H_k^{(0)}$ is denoted by Q_k^0.

CHAPTER 3

Vector-valued integrable systems with delta-potentials

3.1. Introduction

Given any affine root system Σ, Gutkin and Sutherland [**36**], [**77**] defined a quantum integrable system whose Hamiltonian $-\Delta+\mathcal{V}$ has a potential $\mathcal{V}$ expressible as a weighted sum of delta-functions at the affine root hyperplanes of Σ. For the affine root system of type A, the quantum integrable system essentially reduces to the quantum Bose-gas on the circle with pair-wise delta-function interactions, which has been subject of intensive studies over the past 40 years.

The special case of the impenetrable Bose-gas on the circle was exactly solved by relating the model to the free Fermi-gas on the circle (see Girardeau [**29**]). Soon afterwards fundamental progress was made for arbitrary pair-wise delta function interactions by Lieb & Liniger [**56**], Yang [**80**] and Yang & Yang [**81**], leading to the derivation of the associated Bethe ansatz equations and Bethe ansatz eigenfunctions.

One of the aims of this chapter is to formulate vector-valued (i.e. the wave functions of the quantum systems takes values in a certain representation space) versions of Gutkin's and Sutherland's quantum integrable systems associated to affine root systems. We furthermore generalize and extend the results of Gutkin [**32**] to the vector-valued system.

Quantum Calogero-Moser systems are root system generalizations of quantum Bose-gases on the line or circle with long range pair-wise interactions. In special cases quantum Calogero-Moser systems naturally arose from harmonic analysis on symmetric spaces. A decisive role in the studies of quantum Calogero-Moser systems has been played by certain non-bosonic analogs of these systems, which are defined in terms of Dunkl-type commuting differential-reflection operators. Degenerate double affine Hecke algebras naturally appear here as the fundamental objects governing the algebraic relations between the Dunkl-type operators and the natural Weyl group action.

In this chapter we define vector-valued Dunkl-type commuting differential-reflection operators associated to the root system generalizations of the quantum Bose-gas with delta-function interactions. We furthermore show that the Dunkl-type operators, together with the natural affine Weyl group action (cf. (2.2.8)) realize a faithful representation of the Hecke algebra H on certain vector-valued function spaces. These results show that these quantum integrable systems with values in the trivial representation (also called the *scalar* case) naturally fit into the class of quantum Calogero-Moser integrable systems, a point of view which also has been advertised from the perspective of harmonic analysis in [**40**, Sect. 5].

The quantum integrable systems under consideration for affine root systems Σ of classical type still have reasonable physical interpretations in terms of interacting one-dimensional quantum bosons. In these cases various results of the present chapter can be found in the vast physics literature on this subject. We will give the precise connections to the literature in the main body of the text.

The contents of the chapter is as follows. Section 2 is meant to introduce the quantum integrable systems and to state and clarify the results on the associated spectral problem. We formulate the spectral problem for the quantum integrable systems under consideration as an explicit boundary value problem.

In Section 3 we formulate for the scalar case the analog of Girardeau's equivalence between the impenetrable Bose-gas and the free Fermi-gas on the circle for the quantum integrable systems under consideration.

In Section 4 we introduce vector-valued Dunkl-type commuting differential-reflection operators and show that they realize, together with the natural affine Weyl group action (2.2.8), a faithful realization of the Hecke algebra H (corresponding to the appropriate lattice and multiplicity function) as defined in the previous chapter. In Section 2.5 we showed that integral-reflection operators, together with the ordinary directional derivatives, yield an realization of H. In Section 5 it is shown that Gutkin's [**32**] propagation operator can be extended to vector-valued systems. It is furthermore shown that this operator establishes an equivalence between these two realizations of H. We furthermore show that the Dunkl operators naturally act on a space of vector-valued functions with higher order normal derivative jumps over the affine root hyperplanes.

In Section 6 we return to the boundary value problem of Section 2. Using the Hecke-type algebra H we extend (and in the scalar case also refine) Gutkin's [**32**] generalization of Girardeau's equivalence between the boundary value problem for the impenetrable Bose gas and the boundary value problem for the free Fermi-gas as formulated in Section 3 to vector-valued systems. The results in this section entail that the boundary value problem is equivalent to a boundary value problem with trivial boundary value conditions, at the cost of having to deal with the non-standard affine Weyl group action Q as defined in Section 2.2.

In Section 7 we study the reformulated boundary value problem for generic spectral parameters. It is shown that, genericallly, a spectral parameter is contained in the spectrum of the system if and only certain Bethe ansatz equations, a family of transcendental equations, are satisfied. Section 8 deals with the case of positive multiplicity function and the integrable system taking values in a finite dimensional unitary representation. With the aid of the Bethe ansatz equations it is shown that, generically, the spectrum of the system is purely imaginary. For the scalar case a finer analysis shows that these results holds not only generically but always. The details of this analysis one can find in the next chapter.

3.2. The boundary value problem

We use the notations and definitions from the previous chapter implicitly. In particular X always denotes a lattice between $Q^\vee$ and $P^\vee$. In this section we attach to any finite dimensional representation U of the affine Weyl group W_X a vector-valued integrable system with delta-potential. Then we reformulate the spectral problem of this integrable system in terms of a concrete boundary value problem. For U the trivial representation we recover Gutkin's [**32**] reformulation of the spectral problem for periodic integrable systems with delta-potentials in terms of a concrete boundary value problem.

REMARK 3.2.1. The space of functions, boundary value problems, Hamiltonians and operators we consider will thus be dependent of the representation U of W_X. For the case that U is the trivial representation, we reserve the adjective *scalar*. For general U, the adjective *vector-valued* is used.

We also adopt the convention that in any notation depending on a representation U of W_X, we supress the U from this notation if $U = \mathbb{C}_{triv}$. Here $\mathbb{C}_{triv}$ denotes by the trivial represenation of W_X. For example: $C(V, \mathbb{C}_{triv}) = C(V)$.

The quantum integrable system which we will define now in a moment depends on a multiplicity function as defined in Definition 2.2.1. In the context of integrable systems (of classical type) such functions have the interpretation of coupling constants. We fix such a multiplicity function $k : \Sigma \longrightarrow \mathbb{C}$.

If D is a constant coefficient differential operator on V (say acting on $C^\infty(V)$), the U-valued constant coefficient differential $D \otimes \mathrm{id}_U$ (say acting on $C^\infty(V, U) = C^\infty(V) \otimes U$) is also denoted by D, unless stated otherwise.

We define the vector-valued quantum Hamiltonian $\mathcal{H}_k$ by

$$\mathcal{H}_k = -\Delta + \sum_{a \in \Sigma} k_a \delta(a(\cdot)) \otimes s_a, \tag{3.2.1}$$

where δ is the Dirac delta-function. Here we interpret $\mathcal{H}_k$ as a linear map $C(V, U) \to D'(V, U)$, with $D'(V, U) = D'(V) \otimes U$ the space of U-valued distributions on V (the concept of vector-valued distributions as defined by Schwartz [**70**] simplifies due to the fact that U is finite dimensional), as

$$\big(\mathcal{H}_k(f \otimes u)\big)(\phi) := -\int_V f(v)\big(\Delta\phi\big)(v) d\mu_E(v) \otimes u + \sum_{a \in \Sigma} \frac{k_a}{\|Da^\vee\|} \int_{V_a} f(v)\phi(v) d_a v \otimes s_a u \tag{3.2.2}$$

for a test function ϕ, with $d\mu_E(v)$ the Euclidean volume measure on V (see also the paragraph coming before Definition 2.5.6) and $d_a v$ ($a \in \Sigma^+$) the corresponding volume measure on the root hyperplane V_a.

The quantum Hamiltonian $\mathcal{H}_k$ and the associated quantum physical system has been studied for the scalar case in e.g. see [**77**], [**36**], [**32**]. A key step in these investigations is the reformulation of the spectral problem for $\mathcal{H}_k$ in terms of an explicit boundary value problem for the Laplacean Δ on V. We will extended this reformulation to the general vector-valued case in a moment.

The general vector-valued case has been studied for finite classical root systems in the context of quantum mechanical systems of particles with spin in [**3**] (type A), [**31**](type A, fermionic theory), [**53**] (bosonic theory).

DEFINITION 3.2.2. *For a compact subset A of V with nonempty interior we define $C^1(A)$ to be the vector space of functions f on A that have a continuously differentiable extension to some open $A_f \supset A$. Consider the vector subspace*

$$B^1(V) = \prod_{C\in\mathcal{C}} C^1(\overline{C})$$

of $B(V)$. Let $CB^1(V) = C(V) \cap B^1(V)$.

DEFINITION 3.2.3. *Let $C^{1,(k)}(V,U)$ be the space of functions $f \in CB^1(V,U)$ which satisfy the derivative jump conditions*

$$\big(\partial_{Da^\vee} f\big)(v + 0Da^\vee) - \big(\partial_{Da^\vee} f\big)(v - 0Da^\vee) = 2k_a s_a\big(f(v)\big) \tag{3.2.3}$$

(as identities in U) for sub-regular vectors $v \in V_a$ ($a \in \Sigma^+$).

REMARK 3.2.4. Note that the convention (2.2.6) breaksdown for $Z(V) = C^{1,(k)}(V)$, i.e. we have in general $C^{1,(k)}(V,U) \neq C^{1,(k)}(V) \otimes U$.

PROPOSITION 3.2.5. *For $f \in CB^1(V,U)$ and $E \in \mathbb{C}$ the following two statements are equivalent.*
(i) *$\mathcal{H}_k f = Ef$ as U-valued distributions on V.*
(ii) *$f \in C^{1,(k)}(V,U)$ and $\Delta f|_{V_{reg}} = -Ef|_{V_{reg}}$ as U-valued distributions on V_{reg}.*
A function $f \in CB^1(V,U)$ satisfying these equivalent conditions is smooth on V_{reg}.

PROOF. The first part of the proposition follows from a straightforward application of Green's identity (cf. the proof of [**32**, Theorem 2.7]). Since the constant differential operator $\Delta + E$ is (hypo)elliptic is, $(\Delta + E) \otimes \mathrm{id}_U$ only acts on the first component of $CB^1(V) \otimes U$ and U is finite dimensional, the last statement of the proposition follows. □

The quantum physical system with quantum Hamiltonian $\mathcal{H}_k$ is known to be integrable. The spectral problem for the associated quantum quantum conserved integrals can be translated into the following boundary value problem (for the scalar case, see also [**36**], [**32**]).

DEFINITION 3.2.6. *Fix a spectral parameter $\lambda \in V_{\mathbb{C}}^*$. We denote $\mathrm{BVP}_k(\lambda, U)$ for the space of functions $f \in C^{1,(k)}(V,U)$ solving (in distributional sense) the system*

$$p(\partial)f\big|_{V_{reg}} = p(\lambda)f\big|_{V_{reg}} \qquad \forall\, p \in S(V)^{W_0} \tag{3.2.4}$$

of differential equations away from the root hyperplane configuration $V_{irreg} = \bigcup_{a\in\Sigma^+} V_a$.

REMARK 3.2.7. The W_0-invariant constant coefficient differential operator $p_2(\partial)$ associated to the polynomial $p_2(\cdot) = \|\cdot\|^2 \in P(V^*)^{W_0}$ is the Laplacean Δ on V. Now Proposition 3.2.5 implies that a functions $f \in \mathrm{BVP}_k(\lambda, U)$ is smooth on V_{reg} and satisfies the differential equations (3.2.4) in the strong sense. The fact that f is an eigenfunction of all W_0-invariant constant differential operators on V_{reg} in fact implies that $f_{|C}$ is the restriction of a (necessarily unique) analytic U-valued function on V for all alcoves $C \in \mathcal{C}$, see [**75**].

Recall the π-action (2.2.8) of W_X on the space $\mathrm{Fun}(V,U)$ of functions from V to U, i.e.

$$(\pi(w)f)(v) = w(f^w(v)) = w(f(w^{-1}v)) \quad (w \in W_X, f \in \mathrm{Fun}(V,U), v \in V).$$

A function $f \in C^{1,(k)}(V,U)$ automatically satisfies the jump conditions (3.2.3) for $v \in V_b$ sub-regular and $b \in \Sigma^-$. Hence the space $C^{1,(k)}(V,U)$ does not dependent on the choice of positive roots Σ^+ in Σ. Since also $\omega(\Sigma) = \Sigma$ ($\omega \in \Omega_X$) holds, we can and will interpret $C^{1,(k)}(V,U)_\pi$ and $CB^1(V,U)_\pi$ as W_X-submodules of $C^\infty(V_{reg},U)_\pi$. The subspace $\mathrm{BVP}_k(\lambda,U) \subset C^{1,(k)}(V,U)$ is a W_X-submodule of $C^{1,(k)}(V,U)$ under the π-action because of Proposition 2.2.7 and (2.3.11).

The central theme of this chapter is the study of the subspace $\mathrm{BVP}_k(\lambda,U)_\pi^W \subset \mathrm{BVP}_k(\lambda,U)$ of $\pi(W)$-invariant solutions.

EXAMPLE 3.2.8 (Scalar free case $k \equiv 0$). A function $f \in \mathrm{BVP}_0(\lambda)$ (recal the convention from Remark 3.2.1) is a weak eigenfunction of the Laplacean $\Delta = p_2(\partial)$ on V with eigenvalue $p_2(\lambda)$, cf. the proof of [**32**, Thm. 2.7]. The regularity theorem for elliptic differential operators now implies that f is smooth. Consequently $\mathrm{BVP}_0(\lambda)^W$ is the solution space to the spectral problem for the free bosonic quantum integrable system on $V/Q^\vee$ associated to the Laplacean Δ on V. It is easy to show that $\mathrm{BVP}_0(\lambda)^W$ is zero-dimensional unless $\lambda \in 2\pi i P$, in which case it is spanned by the plane wave

$$\phi_{\lambda,0} = \frac{1}{\#W_0} \sum_{w \in W_0} e^{w\lambda}$$

(cf. with the analysis in the impenetrable case $k \equiv \infty$ in Section 3.3).

For any abelian group A, we denote by $\widehat{A}$ the unitary dual $\mathrm{Hom}_{\mathbb{Z}}(A,S^1)$ of A.

For root systems Σ_0 with property $Q^\vee \neq P^\vee$ (i.e. all types, except Σ_0 of type E_8, F_4 and G_2) the space $\mathrm{BVP}_k(\lambda,U)_\pi^W$ decomposes into nontrivial $\widehat{X/Q^\vee}$-isotypical components. To make this statement precise we start with the following lemma.

LEMMA 3.2.9. *The group Ω_X is a finite abelian group isomorphic to $X/Q^\vee$. An isomorphim $\jmath : X/Q^\vee \xrightarrow{\sim} \Omega_X$ of groups is defined as follows: $\jmath(x+Q^\vee) = \omega_x$, with $\omega_x \in \tau_x W$ the unique representative in Ω_X. In particular $x + Q^\vee \mapsto \omega_x W$ defines an isomorphism $X/Q^\vee \xrightarrow{\sim} W_X/W$ of groups.*

DEFINITION 3.2.10. *Let M be a W_X-module, For a subgroup G of W_X the subpace of elements in M fixed by G is denoted by M^G, i.e. $M^G = \{m \in M | gm = m \quad \forall g \in G\}$. Let $\chi \in \widehat{X/Q^\vee}$. We set*

$$\begin{aligned} M^{W,\chi} &:= \{m \in M^W | \tau_x m = \chi(x+Q^\vee)m \; \forall x \in X\} \\ &= \{m \in M | w\tau_x m = \chi(x+Q^\vee)m \; \forall w \in W_0, x \in X\}. \end{aligned}$$

In particular, $M^{W_X} = M^{W,triv}$ *and the chain of inclusions*

$$M^{W,\chi} \subset M^W \subset M^{W_0} \subset M \tag{3.2.5}$$

holds.

LEMMA 3.2.11. *For a* W_X*-module* M *the subspace* M^W *of* M *is a* W_X*-submodule and decomposes into* $(\widehat{X/Q^\vee})$*-isotypical components, i.e.*

$$M^W = \bigoplus_{\chi \in (\widehat{X/Q^\vee})} M^{W,\chi}, \tag{3.2.6}$$

PROOF. That M^W is a W_X-module, follows because W is a normal subgroup of W_X. The decomposition (3.2.6) follows from the last statement of Lemma 3.2.9 and general representation theoretic considerations. □

Let us apply Lemma 3.2.11 to the W_X-module $\mathrm{BVP}_k(\lambda, U)_\pi$. We get

$$\mathrm{BVP}_k(\lambda, U)_\pi^W = \bigoplus_{\chi \in (\widehat{X/Q^\vee})} \mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi}, \tag{3.2.7}$$

with

$$\begin{aligned}\mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi} &= \{f \in \mathrm{BVP}_k(\lambda, U)_\pi^W | \pi(\tau_x) f = \chi(x + Q^\vee) f \ \forall x \in X\} \\ &= \{f \in \mathrm{BVP}_k(\lambda, U) | \pi(w\tau_x) f = \chi(x + Q^\vee) f \ \forall w \in W_0, x \in X\}.\end{aligned}$$

Call a U-valued function f, (W, χ)-invariant if $\pi(w\tau_x) f = \chi(x + Q^\vee) f$ for all $w \in W_0$ and $x \in X$.

EXAMPLE 3.2.12. Consider the case: χ is trivial and $\tau(X) \subset W_X$ acts trivial on U. In this case

$$\mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi} = \mathrm{BVP}_k(\lambda, U)_\pi^{W_X} = \{f \in \mathrm{BVP}_k(\lambda, U) |\, f^{\tau_x} = f \quad \forall x \in X\}_\pi^{W_0}.$$

Hence in this case the study of $\pi(W, \chi)$-invariant solutions amounts to studying the associated bosonic theory under X-periodicity constraints (or equivalently, we view the vector-valued quantum system on the torus V/X).

The quantum Hamiltonian (3.2.1) for Σ_0 of type A_n takes the explicit form

$$-\Delta + k \sum_{m \in \mathbb{Z}} \sum_{1 \le i \ne j \le n+1} \delta(x_i - x_j + m)$$

in the scalar case. Here we have embedded V into $\mathbb{R}^{n+1}$ as the hyperplane defined by $x_1 + \cdots + x_{n+1} = 0$. Although studying the (W, χ)-invariant ($\chi \in \widehat{X/Q^\vee}$) solutions to the boundary value problem is similar to the study of the spectral problem for the system describing $n+1$ quantum spinless bosons on the circle with pair-wise delta-function interactions, they do not follows from each other directly. By generalizing the constructions of this chapter to more general root systems [**57**], a precise connection can be made. This system on the circle has been extensively studied in the physical literature, see e.g. [**29, 56, 80, 81, 26, 47**] (scalar case) and [**23**],[**80**] (spin-$\frac{1}{2}$ fermions). The upgrade to other classical root systems amounts to adding particular reflection terms to the physical model, see e.g. [**72**], [**9**], [**28**], [**43**], [**53**] and [**62**].

We call the spectral parameter $\lambda \in V_{\mathbb{C}}^* = V^* + iV^*$ regular if $\lambda(\alpha^\vee) \neq 0$ for all $\alpha \in \Sigma_0$. We call λ singular otherwise. The set of regular spectral parameters is W_0-invariant and is denoted by $V_{\mathbb{C},reg}^*$. Furthermore, λ is called real (respectively purely imaginary) if $\lambda \in V^*$ (respectively $\lambda \in iV^*$). The set of all regular real spectral parameters is denoted by V_{reg}^*.

DEFINITION 3.2.13. *Let $\chi \in \widehat{X/Q^\vee}$. The* spectrum *associated to χ is the set*

$$\mathcal{S}_k(U,\chi) := \{\lambda \in V_{\mathbb{C}}^* | \mathrm{BVP}_k(\lambda,U)_\pi^{W,\chi} \neq \{0\}\}.$$

The regular spectrum *associated to χ is the set $\mathcal{S}_k^{reg}(U,\chi) = \mathcal{S}_k(U,\chi) \cap V_{\mathbb{C},reg}^*$. When $X = Q^\vee$ we write $\mathcal{S}_k(U)$ (respectively $\mathcal{S}_k^{reg}(U)$) for the spectrum (respectively the regular spectrum).*

Note that by (3.2.7)

$$\mathcal{S}_k(U) = \dot{\bigcup_{\chi \in \widehat{X/Q^\vee}}} S_k(U,\chi). \tag{3.2.8}$$

A spectral parameter $\lambda \in V_{\mathbb{C}}^*$ is called generic if

$$\lambda(\alpha^\vee) \neq \pm k_\alpha \quad \forall \alpha \in \Sigma_0. \tag{3.2.9}$$

The set of generic spectral parameters $\lambda \in V_{\mathbb{C}}^*$ is W_0-invariant because k is W_0-invariant. Note that "generic" and "regular" are concepts that do not depend on U, X and χ.

Two of the main results of this chapter are Theorem 3.7.16 and Theorem 3.8.3. Theorem 3.7.16 states that for generic regular $\lambda \in V_{\mathbb{C}}^*$ and $\chi \in \widehat{X/Q^\vee}$, the spectral parameter λ is contained in the regular spectrum $\mathcal{S}_k^{reg}(U,\chi)$ iff λ satisfies certain Bethe ansatz equations (3.7.22), a family of transcendental equations.

Theorem 3.8.3 states that if U is a *unitary* finite dimensional representation of W_X and k a *strictly positive* multiplicity function, then a necessary condition for a generic regular spectral parameter λ to be in a $\mathcal{S}_k^{reg}(U,\chi)$ is that λ is purely imaginary, i.e. $\lambda \in iV^*$.

3.3. Generalization of Girardeau's isomorphism

Let $C^\omega(V)$ be the space of real analytic functions on V, which we consider as a W-module with respect to the usual action (2.2.7). Consider for $\lambda \in V_{\mathbb{C}}^*$ the space

$$E(\lambda) = \{f \in C^\omega(V) \mid p(\partial)f = p(\lambda)f \quad \forall\, p \in S(V)_{\mathbb{C}}^{W_0}\}, \tag{3.3.1}$$

which is a W-submodule of $C^\omega(V)$ under the usual action (2.2.7). We observed in Example 3.2.8 that

$$E(\lambda) = \mathrm{BVP}_0(\lambda), \qquad \lambda \in V_{\mathbb{C}}^*. \tag{3.3.2}$$

In this section we give a convenient description of the solution space $\mathrm{BVP}_k(\lambda)^W$ of the boundary value problem (Definition 3.2.6) in terms of the space of invariants in $E(\lambda)$

with respect to the k-dependent W-action Q_k (cf. Theorem 2.5.12). We will view this result as a natural generalization of Girardeau's [**29**] equivalence between the impenetrable quantum Bose-gas and the free quantum Fermi-gas on the circle to arbitrary root systems and to arbitrary multiplicity functions k. In order to convey the essence, while the technical details are kept to a minimum, we restrict ourselves in this expository section to the scalar case. Later in this chapter the general vector-valued case is considered.

We start by generalizing Girardeau's [**29**] results on the impenetrable quantum Bose-gas on the circle (which relates to the extremal case $k \equiv \infty$ and to the affine root system of type A) to arbitrary affine root systems.

We denote $E(\lambda)^{-W}$ for the space of functions $f \in E(\lambda)$ satisfying $f(w^{-1}v) = (-1)^{l(w)}f(v)$ for all $w \in W$ and $v \in V$. Since translations $\mu \in Q^\vee \subset W$ have even length, $E(\lambda)^{-W}$ consists of $Q^\vee$-translation invariant functions. In particular, $E(\lambda)^{-W}$ is the solution space to the spectral problem for free fermionic quantum integrable system on $V/Q^\vee$ associated to the Laplacean Δ on V.

Following the analogy with Girardeau's [**29**] analysis of the impenetrable quantum bosons on the circle, we define now a linear map $G : C^\omega(V) \to C(V)^W$ by

$$\big(Gf\big)(w^{-1}v) := f(v), \qquad w \in W,\ v \in \overline{C_+}. \tag{3.3.3}$$

The map G is injective: for $g \in C(V)^W$ in the image of G, the function $G^{-1}g$ is the unique real analytic continuation of $g|_{C_+}$ to V.

For $k \equiv \infty$ we interpret the boundary conditions (3.2.3) as $f|_{V_a} \equiv 0$ for all $a \in \Sigma^+$. The solution spaces $\mathrm{BVP}_\infty(\lambda)^W$ of the associated boundary value problem (see Definition 3.2.6) can now be analyzed as follows.

PROPOSITION 3.3.1. *For $\lambda \in V_{\mathbb{C}}^*$ the map G restricts to a linear isomorphism $G : E(\lambda)^{-W} \xrightarrow{\sim} \mathrm{BVP}_\infty(\lambda)^W$.*

PROOF. A function $f \in E(\lambda)^{-W}$ vanishes on the root hyperplanes V_a ($a \in \Sigma^+$), hence so does $g := G(f) \in C(V)^W$. The function g furthermore satisfies the differential equations (3.2.4), hence $g \in \mathrm{BVP}_\infty(\lambda)^W$.

For $g \in \mathrm{BVP}_\infty(\lambda)^W$ we define $f = \widetilde{G}(g) \in C(V)^{-W}$ by $f(w^{-1}v) := (-1)^{l(w)}g(v)$ for $w \in W$ and $v \in \overline{C_+}$. This is well defined since g vanishes on the root hyperplanes V_a ($a \in \Sigma^+$). Since f is W-alternating we have $f \in C^{1,(0)}(V)$. The function f satisfies the differential equations (3.2.4), hence $f \in \mathrm{BVP}_0(\lambda)^{-W} = E(\lambda)^{-W}$, where the last equality follows from (3.3.2). The proof is now completed by observing that $\widetilde{G} : \mathrm{BVP}_\infty(\lambda)^W \to E(\lambda)^{-W}$ is the inverse of the map $G : E(\lambda)^{-W} \to \mathrm{BVP}_\infty(\lambda)^W$. $\square$

For root system Σ_0 of type A, Proposition 3.3.1 is due to Girardeau [**29**].

For the generalization of Proposition 3.3.1 to arbitrary multiplicity function k it is convenient to reinterpret the space $E(\lambda)^{-W}$ as follows. Given a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_{l(w)}}$ for $w \in W\backslash\{e\}$, the operator

$$Q_\infty(w) = \lim_{k\to\infty} k_w^{-1}Q_k(w) \tag{3.3.4}$$

is well-defined because of the specific form of Q_k. Here $k_w := k_{a_{i_1}} k_{a_{i_2}} \ldots k_{a_{i_{l(w)}}}$ for a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}}$. Recall the integral operators $I(a)$ from Chapter 2 (cf. (2.3.2)) acting on $C(V)$. Then

$$Q_\infty(w) = I(a_{i_1}) I(a_{i_2}) \cdots I(a_{i_{l(w)}}) \qquad (w \in W \backslash \{e\})$$

holds, where $w = s_{i_1} s_{i_2} \cdots s_{i_{l(w)}}$ is a reduced expression. In fact, the integral operators $I(a)$ ($a \in I$) satisfy the braid relations of Σ with respect to the fixed basis I of Σ. Furthermore the limit (3.3.4) shows that the quadratic relations $I(a)^2 = 0$ holds for $a \in I$ (see also [**34**]).

Denote

$$E(\lambda)^W_{Q_\infty} := \{ f \in E(\lambda) \,|\, Q_\infty(w) f = 0, \quad \forall\, w \in W \setminus \{e\} \}.$$

We now have the following simple observation.

LEMMA 3.3.2. *For $f \in C(V)$ and $b \in \Sigma$ we have $s_b f = -f$ if and only if $I(b) f = 0$. In particular, $E(\lambda)^{-W} = E(\lambda)^W_{Q_\infty}$ for all $\lambda \in V^*_{\mathbb{C}}$.*

PROOF. It is immediate that $I(b) f = 0$ if $s_b f = -f$. Conversely, suppose $I(b) f = 0$. It follows from (2.3.4) and (2.5.8) that

$$\partial_{Db^\vee} I(b) = 1 + s_b \tag{3.3.5}$$

holds as operators on $C^\infty(V)$ (or $C(V)$, which is easily seen). Applied to f we obtain $s_b f = -f$. □

By Lemma 3.3.2, Proposition 3.3.1 can be reformulated as the statement that the map G yields an isomorphism

$$G : E(\lambda)^W_{Q_\infty} \xrightarrow{\sim} \mathrm{BVP}_\infty(\lambda)^W. \tag{3.3.6}$$

The generalization of (3.3.6) for arbitrary multiplicity function k is now the statement that the map G restricts to a linear isomorphism

$$G : E(\lambda)^W_{Q_k} \xrightarrow{\sim} \mathrm{BVP}_k(\lambda)^W \tag{3.3.7}$$

for arbitrary positive multiplicity function k, where $E(\lambda)^W_{Q_k}$ is the subspace of $Q_k(W)$-invariant functions in $E(\lambda)$. The proof of the isomorphism (3.3.7) (for the vector valued version) will be given in Section 3.6.

In order to reveal the full symmetry structures underlying the isomorphism (3.3.7), we will consider the upgrade of the map G to a k-dependent linear endomorphism T_k of $C(V)$ which intertwines the $Q_k(W)$-action with the usual W-action (2.2.7), and which acts as G when applied to $Q_k(W)$-invariant functions. The map which does the job for the scalar case is Gutkin's [**32**] propagation operator, defined by $\big(T_k f\big)(w^{-1} v) = \big(Q_k(w) f\big)(v)$ for $w \in W$ and $v \in \overline{C_+}$ (see also (3.5.1)) The propagation operator T_k now restricts to an isomorphism

$$T_k : E(\lambda) \xrightarrow{\sim} \mathrm{BVP}_k(\lambda) \tag{3.3.8}$$

for all $\lambda \in V_{\mathbb{C}}^*$ (cf. [**32**] and Theorem 3.6.4), which implies (3.3.7) by restricting to the subspaces of W-invariant functions. In Section 3.6 we generalize (3.3.8) and (3.3.7) (cf. Theorem 3.6.4) to the vector-valued case.

3.4. Dunkl operators

It is well known that conserved integrals for quantum integrable systems of Calogero-Moser type can be conveniently expressed in terms of Dunkl-type operators, which are explicit commuting first-order differential-reflection operators, see e.g. [**38**], [**19**] and [**8**]. The Dunkl operators, together with the usual Weyl group action, form a faithful representation of suitable degenerations of affine Hecke algebras, see [**63**, Cor. 2.9]. The exploration of these structures has been instrumental in solving the corresponding quantum integrable systems.

In this section we derive the Dunkl-type operators and the underlying Hecke algebra structures for the vector-valued quantum integrable systems with delta-potentials as introduced in Section 3.2.

In this section X denotes as usual a lattice between $Q^\vee$ and $P^\vee$, and U a finite dimensional representation of W_X.

We initially define the Dunkl operators as explicit differential-reflection operators on the space $C^\infty(V_{reg}, U) = C^\infty(V_{reg}) \otimes U$ of smooth U-valued functions on V_{reg}. In Section 3.6 we obtain the key result that these Dunkl operators act on the solution space $\mathrm{BVP}_k(\lambda, U)$ to the boundary value problem. Together with the π-action (2.2.8) of W_X, the space $\mathrm{BVP}_k(\lambda, U)$ becomes a module over the associated graded algebra H_k^X of Cherednik's (suitably filtered) degenerate double affine Hecke algebra (see [**9**] and Theorem 2.2.2).

We denote $\chi : \mathbb{R} \setminus \{0\} \to \{0, 1\}$ for the characteristic function of the interval $(-\infty, 0)$, so $\chi(x) = 1$ if $x < 0$ and $\chi(x) = 0$ if $x > 0$. For $a \in \Sigma$ the function $\chi_a(v) := \chi(a(v))$ ($v \in V_{reg}$) defines a smooth function on V_{reg}, which is constant on the alcoves C of V_{reg}. In fact, for $w \in W$ and $a \in \Sigma^+$ we have

$$\chi_a|_{w^{-1}C_+} \equiv \begin{cases} 1 & \text{if } wa \in \Sigma^- \\ 0 & \text{if } wa \in \Sigma^+, \end{cases} \tag{3.4.1}$$

hence χ_a is nonzero on a given alcove $w^{-1}C_+$ ($w \in W$) for only finitely many positive roots $a \in \Sigma^+$. The vector-valued Dunkl-type operators

$$\mathcal{D}_v^k = \partial_v + \sum_{a \in \Sigma^+} k_a Da(v) \chi_a(\cdot) \pi(s_a) \qquad (v \in V), \tag{3.4.2}$$

thus define linear operators on $C^\infty(V_{reg}, U)$, which depend linearly on $v \in V$, and where $\pi(s_a) = s_a \otimes s_a$ (cf. subsection 2.2.3). For $f \in C^\infty(V_{reg}, U)$ and $w \in W$ we have by (3.4.1)

$$\mathcal{D}_v^k f|_{w^{-1}C_+} = \Big(\partial_v f + \sum_{a \in \Sigma^+ \cap w^{-1}\Sigma^-} k_a Da(v) \pi(s_a) f\Big)\Big|_{w^{-1}C_+}. \tag{3.4.3}$$

In particular, for the fundamental alcove C_+ we simply have

$$\mathcal{D}_v^k f|_{C_+} = \partial_v f|_{C_+}. \tag{3.4.4}$$

The Dunkl operators $\mathcal{D}_v^k$ ($v \in V$) and the π-action (2.2.8) of W_X on $C^\infty(V_{reg}, U)$ satisfy the following fundamental commutation relations.

THEOREM 3.4.1. **(i)** *We have the cross relation*

$$\pi(s_a)\mathcal{D}_v^k = \mathcal{D}_{s_{Da}v}^k \pi(s_a) + k_a Da(v), \qquad v \in V,\ a \in I.$$

(ii) *The Dunkl operators $\mathcal{D}_v^k$ ($v \in V$) pair-wise commute.*
(iii) *We have the cross relation*

$$\pi(\omega)\mathcal{D}_v^k - \mathcal{D}_{D\omega(v)}^k \pi(\omega) = 0 \qquad v \in V,\ \omega \in \Omega_X.$$

PROOF. **(i)** Fix $v \in V$ and $a \in I$. By a direct computation we have

$$\pi(s_a)\mathcal{D}_v^k \pi(s_a) = \partial_{s_{Da}v} + \sum_{b \in s_a\Sigma^+} k_b Db(s_{Da}v)\chi_b(\cdot)\pi(s_b).$$

Since $s_a\Sigma^+ = \big(\Sigma^+ \setminus \{a\}\big) \cup \{-a\}$ we obtain

$$\begin{aligned} \pi(s_a)\mathcal{D}_v^k &= \mathcal{D}_{s_{Da}v}^k \pi(s_a) - k_a Da(s_{Da}v)\big(\chi_a(\cdot) + \chi_{-a}(\cdot)\big) \\ &= \mathcal{D}_{s_{Da}v}^k \pi(s_a) + k_a Da(v), \end{aligned}$$

which is the desired cross relation.

(ii) We derive the commutativity of the Dunkl operators $\mathcal{D}_v^k$ ($v \in V$) as a direct consequence of (3.4.4) and the cross relation. Let $f \in C^\infty(V_{reg}, U)$ and $v, v' \in V$. We show by induction on the length $l(w)$ of $w \in W$ that

$$[\mathcal{D}_v^k, \mathcal{D}_{v'}^k] f|_{w^{-1}C_+} = 0. \tag{3.4.5}$$

By (3.4.4), equation (3.4.5) is obviously valid for $w = e$ the unit element of W_X. To prove the induction step, it suffices to show that

$$\pi(s_a)[\mathcal{D}_v^k, \mathcal{D}_{v'}^k] = [\mathcal{D}_{s_{Da}v}^k, \mathcal{D}_{s_{Da}v'}^k]\pi(s_a) \tag{3.4.6}$$

for all $a \in I$. For the proof of (3.4.6), first observe that

$$\pi(s_a)\mathcal{D}_v^k\mathcal{D}_{v'}^k - \mathcal{D}_{s_{Da}v}^k\mathcal{D}_{s_{Da}v'}^k\pi(s_a) = k_a\big(Da(v')\mathcal{D}_v^k + Da(v)\mathcal{D}_{v'}^k - Da(v)Da(v')\mathcal{D}_{Da^\vee}^k\big) \tag{3.4.7}$$

for all $a \in I$, which follows from applying the cross relation twice. Now (3.4.6) follows from the fact that the right hand side of (3.4.7) is symmetric in v and v'.
(iii) Let $f \in C^\infty(V_{reg}, U), v \in V, \omega \in \Omega_X$ and $\xi \in V_{reg}$. Then

$$\begin{aligned} (\pi(\omega)\mathcal{D}_v^k f)(\xi) &= \omega((\mathcal{D}_v^k f)(\omega^{-1}\xi)) \\ &= (\partial_{(D\omega)v}\pi(\omega)f)(\xi) + \sum_{a \in \Sigma^+} k_a (Da)(v)\chi_a(\omega^{-1}\xi)(\pi(\omega s_a)f)(\xi) \\ &= (\partial_{(D\omega)v}\pi(\omega)f)(\xi) + \sum_{a \in \Sigma^+} k_a (Da)(v)\chi_{\omega a}(\xi)(\pi(s_{\omega a}\omega)f)(\xi) \end{aligned}$$

holds. In the second equality we have used (2.3.11), in the third $\omega s_a = s_{\omega a}\omega$ and $\chi_a(\omega^{-1}\xi) = \chi(a(\omega^{-1}\xi)) = \chi_{\omega a}(\chi)$.

Using that $\omega(\Sigma^+) = \Sigma^+$, that k is a W_X-invariant function and summing over $b = \omega a \in \omega\Sigma^+ = \Sigma^+$ gives

$$(\pi(\omega)\mathcal{D}_v^k f)(\xi) = (\partial_{(D\omega)v}\pi(\omega)f)(\xi) + \sum_{b\in\Sigma^+} k_b(D(\omega^{-1}b))(v)\chi_b(\xi)(\pi(s_b\omega)f)(\xi).$$

Use $D(\omega^{-1}b)(v) = (Db)(D\omega(v))$ to conclude the proof. □

By Theorem 3.4.1**(ii)**, the assignment $v \mapsto \mathcal{D}_v^k$ uniquely extends to an algebra morphism $S(V)_{\mathbb{C}} \to \mathrm{End}(C^\infty(V_{reg}, U))$. We denote $p(\mathcal{D}^k)$ for the differential-reflection operator on $C^\infty(V_{reg}, U)$ associated to $p \in S(V)_{\mathbb{C}}$.

We are now in position to give a proof of Theorem 2.2.2.

PROOF OF THEOREM 2.2.2. We take for $X = P^\vee$ and U the trivial representation in the proof. Suppose that $\sum_{w\in W^e} p_w(\mathcal{D}^k)w = 0$ as endomorphisms of $C^\infty(V_{reg})$ with only finitely many $p_w \in S(V)_{\mathbb{C}}$'s non zero. We show that all p_w's are zero. Let B be an open ball in C_+ with the property that the closed balls $\overline{\omega(B)}$ ($\omega \in \Omega$) are pairwise disjoint. Equation (3.4.4) implies

$$\sum_{w\in W^e} p_w(\partial)(wf)|_B \equiv 0, \qquad f \in C^\infty(V_{reg}). \tag{3.4.8}$$

Applying (3.4.8) to functions f of the form $w^{-1}g$ with $w \in W^e$ and $g \in C^\infty(V_{reg})$ having support in the open ball B, we conclude that $p_w(\partial) = 0$ as constant coefficient differential operator on smooth functions in B, hence $p_u = 0$.

Let $\widetilde{H}_k^e$ be the complex unital associative algebra generated by $v \in V$, s_a ($a \in I$) and ω ($\omega \in \Omega$) with defining relations as in **(b)**, **(c)** and **(d)** . By Theorem 3.4.1, the assignment $v \mapsto \mathcal{D}_v^k$, together with the action (2.2.7) of W^e, uniquely defines an algebra morphism $\pi_k : \widetilde{H}_k^e \to \mathrm{End}(C^\infty(V_{reg}))$. By the previous paragraph and by the cross relations in $\widetilde{H}_k^e$ it follows that π_k is injective and that $\widetilde{H}_k^e \simeq S(V)_{\mathbb{C}} \otimes \mathbb{C}[W^e]$ as vector spaces (the Poincaré-Birkhoff-Witt Theorem for $\widetilde{H}_k^e$). The theorem follows now immediately. □

THEOREM 3.4.2. *The assignment $v \mapsto \mathcal{D}_v^k$, together with the π-action* (2.2.8) *of W_X, defines a faithful representation $\pi_k : H_k^X \longrightarrow \mathrm{End}(C^\infty(V_{reg}, U))$.*

PROOF. That π_k defines a representation follows immediately from Theorem 3.4.1. The faithfulness of π_k in the scalar case for $X = P^\vee$ is the first part of the proof of Theorem 2.2.2. The vector-valued case and general $Q^\vee \subset X \subset P^\vee$ follows analogously. □

We will use the notation M_{π_k} to indicate that a subspace $M \subset C^\infty(V_{reg}, U)$ is a W_X-submodule or H_k^X-submodule of $C^\infty(V_{reg}, U)$ with respect to the π_k-action (cf. the paragraph following (2.2.8)).

For trivial multiplicity parameters $k \equiv 0$, the operator $p(\mathcal{D}^0)$ ($p \in S(V)_{\mathbb{C}}$) on $C^\infty(V_{reg}, U)$ is the constant-coefficient differential operator $p(\partial)$ on $C^\infty(V_{reg}, U)$. We have the following striking fact when $p \in S(V)_{\mathbb{C}}$ is W_0-invariant.

COROLLARY 3.4.3. *For $p \in S(V)_{\mathbb{C}}^{W_0}$ we have $p(\mathcal{D}^k) = p(\partial)$ as operators on $C^\infty(V_{reg}, U)$.*

PROOF. Let $p \in S(V)_{\mathbb{C}}^{W_0}$ and $f \in C^\infty(V_{reg}, U)$. By (3.4.4) we have $p(\mathcal{D}^k)f|_{C_+} = p(\partial)f|_{C_+}$. Let $w \in W$ and $v \in C_+$. By Proposition 2.2.7 applied twice (once with multiplicity function k, once with $k \equiv 0$), we have

$$\begin{aligned} w\big(\big(p(\mathcal{D}^k)f\big)(w^{-1}v)\big) &= \big(p(\mathcal{D}^k)(\pi(w)f)\big)(v) \\ &= \big(p(\partial)(\pi(w)f)\big)(v) = w\big(\big(p(\partial)f\big)(w^{-1}v)\big), \end{aligned}$$

hence $p(\mathcal{D}^k)f = p(\partial)f$.

□

REMARK 3.4.4. The Dunkl operators $\mathcal{D}_v^k$, Theorem 3.4.1, Theorem 3.4.2 and Corollary 3.4.3 have their obvious analogs in the context of finite root systems. In that case, the Dunkl-type operators are

$$\partial_v + \sum_{\alpha \in \Sigma_0^+} k_\alpha \alpha(v) \chi_\alpha(\cdot) \pi(s_\alpha), \qquad v \in V$$

realizing, together with the usual restriction of the π-action to W_0, an action of the degenerate affine Hecke algebra $H_k^{(0)}$ on the space of smooth U-valued functions on $V \setminus \bigcup_{\alpha \in \Sigma_0^+} V_\alpha$. For Σ_0 a root system of classical type. these operators were constructed using solutions of classical Yang-Baxter equations and reflection equations in [**66, 60, 62**] (type A) and [**53**] in the scalar case. The above construction of the Dunkl-type operators fits into Cherednik's [**9**] general framework relating root system analogs of r-matrices to (degenerate) affine Hecke algebras and Dunkl operators.

3.5. The propagation operator

In this section we generalize Gutkin's [**32**] propagation operator from the scalar case to the vector-valued case. It will be shown that this operator intertwines the Q_k-action (cf. Theorem 2.5.13) and the π_k-action which is defined in the previous section in terms of Dunkl-type differential-reflection operators.

THEOREM 3.5.1. *The following defines uniquely an endomorphism of $C(V, U)$:*

$$(T_k f)_{|w^{-1}C_+} = (\pi(w^{-1})(Q_k(w)f))_{|w^{-1}C_+} \quad (w \in W) \tag{3.5.1}$$

for $f \in C(V, U)$. The linear endomorphism T_k of $C(V, U)$ is called the propagation operator. In particular, T_0 is the identity operator on $C(V, U)$.

Before giving the proof consider the following integral-reflection endomorphisms of $C(V,U)$:

$$Q_{k,b} = s_b \otimes s_b + k_b I(b) \otimes \mathrm{id}_U \quad (b \in \Sigma). \tag{3.5.2}$$

In particular $Q_k(s_a) = Q_{k,a}$ for $a \in I$. We also have

$$(Q_{k,b} f)(v) = s_b(f(v)) \quad (b \in \Sigma, v \in V_b). \tag{3.5.3}$$

PROOF OF THEOREM 3.5.1. Let $f \in C(V,U)$. It is clear that $T_k f \in B(V,U)$. Whence it suffices to show that $T_k f$, considered as a multi-valued U-valued function (cf. Remark 2.5.2), is single-valued. Let $v \in V$ and $\overline{C}, \overline{C'} \in \mathcal{C}_v$ (see Section 2.5 for the definition of $\mathcal{C}_v$). There is a sequence $C_0 = C, C_1, \ldots, C_r = C'$ of chambers in $\mathcal{C}_v$ such that $v \in \overline{C}$ for some r such that C_j, C_{j+1} ($j = 0, \ldots, r-1$) are adjacent. Then

$$\lim_{C_j \ni x \to v} (T_k f)(x) = \lim_{C_{j+1} \ni x \to v} (T_k f)(x)$$

follows from (3.5.3). Whence $T_k f \in B(V,U)$ is single-valued and therefore $T_k f \in C(V,U)$. □

The operator defined in the previous theorem is the U-valued version of Gutkin's propagation operator (cf. [**32**, Theorem 2.6]). By construction $T_k : C(V,U)_Q \longrightarrow C(V,U)_\pi$ is W-equivariant.

LEMMA 3.5.2. *The propagation operator* $T_k : C(V,U)_Q \longrightarrow C(V,U)_\pi$ *is a linear endomorphism of* W_X*-modules.*

PROOF. Because W_X is generated by the subgroups W and Ω_X (cf. Subsection 2.2.2.) and by the W-equivariance of T_k, it suffices to show that T_k is Ω_X-equivariant, that is,

$$T_k(Q_k(\omega) f) = \pi(\omega) T_k f \quad (\omega \in \Omega_X, f \in C(V,U)). \tag{3.5.4}$$

If $w_1 \in W$, $\omega \in \Omega_X$, $v \in \overline{C}_+$ and $f \in C(V,U)$, then

$$\begin{aligned}
(T_k(Q_k(\omega)f))(w_1^{-1}v) &= w_1^{-1}((Q_k(w_1)Q_k(\omega)f)(v)) \\
&= w_1^{-1}\omega[\omega^{-1}((Q_k(\omega)Q_k(\omega^{-1}w_1\omega)f)(v))] \\
&= w_1^{-1}\omega[(Q_k(\omega^{-1}w_1\omega)f)(\omega^{-1}v)] \\
&= \omega[(T_k f)((\omega^{-1}w_1\omega)^{-1}\omega^{-1}v)] \\
&= (\pi(\omega)(T_k f))(w_1^{-1}v),
\end{aligned}$$

holds. In the third equality we have used $Q_k(\omega) = \pi(\omega)$ as operators on $C(V,U)$, in the fourth that Ω_X normalizes W in W_X and that Ω_X leaves C_+ stable. In other words, T_k is Ω_X-equivariant. □

Observe that the space of U-valued real analytic functions $C^\omega(V,U)$ is a H_k^X-submodule of $C^\infty(V,U)$ under the Q_k-action.

DEFINITION 3.5.3. *For a compact subset A of V with nonempty interior we define $C^\omega(A)$ to be the space of functions f on A that have a (necessarily unique) real analytic extension to V. Let*

$$B^\omega(V) = \prod_{C\in\mathcal{C}} C^\omega(\overline{C}).$$

We have injections $C^\omega(V) \hookrightarrow B^\omega(V) \hookrightarrow B(V)$. Let $CB^\omega(V) = C(V) \cap B^\omega(V)$. The vector-valued versions of these spaces are also considered, using the convention (2.2.6).

Denote $C^{\omega,(k)}(V,U)$ for the space of functions $f \in CB^\omega(V,U)$ satisfying

$$\partial^r_{Db^\vee} f\big(v + 0Db^\vee\big) - \partial^r_{Db^\vee} f\big(v - 0Db^\vee\big) = \big(1-(-1)^r\big)k_b s_b \left(\partial^{r-1}_{Db^\vee} f\big(v+0Db^\vee\big)\right) \tag{3.5.5}$$

for $b \in \Sigma^+$, $v \in V_b$ sub-regular and $r \in \mathbb{Z}_{>0}$. Observe that $C^{\omega,(k)}(V,U)$ is contained in the subspace $C^{1,(k)}(V,U)$ involving normal derivative jump conditions over affine hyperplanes V_b ($b \in \Sigma^+$) up to first order, which we have used in the formulation of the boundary value problems (see Proposition 3.2.5 and Definition 3.2.6).

A function $f \in C^{\omega,(k)}(V,U)$ automatically satisfies the jump conditions (3.5.5) for $v \in V_b$ sub-regular, $r \in \mathbb{Z}_{>0}$ and $b \in \Sigma^-$. Hence the space $C^{\omega,(k)}(V,U)$ does not dependent on the choice of positive roots Σ^+ in Σ. Since also $\omega(\Sigma) = \Sigma$ ($\omega \in \Omega_X$) holds, we can and will interpret $C^{\omega,(k)}(V,U)_{\pi_k}$ and $CB^\omega(V,U)_{\pi_k}$ as W_X-submodules of $C^\infty(V_{reg},U)_{\pi_k}$.

Observe that the propagation operator T_k restricts to a linear map

$$T_k : C^\omega(V,U) \to CB^\omega(V,U).$$

We now obtain the following theorem.

THEOREM 3.5.4. **(i)** *$C^{\omega,(k)}(V,U)_{\pi_k} \subseteq C^\infty(V_{reg},U)_{\pi_k}$ is a H_k^X-submodule.*
(ii) *The propagation operator T_k restricts to an isomorphism*

$$T_k : C^\omega(V,U)_{Q_k} \xrightarrow{\sim} C^{\omega,(k)}(V,U)_{\pi_k}$$

of H_k^X-modules.

PROOF. We first show that T_k restricts to a linear isomorphism $T_k : C^\omega(V,U) \xrightarrow{\sim} C^{\omega,(k)}(V,U)$. For this we use the commutation relations

$$s_a \cdot \big(Da^\vee\big)^r - (-1)^r \big(Da^\vee\big)^r \cdot s_a = \big(1-(-1)^r\big)k_a\big(Da^\vee\big)^{r-1}, \qquad a \in I,\ r \in \mathbb{Z}_{>0} \tag{3.5.6}$$

in H_k^X, which follows from (2.2.11) applied to $q = (Da^\vee)^r \in S(V)_{\mathbb{C}}$.

Let $\phi \in C^\omega(V,U)$ and denote $f = T_k\phi \in CB^\omega(V,U)$. We show that f satisfies the derivative jumps (3.5.5) over sub-regular $v \in V_b$ ($b \in \Sigma^+$) for all $r \in \mathbb{Z}_{>0}$. In view of the W-equivariance of the propagation operator T_k, it suffices to derive the derivative jumps for f over sub-regular vectors $v \in V_a \cap \overline{C_+}$ ($a \in I$). Fix $a \in I$, $v \in V_a \cap \overline{C_+}$ sub-regular and $r \in \mathbb{Z}_{>0}$. For $\epsilon > 0$ small we have $v + tDa^\vee = s_a(v - tDa^\vee) \in C_+$ for $0 < t < \epsilon$. Hence

$$\partial^r_{Da^\vee} f(v + 0Da^\vee) = \partial^r_{Da^\vee}\phi(v) = s_a(Q_k(s_a)(\partial^r_{Da^\vee}\phi)(v)), \tag{3.5.7}$$

where the second equality follows from (3.5.3). On the other hand,

$$\partial^r_{Da^\vee} f(v-0Da^\vee) = (-1)^r \partial^r_{Da^\vee}(f^{s_a})(v+0Da^\vee) = s_a\big((-1)^r\partial^r_{Da^\vee}(Q_k(s_a)\phi)(v)\big). \tag{3.5.8}$$

Combining (3.5.7) and (3.5.8) now yields

$$\begin{aligned}
&\partial^r_{Da^\vee} f(v+0Da^\vee) - \partial^r_{Da^\vee} f(v-0Da^\vee)\\
&\quad= s_a\Big(\Big(\big(Q_k(s_a)\partial^r_{Da^\vee} - (-1)^r\partial^r_{Da^\vee}Q_k(s_a)\big)\phi\Big)(v)\Big)\\
&\quad= s_a\Big(\big(1-(-1)^r\big)k_a\partial^{r-1}_{Da^\vee}\phi(v)\Big)\\
&\quad= \big(1-(-1)^r\big)k_a s_a\left(\partial^{r-1}_{Da^\vee} f(v+0Da^\vee)\right),
\end{aligned}$$

where the second equality follows from (the Q_k-image of) (3.5.6). Thus $f \in C^{\omega,(k)}(V,U)$.

The propagation operator $T_k : C^\omega(V,U) \longrightarrow C^{\omega,(k)}(V,U)$ is clearly injective. We now proceed to prove surjectivity. Let $f \in C^{\omega,(k)}(V,U)$ and denote ϕ for the unique U-valued real analytic function on V satisfying $\phi|_{C_+} = f|_{C_+}$. The function $g := f - T_k\psi \in C^{\omega,(k)}(V,U)$ satisfies $g|_{\overline{C_+}} \equiv 0$, hence the continuity of g and the derivative jump conditions (3.5.5) for g imply that

$$\big(\partial^r_{Da^\vee} g\big)(v-0Da^\vee) = 0$$

for $r \in \mathbb{Z}_{\geq 0}$, $a \in I$ and $v \in V_a \cap \overline{C_+}$ sub-regular. Since $g|_C$ has an extension to an U-valued analytic function on the whole Euclidean space V for any alcove $C \in \mathcal{C}$, we conclude that $g|_{\overline{C}} \equiv 0$ for the neighboring alcoves $C = s_aC_+$ ($a \in I$) of C_+. Continuing inductively, we conclude that $g \equiv 0$ on V, hence $f = T_k\psi$.

It remains to show that the isomorphism

$$T_k : C^\omega(V,U)_{Q_k} \xrightarrow{\sim} C^{\omega,(k)}(V,U)_{\pi_k}$$

of W_X-modules is in fact an isomorphism of H_k^X-modules. For this it suffices to show that

$$T_k\big(\partial_v f\big)|_{V_{reg}} = \mathcal{D}^k_v(T_k f|_{V_{reg}}) \tag{3.5.9}$$

for $v \in V$ and $f \in C^\omega(V,U)$. To prove (3.5.9) we use the commutation relation

$$w\cdot v = \big((Dw)v\big)\cdot w + \sum_{a\in\Sigma^+\cap w^{-1}\Sigma^-} k_a Da(v) w s_a \tag{3.5.10}$$

in H_k^X, which can be easily proved by induction on the length $l(w)$ of $w \in W$ using the cross relations in H_k^X (see Theorem 2.2.2**(c)**). Fix $w \in W$ and $v' \in C_+$. By (3.5.10) and

Theorem 2.5.13 we have

$$
\begin{aligned}
w(T_k(\partial_v f)(w^{-1}v')) \\
&= Q_k(w)(\partial_v f)(v') \\
&= \partial_{(Dw)v}(Q_k(w)f)(v') + \sum_{a\in\Sigma^+\cap w^{-1}\Sigma^-} k_a Da(v) Q_k(ws_a)f(v') \\
&= (\partial_{(Dw)v}(\pi(w)T_k f))(v') + \sum_{a\in\Sigma^+\cap w^{-1}\Sigma^-} k_a Da(v)(\pi(ws_a)T_k f)(v') \\
&= (\pi(w)\partial_v(T_k f))(v') + \sum_{a\in\Sigma^+\cap w^{-1}\Sigma^-} k_a Da(v) ws_a(T_k f(s_a w^{-1}v')) \\
&= w(\partial_v(T_k f)(w^{-1}v')) + w\sum_{a\in\Sigma^+\cap w^{-1}\Sigma^-} k_a Da(v) s_a(T_k f(s_a w^{-1}v')) \\
&= w(\mathcal{D}_v^k(T_k f)(w^{-1}v')),
\end{aligned}
$$

where the last equality follows from (3.4.3), and hence

$$T_k(\partial_v f)(w^{-1}v') = \mathcal{D}_v^k(T_k f)(w^{-1}v').$$

□

REMARK 3.5.5. The assertion [**32**, Theorem 2.7] that, in Gutkin's notation, the propagation operator (corresponding to the scalar case) T_k is an automorphism of the W-module $CB^\infty(V)$ seems to be incorrect (see Remark 2.5.5). In [**32**], this result is used to link $\mathrm{BVP}_k(\lambda)$ to $E(\lambda)$ (see (3.3.1)). We will show in the next section that Theorem 3.5.4**(ii)** suffices to provide this link.

REMARK 3.5.6. Theorem 3.5.4 has an obvious analog in the context of finite root systems (compare with Remark 3.4.4). For the scalar case and Σ_0 of type A, the intertwining properties of the propagation operator with respect to the degenerate affine Hecke algebra actions were considered in [**43**] and the normal derivative jump conditions of higher order were considered in [**33**].

COROLLARY 3.5.7. *Fix $v \in V$. The Dunkl operator $\mathcal{D}_v^k$ is a linear operator on $C^{\omega,(k)}(V,U)$ satisfying $\mathcal{D}_v^k\big(T_k f\big) = T_k\big(\partial_v f\big)$ for all $f \in C^\omega(V,U)$.*

In the following proposition we relate the Dunkl operators $\mathcal{D}_v^k$ to the quantum Hamiltonian $\mathcal{H}_k$ (see (3.2.1) and (3.2.2)). Recall that $p_2(\partial) = \Delta$ for the W_0-invariant polynomial $p_2 = \|\cdot\|^2$ on V^*.

PROPOSITION 3.5.8. *For $f \in C^{\omega,(k)}(V,U)$ we have*

$$-p_2(\mathcal{D}^k)f = \mathcal{H}_k f \tag{3.5.11}$$

as U-valued distributions on V.

PROOF. Fix $f \in C^{\omega,(k)}(V,U)$, then $p_2(\mathcal{D}^k)f \in C^{\omega,(k)}(V,U) \subseteq C(V,U)$ and $p_2(\mathcal{D}^k)f_{|V_{reg}} = \Delta f_{|V_{reg}}$ by Corollary 3.4.3. Furthermore, f satisfies the first order normal derivative jumps (3.2.3) over the affine hyperplanes V_a ($a \in \Sigma^+$). The identity (3.5.11) then follows from a standard argument using Green's identity, cf. (the proof of) Proposition 3.2.5. □

By Proposition 3.5.8 it is justified to interpret the quantum Hamiltonian $\mathcal{H}_k$ on $C^{\omega,(k)}(V,U)$ as the operator $-p_2(\mathcal{D}^k)$ on $C^{\omega,(k)}(V,U)$. The complete integrability of the quantum system is then directly reflected by the commutativity of the Dunkl operators $\mathcal{D}_v^k$ ($v \in V$). More precisely, the space $C^{\omega,(k)}(V,U)_\pi^W$ serves as an algebraic model for the Hilbert space of quantum states associated to the vector-valued bosonic quantum system on $V/Q^\vee$ with Hamiltonian $\mathcal{H}_k = -p_2(\mathcal{D}^k)$. The pair-wise commuting operators $p(\mathcal{D}^k)$ ($p \in S(V)_{\mathbb{C}}^{W_0}$) on $C^{\omega,(k)}(V,U)_\pi^W$ are the corresponding quantum conserved integrals.

3.6. The boundary value problem revisited

The operator $p(\mathcal{D}^k)$ ($p \in S(V)_{\mathbb{C}}^{W_0}$) on $C^{\omega,(k)}(V,U)$ satisfies

$$p(\mathcal{D}^k)f|_{V_{reg}} = p(\partial)f|_{V_{reg}}, \qquad f \in C^{\omega,(k)}(V,U)$$

by Corollary 3.4.3. This key observation leads to an explicit connection between the spectral problem of the operators $p(\mathcal{D}^k)$ ($p \in S(V)_{\mathbb{C}}^{W_0}$) and the boundary value problem as formulated in Definition 3.2.6. We will first do the analysis for the spectral problem of the quantum Hamiltonian $\mathcal{H}_k$ (defined by (3.2.1) and (3.2.2)).

For $E \in \mathbb{C}$ we write $\mathcal{E}(E,U)$ for the space of functions $f \in C^\omega(V,U)$ satisfying $\Delta f = -Ef$ on V (cf. Example 3.2.8). By Proposition 2.2.7, $\mathcal{E}(E,U)_{Q_k} \subseteq C^\omega(V,U)_{Q_k}$ is a H_k^X-submodule. Denote $\mathcal{E}_k(E,U)$ for the space of functions $f \in CB^\omega(V,U)$ satisfying $\mathcal{H}_k f = Ef$ as U-valued distributions on V.

THEOREM 3.6.1. *Fix $E \in \mathbb{C}$.*
(i) *We have*

$$\mathcal{E}_k(E,U) = \{f \in C^{\omega,(k)}(V,U)_{\pi_k} \,|\, p_2(\mathcal{D}^k)f = -Ef\}, \tag{3.6.1}$$

hence $\mathcal{E}_k(E,U)_{\pi_k} \subseteq C^{\omega,(k)}(V,U)_{\pi_k}$ is a H_k^X-submodule.
(ii) *The propagation operator T_k restricts to an isomorphism*

$$T_k : \mathcal{E}(E,U)_{Q_k} \xrightarrow{\sim} \mathcal{E}_k(E,U)_{\pi_k}$$

of H_k^X-modules.

PROOF. **(i)** We first show that $\mathcal{E}_k(E,U) \subset C^{\omega,(k)}(V,U)$. Fix $f \in \mathcal{E}_k(E,U)$. By Proposition 3.2.5, $f \in C^{1,(k)}(V,U) \cap CB^\omega(V,U)$ and $\Delta f|_{V_{reg}} = -Ef|_{V_{reg}}$. Let ψ be the unique U-valued analytic function on V satisfying $\psi|_{C_+} = f|_{C_+}$, then $\psi \in \mathcal{E}(E,U)$. By Theorem 3.5.4 and Corollary 3.4.3 we conclude that $T_k\psi \in C^{\omega,(k)}(V,U)$ and $\Delta(T_k\psi)|_{V_{reg}} = -E(T_k\psi)|_{V_{reg}}$. Hence

$$g := f - T_k\psi \in C^{1,(k)}(V,U) \cap CB^\omega(V,U)$$

satisfies $\Delta g|_{V_{reg}} = -Eg|_{V_{reg}}$ and has the additional property that $g|_{\overline{C_+}} \equiv 0$. Fix $v \in V_a \cap \overline{C_+}$ ($a \in I$) sub-regular. The nontrivial normal derivative jump condition (3.2.3) for g at v

trivializes since $g|_{\overline{C_+}} \equiv 0$, hence g is continuously differentiable in an open neighborhood N of v. Denoting by $\Delta + E$ the (hypo)elliptic constant coefficient differential operator, we see that $g|_N$ is a distributional solution of the U-valued operator $(\Delta + E) \otimes \mathrm{id}_U$ on N, hence $g|_N$ is smooth (cf. Example 3.2.8 and (proof of) Proposition 3.2.5). Since $g|_{\overline{C_+}} \equiv 0$, we conclude that

$$\partial^r_{Da^\vee} g(v - 0Da^\vee) = \partial^r_{Da^\vee} g(v + 0Da^\vee) = 0, \qquad r \in \mathbb{Z}_{\geq 0}.$$

As in the proof of Theorem 3.5.4 we conclude that $g|_{\overline{s_a C_+}} \equiv 0$ for $a \in I$ (alternatively, this is a direct consequence of Holmgren's uniqueness Theorem). Continuing inductively, we conclude that $g \equiv 0$ on V. Hence $f = T_k \psi \in C^{\omega,(k)}(V, U)$.

Formula (3.6.1) now follows from Proposition 3.5.8. Since $p_2(\mathcal{D}^k) = \pi_k(p_2)$, Proposition 2.2.7 implies that $\mathcal{E}(E, U)_{\pi_k} \subset C^{\omega,(k)}(V, U)_{\pi_k}$ is a H_k^X-submodule.
(ii) This follows from Theorem 3.5.4, (3.6.1) and the fact that $Q_k(p_2) = p_2(\partial) = \Delta$. □

We now extend these results to the solution spaces $\mathrm{BVP}_k(\lambda, U)$ of the boundary value problem (Definition 3.2.6). For a H_k^X-module M and $\lambda \in V_{\mathbb{C}}^*$ we define

$$M_\lambda := \{m \in M \mid p \cdot m = p(\lambda) m \quad \forall\, p \in S(V)_{\mathbb{C}}^{W_0}\}, \tag{3.6.2}$$

which is a H_k^X-submodule of M in view of Proposition 2.2.7. By Remark 2.2.8 the module M_λ consists of the vectors $m \in M$ transforming according to the central character $\lambda \in V_{\mathbb{C}}^*$ for the action of the center of the degenerate affine Hecke algebra $H_k^{(0)} \subseteq H_k^X$.

COROLLARY 3.6.2. *Let $\lambda \in V_{\mathbb{C}}^*$. The space $\mathrm{BVP}_k(\lambda, U)$ is the H_k^X-submodule $C^{\omega,(k)}(V, U)_{\pi_k,\lambda}$ of $C^{\omega,(k)}(V, U)$.*

PROOF. By Corollary 3.4.3 and Theorem 3.5.4 we have

$$C^{\omega,(k)}(V, U)_{\pi_k,\lambda} = \{f \in C^{\omega,(k)}(V, U) \mid p(\partial) f|_{V_{reg}} = p(\lambda) f|_{V_{reg}} \quad \forall\, p \in S(V)_{\mathbb{C}}^{W_0}\}, \tag{3.6.3}$$

hence $C^{\omega,(k)}(V, U)_{\pi_k,\lambda} \subseteq \mathrm{BVP}_k(\lambda, U)$. By Proposition 3.2.5 and Remark 3.2.7 we have

$$\mathrm{BVP}_k(\lambda, U) \subseteq \mathcal{E}_k(-p_2(\lambda), U).$$

Theorem 3.6.1 and (3.6.3) now implies that $\mathrm{BVP}_k(\lambda, U) \subseteq C^{\omega,(k)}(V, U)_{\pi_k,\lambda}$. □

Consider the space $E(\lambda, U) \subset C^\omega(V, U)$. Since $Q_k(p) = p(\partial)$ ($p \in S(V)_{\mathbb{C}}$), we see that $E(\lambda, U)_{Q_k}$ is the H_k^X-module $C^\omega(V, U)_{Q_k,\lambda}$. Note that the equality (3.3.2) generalizes immediately to

$$E(\lambda, U) = \mathrm{BVP}_0(\lambda, U).$$

Lemma 3.2.11 applied to the W_X-module $E(\lambda, U)_{Q_k}$ gives the decomposition

$$E(\lambda, U)_{Q_k}^W = \bigoplus_{\chi \in (\widehat{X/Q^\vee})} E(\lambda, U)_{Q_k}^{W,\chi}, \tag{3.6.4}$$

with the χ-isotypical component $E(\lambda,U)^{W,\chi}_{Q_k}$ being defined as

$$\begin{aligned} E(\lambda,U)^{W,\chi}_{Q_k} &= \{\psi\in E(\lambda,U)^W_{Q_k} \,|\, Q_k(\tau_x)\psi=\chi(x+Q^\vee)\psi \ \forall x\in X\} \\ &= \{\psi\in E(\lambda,U) \,|\, Q_k(w\tau_x)\psi=\chi(x+Q^\vee)\psi \,\forall w\in W_0, x\in X\}. \end{aligned} \tag{3.6.5}$$

We now generalize the isomorphism (3.3.7) to the vector-valued case. Before doing this we recall some concepts from parabolic subgroup theory of affine Weyl groups (cf. [**45**]). Let J be a subset of I. The subgroup W_J of W generated by s_a ($a\in J$) is called a *standard parabolic subgroup* of W. The isotropy W_v of a v ($v\in V$) is by definition the following subgroup in W:

$$W_v=\{w\in W|wv=v\}.$$

A well known fact states that for $v\in\overline{C_+}$, the isotropy subgroup W_v is a standard parabolic subgroup of v. Moreover,

$$W_v=\langle s_a|a\in I, s_a(v)=v\rangle=\langle s_a|a\in I, a(v)=0\rangle \quad (v\in\overline{C_+}). \tag{3.6.6}$$

Consider the following subspace of $C^\omega(V,U)$,

$$C^\omega_+(V,U)=\{\psi\in C^\omega(V,U)|s_a(\psi(v))=\psi(v) \quad \forall v\in V_a\cap\overline{C_+}, a\in I\}.$$

LEMMA 3.6.3. *The following*

$$(G\psi)(w^{-1}v)=w^{-1}(\psi(v)), \quad w\in W, v\in\overline{C_+} \tag{3.6.7}$$

defines a linear map $G: C^\omega_+(V,U)\longrightarrow C(V,U)^W_\pi$.

PROOF. It is clear that (3.6.7) defines a linear map $G: C^\omega_+(V,U)\longrightarrow B(V,U)^W_\pi$. We must show that the image of G lies in the subspace $C(V,U)^W_\pi\subset B(V,U)^W_\pi$ (cf. Remark 2.5.2). Let $\psi\in C^\omega_+(V,U)$. It suffices to show that $G\psi$ is single-valued. This follows because by (3.6.6) we have that $w_1^{-1}\psi(v)=w_2^{-1}\psi(v)$ for $v\in\overline{C_+}$ and $w_1^{-1}v=w_2^{-1}v$. □

Note that in the scalar case $C^\omega_+(V,U)$ (respectively (3.6.7)) reduces to $C^\omega(V)$ (respectively (3.3.3)).

THEOREM 3.6.4. *Let* $\lambda\in V^*_{\mathbb{C}}$.
(i) *The propagation operator T_k restricts to an isomorphism*

$$T_k: E(\lambda,U)_{Q_k}\xrightarrow{\sim}\mathrm{BVP}_k(\lambda,U)_{\pi_k}$$

of left H^X_k-modules.
(ii) *The map G* (3.6.7) *restricts to an isomorphism* $G: E(\lambda,U)^{W,\chi}_{Q_k}\xrightarrow{\sim}\mathrm{BVP}_k(\lambda,U)^{W,\chi}_{\pi_k}$ *for all* $\chi\in\widehat{X/Q^\vee}$.

PROOF. **(i)** The restriction of the propagation operator T_k to the H^X_k-module $E(\lambda,U)_{Q_k}=C^\omega(V,U)_{Q_k,\lambda}$ defines an isomorphism

$$T_k: E(\lambda,U)_{Q_k}\xrightarrow{\sim}C^{\omega,(k)}(V,U)_{\pi_k,\lambda}$$

of H^X_k-modules in view of Theorem 3.5.4. Corollary 3.6.2 now completes the proof.

(ii) Note that $E(\lambda,U)^W_{Q_k} \subset C^\omega_+(V,U)$ holds, which follows from (3.5.3). Now use **(i)** and the fact that the propagation map T_k acts on $Q_k(W)$-invariant functions in the same way as the map G (3.6.7), or in other words, the following diagram

$$\begin{array}{ccc} C^\omega(V,U)^W_Q & \hookrightarrow & C^\omega_+(V,U) \\ T_k \downarrow \wr & & \downarrow G \\ C^{\omega,(k)}(V,U)^W_\pi & \hookrightarrow & C(V,U)^W_\pi \end{array}$$

is a commutative diagram.

□

COROLLARY 3.6.5. *Let* $\lambda \in V^*_{\mathbb{C}}$. *The map* G (3.6.7) *restricts to an isomorphism* $G : E(\lambda,U)^{W_X}_{Q_k} \longrightarrow \mathrm{BVP}_k(\lambda,U)^{W_X}_{\pi_k}$.

Theorem 3.6.4**(ii)** can be used to connect the χ-isotypical ($\chi \in \widehat{X/Q^\vee}$) component of the solution space $\mathrm{BVP}_k(\lambda,U)^W_\pi$ to the boundary value problem to the χ-isotypical component of the space of invariants $E(\lambda,U)^W_Q$, where $E(\lambda,U)$ now is the solution space to the boundary value problem with zero normal derivative jumps over sub-regular vectors.

3.7. The Bethe ansatz equations

As usual X denotes a lattice between $Q^\vee$ and $P^\vee$ and U denotes always a finite dimensional representation of W_X, unless stated otherwise. In this section we show that for $\chi \in \widehat{X/Q^\vee}$ the space $E(\lambda,U)^{W,\chi}_{Q_k}$ is not the null space (for generic regular λ) if and only if the spectral parameter $\lambda \in V^*_{\mathbb{C}}$ satisfies certain transcendental equations. These equations will be called Bethe ansatz equations (in short, BAE). For root system of type **A** and U the regular representation $\mathrm{Fun}(W_0,\mathbb{C})$ (and the translation part X of $W_X = W_0 \ltimes X$ acting trivial on U) these equations are essentially the equations as [**80**, Equations (9)]

In the scalar cases we recover for Σ_0 of type **A** the Bethe ansatz equations of Lieb and Liniger [**56**], and for Σ_0 of type **D** the Bethe ansatz equations of Gaudin [**26**], [**28**]. In the scalar case it is possible to prove stronger results, see Chapter 4.

In this section λ always denotes a *generic regular* spectral parameter (cf. (3.2.9)).

For a simple root $a \in I$ we set

$$I^a(\lambda) = \frac{\lambda(Da^\vee)s_a - k_a}{\lambda(Da^\vee) + k_a} \in \mathbb{C}[W_X]. \tag{3.7.1}$$

THEOREM 3.7.1. *Let* $w = s_{i_1} \dots s_{i_t}\omega \in W_X$ *be an expression for* w *in simple reflections* s_{a_i} *and with* $\omega \in \Omega_X$. *The following element in* $\mathbb{C}[W_X]$ *is well-defined,*

$$I_w(\lambda) = I^{a_{i_1}}((D(s_{i_2}\dots s_{i_t}))\lambda)I^{a_{i_2}}((D(s_{i_3}\dots s_{i_t}))\lambda)\dots I^{a_{i_t}}(\lambda)\omega. \tag{3.7.2}$$

In particular, the $I_w(\lambda)$ *satisfies*

$$I_{w'w}(\lambda) = I_{w'}((Dw)\lambda)I_w(\lambda) \quad (w, w' \in W_X). \tag{3.7.3}$$

PROOF. This is analogous to the argument in [**64**, Section 4.1] (see also [**63**, Section 1] and [**11**]). Here we give only the main arguments and refer to [**64**] and [**63**] for more details.

Let $\mu \in V_{\mathbb{C}}^*$. If N is a finite dimensional H_k^X-module we define $N^\mu = \{n \in N | qn = q(\mu)n \,\forall q \in S(V)_{\mathbb{C}}\}$. We denote by $\mathbb{C}_\mu$ the following one-dimensional representation of $S(V)_{\mathbb{C}}$, given by $qc = q(\mu)c$ for all $q \in S(V)_{\mathbb{C}}$ and $c \in \mathbb{C}$. For any μ we call $P(\mu) = \mathrm{Ind}_{S(V)_{\mathbb{C}}}^{H_k^X}(\mathbb{C}_\mu) = H_k^X \otimes_{S(V)_{\mathbb{C}}} \mathbb{C}_\mu$ the (minimal) principal series representation of H_k^X with central character $W_0\mu$ (for the action of the center of the degenerate affine Hecke algebra $H_k^{(0)} \subseteq H_k^X$).

Then $P(\mu)$ is isomorphic to the regular representation of $\mathbb{C}[W_X]$ when restricted to $\mathbb{C}[W_X] \subset H_k^X$ (and has central character $W_0\mu$). It has the the following universal property: for a finite dimensional module N of H_k^X and a $n \in N^\mu$ there exists a unique H_k^X-module morphism $P(\mu) \to N$ such that $e \otimes_{S(V)_{\mathbb{C}}} 1 \mapsto n$.

Define $J_a = s_a \cdot Da^\vee - k_a \in H_k^X$ $(a \in I)$ (the operator version of J_{a_0} (see (4.4.4)) will also be usefull in the next chapter). Then one shows in the same way as [**64**, Theorem 4.2**(ii)**] that the J_a $(a \in I)$ satisfy the braid relations of Σ (but not the quadratic relations). In particular $J_w := \omega J_{a_{i_1}} \ldots J_{a_{i_{l(w)}}}$ is well-defined for a reduced expression $w = \omega s_{i_1} \ldots s_{i_{l(w)}}$, and furthermore $J_w q = q^{Dw} J_w$ for all $q \in S(V)_{\mathbb{C}}$. By the previous paragraph $e \otimes_{S(V)_{\mathbb{C}}} 1 \mapsto J_w \otimes_{S(V)_{\mathbb{C}}} 1$ uniquely defines a H_k^X-module morphism $P((Dw)\lambda) \to P(\lambda)$. Denote by J_w^λ the (unique) element in $\mathbb{C}[W_X]$ such that $J_w \otimes_{S(V)_{\mathbb{C}}} 1 = J_w^\lambda \otimes_{S(V)_{\mathbb{C}}} 1$ holds in $P(\lambda)$. Consider the following maps,

$$P((D(ww'))\lambda) \to P((Dw')\lambda) \to P(\lambda).$$

Under the composition of these maps, $e \otimes_{S(V)_{\mathbb{C}}} 1$ is mapped to $(J_w J_{w'}) \otimes_{S(V)_{\mathbb{C}}} 1$ and also to $(J_w^{(Dw')\lambda} J_{w'}^\lambda) \otimes_{S(V)_{\mathbb{C}}} 1$. If $l(ww') = l(w) + l(w')$, then $J_{ww'} = J_w J_{w'}$ and therefore by (the uniqueness part of the) universal property above, we have then $J_{ww'}^\lambda = J_w^{(Dw')\lambda} J_{w'}^\lambda$. When normalized as

$$\tilde{J}_w^\lambda = \frac{J_w^\lambda}{\prod\limits_{a \in \Sigma^+ \cap w^{-1}\Sigma^-} (\lambda(Da^\vee) + k_a)},$$

they satisfy $\tilde{J}_{ww'}^\lambda = \tilde{J}_w^{(Dw')\lambda} \tilde{J}_{w'}^\lambda$ for *all* $w, w' \in W_X$. To conclude the proof observe that $I_w(\lambda) = \tilde{J}_w^\lambda$. □

REMARK. Note the similarity of relation (3.7.3) with the cocycle relations (cf. [**64**, Section 4.1]).

For later use the relation

$$I_w(\lambda)^{-1} = I_{w^{-1}}((Dw)\lambda) \qquad \forall w \in W_X, \tag{3.7.4}$$

will be useful, and follows directly from the relations (3.7.3) and $I_e(\lambda) = e$.

COROLLARY 3.7.2. *The map* $\mathbb{C}[X] \to \mathbb{C}[W_X]$ *defined by* $x \mapsto I_x(\lambda) := I_{\tau_x}(\lambda)$ *(*$x \in X$*) is a unit preserving algebra morphism. In particular, for a fixed* λ*, the subset* $\{I_x(\lambda) | x \in X\}$ *of* $\mathbb{C}[W_X]$ *is an abelian subgroup of the group of invertible elements of* $\mathbb{C}[W_X]$.

PROOF. This is a direct consequence of the relations (3.7.3) and (3.7.4). □

REMARK 3.7.3. It is possible to define $E(\lambda, N)_{Q_k^0}$ for finite dimensional representation N of W_0, using the corresponding Q_k^0-representation of W_0 on $C(V, N)$ (cf. Remark 2.5.14 and [**40**]). We denote the corresponding subspace of W_0-invariant elements by $E(\lambda, N)_{Q_k^0}^{W_0}$. The following vector space identity is obvious by construction,

$$E(\lambda, U)_{Q_k}^{W_0} = E(\lambda, \mathrm{Restr}_{W_0}^U)_{Q_k^0}^{W_0},$$

and will be used implicitely in the whole thesis (cf. Definition 3.2.10). Here $\mathrm{Restr}_{W_0}^U$ denotes the restriction of U to the subgroup W_0 of W_X.

Consider the group algebra $\mathbb{C}[W_0]$ over $\mathbb{C}$ as the regular representation of W_0. The corresponding Q^0-representation of W_0 on $C^\infty(V) \otimes \mathbb{C}[W_0]$ from Chapter 2 (cf. Remark 2.5.14) will be denoted by Q_k^u (u stands for *universal*).

The following vector space embeddings

$$C^\infty(V) \otimes \mathbb{C}[W_0] \subset C^\infty(V) \otimes \mathbb{C}[W] \subset C^\infty(V) \otimes \mathbb{C}[W_X],$$

will be used implicitly, with $\otimes$ denoting the algebraic tensor product over $\mathbb{C}$ (compare this with (3.2.5)).

Before analyzing the spaces $E(\lambda, U)_{Q_k}^{W,\chi}$ ($\chi \in \widehat{X/Q^\vee}$) it is useful to first describe the space $E(\lambda, U)_{Q_k}^{W_0}$. This can be done in a universal way.

DEFINITION 3.7.4. *Let* $\mu \in V_{\mathbb{C}}^*$. *The* universal eigenfunction with spectral parameter μ *is the* $\mathbb{C}[W_0]$*-valued function*

$$\Psi_\mu = \Psi_{\mu,k} = \frac{1}{\#W_0} \sum_{w \in W_0} Q_k^u(w)(e^\mu \otimes e) \in E(\mu, \mathbb{C}[W_0])_{Q_k^u}^{W_0}. \tag{3.7.5}$$

For regular $\mu \in V_{\mathbb{C}}^*$ the operator-valued function Ψ_μ admits the following plane wave decomposition

$$\Psi_\mu = \frac{1}{\#W_0} \sum_{w \in W_0} e^{w\mu} \otimes c_w(\mu), \tag{3.7.6}$$

for certain elements $c_w(\mu) = c_{w,k}(\mu)$ in $\mathbb{C}[W_0]$.

The normalized intertwiners $I_w(\lambda)$ allows us to manipulate with the $c_w(\lambda)$ in an efficient way because of the relations (3.7.3). To connect $I_w(\lambda)$ and $c_w(\lambda)$ we introduce the following c-function by

$$\tilde{c}_k(\mu) = \prod_{a \in \Sigma_0^+} \frac{\mu(\alpha^\vee) + k_\alpha}{\mu(\alpha^\vee)}, \tag{3.7.7}$$

considered as rational function of $\mu \in V_{\mathbb{C}}^*$, cf. [**28**], [**40**].

LEMMA 3.7.5. *We have*

$$c_w(\lambda) = \tilde{c}_k(\lambda) I_w(\lambda) \quad \forall w \in W_0. \tag{3.7.8}$$

PROOF. Let $\mu \in V^*_{\mathbb{C},reg}$. Then

$$Q_{k,a}(e^\mu \otimes u) = s_a(e^\mu) \otimes \left(\frac{\mu(Da^\vee)s_a - k_a}{\mu(Da^\vee)} \right) u + e^\mu \otimes \frac{k_a}{\mu(Da^\vee)} u \quad (a \in \Sigma, u \in U) \tag{3.7.9}$$

holds, which follows by a direct computation. A simple calculation then shows

$$Q_k^u(w_0)(e^\mu \otimes a) = e^{w_0\mu} \otimes \tilde{c}_k(\mu) I_{w_0}(\mu) a \quad (a \in \mathbb{C}[W_0])$$

modulo terms $e^{v\mu}$ with $v \in W_0$ and $v < w$ in the Bruhat ordering (for the definition of the Bruhat order, see [**45**, Section 5.5]). Since $Q_k^u(w)(e^\mu)$ $(w \in W_0)$ only consists of terms $e^{v\mu}$ with $v \in W_0$ and $v \leq w$ in the Bruhat ordering, we conclude that (3.7.8) is true for $w = w_0$.

We also have

$$c_{s_\alpha w}(\lambda) = I_{s_\alpha}(w\lambda) c_w(\lambda) \quad \forall w \in W_0, \alpha \in I_0, \tag{3.7.10}$$

which follows from the decomposition (3.7.6), $Q_k^u(s_\alpha)\Psi_\lambda = \Psi_\lambda$ $(\alpha \in I_0)$ and (3.7.9). The lemma follows from

$$c_{ww_0} = I_w(w_0\lambda) c_{w_0}(\lambda) = \tilde{c}_k(\lambda) I_{ww_0}(\lambda) \quad (\forall w \in W_0),$$

with the first equality following by induction on the length of w and (3.7.10), and the second equality from $c_{w_0}(\lambda) = \tilde{c}_k(\lambda) I_{w_0}(\lambda)$ and (3.7.3). □

LEMMA 3.7.6. *Let H be a finite group and N a finite dimensional representations of H. Consider the group algebra $\mathbb{C}[H]$ as the regular representation of H. The assignment*

$$n \mapsto \sum_{h \in H} h \otimes hn \tag{3.7.11}$$

defines a vecor space isomorphism from N to $(\mathbb{C}[H] \otimes N)^H$ (the space of H-invariant elements in the tensor product representation).

PROOF. It is obvious that the map (3.7.11) is injective. For the surjectivity, let $m = \sum_h h \otimes n_h \in (\mathbb{C}[H] \otimes N)^H$, for certain element n_h in N. We have to show that $n_h = n_e$ for all $h \in H$, with e denoting the unit element of H. To show this we start with observing that

$$\sum_{h \in H} h \otimes n_h = m = g^{-1}m = \sum_{h \in H} g^{-1}h \otimes g^{-1}n_h = \sum_{h \in H} h \otimes g^{-1}n_{gh}$$

holds for all $g \in H$. Whence $n_h = g^{-1}n_{gh}$ for all $g, h \in H$. Taking $h = e$ we conclude that $n_g = n_e$ for all $g \in H$, finishing the proof. □

The following lemma explains the universality of Ψ_λ.

LEMMA 3.7.7. *Let N be a finite dimensional representation of W_0 and $\mu \in V_{\mathbb{C}}^*$. The map $N \to E(\mu, N)_{Q_k^0}^{W_0}$, defined by $u \mapsto \psi_\mu^n$, and with*

$$\psi_\mu^n = \psi_{\mu,k}^n = \frac{1}{\#W_0} \sum_{w \in W_0} Q_k^0(w)(e^\mu \otimes n), \tag{3.7.12}$$

is a vector space isomorphism.

PROOF. Since the representation theory of the finite group W_0 does not admit non-trivial continuous deformations, the lemma will follow from the case $k \equiv 0$ and regular μ. Let μ then be regular. Then it is immediate that $E(\mu, N)_{Q_0^0}$ is the tensor product representation of the regular representation of W_0 and U. Moreover,

$$Q_0^0(w')(e^{w\mu} \otimes n) = e^{w'w\mu} \otimes w'n \qquad (w, w' \in W_0, n \in U).$$

Now apply Lemma 3.7.6 to conclude the proof. □

For regular $\mu \in V_{\mathbb{C}}^*$ and $n \in N$ (with the same conditions as in Lemma 3.7.7), the N-valued function ψ_λ^n allows the following plane wave decomposition (compare with (3.7.6))

$$\psi_\mu^n = \frac{1}{\#W_0} \sum_{w \in W_0} e^{w\mu} \otimes c_w(\mu) n. \tag{3.7.13}$$

The following theorem is a special of the main result Theorem 3.7.14 (corresponding to the case $X = Q^\vee$) and is used in the proof of Theorem 3.7.14.

THEOREM 3.7.8. *Let $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter and u a nonzero element in U. Then $\psi_\lambda^u \in E(\lambda, U)_{Q_k}^W$ if and only if u is a simultaneous eigenvector of the family of commuting operators $I_x(\lambda)$ ($x \in Q^\vee$) with eigenvalue $e^{\lambda(x)}$, i.e.*

$$I_x(\lambda) u = e^{\lambda(x)} u \qquad \forall x \in Q^\vee. \tag{3.7.14}$$

The equations (3.7.14) are called *Bethe ansatz equations*.

PROOF. Note that because of Corollary 3.7.2, the $I_x(\lambda)$ ($x \in Q^\vee$) indeed form a family of commuting operators on U.

Since $\psi_\lambda^u \in E(\lambda, U)_{Q_k}^{W_0}$, it suffices to determine what the conditions on λ and u are such that

$$Q_k(s_0)\psi_\lambda^u = \psi_\lambda^u. \tag{3.7.15}$$

Using (3.7.9) (applied to $a = a_0$) and the plane wave decomposition (3.7.13) it is easily shown that (3.7.15) holds iff

$$c_{s_\varphi w}(\lambda) u = e^{w\lambda(\varphi^\vee)} I^{a_0}(w\lambda) c_w(\lambda) u \quad \forall w \in W_0.$$

Using (3.7.8) and Theorem 3.7.1 this can be reformulated as

$$e^{w\lambda(\varphi^\vee)} I^{a_0}(w\lambda) I_w(\lambda) u = I_{s_\varphi}(w\lambda) I_w(\lambda) u \quad \forall w \in W_0. \tag{3.7.16}$$

Let $\alpha = w^{-1}\varphi$ and $b = w^{-1}a_0 = -\alpha + \delta \in \Sigma$. Using $ws_\alpha = s_\varphi w$, the relations (3.7.3) and $s_0 w = ws_b$, one immediately sees:

$$I_{s_\varphi}(w\lambda)I_w(\lambda) = I_w(s_\alpha\lambda)I_{s_\alpha}(\lambda)$$

and

$$I_{s_0}(w\lambda)I_w(\lambda) = I_w(s_\alpha\lambda)I_{s_{-\alpha+\delta}}(\lambda) = I_w(s_\alpha\lambda)I_{s_\alpha}I_{-\alpha^\vee},$$

where the second equality follows from (3.7.3) and $s_{-\alpha+\delta} = s_\alpha\tau_{-\alpha^\vee}$. Whence (3.7.16) is equivalent with

$$I_{w^{-1}\varphi^\vee}(\lambda)u = e^{\lambda(w^{-1}\varphi^\vee)}u \qquad \forall w \in W_0.$$

Since all long roots in Σ_0 are conjugate to φ we have $\psi_\lambda^u \in E(\lambda, U)_{Q_k}^W$ if and only if

$$I_{\beta^\vee}(\lambda)u = e^{\lambda(\beta^\vee)}u \qquad \forall \text{ long roots } \beta \in \Sigma_0.$$

Now observe that the BAE (3.7.14) are "additive" in x. By this we mean: if (3.7.14) is satisfied for $x = x_1$ and $x = x_2$, then it is also satisfied for $x = -x_1$ and $x = x_1 + x_2$. Together with the fact that $Q^\vee$ is generated by short co-roots we conclude that $Q(a_0)\psi_\lambda^u = \psi_\lambda^u$ if and only if λ and u satisfy the Bethe ansatz equations (3.7.14). □

In the scalar case Theorem 3.7.8 is equivalent to the following (use Lemmas 4.2.5, 4.2.2 and 3.7.7).

COROLLARY 3.7.9. *Let $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter. Then $E(\lambda)_{Q_k}^W$ is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is solution of the Bethe ansatz equations*

$$\prod_{\alpha\in\Sigma_0^+}\left(\frac{\lambda(\alpha^\vee) - k_\alpha}{\lambda(\alpha^\vee) + k_\alpha}\right)^{\alpha(x)} = e^{\lambda(x)} \qquad \forall\, x \in Q^\vee.$$

If $E(\lambda)_{Q_k}^W$ is one-dimensional then

$$\psi_\lambda = \frac{1}{\#W_0}\sum_{w\in W_0}\widetilde{c}_k(w\lambda)e^{w\lambda} \tag{3.7.17}$$

is the unique function in $E(\lambda)_{Q_k}^W$ normalized by $\psi_\lambda(0) = 1$.

In the next chapter we will analyze the scalar case more throughly and are able to prove a stronger result, cf. Theorems 4.6.1 and 4.6.2.

Reformulated in terms of the original boundary value problem (cf. Definition 3.2.6) the previous Theorem 3.7.8 can be reformulated as follows.

THEOREM 3.7.10. *Let $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter. Then $\mathrm{BVP}_k(\lambda, U)_\pi^W$ is nonzero iff $T_k\psi_\lambda^u \in \mathrm{BVP}_k(\lambda, U)_\pi^W$ for a nonzero $u \in U$. The latter holds iff (λ, u) is a solution of the Bethe ansatz equations* (3.7.14)*. In particular: $\lambda \in \mathcal{S}_k^{reg}(U)$ iff (λ, u) satisfies* (3.7.14) *for a nonzero $u \in U$.*

PROOF. Follows from Definition 3.2.13, Corollary 3.6.5 and Theorem 3.7.8. □

We now investigate the conditions on λ under which the χ-isotypical component (with $\chi \in \widehat{X/Q^\vee}$) $E(\lambda, U)^{W,\chi}_{Q_k}$ of $E(\lambda, U)^W_{Q_k}$ is not the null space. Before we state and prove the main result Theorem 3.7.14 of this section, we introduce more notations and results concerning the groups $X/Q^\vee$ and Ω_X.

Let $\xi_1, \ldots, \xi_n \in P^\vee$ denote the *fundamental co-weights* with respect to the simple roots $I_0 = \{a_1, \ldots, a_n\}$, defined as the vectors in V satisfying

$$a_i(\xi_j) = \delta_{ij} \qquad \forall i, j \in \{1, 2, \ldots, n\}.$$

A co-weight $\xi \in P^\vee$ is called *minuscule* if $\xi \in (\overline{C_+} \cap P^\vee)\backslash\{0\}$. Alternatively, a co-weight $\xi \in P^\vee$ is minuscule if and only if $\xi \neq 0$ and $0 \leq \alpha(\xi) \leq 1$ for all $\alpha \in \Sigma_0^+$. Let $\varphi = m_1a_1 + m_2a_2 + \cdots + m_na_n$ be the expansion of the highest root as linear combination of basis elements. Then $m_i \in \mathbb{N} = \{1, 2, \ldots\}$. Consider the subset $O^* := \{r | m_r = 1\}$ of $\{1, 2, \ldots, n\}$. The set of minuscule co-weights is known to be equal to $\{\xi_r | \, r \in O^*\}$ (see [**64**, Proposition 3.3]).

Let $J_r := I_0\backslash\{a_r\}$ $(r = 1, 2, \ldots, n)$ and consider the standard parabolic subgroup W_{0,J_r} generated by the simple reflections s_α $(\alpha \in J_r)$. It is the isotropic subgroup of ξ_r in W_0. The longest element in W_{0,J_r} is denoted by w_{J_r}.

PROPOSITION 3.7.11. *The group* Ω *equals* $\{\omega_r := w_0 w_{J_r} \tau_{-\xi_r} | \, r \in O^*\} \cup \{1\}$. *In particular the set of all minuscule co-weights is a complete set of representatives of* $(P^\vee/Q^\vee)\backslash\{0\}$.

For a proof see [**64**, Proposition 3.4]. Proposition 3.7.11 together with $\Omega_X = W_X \cap \Omega$, $\omega_r^{-1} W = \tau_{\xi_r} W$ $(r \in O^*)$ gives

COROLLARY 3.7.12. *The group* Ω_X *equals* $\{\omega_r | \, r \in O^*_X\} \cup \{1\}$, *where* $O^*_X = \{r \in O^* | \, \xi_r \in X\}$. *In particular the set* $\{\xi_r | \, r \in O^*_X\} \cup \{0\}$ *is a complete set of representatives of* $X/Q^\vee$.

LEMMA 3.7.13. *Let* $\lambda \in V^*_{\mathbb{C}}$ *be a generic regular spectral parameter and* $r \in O^*_X$. *We have*

$$\pi(\omega_r)\psi^u_\lambda = \chi(-\xi_r + Q^\vee)\psi^u_\lambda \iff I_x(\lambda)u = \chi(x + Q^\vee)e^{\lambda(x)}u \qquad \forall x \in W_0\xi_r. \tag{3.7.18}$$

PROOF. First observe that for $v \in W_0$ and $x \in X$ we have

$$\begin{aligned}\pi(v\tau_x)\psi^u_\lambda &= \frac{1}{\#W_0}\pi(v)\sum_{w \in W_0} e^{-w\lambda(x)}e^{w\lambda} \otimes \tau_x c_w(\lambda)u \\ &= \frac{1}{\#W_0}\sum_{w \in W_0} e^{-w\lambda(x)}e^{vw\lambda} \otimes v\tau_x c_w(\lambda)u.\end{aligned}$$

In particular, if also $v\tau_x \in \Omega_X$ we get, using that λ is regular,

$$\pi(v\tau_x)\psi^u_\lambda = \chi(x + Q^\vee)\psi^u_\lambda \iff e^{-w\lambda(x)}v\tau_x c_w(\lambda)u = \chi(x + Q^\vee)c_{vw}(\lambda)u \quad \forall w \in W_0. \tag{3.7.19}$$

Put $v_r = w_0 w_{J_r}$. Apply (3.7.19) with $v = v_r$ and $x = -\xi_r$ to conclude

$$\pi(\omega_r)\psi_\lambda^u = \chi(-\xi_r + Q^\vee)\psi_\lambda^u \iff$$
$$e^{-w\lambda(-\xi_r)}\omega_r c_w(\lambda)u = \chi(-\xi_r + Q^\vee)c_{v_r w}(\lambda)u \qquad \forall w \in W_0. \quad (3.7.20)$$

Whence, using (3.7.8) and (3.7.4) we see that $\pi(\omega_r)\psi_\lambda^u = \chi(-\xi_r + Q^\vee)\psi_\lambda^u$ if and only if

$$e^{\lambda(w^{-1}\xi_r)}u = \chi(-\xi_r + Q^\vee)I_{w^{-1}}(w\lambda)\omega_r^{-1}I_{v_r w}(\lambda)u \quad \forall w \in W_0 \qquad (3.7.21)$$

holds. Use (3.7.20), (3.7.21), $-w^{-1}\xi_r + Q^\vee = -\xi_r + Q^\vee$ (cf. paragraph following (2.2.5)) and the identity

$$I_{w^{-1}\xi_r}(\lambda) = I_{w^{-1}}(w\lambda)\omega_r^{-1}I_{v_r w}(\lambda)$$

to conclude (3.7.18). □

We are able to prove the main result of this section.

THEOREM 3.7.14. *Let $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter, u a nonzero element in U and $\chi \in \widehat{X/Q^\vee}$. Then $\psi_\lambda^u \in E(\lambda, U)_{Q_k}^{W,\chi}$ if and only if u is a simultaneous eigenvector of the family of commuting operator $I_x(\lambda)$ ($x \in X$) with eigenvalue $\chi(x + Q^\vee)e^{\lambda(x)}$, i.e.*

$$I_x(\lambda)u = \chi(x + Q^\vee)e^{\lambda(x)}u \qquad \forall x \in X. \qquad (3.7.22)$$

The equations (3.7.22) are called *Bethe ansatz equations* (associated to X and χ).

PROOF. We start with the identity

$$E(\lambda, U)_{Q_k}^{W,\chi} = \{\psi \in E(\lambda, U)_{Q_k}^W \,|\, \pi_k(\omega_r)\psi = \chi(-\xi_r + Q^\vee)\psi \quad \forall r \in O_X^*\}. \quad (3.7.23)$$

This follows from (3.6.5), $W_X = W \rtimes \Omega_X$, Corollary 3.7.12, $\omega_r = w_0 w_{J_r}\tau_{-\xi_r}$ and $\pi_k(\omega_r) = Q_k(\omega_r)$ ($r \in O_X^*$) as elements in $\mathrm{End}(C^\infty(V, U))$. By Theorem 3.7.8 and Lemma 3.7.13 then $\psi_\lambda^u \in E(\lambda, U)_{Q_k}^{W,\chi}$ if and only if the BAE (3.7.22) are satisfied for $x \in Q^\vee \cup \{w\xi_r | w \in W_0, r \in O_X^*\}$. To conclude the proof observe that the BAE (3.7.22) are "additive" in $x \in X$ (cf. proof of Theorem 3.7.8) and X is generated as an abelian group by $Q^\vee$ and $\{w\xi_r | w \in W_0, r \in O_X^*\}$ (cf. second statement of Corollary 3.7.12). □

COROLLARY 3.7.15. *Every solution $(\lambda, u) \in V_{\mathbb{C}}^* \times U$ of the Bethe ansatz equations* (3.7.14) *has a unique decomposition $(\lambda, u) = (\lambda, \sum_\chi u_\chi)$ with $(\lambda, u_\chi) \in V_{\mathbb{C}}^* \otimes U$ a solution of the Bethe ansatz equations* (3.7.22)*.*

PROOF. This follows from the decomposition (3.6.4) and Theorem (3.7.14). □

Reformulated in terms of the original boundary value problem (cf. Definition 3.2.6) the previous theorem takes the following form.

THEOREM 3.7.16. *Let $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter and $\chi \in \widehat{X/Q^\vee}$. Then $\mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi}$ is nonzero iff $T_k\psi_\lambda^u \in \mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi}$ for a nonzero $u \in U$. The latter holds iff (λ, u) satisfy the Bethe ansatz equations* (3.7.22)*. In particular: $\lambda \in \mathcal{S}_k^{reg}(U, \chi)$ iff (λ, u) satisfies* (3.7.14) *for a nonzero $u \in U$.*

PROOF. Follows from Definition 3.2.13, Theorem 3.6.4**(ii)** and Theorem 3.7.14. $\square$

3.8. The spectrum

In this section we restrict ourself to finite dimensional *unitary* representations U of W_X (with inner product denoted by $(\cdot,\cdot)_U$) and *strictly positive* multiplicity functions k. As in the previous section we always assume that λ denotes a generic regular spectral parameter. The main result of this section states that if the space $E(\lambda,U)_Q^W$ is nonempty, λ must be purely imaginary, i.e. $\lambda \in iV^*$. Note that a purely imaginary spectral parameter $\mu \in iV^*$ is always generic under the assumption $k > 0$.

LEMMA 3.8.1. *The spectrum $\mathcal{S}_k(U,\chi)$ and the regular spectrum $\mathcal{S}_k^{reg}(U,\chi)$ are W_0-invariant.*

PROOF. This is an immediate consequence of $E(\mu,U) = E(w\mu,U)$ ($w \in W_0$, $\mu \in V_{\mathbb{C}}^*$), Corollary 3.6.5 and the W_0-invariance of $V_{\mathbb{C},reg}^*$. $\square$

In particular we get for a $w \in W_0$:

$$\psi_\lambda^u \in E(\lambda,U)_{Q_k}^W \text{ for a } 0 \neq u \in U \iff \psi_{w\lambda}^{u'} \in E(\lambda,U)_{Q_k}^W \text{ for a } 0 \neq u' \in U.$$

Using Theorem 3.7.8 we can actually be more specific.

COROLLARY 3.8.2. *Let $u \in U$. Then for all $w \in W_0$ we have*

$$\psi_\lambda^u \in E(\lambda,U)_{Q_k}^W \iff \psi_{w\lambda}^{I_w(\lambda)u} \in E(\lambda,U)_{Q_k}^W.$$

THEOREM 3.8.3 (Purely imaginary regular spectrum). *Assume that λ is a generic spectral parameter in the regular spectrum $\mathcal{S}_k^{reg}(U,\chi)$ for a $\chi \in \widehat{X/Q^\vee}$. Then λ is purely imaginary, i.e. $\lambda \in iV^*$.*

PROOF. Since $E(\lambda,U)_{Q_k}^{W,\chi} \subset E(\lambda,U)_{Q_k}^W$, the general case will from the special case $X = Q^\vee$, which we assume in the rest of the proof. Recall from Section 2 that the set of generic $\lambda \in V^*$ is W_0-invariant. Whence by by Lemma 3.8.1 we may assume that the real part of λ lies in the closure of the positive chamber: $\lambda = \mu + i\nu$, with $\mu \in \overline{V_+^*}$ and $\nu \in V^*$. The essence of the proof is that the operator $I_{\varphi^\vee}(\lambda)$ is a contraction on U with respect to the invariant form $(\cdot,\cdot)_U$ on U, i.e.

$$(I_{\varphi^\vee}(\lambda)u, I_{\varphi^\vee}(\lambda)u)_U \leq (u,u)_U \quad \forall u \in U. \tag{3.8.1}$$

Since $\tau_{\varphi^\vee} = s_0 s_\varphi$ (cf. (2.2.2)) and the relations (3.7.3) we have

$$I_{\varphi^\vee}(\lambda) = I^{a_0}(s_\varphi\lambda) I_{s_\varphi}(\lambda).$$

Now let $s_\varphi = s_{i_1}\ldots s_{i_r}$ be a reduced expression. Applying once again (3.7.3) gives

$$I_{\varphi^\vee}(\lambda) = I^{a_0}(s_\varphi\lambda) I^{a_{i_1}}(s_{i_2}\ldots s_{i_r}\lambda) I^{a_{i_2}}(s_{i_3}\ldots s_{i_r}\lambda)\ldots I^{a_{i_r}}(\lambda) \tag{3.8.2}$$

Every term on the right hand side is of the form

$$\frac{\lambda(Da^\vee)s_a - k_a}{\lambda(Da^\vee) + k_a} \qquad \left(a \in \left(\Sigma_0^+ \cap s_\varphi\Sigma_0^-\right) \cup \{-a_0\}\right). \tag{3.8.3}$$

To show (3.8.1) it therefore suffices to show that (3.8.3) is a contraction. Let $a \in \left(\Sigma_0^+ \cap s_\varphi \Sigma_0^-\right) \cup \{-a_0\}$ and put $\alpha = Da \in \Sigma_0^+$. Then

$$|\lambda(\alpha^\vee) + k_a|^2 = \mu(\alpha^\vee)^2 + k_a^2 + \nu(\alpha^\vee)^2 + 2k_a\mu(\alpha^\vee), \tag{3.8.4}$$

and with every term on the right hand side real and non-negative because $\mu \in \overline{V_+^*}$ and $k > 0$.

Furthermore, if $u \in U$, then

$$\begin{aligned} \|(\lambda(\alpha^\vee)s_a - k_a)u\|_U^2 = &\left(\mu(\alpha^\vee)^2 + k_a^2 + \nu(\alpha^\vee)^2\right)(u,u)_U \\ &- k_a((\lambda(\alpha^\vee)s_a u, u)_U + (u, \lambda(\alpha^\vee)s_a u)_U) \end{aligned} \tag{3.8.5}$$

By the Cauchy-Schwarz inequality and unitary of U we have $|(s_a u, u)_U| \leq (u,u)_U$. By unitarity of U follows also $(s_a u, u)_U = (u, s_a u)_U \in \mathbb{R}$. Whence the last term in (3.8.5) is real and can be estimated by

$$-2k_a(s_a u, u)_U \mu(\alpha^\vee) \leq 2k_a\|u\|_U^2 \mu(\alpha^\vee). \tag{3.8.6}$$

It follows that

$$\begin{aligned} \|(\lambda(\alpha^\vee)s_a - k_a)u\|_U^2 \leq (\mu(\alpha^\vee)^2 + k_a^2 + \nu(\alpha^\vee)^2 + 2k_a\mu(\alpha^\vee))\|u\|_U^2 \\ |\lambda(\alpha^\vee) + k_a|^2 \, \|u\|_U^2, \end{aligned}$$

where the equality follows from (3.8.4). Hence the operators (3.8.3) are contractions, and therefore also $I_{\varphi^\vee}(\lambda)$.

Since λ is in $\mathcal{S}_k^{reg}(U)$ and generic, by Theorem 3.7.8 $I_{\varphi^\vee}(\lambda)$ has a eigenvector $u \in U$ with eigenvalue $e^{\lambda(\varphi^\vee)}$. Since $I_{\varphi^\vee}(\lambda)$ is a contraction, it follows that the modulus of this eigenvalue satisfies $e^{\mu(\varphi^\vee)} = \left|e^{\lambda(\varphi^\vee)}\right| \leq 1$. But also $e^{\mu(\varphi^\vee)} \geq 1$ since $\mu \in \overline{V^*}$. Thus $\left|e^{\lambda(\varphi^\vee)}\right| = 1$, implying that $\lambda(\varphi^\vee)$ is purely imaginary. Since $\varphi^\vee = \sum_{j=1}^n n_j a_j^\vee$ with n_j strictly positive integers and since the real part of λ lies in $\overline{V_+^*}$, we conclude that $\lambda(a_j^\vee)$ is purely imaginary for all co-roots $a_j^\vee$ ($j = 1, \ldots, n$). This implies $\lambda \in iV^*$ and concludes the proof. □

LEMMA 3.8.4. *Let $\lambda \in iV_{reg}^*$ and $w \in W_X$. Then the operator $I_w(\lambda)$ is a unitary operator on U, i.e.*

$$(I_w(\lambda)u, I_w(\lambda)u)_U = (u,u)_U \qquad \forall u \in U.$$

PROOF. With a similar argument as in the proof of Theorem 3.8.3 one easily shows that

$$I^a(\nu) = \frac{\nu(Da^\vee)s_a - k_a}{\nu(Da^\vee) + k_a} \quad (a \in I)$$

is a unitary operator on U if $\nu \in iV_{reg}^*$. Now observe that $I_w(\lambda)$ is a product of operators of the form $I^a(\nu)$ and $I_\omega(\nu) = \omega$ (as operators on U for $\omega \in \Omega_X$) with $\nu \in iV_{reg}^*$, and these are unitary operators on U. □

Using the previous lemma, Theorem 3.7.14 can in this context be strengthened as follows.

THEOREM 3.8.5. *Let U be a finite dimensional unitary representation of W_X and k a strictly positive multiplicity function. Let also $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter, u a nonzero element of U and $\chi \in \widehat{X/Q^\vee}$. Then $\psi_\lambda^u \in E(\lambda, U)_{Q_k}^{W,\chi}$ if and only if λ is purely imaginary and (λ, u) is a solution of the Bethe ansatz equations* (3.7.22).

Reformulated in terms of the original boundary value problem (cf. Definition 3.2.6) the previous theorem takes the following form.

THEOREM 3.8.6. *Let U be a finite dimensional unitary representation of W_X and k a strictly positive multiplicity function. Let also $\lambda \in V_{\mathbb{C}}^*$ be a generic regular spectral parameter and $\chi \in \widehat{X/Q^\vee}$. Then $\mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi}$ is nonzero iff $T_k\psi_\lambda^u \in \mathrm{BVP}_k(\lambda, U)_\pi^{W,\chi}$ for a nonzero $u \in U$. The latter holds iff λ is purely imaginary and (λ, u) is a solution of the Bethe ansatz equations* (3.7.22).

CHAPTER 4

Periodic scalar integrable systems with delta-potentials

4.1. Introduction

In Chapter 3 we attached to any affine root system Σ and a finite dimensional representation of the corresponding affine Weyl group a quantum integrable system with values in this representation. In this chapter we restrict ourself to the scalar system, i.e. our quantum system takes values in the trivial representation.

Our aim in this chapter is two-fold. The first one is to show that the results of Section 3.8 holds without the assumption that the spectral parameters have to be generic. Second, we we will show that the solutions of the Bethe ansatz equations are controlled by a strictly convex master function, generalizing the results of Yang & Yang [**81**] for the special case of the impenetrable Bose-gas on the circle.

We now give a summary of every section separately. In Section 4.2 we revisit the intertwiner operators from Section 3.7. They act as scalar multiplication on the trivial representation and we make these scalars explicit. In Section 4.3 we study the reformulated boundary value problem (see Theorem 3.6.4). In particular the invariants under Q-action of the finite Weyl group are analyzed for general spectral parameters (cf. Lemma 3.7.7 and (3.7.13)). This leads in Section 4.4 to the derivation of the Bethe ansatz equations for $X = Q^\vee$. It is furthermore shown that the spectrum of the quantum system under consideration is purely imaginary. In Section 4.5 we introduce the master function and show that it is strictly convex. This allows us to proof that the boundary value problem has solutions if and only if the associated spectral value is a *regular* solution of the Bethe ansatz equations. In case of root system of type A, this is known as the Pauli principle for the interacting bosons. In Section 4.6 we show that the results of Sections 4.4 and 4.5 holds for general lattices $Q^\vee \subset X \subset P^\vee$.

In Section 4.7 some elementary facts about the connection between lattices and cosets are given. It is al preparatory to Section 4.8, where we continue the study of the master function from Section 4.5, leading to a natural parametrization of the solutions of the Bethe ansatz equations. In Section 4.9 the solutions of the Bethe ansatz equations are further analyzed, which leads to estimates for the momenta gaps.

In this chapter the notations from Chapter 2 and Chapter 3 are used. In particular X always denotes a lattice between the co-root lattice $Q^\vee$ and the co-weight lattice $P^\vee$, and $\mathbb{C}_{triv}$ denotes the trivial representation of $\mathbb{C}[W_X]$ (and occasionally of $\mathbb{C}[W_0]$). Unless stated explicitly otherwise, we fix a strictly positive multiplicity function $k : \Sigma \to \mathbb{R}_{>0}$. *In particular all the results of Section 3.8 apply*.

4.2. The intertwiners revisited

In this section λ always denotes a generic spectral parameter, unless stated otherwise.

An element $I_w(\lambda) \in \mathbb{C}[W_X]$ ($w \in W_X$) from Section 3.7 acts as scalar multiplication on $\mathbb{C}_{triv}$. *In this chapter we identify $I_w(\lambda)$ with this scalar*.

For a $w \in W^e$ write $\Sigma(w) = \Sigma^+ \cap w^{-1}\Sigma^-$.

LEMMA 4.2.1. **(i)** $\Sigma(\tau_{\varphi^\vee}) = (\Sigma^+ \cap s_\varphi \Sigma^-) \cup \{\varphi + \delta\}$.
(ii) *Let $\alpha \in \Sigma_0$. Then:*

$$\alpha(\varphi^\vee) = \begin{cases} 2 & \text{if } \alpha = \varphi, \\ 1 & \text{if } \alpha \in \left(\Sigma_0^+ \cap s_\varphi \Sigma_0^-\right) \setminus \{\varphi\}, \\ 0 & \text{if } \alpha \notin \Sigma_0^+ \cap s_\varphi \Sigma_0^-. \end{cases} \tag{4.2.1}$$

PROOF. **(i)** This follows from $\tau_{\varphi^\vee} = s_0 s_\varphi$ (cf. (2.2.2)) and [**59**, (2.2.4).(ii)].

(ii) Use $s_\alpha(\varphi) = \varphi - \varphi(\alpha^\vee)\alpha$ and that φ is the highest root to conclude that $\varphi(\alpha^\vee) \in \mathbb{Z}_+$, and whence also $\alpha(\varphi^\vee) \in \mathbb{Z}_+$. Let $\alpha \in \Sigma^+$ and $\alpha \notin \Sigma^+ \cap s_\varphi \Sigma^-$. Then $s_\varphi(\alpha) = \alpha - \alpha(\varphi^\vee)\varphi \in \Sigma^+$. Since φ is the highest root and $\alpha(\varphi^\vee) \in \mathbb{Z}_+$, we must have $\alpha(\varphi^\vee) = 0$.

Now assume that $\alpha \in (\Sigma_0^+ \cap s_\varphi \Sigma_0^-) \setminus \{\varphi\}$. Together with $\alpha(\varphi^\vee) \in \mathbb{Z}_+$ this gives $\alpha(\varphi^\vee) \in \mathbb{Z}_{\geq 1}$. Now consider the following identity,

$$s_\varphi(\alpha) = \alpha - \alpha(\varphi^\vee)\varphi = (\alpha - \varphi) + (1 - \alpha(\varphi^\vee))\varphi.$$

Because φ is the highest root and $\alpha \lneqq \varphi$, we must have $1 - \alpha(\varphi^\vee) = 0$, i.e. $\alpha(\varphi^\vee) = 1$.

The case $\alpha = \varphi$ is trivial. □

LEMMA 4.2.2. *The following holds for $x \in Q^\vee \subset X$:*

$$I_x(\lambda) = \prod_{\alpha \in \Sigma_0^+} \left(\frac{\lambda(\alpha^\vee) - k_\alpha}{\lambda(\alpha^\vee) + k_\alpha} \right)^{\alpha(x)}. \tag{4.2.2}$$

PROOF. Note that

$$I^a(\lambda) = \frac{\lambda(Da^\vee) - k_a}{\lambda(Da^\vee) + k_a} \qquad (a \in I). \tag{4.2.3}$$

The defining relation (3.7.2) of $I_w(\lambda)$ gives therefore

$$I_w(\lambda) = \prod_{a \in \Sigma(w) = \Sigma^+ \cap w^{-1}\Sigma^-} \frac{\lambda(Da^\vee) - k_a}{\lambda(Da^\vee) + k_a} \qquad (w \in W_X). \tag{4.2.4}$$

Taking $w = \tau_{\varphi^\vee}$ and using Lemma 4.2.1 we see that (4.2.2) holds for $x = \varphi^\vee$.

The following identity

$$I_{wx}(\lambda) = I_w(w^{-1}\lambda) I_x(w^{-1}\lambda) I_{w^{-1}}(\lambda) = I_x(w^{-1}\lambda) \qquad (x \in X, w \in W_0) \tag{4.2.5}$$

follows from $\tau_{wx} = w\tau_x w^{-1}$ as elements in W_X, the relations (3.7.3), (3.7.4) and the commutativity of the $I_w(\lambda)$. Observe that the expression

$$\left(\frac{\lambda(x) - k_\alpha}{\lambda(x) + k_\alpha}\right)^{\alpha(x)} \qquad (\alpha \in \Sigma_0, x \in Q^\vee) \tag{4.2.6}$$

is invariant under $\alpha \mapsto -\alpha$. Together with the obvious identity

$$\Sigma_0^+ = \big((w\Sigma_0^+) \cap \Sigma_0^+\big) \cup -((w\Sigma_0^+) \cap \Sigma_0^-) \qquad (w \in W_0),$$

(4.2.5) and the fact that (4.2.2) holds for $x = \varphi^\vee$, we conclude that (4.2.2) holds for $x = w\varphi^\vee$ ($w \in W_0$). Since all short co-roots are conjugate in $\Sigma_0^\vee$ under W_0, we see that (4.2.2) holds for all short co-roots $x \in \Sigma_0^\vee$. To conclude the proof observe that the (4.2.2) are “additive” in $x \in Q^\vee$ (cf. proof of Theorem 3.7.8) and $Q^\vee$ is generated by the short co-roots in $\Sigma_0^\vee$. □

Let J be a subset of the simple roots I_0. We denote $\Sigma_0^J \subset \Sigma_0$ for the parabolic root subsystem associated to J. We write N_J for the cardinality of the corresponding set $\Sigma_0^{J,+} := \Sigma_0^J \cap \Sigma_0^+$ of positive roots in Σ_0^J.

LEMMA 4.2.3. *The identity* (4.2.2) *holds for all* $x \in X$.

PROOF. Because the equations (4.2.2) are “additive” in $x \in X$ (cf. proof of Theorem 3.7.14), the fact that X is generated as an abelian group by $Q^\vee$ and $\{w\xi_r | w \in W_0,\, r \in O_X^*\}$ and Lemma 4.2.2, it suffices to show that (4.2.2) holds for $x = w\xi_r$ ($w \in W_0$, $r \in O_X^*$). Fix a $r \in O_X^*$. Let us denote by η a generic regular spectral paramter. For an $u \in W_0$ we have by (4.2.4)

$$\begin{aligned} I_{w_0}(u^{-1}\eta) &= \prod_{\alpha \in \Sigma_0^+} \frac{\eta(u\alpha^\vee) - k_\alpha}{\eta(u\alpha^\vee) + k_\alpha} \\ &= \prod_{\beta \in \Sigma_0^+ \cap u\Sigma_0^+} \frac{\eta(\beta^\vee) - k_\beta}{\eta(\beta^\vee) + k_\beta} \prod_{\beta \in -\left(\Sigma_0^- \cap u\Sigma_0^+\right) = \Sigma_0^+ \cap u\Sigma_0^-} \left(\frac{\eta(\beta^\vee) - k_\beta}{\eta(\beta^\vee) + k_\beta}\right)^{-1}. \end{aligned} \tag{4.2.7}$$

Note that $I_\omega(\lambda) = 1$ ($\omega \in \Omega_X$). Using $\tau_{\xi_r} = \omega_r^{-1} w_0 w_{J_r}$, (3.7.3) and (4.2.7) gives

$$I_{\xi_r}(\lambda) = I_{w_0}(w_{J_r}\lambda) I_{w_{J_r}}(\lambda) = \prod_{\alpha \in \Sigma_0^+ \cap w_{J_r}\Sigma_0^+} \frac{\eta(\alpha^\vee) - k_\alpha}{\eta(\alpha^\vee) + k_\alpha}. \tag{4.2.8}$$

Note that ξ_r is a minuscule co-weight, and whence $\alpha(\xi_r) = 1$ for all $\alpha \in \Sigma_0^+ \backslash \Sigma_0^{J_r,+}$. Standard parabolic theory of finite Weyl groups implies also the following,

$$\Sigma_0^+ \cap w_{J_r}\Sigma_0^+ = \Sigma_0^+ \backslash \Sigma_0^{J_r,+}.$$

Together with (4.2.8) we see that (4.2.2) holds for $x = \xi_r$. By (4.2.5) we get $I_{w\xi_r} = I_{\xi_r}(w^{-1}\lambda)$ ($w \in W_0$). We conclude as in the proof of Lemma that (4.2.2) holds for $x = w\xi_r$ ($w \in W_0$). □

COROLLARY 4.2.4. *The $I_x(\lambda)$ is independent of the choice of positive system Σ_0^+.*

PROOF. This follows because the expression (4.2.6) does not change when α is replaced by $-\alpha$. $\square$

Note that the coefficients $c_w(\mu) \in \mathbb{C}[W_0]$ ($w \in W_0$, $\mu \in V^*_{\mathbb{C},reg}$) in the plane wave decomposition (3.7.6) of the universal eigenfunctin Ψ_μ acts scalar multiplication on $\mathbb{C}_{triv}$. *In this chapter we identify $c_w(\mu)$ with this scalar*. We then have the following.

LEMMA 4.2.5. *Let $w \in W_0$ and μ a regular spectral parameter in $V^*_{\mathbb{C}}$. Then $c_w(\mu) = \tilde{c}_k(w\mu)$ and whence $I_w(\mu) = \tilde{c}_k(w\mu)/\tilde{c}_k(\mu)$.*

PROOF. Because $c_w(\mu)$ and $\tilde{c}_k(\mu)$ are both analytic in $\mu \in V^*_{\mathbb{C},reg}$, it suffices to show $c_w(\mu) = \tilde{c}_k(w\mu)$ for μ in the dense subset of generic regular spectral parameters. But if μ is generic and regular, then $c_w(\mu) = \tilde{c}_k(w\mu)$ follows easily from (4.2.4) and the definition (3.7.7) of $\tilde{c}_k$. $\square$

4.3. Invariants in $E(\lambda)$

In this section we restrict ourself to the W_0-theory. We analyze the subspace $E(\lambda)^{W_0}_{Q_k^0}$ of W_0-invariants of $E(\lambda)_{Q_k^0}$ (cf. (3.2.5) and Remark 3.7.3). First we recall some well known properties of the space $E(\lambda)$ from [**75**] and [**40**]. For technical purposes it is convenient to introduce the following terminology.

DEFINITION 4.3.1. *Let J be a subset of the simple roots I_0. The spectral parameter $\lambda \in V^*_{\mathbb{C}}$ is called J-standard if $\lambda \in V^* \oplus i\overline{V^*_+}$ and if the isotropic sub-group of λ in W_0 is the standard parabolic sub-group $W_{0,J}$ generated by the simple reflections s_α ($\alpha \in J$).*

LEMMA 4.3.2. *Let $\lambda \in V^*_{\mathbb{C}}$. The W_0-orbit of λ contains a J-standard spectral parameter for some subset $J \subseteq I_0$.*

PROOF. Taking a W_0-translate of λ we may assume that $\lambda = \mu + i\nu$ with $\mu \in V^*$ and $\nu \in \overline{V^*_+}$. The isotropy group of ν in W_0 is a standard parabolic sub-group $W_{0,K} \subset W_0$ for some subset $K \subseteq I_0$. Write $V^* = V^*_K \oplus (V^*_K)^\perp$ with $V^*_K = \mathrm{span}_{\mathbb{R}}\{\alpha \,|\, \alpha \in K\}$ and $(V^*_K)^\perp$ its orthocomplement in V^*. Set

$$V^*_{K,+} = \{\xi \in V^*_K \,|\, \xi(\alpha^\vee) > 0 \quad \forall\, \alpha \in K\},$$

which we view as the fundamental chamber for the action of the standard parabolic sub-group $W_{0,K}$ on V^*_K. Taking a $W_{0,K}$-translate of λ we may assume that $\lambda = \mu + \mu' + i\nu$ with $\mu \in \overline{V^*_{K,+}}$, $\mu' \in (V^*_K)^\perp$, and $\nu \in \overline{V^*_+}$ as before. The isotropy sub-group of λ in W_0 then equals the isotropy sub-group of μ in $W_{0,K}$, which is a standard parabolic sub-group $W_{0,J}$ for some subset $J \subseteq K$ since $\mu \in \overline{V^*_{K,+}}$. $\square$

Observe that a J-standard spectral parameter λ is regular if and only if $J = \emptyset$. Note furthermore that the module $E(\lambda)$ ($\lambda \in V^*_{\mathbb{C}}$) only depends on the orbit $W_0\lambda$. When analyzing the module $E(\lambda)$, we thus may assume without loss of generality that λ is J-standard for some subset $J \subseteq I_0$. In particular, we will now assume this condition for the remainder of this section.

For $j \in \mathbb{Z}_{\geq 0}$ we denote $P^{(j)}(V)_{\mathbb{C}}$ (respectively $P^{(\leq j)}(V)_{\mathbb{C}}$) for the homogeneous polynomials $p \in P(V)_{\mathbb{C}}$ of degree j (respectively the polynomials $p \in P(V)_{\mathbb{C}}$ of degree $\leq j$). The W_0-action (2.2.7) on $P(V)_{\mathbb{C}}$ respects the natural grading $P(V)_{\mathbb{C}} = \bigoplus_{j=0}^{\infty} P^{(j)}(V)_{\mathbb{C}}$. Furthermore,

$$E_J(0) = \{f \in P(V)_{\mathbb{C}} \,|\, p(\partial)f = p(0)f \quad \forall\, p \in S(V)^{W_{0,J}}\}$$

is a graded $W_{0,J}$-submodule of $P(V)_{\mathbb{C}}$, isomorphic to the regular representation of $W_{0,J}$ (see e.g. [**75**, Theorem 1.2] and references therein). We write $E_J^{(j)}(0) = E_J(0) \cap P^{(j)}(V)_{\mathbb{C}}$ and $E_J^{(\leq j)}(0) = E_J(0) \cap P^{(\leq j)}(V)_{\mathbb{C}}$.

Denote by W_0^J the minimal coset representatives of $W_0/W_{0,J}$. Steinberg [**75**] established the decomposition

$$E(\lambda) = \bigoplus_{u \in W_0^J} u\big(E_J(0)e^{\lambda}\big). \tag{4.3.1}$$

Furthermore, we have $E(\lambda) = \bigoplus_{j=0}^{\infty} E^{(j)}(\lambda)$ with $E^{(j)}(\lambda)$ the W_0-submodule

$$E^{(j)}(\lambda) = \bigoplus_{u \in W_0^J} u\big(E_J^{(j)}(0)e^{\lambda}\big).$$

We denote $E^{(\leq j)}(\lambda) = \bigoplus_{r=0}^{j} E^{(r)}(\lambda)$.

The vector space $E(\lambda)_{Q^0}^{W_0}$ is one-dimensional for all spectral values $\lambda \in V_{\mathbb{C}}^*$ by Lemma 3.7.7. In fact, by (3.5.3) the function

$$\psi_{\lambda,k} = \frac{1}{\#W_0} \sum_{w \in W_0} Q_k^0(w)e^{\lambda} \tag{4.3.2}$$

satisfies $\psi_{\lambda,k}(0) = 1$ and spans $E(\lambda)_{Q_k^0}^{W_0}$. On the other hand, by (4.3.1) there exist unique polynomials $p_u^{\lambda} \in E_J(0)$ ($u \in W_0^J$) such that

$$\psi_{\lambda,k}(v) = \sum_{u \in W_0^J} p_u^{\lambda}(u^{-1}v)e^{u\lambda(v)}, \qquad v \in V. \tag{4.3.3}$$

By (4.3.1) we have

$$E(\lambda) = \bigoplus_{w \in W_0} \mathbb{C}e^{w\lambda}, \qquad \lambda \in V_{\mathbb{C}}^* \text{ regular}, \tag{4.3.4}$$

so the polynomials p_w^{λ} ($w \in W_0$) are constants for regular λ.

In fact we have

$$\psi_{\lambda,k} = \frac{1}{\#W_0} \sum_{w \in W_0} \widetilde{c}_k(w\lambda)e^{w\lambda}, \qquad \lambda \in V_{\mathbb{C}}^* \text{ regular}, \tag{4.3.5}$$

where the c-function $\widetilde{c}_k$ is given by (3.7.7). This follows from $\psi_{\lambda,k} = \psi^1_{\lambda,k}$ (with $1 \in \mathbb{C}_{triv}$), (3.7.5) and Lemma 4.2.5 (see also [**28**] and [**40**, Section 2]). In the remainder of the chapter it will actually be more convenient to work with the regularized c-function

$$c_k(\mu) := \prod_{\substack{\alpha\in\Sigma_0^+ \\ \mu(\alpha^\vee)\neq 0}} \frac{\mu(\alpha^\vee)+k_\alpha}{\mu(\alpha^\vee)}, \qquad \mu \in V_{\mathbb{C}}^* \tag{4.3.6}$$

which is equal to $\widetilde{c}_k(\mu)$ for regular μ. We can then write

$$p_w^\lambda = \frac{1}{\#W_0} c_k(w\lambda), \qquad \lambda \in V_{\mathbb{C}}^* \text{ regular.}$$

For singular λ an explicit expression for $p_u^\lambda \in E_J(0)$ ($u \in W_0^J$) is not known. For our purposes it suffices to have explicit expressions for the highest and the next to highest homogeneous components of p_u^λ, which we will now proceed to derive.

We let

$$\delta_J = \frac{1}{2}\sum_{\alpha\in\Sigma_0^{J,+}} \alpha \in V^*.$$

Recall that the minimal coset representatives W_0^J of W_0/W_0^J can be characterized by

$$W_0^J = \{u \in W_0 \,|\, u(\Sigma_0^{J,+}) \subseteq \Sigma_0^+\}.$$

The following lemma now gives a derivational expression for p_u^λ ($u \in W_0^J$).

LEMMA 4.3.3. *Let $\lambda \in V_{\mathbb{C}}^*$ be J-standard. For $u \in W_0^J$ we have*

$$p_u^\lambda = K_J^{-1} \frac{d^{N_J}}{dt^{N_J}}\bigg|_{t=0} \left(\sum_{v\in W_{0,J}} d_u(t) e_{uv}(t)(-1)^{l(v)} e^{tv\delta_J} \right)$$

with coefficients

$$d_u(t) = \prod_{\alpha\in\Sigma_0^+\setminus u(\Sigma_0^{J,+})} \big(u\delta_J(\alpha^\vee)t + u\lambda(\alpha^\vee)\big)^{-1},$$

$$e_{uv}(t) = \prod_{\alpha\in\Sigma_0^+} (uv\delta_J(\alpha^\vee)t + u\lambda(\alpha^\vee) + k_\alpha)$$

and with strictly positive constant $K_J = N_J!\#W_0 \prod_{\alpha\in\Sigma_0^{J,+}} \delta_J(\alpha^\vee)$.

PROOF. By (4.3.2), $\psi_\mu^k(v')$ ($v' \in V$) depends analytically on the spectral parameter $\mu \in V_{\mathbb{C}}^*$. In particular, $\psi_{\lambda_t,k}(v')$ with $\lambda_t := \lambda + t\delta_J \in V_{\mathbb{C}}^*$ depends analytically on $t \in \mathbb{C}$, and we have the (point-wise) limit

$$\lim_{t\to 0} \psi_{\lambda_t,k} = \psi_{\lambda,k}. \tag{4.3.7}$$

For $\epsilon > 0$ we write

$$U_\epsilon^0 = \{t \in \mathbb{C} \,|\, 0 < |t| < \epsilon\}, \qquad U_\epsilon = \{t \in \mathbb{C} \,|\, |t| < \epsilon\}.$$

There exists an $\epsilon > 0$ such that λ_t is regular for $t \in U_\epsilon^0$, hence

$$\psi_{\lambda_t,k} = \frac{1}{\#W_0} \sum_{w \in W_0} \left(\prod_{\alpha \in \Sigma_0^+} \frac{w\lambda_t(\alpha^\vee) + k_\alpha}{w\lambda_t(\alpha^\vee)} \right) e^{w\lambda_t}, \qquad t \in U_\epsilon^0$$

by (4.3.5). Splitting the sum into a double sum $w = uv$ with $u \in W_0^J$ and $v \in W_{0,J}$ and using

$$\begin{aligned} \prod_{\alpha \in \Sigma_0^+} uv\lambda_t(\alpha^\vee) &= (-1)^{l(u)+l(v)} t^{N_J} \prod_{\alpha \in \Sigma_0^{J,+}} \delta_J(\alpha^\vee) \prod_{\beta \in \Sigma_0^+ \setminus \Sigma_0^{J,+}} \lambda_t(\beta^\vee) \\ &= (-1)^{l(v)} t^{N_J} \prod_{\alpha \in \Sigma_0^{J,+}} \delta_J(\alpha^\vee) \prod_{\beta \in \Sigma_0^+ \setminus u(\Sigma_0^{J,+})} u\lambda_t(\beta^\vee), \end{aligned}$$

we obtain

$$t^{N_J} \psi_{\lambda_t}^k = K_J^{-1} N_J! \sum_{u \in W_0^J} \sum_{v \in W_{0,J}} d_u(t) e_{uv}(t) (-1)^{l(v)} e^{tuv\delta_J + u\lambda} \tag{4.3.8}$$

as analytic functions in $t \in U_\epsilon$ (note that $d_u(t)$ is analytic at $t \in U_\epsilon$). By (4.3.7), $\psi_{\lambda,k}$ is the N_Jth term in the power series expansion of (4.3.8) at $t = 0$, which yields the desired result. □

Define the strictly positive constant C_J^k by

$$C_J^k = \frac{1}{\#W_0} \prod_{\alpha \in \Sigma_0^{J,+}} \frac{k_\alpha}{\delta_J(\alpha^\vee)}.$$

The highest and next to highest homogeneous terms of $p_u^\lambda \in E_J(0)$ ($u \in W_0^J$) can now be explicitly computed as follows.

PROPOSITION 4.3.4. *Let $\lambda \in V_{\mathbb{C}}^*$ be J-standard and $u \in W_0^J$.*
(i) *The highest homogeneous term h_u^λ of $p_u^\lambda \in E_J(0)$ is of degree N_J and is explicitly given by*

$$h_u^\lambda = C_J^k c_k(u\lambda) \prod_{\alpha \in \Sigma_0^{J,+}} \alpha.$$

(ii) *Suppose that λ is singular (i.e. $J \neq \emptyset$). The next to highest homogeneous term n_u^λ of $p_u^\lambda \in E_J(0)$ is*

$$n_u^\lambda = \partial_{u^{-1}\rho_{u\lambda}^k}\left(h_u^\lambda\right) = C_J^k c_k(u\lambda) \sum_{\beta \in \Sigma_0^{J,+}} u\beta(\rho_{u\lambda}^k) \prod_{\alpha \in \Sigma_0^{J,+} \setminus \{\beta\}} \alpha$$

with

$$\rho_\mu^k = \sum_{\alpha \in \Sigma_0^+} \frac{\alpha^\vee}{\mu(\alpha^\vee) + k_\alpha} \in V_{\mathbb{C}}. \tag{4.3.9}$$

REMARK 4.3.5. The formula for n_u^λ should be read as an identity between analytic functions in $k_\alpha > 0$ (the possible singularities are easily seen to be removable).

PROOF. **(i)** Observe that $e_{uv}(0) = e_u(0)$ is independent of $v \in W_{0,J}$, and

$$d_u(0)e_u(0) = c_k(u\lambda) \prod_{\alpha\in\Sigma_0^{J,+}} k_\alpha.$$

Combined with Lemma 4.3.3 we conclude that the highest homogeneous term h_u^λ of p_u^λ is given by

$$\begin{aligned} h_u^\lambda &= \frac{C_J^k}{N_J!} c_k(u\lambda) \frac{d^{N_J}}{dt^{N_J}}\Big|_{t=0} \sum_{v\in W_{0,J}} (-1)^{l(v)} e^{tv\delta_J} \\ &= \frac{C_J^k}{N_J!} c_k(u\lambda) \sum_{v\in W_{0,J}} (-1)^{l(v)} \big(v\delta_J\big)^{N_J}. \end{aligned} \tag{4.3.10}$$

On the other hand, by the Weyl denominator formula for Σ_0^J we have

$$\frac{d^{N_J}}{dt^{N_J}}\Big|_{t=0} \sum_{v\in W_{0,J}} (-1)^{l(v)} e^{tv\delta_J} = \frac{d^{N_J}}{dt^{N_J}}\Big|_{t=0} e^{t\delta_J} \prod_{\alpha\in\Sigma_0^{J,+}} \big(1-e^{-t\alpha}\big) = N_J! \prod_{\alpha\in\Sigma_0^{J,+}} \alpha.$$

Combined with the first equality in (4.3.10) we obtain the desired expression for h_u^λ.
(ii) The next to highest homogeneous term n_u^λ of p_u^λ is

$$\begin{aligned} n_u^\lambda = \frac{N_J}{K_J} \Big\{ & d_u'(0)e_u(0) \sum_{v\in W_{0,J}} (-1)^{l(v)} \big(v\delta_J\big)^{N_J-1} \\ & + d_u(0) \sum_{v\in W_{0,J}} (-1)^{l(v)} e_{uv}'(0) \big(v\delta_J\big)^{N_J-1} \Big\} \end{aligned}$$

in view of Lemma 4.3.3, where the prime denotes the t-derivative. The first $W_{0,J}$-sum in this expression is identically zero since it is a $W_{0,J}$-alternating polynomial of degree $< N_J$. By a direct calculation the remaining expression can be rewritten as

$$n_u^\lambda = \frac{C_J^k}{(N_J-1)!} c_k(u\lambda) \sum_{v\in W_{0,J}} (-1)^{l(v)} (v\delta_J)(u^{-1}\rho_{u\lambda}^k) \big(v\delta_J\big)^{N_J-1}.$$

The desired expression for n_u^λ now follows from (4.3.10). □

4.4. The Bethe ansatz equations

In the this and the next section we take $X = Q^\vee$. This and the next section are devoted to proving the following main results on the solution space of the boundary value problem.

THEOREM 4.4.1. *Let $\lambda \in V_{\mathbb{C}}^*$. The space $\mathrm{BVP}_k(\lambda)^W$ of W-invariant solutions to the boundary value problem is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is a purely imaginary, regular solution of the Bethe ansatz equations*

$$\prod_{\alpha\in\Sigma_0^+}\left(\frac{\lambda(\alpha^\vee)-k_\alpha}{\lambda(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)} = e^{\lambda(x)} \qquad \forall\, x\in Q^\vee. \tag{4.4.1}$$

If $\mathrm{BVP}_k(\lambda)^W$ is one-dimensional, then there exists a unique $\phi_{\lambda,k}\in \mathrm{BVP}_k(\lambda)^W$ normalized by $\phi_{\lambda,k}(0)=1$. The solution $\phi_{\lambda,k}$ is the unique W-invariant function satisfying

$$\phi_{\lambda,k}(v) = \frac{1}{\#W_0}\sum_{w\in W_0}\widetilde{c}_k(w\lambda)e^{w\lambda(v)}, \qquad v\in\overline{C_+}. \tag{4.4.2}$$

REMARK 4.4.2. **(i)** By Lemma 4.2.1 and (the proof of) Lemma 4.2.2 the Bethe ansatz equations (4.4.1) can be rewritten as

$$e^{w\lambda(\varphi^\vee)} = \frac{w\lambda(\varphi^\vee)-k_\varphi}{w\lambda(\varphi^\vee)+k_\varphi}\prod_{\alpha\in\Sigma_0^+\cap s_\varphi\Sigma_0^-}\frac{w\lambda(\alpha^\vee)-k_\alpha}{w\lambda(\alpha^\vee)+k_\alpha} \qquad \forall w\in W_0. \tag{4.4.3}$$

(ii) The Bethe ansatz equations (4.4.1) are independent of the choice of positive system Σ_0^+ by Corollary 4.2.4.

By Theorem 3.6.4**(ii)**, Theorem 4.4.1 is equivalent to the following theorem.

THEOREM 4.4.3. *Let $\lambda \in V_{\mathbb{C}}^*$. The space $E(\lambda)_{Q_k}^W$ is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is a purely imaginary, regular solution of the Bethe ansatz equations* (4.4.1). *If $E(\lambda)_{Q_k}^W$ is one-dimensional then $\psi_{\lambda,k}$ (cf.* (4.3.5)*) is the unique function in $E(\lambda)_{Q_k}^W$ normalized by $\psi_{\lambda,k}(0)=1$.*

Theorem 4.4.3 is proved in this section under the assumption that λ is regular. The assertion that λ is necessarily regular is proved in Section 4.5. Note that Theorem 3.8.5 (for $U=\mathbb{C}_{triv}$ and $X=Q^\vee$) is the generic regular version of Theorem 4.4.3. This is also the case for general lattices X, cf. Section 4.7.

We show that $E(\lambda)_Q^W\neq\{0\}$ implies that the spectral parameter λ is a purely imaginary solution of the Bethe ansatz equations (4.4.1).

From the results of the previous section it is clear that $E(\lambda)_Q^W$ is one-dimensional or zero-dimensional. In fact it is one-dimensional if and only if $Q_k(s_0)\psi_{\lambda,k}=\psi_{\lambda,k}$, in which case we have

$$E(\lambda)_Q^W = E(\lambda)_Q^{W_0} = \mathrm{span}_{\mathbb{C}}\{\psi_{\lambda,k}\}$$

(cf. with the analysis in Section 3.7). It is convenient to reformulate these observations in terms of

$$\mathcal{J}_k = \partial_{\varphi^\vee}Q_k(s_0)+k_\varphi \tag{4.4.4}$$

(viewed as an operator on e.g. $C^\infty(V)$ or $E(\lambda)$). The operator $\mathcal{J}_k$ is precisely $Q_k(-J_{a_0})$, with J_{a_0} as defined in the proof of Theorem 3.7.1. The following elementary commutation relations

$$\mathcal{J}_k\partial_v = \partial_{s_\varphi v}\mathcal{J}_k, \qquad \forall\, v\in V$$

follows from the cross relations **(c)** in Theorem 2.2.2. The equality $Q_k(s_0)\psi_{\lambda,k} = \psi_{\lambda,k}$ clearly implies $\mathcal{J}_k\psi_{\lambda,k} = (\partial_{\varphi^\vee} + k_\varphi)\psi_{\lambda,k}$.

LEMMA 4.4.4. *If λ is regular, then $\mathcal{J}_k\psi_{\lambda,k} = (\partial_{\varphi^\vee} + k_\varphi)\psi_{\lambda,k}$ implies $Q_k(s_0)\psi_{\lambda,k} = \psi_{\lambda,k}$.*

PROOF. By (4.3.4) we have a unique expansion

$$Q_k(s_0)\psi_{\lambda,k} - \psi_{\lambda,k} = \sum_{w\in W_0} d_w e^{w\lambda}$$

with $d_w \in \mathbb{C}$. We conclude from the equality $\mathcal{J}_k\psi_{\lambda,k} = (\partial_{\varphi^\vee} + k_\varphi)\psi_{\lambda,k}$ that $w\lambda(\varphi^\vee)d_w = 0$ for all $w \in W_0$. Since λ is regular, this implies $d_w = 0$ for all $w \in W_0$. □

For $p \in P(V)_\mathbb{C} \simeq S(V^*)_\mathbb{C}$ we write $p(\partial^\mu)$ for the associated constant coefficient differential operator acting on smooth functions in $\mu \in V_\mathbb{C}^*$.

LEMMA 4.4.5. *Let $p \in P(V)_\mathbb{C} \simeq S(V^*)_\mathbb{C}$. For $w \in W_0$ we have*

$$\mathcal{J}_k\big(p(w^{-1}\cdot)e^{w\mu}\big)(v) = -p(\partial^\mu)\big((\mu(w^{-1}\varphi^\vee) + k_\varphi)e^{\mu(w^{-1}\varphi^\vee)}e^{\mu(w^{-1}s_\varphi v)}\big),$$
$$\big(\partial_{\varphi^\vee} + k_\varphi\big)\big(p(w^{-1}\cdot)e^{w\mu}\big)(v) = p(\partial^\mu)\big((\mu(w^{-1}\varphi^\vee) + k_\varphi)e^{\mu(w^{-1}v)}\big),$$

where we view the left hand sides as functions in $v \in V$ and the right hand sides as functions in $\mu \in V_\mathbb{C}^*$. In particular,

$$\mathcal{J}_k\big(P^{(\le j)}(V)_\mathbb{C}\, e^\mu\big) \subseteq P^{(\le j)}(V)_\mathbb{C}\, e^{s_\varphi\mu}, \quad \big(\partial_{\varphi^\vee} + k_\varphi\big)\big(P^{(\le j)}(V)_\mathbb{C}\, e^\mu\big) \subseteq P^{(\le j)}(V)_\mathbb{C}\, e^\mu$$

for $j \in \mathbb{Z}_{\ge 0}$ and $\mu \in V_\mathbb{C}^*$.

PROOF. Observe that

$$\big(p(w^{-1}\cdot)e^{w\mu}\big)(v) = p(\partial^\mu)\big(e^{\mu(w^{-1}v)}\big),$$

and $p(\partial^\mu)$ (acting on $\mu \in V_\mathbb{C}^*$) clearly commutes with $\mathcal{J}_k$ and $(\partial_{\varphi^\vee} + k_\varphi)$ (which act on $v \in V$). Thus it suffices to prove the lemma for $p \equiv 1$, in which case the second formula is trivial. To prove the first formula for $p \equiv 1$ we may assume without loss of generality that $w = e$ is the unit element of W_0. Suppose that $\mu \in V_\mathbb{C}^*$ is regular. Then

$$\mathcal{J}_k(e^\mu) = -\big(\mu(\varphi^\vee) + k_\varphi\big)e^{\mu(\varphi^\vee)}e^{s_\varphi\mu}$$

holds by (3.7.9) (applied with $a = a_0$). In this formula the regularity constraint on μ can be removed by continuity. □

We denote $\pi_\lambda^{(j)} : E(\lambda) \to E^{(j)}(\lambda)$ for the projection onto $E^{(j)}(\lambda)$ along the decomposition $E(\lambda) = \bigoplus_{r=0}^\infty E^{(r)}(\lambda)$. Observe that

$$\mathrm{Id}_{E(\lambda)} = \sum_{j=0}^{N_J} \pi_\lambda^{(j)} \tag{4.4.5}$$

if λ is J-standard in view of Proposition 4.3.4**(i)**. In this section we consider the constraint on λ such that

$$\pi_\lambda^{(j)}\big(\mathcal{J}_k\psi_{\lambda,k}\big)=\pi_\lambda^{(j)}\big((\partial_{\varphi^\vee}+k_\varphi)\psi_{\lambda,k}\big) \tag{4.4.6}$$

for the highest degree component $j=N_J$.

The map $u\mapsto u^J$, where $u^J\in W_0^J$ is obtained from the unique decomposition

$$s_\varphi u=u^Ju_J,\qquad u^J\in W_0^J,\ u_J\in W_{0,J}, \tag{4.4.7}$$

defines an involution on W_0^J. Observe that

$$(u^J)_J=(u_J)^{-1},\qquad u\in W_0^J. \tag{4.4.8}$$

Recall that c_k denotes the regularized c-function (4.3.6).

LEMMA 4.4.6. *Suppose that $\lambda\in V_{\mathbb{C}}^*$ is J-standard.*
(i) *The equation* (4.4.6) *for $j=N_J$ holds if and only if λ satisfies the equations*

$$c_k(s_\varphi u\lambda)(u\lambda(\varphi^\vee)-k_\varphi)e^{-u\lambda(\varphi^\vee)}(-1)^{l(u_J)}=c_k(u\lambda)(u\lambda(\varphi^\vee)+k_\varphi),\quad \forall\, u\in W_0^J. \tag{4.4.9}$$

(ii) *For $u\in W_0^J$ and for multiplicity functions k such that $c_k(u\lambda)\neq 0$, we have*

$$\frac{c_k(s_\varphi u\lambda)}{c_k(u\lambda)}=(-1)^{l(u_J)}\prod_{\alpha\in\Sigma_0^+\cap s_\varphi\Sigma_0^-}\frac{u\lambda(\alpha^\vee)-k_\alpha}{u\lambda(\alpha^\vee)+k_\alpha}.$$

PROOF. **(i)** By (4.3.3), Lemma 4.4.5 and Proposition 4.3.4**(i)** we have

$$\begin{aligned}
\pi_\lambda^{(N_J)}(\mathcal{J}_k\psi_{\lambda,k})&=-C_J^k\sum_{u\in W_0^J}c_k(u\lambda)(u\lambda(\varphi^\vee)+k_\varphi)e^{u\lambda(\varphi^\vee)}e^{s_\varphi u\lambda}\prod_{\alpha\in\Sigma_0^{J,+}}s_\varphi u\alpha,\\
\pi_\lambda^{(N_J)}\big((\partial_{\varphi^\vee}+k_\varphi)\psi_{\lambda,k}\big)&=C_J^k\sum_{u\in W_0^J}c_k(u\lambda)(u\lambda(\varphi^\vee)+k_\varphi)e^{u\lambda}\prod_{\alpha\in\Sigma_0^{J,+}}u\alpha.
\end{aligned} \tag{4.4.10}$$

The proof now follows by equating the coefficients of $e^{u\lambda}\prod_{\alpha\in\Sigma_0^{J,+}}u\alpha$ ($u\in W_0^J$) in (4.4.10) using (4.4.7).
(ii) We first compare the denominators of $c_k(u\lambda)$ and $c_k(s_\varphi u\lambda)=c_k(u^J\lambda)$. If $\mu\in V_{\mathbb{C}}^*$ is regular then

$$\begin{aligned}
\prod_{\alpha\in\Sigma_0^+\setminus u^J\Sigma_0^{J,+}}u^J\mu(\alpha^\vee)&=\prod_{\alpha\in\Sigma_0^+}u^J\mu(\alpha^\vee)\prod_{\beta\in uu_J^{-1}\Sigma_0^{J,+}}(uu_J^{-1}\mu(\beta^\vee))^{-1}\\
&=(-1)^{l(u_J)}\prod_{\alpha\in\Sigma_0^+}s_\varphi uu_J^{-1}\mu(\alpha^\vee)\prod_{\beta\in u\Sigma_0^{J,+}}(uu_J^{-1}\mu(\beta^\vee))^{-1}\\
&=(-1)^{l(u_J)+1}\prod_{\alpha\in\Sigma_0^+\setminus u\Sigma_0^{J,+}}uu_J^{-1}\mu(\alpha^\vee).
\end{aligned}$$

Taking the limit $\mu\to\lambda$ we obtain

$$\prod_{\alpha\in\Sigma_0^+\setminus u^J\Sigma_0^{J,+}}u^J\lambda(\alpha^\vee)=(-1)^{l(u_J)+1}\prod_{\alpha\in\Sigma_0^+\setminus u\Sigma_0^{J,+}}u\lambda(\alpha^\vee).$$

A similar (and easier) computation leads to the comparative formula

$$\prod_{\alpha\in\Sigma_0^+\setminus u^J\Sigma_0^{J,+}}\big(u^J\lambda(\alpha^\vee)+k_\alpha\big)= \\ -\left(\prod_{\beta\in\Sigma_0^+\cap s_\varphi\Sigma_0^-}\frac{u\lambda(\beta^\vee)-k_\beta}{u\lambda(\beta^\vee)+k_\beta}\right)\prod_{\alpha\in\Sigma_0^+\setminus u\Sigma_0^{J,+}}\big(u\lambda(\alpha^\vee)+k_\alpha\big)$$

for the numerators of $c_k(u\lambda)$ and $c_k(u^J\lambda)$. Combining both formulas leads to the desired result. □

The set of solutions $\lambda\in iV^*$ of the Bethe ansatz equations (4.4.1) is denoted by BAE_k.

PROPOSITION 4.4.7. *Suppose that $\lambda\in V_{\mathbb{C}}^*$ is J-standard. The equation* (4.4.6) *for $j=N_J$ holds if and only if $\lambda\in\mathrm{BAE}_k$.*

PROOF. We first show that λ is purely imaginary if λ satisfies the equation (4.4.9) (see also (the proof of) Theorem 3.8.3). Let $\mu=u\lambda$ ($u\in W_0^J$) be the element in the W_0-orbit of λ having its real part in $\overline{V_+^*}$. Then $c_k(\mu)\neq 0$ since the multiplicity function k is strictly positive, hence (4.4.9) and Lemma 4.4.6**(ii)** imply

$$e^{\mu(\varphi^\vee)}=\frac{\mu(\varphi^\vee)-k_\varphi}{\mu(\varphi^\vee)+k_\varphi}\prod_{\alpha\in\Sigma_0^+\cap s_\varphi\Sigma_0^-}\frac{\mu(\alpha^\vee)-k_\alpha}{\mu(\alpha^\vee)+k_\alpha}. \tag{4.4.11}$$

The modulus of the left hand (respectively right hand side) of (4.4.11) is ≥ 1 (respectively ≤ 1) since the real part of μ is in $\overline{V_+^*}$ and the multiplicity function k is strictly positive. An argument similar to the argument in the last paragraph of the proof of Theorem 3.8.3 shows that $\lambda\in iV^*$.

Combined with Lemma 4.4.6**(i)** it follows that λ satisfies (4.4.6) for $j=N_J$ if and only if λ is a purely imaginary solution of the equations (4.4.9). For purely imaginary λ we have $c_k(u\lambda)\neq 0$ for all $u\in W_0^J$ due to the strict positivity of the multiplicity function k. The proof now follows from Lemma 4.4.6**(ii)** and Remark 4.4.2**(i)**. □

As an immediate result we obtain the following "regular part" of Theorem 4.4.1.

COROLLARY 4.4.8. *Suppose that $\lambda\in V_{\mathbb{C}}^*$ is regular. The space $E(\lambda)_Q^W$ is zero-dimensional or one-dimensional. It is one-dimensional if and only if $\lambda\in\mathrm{BAE}_k$. In that case $E(\lambda)_Q^W$ is spanned by $\psi_{\lambda,k}$* (3.7.17).

PROOF. By the observations at the beginning of the section it suffices to show that $E(\lambda)_Q^W\neq\{0\}$ iff $\lambda\in\mathrm{BAE}_k$.

Since $\mathrm{BAE}_k\subset iV^*$ is a W_0-invariant subset and $E(\lambda)_Q^W$ only depends on the W_0-orbit of λ, we may assume without loss of generality that λ is $\emptyset$-standard. If $E(\lambda)_Q^W\neq\{0\}$ then (4.4.6) holds, hence $\lambda\in\mathrm{BAE}_k$ by Proposition 4.4.7. Conversely, suppose that $\lambda\in\mathrm{BAE}_k$. Since λ is regular we have $\mathrm{Id}_{E(\lambda)}=\pi_\lambda^{(0)}$ by (4.4.5), hence $\mathcal{J}_k\psi_{\lambda,k}=$

$\big(\partial_{\varphi^\vee} + k_\varphi\big)\psi_{\lambda,k}$ by Proposition 4.4.7. By Lemma 4.4.4 this implies $Q_k(s_0)\psi_{\lambda,k} = \psi_{\lambda,k}$, hence $0 \neq \psi_{\lambda,k} \in E(\lambda)_Q^W$. □

4.5. The Pauli principle

In this section we complete the proof of Theorem 4.4.1. In view of Proposition 4.4.7 and Corollary 4.4.8 it suffices to show the following root system analog of the Pauli principle.

PROPOSITION 4.5.1. *If* $\lambda \in \mathrm{BAE}_k$ *is singular then* $E(\lambda)_Q^W = \{0\}$.

REMARK 4.5.2. In physics literature the regularity of the spectral parameter λ (see Theorems 4.4.1 and 4.6.1) is usually imposed as an additional requirement, since it automatically ensures (cf. Lemma 3.7.7 and (3.7.13)) that eigenstates admit plane wave expansion within any alcove $C \in \mathcal{C}$. The regularity condition for root system Σ_0 of type A_n can be viewed as a Pauli type principle for the interacting quantum *bosons*, since it implies that the momenta of the quantum bosons are pair-wise different. An actual proof of the regularity of the spectrum was obtained by Izergin and Korepin [**47**] using quantum inverse scattering methods. In this derivation the regularity condition follows from the strict convexity of the master function (introduced by Yang & Yang [**81**]).

It is believed [**47**] that quantum integrable systems governed by a strictly convex master function always have a regularity constraint on the spectrum, although a conceptual understanding is not known as far as we know. Our derivation of the regularity constraint on the spectrum is in accordance to this point of view.

REMARK 4.5.3. In Chapter 5 we give an independent proof of Proposition 4.5.1 that is based on a density argument of the Bethe ansatz eigenfunctions in the Hilbert space of square-integrable functions on the fundamental chamber C_+ (see also Remark 5.6.8).

We first introduce the *master function* for our quantum system.

DEFINITION 4.5.4. *The master function* $S_k : P \times V^* \to \mathbb{R}$ *is defined by*

$$S_k(\mu,\xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle\mu,\xi\rangle + \frac{1}{2}\sum_{\alpha\in\Sigma_0}\|\alpha\|^2\int_0^{\xi(\alpha^\vee)}\arctan\left(\frac{t}{k_\alpha}\right)dt. \tag{4.5.1}$$

We analyze the master function $S_k(\mu,\cdot)$ at a given weight $\mu \in P$. Observe that the Hessian $B_\xi^k : V^* \times V^* \to \mathbb{R}$ of $S_k(\mu,\cdot)$ at $\xi \in V^*$ is independent of μ, and is given explicitly by

$$\begin{aligned} B_\xi^k(\eta,\eta') &= \left(\partial_\eta\partial_{\eta'}S_k(\mu,\cdot)\right)(\xi) \\ &= \langle\eta,\eta'\rangle + \frac{1}{2}\sum_{\alpha\in\Sigma_0}k_\alpha\|\alpha\|^2\frac{\eta(\alpha^\vee)\eta'(\alpha^\vee)}{k_\alpha^2+\xi(\alpha^\vee)^2}, \qquad \eta,\eta' \in V^*. \end{aligned} \tag{4.5.2}$$

By the strict positivity of the multiplicity function k, it follows from (4.5.2) that the Hessian B_ξ^k is positive definite for all $\xi \in V^*$, hence $S_k(\mu,\cdot)$ is strictly convex.

For the proof of Proposition 4.5.1 we may assume without loss of generality that $\lambda \in \mathrm{BAE}_k$ is J-standard (in particular, $\lambda \in i\overline{V_+^*}$). We write $V_J^* \subseteq V^*$ for the real subspace spanned by the subset J of simple roots. Its complement in V is defined by

$$V_J^\perp = \{v \in V \,|\, \xi(v) = 0 \quad \forall \xi \in V_J^*\}.$$

Observe that $V_J^\perp = V$ iff $J = \emptyset$ iff λ is regular.

Consider the linear map $K_\lambda^k : V \to V$ defined by

$$K_\lambda^k(v) = v + \sum_{\alpha \in \Sigma_0} \frac{k_\alpha \alpha(v)\alpha^\vee}{k_\alpha^2 - \lambda(\alpha^\vee)^2}, \qquad v \in V.$$

It follows from (4.5.2) that

$$B_{-i\lambda}^k(\eta_v, \eta_{v'}) = \langle K_\lambda^k(v), v' \rangle, \qquad v, v' \in V$$

with $\eta_v = \langle v, \cdot \rangle \in V^*$. Since $B_{-i\lambda}^k$ is positive definite, $K_\lambda^k : V \xrightarrow{\sim} V$ is a linear isomorphism. Proposition 4.5.1 thus is an immediate consequence of the following lemma.

LEMMA 4.5.5. *Let $\lambda \in iV^*$ be a singular J-standard solution of the Bethe ansatz equations* (4.4.1)*. Then λ satisfies the constraint*

$$\pi_\lambda^{(N_J-1)}\big(\mathcal{J}_k \psi_{\lambda,k}\big) = \pi_\lambda^{(N_J-1)}\big((\partial_{\varphi^\vee} + k_\varphi)\psi_{\lambda,k}\big) \tag{4.5.3}$$

iff $K_\lambda^k(V) \subseteq V_J^\perp$.

PROOF. Fix a singular J-standard solution $\lambda \in i\overline{V_+^*}$ of the Bethe ansatz equations (4.4.1) (in particular $J \neq \emptyset$). By a similar computation as in the proof of Proposition 4.4.7 we obtain from (4.3.3), Lemma 4.4.5 and Proposition 4.3.4,

$$\pi_\lambda^{(N_J-1)}\big((\partial_{\varphi^\vee} + k_\varphi)\psi_{\lambda,k}\big) = C_J^k \sum_{u \in W_0^J} c_k(u\lambda) \sum_{\beta \in \Sigma_0^{J,+}} u\beta(a_{u\lambda}) e^{u\lambda} \prod_{\alpha \in \Sigma_0^{J,+} \setminus \{\beta\}} u\alpha,$$
$$\pi_\lambda^{(N_J-1)}\big(\mathcal{J}_k \psi_\lambda\big) = C_J^k \sum_{u \in W_0^J} c_k(u\lambda) e^{-u^J\lambda(\varphi^\vee)} \sum_{\beta \in \Sigma_0^{J,+}} u\beta(b_{u^J\lambda}) e^{u^J\lambda} \prod_{\alpha \in \Sigma_0^{J,+} \setminus \{\beta\}} u^J u_J \alpha$$

with vectors $a_\mu, b_\mu \in V_{\mathbb{C}}$ ($\mu \in V_{\mathbb{C}}^*$) given by

$$a_\mu = (\mu(\varphi^\vee) + k_\varphi)\rho_\mu^k + \varphi^\vee,$$
$$b_\mu = \big(\mu(\varphi^\vee) - k_\varphi\big)\big(\rho_{s_\varphi\mu}^k + \varphi^\vee\big) - \varphi^\vee,$$

where we have used the involution on W_0^J defined by (4.4.7), as well as (4.4.8). For $u \in W_0^J$ we have

$$\begin{aligned}\sum_{\beta\in\Sigma_0^{J,+}} u\beta(b_{u^J\lambda}) \prod_{\alpha\in\Sigma_0^{J,+}\setminus\{\beta\}} u^J u_J\alpha &= (-1)^{l(u_J)}\left(\sum_{\beta\in\Sigma_0^{J,+}}\frac{u\beta(b_{u^J\lambda})}{u^Ju_J\beta}\right)\prod_{\alpha\in\Sigma_0^{J,+}}u^J\alpha\\ &= \frac{1}{2}(-1)^{l(u_J)}\left(\sum_{\beta\in\Sigma_0^{J}}\frac{uu_J^{-1}\beta(b_{u^J\lambda})}{u^J\beta}\right)\prod_{\alpha\in\Sigma_0^{J,+}}u^J\alpha\\ &= (-1)^{l(u_J)}\sum_{\beta\in\Sigma_0^{J,+}}uu_J^{-1}\beta(b_{u^J\lambda})\prod_{\alpha\in\Sigma_0^{J,+}\setminus\{\beta\}}u^J\alpha.\end{aligned}$$

Consequently (4.5.3) is equivalent to

$$c_k(u\lambda)u\beta(a_{u\lambda}) = (-1)^{l(u_J)}c_k(u^J\lambda)e^{-u\lambda(\varphi^\vee)}s_\varphi u\beta(b_{u\lambda}), \qquad \forall\, u\in W_0^J,\ \forall\,\beta\in\Sigma_0^{J,+}.$$

Since λ is a solution of the Bethe ansatz equations (see (4.4.9) for the convenient equivalent form of the Bethe ansatz equations) this is equivalent to

$$\big(u\lambda(\varphi^\vee)-k_\varphi\big)a_{u\lambda}-\big(u\lambda(\varphi^\vee)+k_\varphi\big)s_\varphi b_{u\lambda}\in u(V_J^\perp), \qquad \forall\, u\in W_0^J. \tag{4.5.4}$$

Note that (4.5.4) only depends on the coset $uW_{0,J}$ ($u\in W_0^J$). Using the explicit expressions for $a_{u\lambda}$ and $b_{u\lambda}$ we can rewrite (4.5.4) as

$$\big(w^{-1}\rho^k_{w\lambda}-w^{-1}s_\varphi\rho^k_{s_\varphi w\lambda}\big)+\left(\frac{w\lambda(\varphi^\vee)^2-k_\varphi^2-2k_\varphi}{w\lambda(\varphi)^2-k_\varphi^2}\right)w^{-1}\varphi^\vee\in V_J^\perp, \quad \forall\, w\in W_0. \tag{4.5.5}$$

We match (4.5.5) to the desired condition $K_\lambda^k(V)\subseteq V_J^\perp$ as follows. Since Σ_0 is an irreducible root system in V^*, the condition $K_\lambda^k(V)\subseteq V_J^\perp$ is equivalent to $K_\lambda^k(w^{-1}\varphi^\vee)\in V_J^\perp$ for all $w\in W_0$, which in turn is equivalent to (4.5.5) if

$$K_\lambda^k(w^{-1}\varphi^\vee)=\big(w^{-1}\rho^k_{w\lambda}-w^{-1}s_\varphi\rho^k_{s_\varphi w\lambda}\big)+\left(\frac{w\lambda(\varphi^\vee)^2-k_\varphi^2-2k_\varphi}{w\lambda(\varphi)^2-k_\varphi^2}\right)w^{-1}\varphi^\vee \tag{4.5.6}$$

for all $w\in W_0$. To prove (4.5.6) we first observe that

$$s_\varphi\rho^k_{s_\varphi w\lambda}=\rho^k_{w\lambda}-2\sum_{\alpha\in\Sigma_0^+\cap s_\varphi\Sigma_0^-}\frac{k_\alpha\alpha^\vee}{k_\alpha^2-w\lambda(\alpha^\vee)^2}$$

by the explicit expression (4.3.9) for ρ^k_μ. Using (4.2.1) this can be rewritten as

$$w^{-1}\rho^k_{w\lambda}-w^{-1}s_\varphi\rho^k_{s_\varphi w\lambda}=2\frac{k_\varphi w^{-1}\varphi^\vee}{w\lambda(\varphi^\vee)^2-k_\varphi^2}+2\sum_{\alpha\in\Sigma_0^+}\frac{k_\alpha\alpha(\varphi^\vee)w^{-1}\alpha^\vee}{k_\alpha^2-w\lambda(\alpha^\vee)^2}.$$

The second term can be rewritten as

$$\begin{aligned}2\sum_{\alpha\in\Sigma_0^+}\frac{k_\alpha\alpha(\varphi^\vee)w^{-1}\alpha^\vee}{k_\alpha^2-w\lambda(\alpha^\vee)^2} &= \sum_{\alpha\in\Sigma_0}\frac{k_\alpha\alpha(\varphi^\vee)w^{-1}\alpha^\vee}{k_\alpha^2-w\lambda(\alpha^\vee)^2}\\ &= \sum_{\alpha\in\Sigma_0}\frac{k_\alpha\alpha(w^{-1}\varphi^\vee)}{k_\alpha^2-\lambda(\alpha^\vee)^2}\\ &= K_\lambda^k(w^{-1}\varphi^\vee)-w^{-1}\varphi^\vee.\end{aligned}$$

Combining the latter two formulas yields (4.5.6). □

4.6. General lattices case

In this section we consider general lattices X satisfying $Q^\vee\subset X\subset P^\vee$.

Recall that a function f is called (W,χ)-invariant ($\chi\in\widehat{X/Q^\vee}$) if $(w\tau_x)f=\chi(x+Q^\vee)f$ for all $w\in W_0$ and $x\in X$. The main results on the solution space of the boundary value problem is the following statement.

THEOREM 4.6.1. *Let $\lambda\in V_{\mathbb{C}}^*$ and $\chi\in\widehat{X/Q^\vee}$. The space* $\mathrm{BVP}_k(\lambda)^{W,\chi}$ *of (W,χ)-invariant solutions to the boundary value problem is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is a purely imaginary, regular solution of the Bethe ansatz equations*

$$\prod_{\alpha\in\Sigma_0^+}\left(\frac{\lambda(\alpha^\vee)-k_\alpha}{\lambda(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)}=\chi(x+Q^\vee)e^{\lambda(x)}\qquad\forall\, x\in X. \tag{4.6.1}$$

If $\mathrm{BVP}_k(\lambda)^{W,\chi}$ *is one-dimensional, then there exists a unique $\phi_{\lambda,k}\in\mathrm{BVP}_k(\lambda)^{W,\chi}$ normalized by $\phi_{\lambda,k}(0)=1$. The solution $\phi_{\lambda,k}$ is the unique (W,χ)-invariant function satisfying*

$$\phi_{\lambda,k}(v)=\frac{1}{\#W_0}\sum_{w\in W_0}\widetilde{c}_k(w\lambda)e^{w\lambda(v)},\qquad v\in\overline{C_+}. \tag{4.6.2}$$

The Bethe ansatz equations (4.6.1) do not depend on the choice of a positive system Σ_0^+ (cf. Remark 4.4.2.**(i)**).

REMARK. The Bethe ansatz eigenfunctions $\phi_{\lambda,k}$ and the necessity of the Bethe ansatz equations (4.6.1) on the allowed spectrum were obtained by Lieb and Liniger [**56**] for root system Σ_0 of type A_n, and soon after generalized to root system Σ_0 of type D_n by Gaudin [**26**], [**28**] (see also [**53**])

By Theorem 3.6.4**(ii)**, Theorem 4.6.1 is equivalent to the following theorem.

THEOREM 4.6.2. *Let $\lambda\in V_{\mathbb{C}}^*$ and $\chi\in\widehat{X/Q^\vee}$. The space $E(\lambda)_{Q_k}^{W,\chi}$ is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is a purely imaginary, regular solution of the Bethe ansatz equations* (4.6.1)*. If $E(\lambda)_{Q_k}^{W,\chi}$ is one-dimensional then $\psi_{\lambda,k}$ (cf.* (4.3.5)*) is the unique function in $E(\lambda)_{Q_k}^{W,\chi}$ normalized by $\psi_{\lambda,k}(0)=1$.*

PROOF OF THEOREM 4.6.2. Observe that $\psi_{\lambda,k} = \psi^1_{\lambda,k}$ (with $1 \in \mathbb{C}_{triv}$) by (3.7.12) and (4.3.2). Since also $E(\lambda)^{W,\chi}_{Q_k} \subset E(\lambda)^{W_0}_{Q_k}$ (see (3.2.5), the first and last statement follows from Lemma 3.7.7.

Assume that λ is a purely imaginary, regular solution of the Bethe ansatz equations (4.6.1). In particular, λ is a generic regular spectral parameter. Whence by Lemma 4.2.3 and Theorem 3.8.5 we have $\psi_{\lambda,k} \in E(\lambda)^{W,\chi}_{Q_k}$, and therefore $E(\lambda)^{W,\chi}_{Q_k}$ is one-dimensional by the first statement

Now assume that $E(\lambda)^{W,\chi}_{Q_k}$ is one-dimensional. Then λ must be purely imaginary and regular by Theorem 4.4.3 and $E(\lambda)^{W,\chi}_{Q_k} \subset E(\lambda)^{W}_{Q_k}$. It satisfies the Bethe ansatz equations (4.6.1) by Theorem 3.8.5 (see also Lemma 4.2.3), concluding the proof of the second statement, and whence of the theorem. □

4.7. Lattices and cosets

This section is preparatory to the next section. For any abelian group G let $G^* := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$. Recall that the unitary dual $\mathrm{Hom}_{\mathbb{Z}}(G, S^1)$ of G is denoted by $\hat{G}$. Assume $A \subset B$, both free abelian groups of the same finite rank. Then B/A is a finite abelian group. Beceause A and B have the same rank, the restriction map of B^* to A^* is injective. We identify B^* with its image in A^* under this map: $B^* \subset A^*$. We have a natural perfect pairing

$$\langle \cdot, \cdot \rangle : A^* \times A \longrightarrow \mathbb{Z}, \quad (l, a) \mapsto l(a).$$

Since A and B have the same rank, $\langle \cdot, \cdot \rangle$ extends uniquelly to a $\mathbb{Z}$-linear map from $A^* \times B \to \mathbb{Q}$ that coincides with $\langle \cdot, \cdot \rangle$ when restricted to $A^* \times A$. This map will also be denoted by $\langle \cdot, \cdot \rangle$. We get the perfect pairing

$$A^*/B^* \times B/A \to S^1, \text{ defined by } (l + B^*, b + A) \mapsto e^{2\pi i \langle l, b\rangle} =: [l + B^*, b + A].$$

Since the pairing is perfect, the map

$$A^*/B^* \to \widehat{B/A} \text{ defined by } l + B^* \mapsto [\cdot, l + B] =: \chi_l$$

is an isomorphism of groups. The inverse $\widehat{B/A} \to A^*/B^*$ of the isomorphism above is given by

$$\chi \mapsto \{l \in A^* |\, \chi_l = \chi\} =: A^{*,\chi}.$$

It is easily checked that the composition maps in both directions are identity maps. In particular $A^{*,1} = B^*$, the "trivial" coset. As an immediate corollary we get

$$A^* = \dot{\bigcup_{\chi \in \widehat{B/A}}} A^{*,\chi} \tag{4.7.1}$$

We will apply the above theory with: $A = Q^\vee$, $B = X$. Then $A^* = P$, $B^* = Y$, and whence by (4.7.1),

$$P = \dot{\bigcup_{\chi \in \widehat{X/Q^\vee}}} Y^\chi, \tag{4.7.2}$$

and where

$$Y^\chi = \{\lambda \in V^* |\, e^{2\pi i\lambda(x)} = \chi(-x + Q^\vee) \quad \forall x \in X\} \tag{4.7.3}$$

(note that $Y^\chi = A^{*,\chi^{-1}}$). In particular $Y^1 = Y$. Furthermore, the Y^χ are precisely the Y-cosets in P. For our purposes the following decomposition turns out to be more useful,

$$P = \dot{\bigcup_{\chi \in \widehat{X/Q^\vee}}} \rho + Y^\chi, \tag{4.7.4}$$

which is the same decomposition as 4.7.2, but parametrized differently because of the ρ-shift of the Y-cosets.

Because W_0 acts trivially on $X/Q^\vee$ (cf. Section 2.2.2) it follows easily that Y^χ ($\chi \in \widehat{X/Q^\vee}$) is W_0-invariant. Whence the Y-cosets of P are W_0-invariant. Since $Q \subset Y$ and $Y + Y^\chi = Y^\chi$ we also have $Q + Y^\chi = Y^\chi$.

The set of solutions $\lambda \in iV^*$ of the Bethe ansatz equations (4.6.1) is denoted by BAE_k^χ (note that $\mathrm{BAE}_k^\chi = \mathrm{BAE}_k$ if $X = Q^\vee$).

In the next section we will show that that the following decomposition is closely related to (4.7.4).

PROPOSITION 4.7.1. *The sets* BAE_k^χ *(*$\chi \in \widehat{X/Q^\vee}$*) are* W_0*-invariant. Furthermore,* BAE_k *admits the disjoint decomposition*

$$\mathrm{BAE}_k = \dot{\bigcup_{\chi \in \widehat{X/Q^\vee}}} \mathrm{BAE}_k^\chi. \tag{4.7.5}$$

PROOF. One easily shows that (4.2.5) also holds when one replaces $I_x(\lambda)$ by the expression on the right hand side of (4.2.2) ($x \in X$, $\lambda \in iV^*$). Since W_0 acts trivially on $X/Q^\vee$, the first statement follows without difficulty from the definition of BAE_k^χ.

The disjointness of the BAE_k^χ is obvious, as well the fact that $\mathrm{BAE}_k^\chi \subset \mathrm{BAE}_k$. Therefore it suffices to show that every $\lambda \in \mathrm{BAE}_k$ lies in BAE_k^χ for a χ. Let $\lambda \in \mathrm{BAE}_k$. Consider the homomorphism $\chi' : X \longrightarrow \mathbb{C}^*$ of groups defined by

$$x \mapsto e^{-\lambda(x)} \prod_{\alpha \in \Sigma_0^+} \left(\frac{\lambda(\alpha^\vee) - k_\alpha}{\lambda(\alpha^\vee) + k_\alpha} \right)^{\alpha(x)}.$$

Since $\lambda \in iV^*$, $\mathrm{Image}(\chi') \subset S^1$. Furthermore, $Q^\vee$ is contained in the kernel of χ' because $\lambda \in \mathrm{BAE}_k$. Whence the assignment $x + Q^\vee \mapsto \chi'(x)$ gives a well-defined map $\chi : X/Q^\vee \to S^1$, and is an element of $\widehat{X/Q^\vee}$. By construction $\lambda \in \mathrm{BAE}_k^\chi$. □

4.8. The master function

The main result in this section consists in the parametrization of BAE_k^χ ($\chi \in \widehat{X/Q^\vee}$) by $\rho + Y^\chi$.

We start with the parametrization of the set BAE_k.

PROPOSITION 4.8.1. *For $\mu \in P$ there exists a unique extremum $\widehat{\mu}_k \in V^*$ of the master function $S_k(\mu,\cdot)$. The assignment $\mu \mapsto i\widehat{\mu}_k$ defines a W_0-equivariant bijection $P \xrightarrow{\sim} \mathrm{BAE}_k$.*

REMARK. For Σ_0 of type A_n, Yang & Yang [**81**] introduced the master function S (also known as the Yang-Yang action) and derived the special case of Proposition 4.8.1 using its strict convexity.

LEMMA 4.8.2. *We have $\lambda \in \mathrm{BAE}_k$ if and only if $\lambda = i\eta$ with $\eta \in V^*$ an extremal vector of the master function $S_k(\mu,\cdot)$ for some $\mu \in P$.*

PROOF. We rewrite the Bethe ansatz equations (4.4.1) in logarithmic form. By a direct computation using the elementary identity

$$e^{-2i\arctan(x)} = \frac{1-ix}{1+ix} \qquad (x \in \mathbb{R}) \tag{4.8.1}$$

the Bethe ansatz equations (4.4.1) for $\lambda \in iV^*$ can be rewritten as

$$-i\lambda(x) + \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{-i\lambda(\alpha^\vee)}{k_\alpha}\right)\alpha(x) = 0 \quad \text{module } 2\pi\mathbb{Z} \tag{4.8.2}$$

for all $x \in Q^\vee$. On the other hand, for $\mu \in P$ the gradient of the master function $S_k(\mu,\cdot) : V^* \to \mathbb{R}$ (see (4.5.1)) is determined by

$$\big(\partial_\xi S_k(\mu,\cdot)\big)(\eta) = \langle \eta - 2\pi\mu + \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{\eta(\alpha^\vee)}{k_\alpha}\right)\alpha, \xi\rangle, \qquad \xi,\eta \in V^*. \tag{4.8.3}$$

Since the root lattice Q is a full lattice in V^* generated by Σ_0, $\eta \in V^*$ is an extremal vector of $S_k(\mu,\cdot)$ if and only if $\big(\partial_y S_k(\mu,\cdot)\big)(\eta) = 0$ for all $y = \alpha \in \Sigma_0$, which by (4.8.3) is equivalent to

$$\eta(x) + \sum_{\alpha\in\Sigma_0} \alpha(x)\arctan\left(\frac{\eta(\alpha^\vee)}{k_\alpha}\right) = 2\pi\mu(x) \tag{4.8.4}$$

for all $x = \alpha^\vee \in \Sigma_0^\vee$. Since $Q^\vee$, as a lattice, is generated by $\Sigma_0^\vee$, the proof follows by comparing (4.8.4) to (4.8.2). □

PROOF OF PROPOSITION 4.8.1. We analyzed the master function $S_k(\mu,\cdot)$ at a given weight $\mu \in P$ in Section 4.5 and showed that $S_k(\mu,\cdot)$ is strictly convex. Furthermore, for all $\mu \in P$,

$$S_k(\mu,\xi) \geq \frac{\|\xi\|^2}{2} - 2\pi\langle\mu,\xi\rangle \to \infty, \qquad \|\xi\| \to \infty,$$

hence $S_k(\mu,\cdot)$ has a unique extremum $\widehat{\mu}_k \in V^*$, which is a global minimum. It now follows from (4.8.3) that $\widehat{\mu}_k$ ($\mu \in P$) is uniquely determined by the equation

$$\widehat{\mu}_k + \sigma^k_{\widehat{\mu}_k} = 2\pi\mu \tag{4.8.5}$$

in V^*, where $\sigma^k_\lambda \in V^*$ ($\lambda \in V^*$) is defined by

$$\sigma^k_\lambda = \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{\lambda(\alpha^\vee)}{k_\alpha}\right)\alpha.$$

Combined with Lemma 4.8.2 it now follows that the map $\mu \mapsto i\widehat{\mu}_k$ is a bijection from the weight lattice P onto BAE_k. The W_0-equivariance of this map is immediate from the equivariance property

$$\big(\partial_{w\xi} S_k(w\mu,\cdot)\big)(w\eta) = \big(\partial_\xi S_k(\mu,\cdot)\big)(\eta), \qquad \forall\, w \in W_0$$

for $\xi,\eta \in V^*$ and $\mu \in P$, completing the proof. □

PROPOSITION 4.8.3. *Let $\chi \in \widehat{X/Q^\vee}$. Then Y^χ is a Y-coset in P, and any coset is of this form (for a unique χ). Furthermore, $\mu \mapsto i\widehat{\mu}_k$ defines a W_0-equivariant bijection $\rho + Y^\chi \xrightarrow{\sim} \mathrm{BAE}^\chi_k$.*

PROOF. The first statement follows from Section 4.7.

We claim that $\lambda \in \mathrm{BAE}^\chi_k$ if $\lambda = i\widehat{\mu}_k$ for a $\mu \in \rho + Y^\chi$. To prove the second statement it suffices to show this because of Proposition 4.8.1, Proposition 4.7.1 and (4.7.4). Let then $\mu \in \rho + Y^\chi$. By (4.8.1) we have

$$e^{-i\sigma^k_\eta(x)} = \prod_{\alpha\in\Sigma_0^+}\left(-\frac{\eta(\alpha^\vee)-k_\alpha}{\eta(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)} = e^{2\pi i\rho(x)}\prod_{\alpha\in\Sigma_0^+}\left(\frac{\eta(\alpha^\vee)-k_\alpha}{\eta(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)} \qquad (x \in X)$$

for all $\eta \in iV^*$. Together with the fact that $\widehat{\mu}_k$ is uniquely determined by the equation (4.8.5) this gives

$$\begin{aligned} e^{i\hat{\mu}_k(x)} &= e^{2\pi i(\rho+\mu')(x) - i\sigma^k_{\hat{\mu}_k}(x)} \\ &= \chi(-x+Q^\vee)e^{2\pi i\rho(x)}e^{2\pi i\rho(x)}\prod_{\alpha\in\Sigma_0^+}\left(\frac{\hat{\mu}_k(\alpha^\vee)-k_\alpha}{\hat{\mu}_k(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)} \qquad \forall x \in X. \end{aligned}$$

Now use that $2\rho \in Q$ and $\mu \in \rho + Y^\chi$ to conclude that these equations are precisely the Bethe ansatz equations (4.6.1) for $\lambda = i\widehat{\mu}_k$, and therefore $\lambda \in \mathrm{BAE}^\chi_k$. □

4.9. Moment gaps

In this section we prove the following proposition, which yields estimates for the location of the deformed weight $\widehat{\mu} = \widehat{\mu}_k \in \mathrm{BAE}_k$ compared to the parametrizing weight $\mu \in P$.

PROPOSITION 4.9.1. *For $\mu \in P^+$ and $\beta \in \Sigma_0^+$ we have*

$$\frac{2\pi\mu(\beta^\vee)}{\left(1+\frac{h_k}{n}\right)} \le \widehat{\mu}_k(\beta^\vee) \le 2\pi\mu(\beta^\vee), \tag{4.9.1}$$

where $h_k = 2\sum_{\alpha\in\Sigma_0} k_\alpha^{-1}$. Furthermore, $\mu \in P^+$ if and only if $\widehat{\mu}_k \in \overline{V^*_+}$.

The lower bound in (4.9.1) shows how far away the spectral values $\widehat{\mu}_k \in V_+^*$ ($\mu \in P^{++}$) are from being singular.

REMARK. Estimates for the momenta gaps of the quantum particles play a role in the study of the thermodynamical limit, see [**56**] and [**81**]. See e.g. [**28**, Sect 4.3.2] for the exact analog of the estimates (4.9.1) for Σ_0 of type A_n.

In view of (4.8.3) and Lemma 4.8.2, the deformed weight $\widehat{\mu}_k \in V^*$ ($\mu \in P$) is the unique solution of (4.8.5).

The following lemma establishes the necessary bounds for σ_λ^k.

LEMMA 4.9.2. *For $\lambda \in \overline{V_+^*}$,*

$$0 \leq \sigma_\lambda^k(\beta^\vee) \leq \frac{h_k}{n}\lambda(\beta^\vee), \qquad \forall\, \beta \in \Sigma_0^+$$

with $h_k = 2\sum_{\alpha\in\Sigma_0} k_\alpha^{-1}$.

PROOF. Fix $\lambda \in \overline{V_+^*}$ and $\beta \in \Sigma_0^+$. Let Σ_0^β be the set of roots $\alpha \in \Sigma_0$ satisfying $\alpha(\beta^\vee) > 0$, then

$$\sigma_\lambda^k(\beta^\vee) = \sum_{\alpha\in\Sigma_0^\beta} \left\{ \arctan\left(\frac{\lambda(\alpha^\vee)}{k_\alpha}\right) - \arctan\left(\frac{\lambda(s_\beta\alpha^\vee)}{k_\alpha}\right)\right\}\alpha(\beta^\vee). \tag{4.9.2}$$

Each term in this sum is positive, hence $\sigma_\lambda^k(\beta^\vee) \geq 0$.

For the second inequality, we use the estimate for $\alpha \in \Sigma_0^\beta$,

$$\arctan\left(\frac{\lambda(\alpha^\vee)}{k_\alpha}\right) - \arctan\left(\frac{\lambda(s_\beta(\alpha^\vee))}{k_\alpha}\right) = \int_{\lambda(s_\beta(\alpha^\vee))/k_\alpha}^{\lambda(\alpha^\vee)/k_\alpha} \frac{dx}{1+x^2} \leq \frac{\lambda(\beta^\vee)\beta(\alpha^\vee)}{k_\alpha},$$

leading to

$$\sigma_\lambda^k(\beta^\vee) \leq \lambda(\beta^\vee) \sum_{\alpha\in\Sigma_0^\beta} \frac{\beta(\alpha^\vee)\alpha(\beta^\vee)}{k_\alpha} = \frac{\lambda(\beta^\vee)}{2} \sum_{\alpha\in\Sigma_0} \frac{\beta(\alpha^\vee)\alpha(\beta^\vee)}{k_\alpha} \tag{4.9.3}$$

in view of (4.9.2). Now note that

$$\xi \mapsto \sum_{\alpha\in\Sigma_0} k_\alpha^{-1}\xi(\alpha^\vee)\alpha$$

defines a W_0-equivariant linear map $V^* \to V^*$. By Schur's lemma it equals $C_k \mathrm{Id}_{V^*}$ for some constant $C_k \in \mathbb{C}$. To determine C_k explicitly we fix a basis $\{e_j\}_{j=1}^n$ of V and we denote $\{\epsilon_j\}_{j=1}^n$ for the corresponding dual basis of V^*. Then

$$C_k n = \sum_{j=1}^n \sum_{\alpha\in\Sigma_0} k_\alpha^{-1}\epsilon_j(\alpha^\vee)\alpha(e_j) = h_k$$

with $h_k = 2\sum_{\alpha\in\Sigma_0} k_\alpha^{-1}$. Combined with (4.9.3) we obtain $\sigma_\lambda^k(\beta^\vee) \leq \frac{h_k}{n}\lambda(\beta^\vee)$. $\square$

COROLLARY 4.9.3. *Let $\mu \in P$. We have $\widehat{\mu}_k \in \overline{V_+^*}$ if and only if $\mu \in P^+$.*

PROOF. Let $\mu \in P$ and suppose that $\widehat{\mu}_k \in \overline{V_+^*}$. Then for all $\beta \in \Sigma_0^+$,

$$2\pi\mu(\beta^\vee) = \widehat{\mu}_k(\beta^\vee) + \sigma^k_{\widehat{\mu}_k}(\beta^\vee) \geq 0$$

by Lemma 4.9.2, hence $\mu \in P^+$.

Conversely, suppose that $\mu \in P^+$ and let $w \in W_0$ such that $w\widehat{\mu}_k \in \overline{V_+^*}$. By Proposition 4.8.1 this implies $\widehat{w\mu}_k \in \overline{V_+^*}$. By the previous paragraph we conclude that $w\mu \in P^+$. On the other hand $P^+ \cap W_0\mu = \{\mu\}$, hence $w\mu = \mu \in P^+$ and $\widehat{\mu}_k = \widehat{w\mu}_k \in \overline{V_+^*}$. $\square$

Proposition 4.9.1 is now a direct consequence of Corollary 4.9.3 and Lemma 4.9.2.

CHAPTER 5

Completeness of the Bethe ansatz eigenfunctions

5.1. Introduction

In this chapter we show that the Bethe ansatz eigenfunctions of the quantum integrable systems with delta-potentials associated to affine root systems studied in the previous chapter are complete in the Hilbert space of square-integrable functions on the fundamental domain for the reflection representation of the affine Weyl group (with respect to Lebesgue measure).

In their fundamental paper [**56**] Lieb and Liniger obtained the eigenfunctions of the quantum Bose-gas on the circle with pair-wise delta-function interactions, given by the formal Hamiltonian (1.1.1). They did not prove that the set of eigenfunctions thus obtained is orthogonal or complete in the Hilbert space of symmetric square-integrable functions on the unit square $[0,1]^n$ with respect to ordinary Lebesgue measure. To attack the completeness problem, Yang and Yang introduced [**81**] a certain function, nowadays called the master function or the Yang-Yang action (see (4.5.1)). Via a variational problem analogous to our analysis in Section 4.8 this allowed them to give a convenient parametrization of the solutions of the Bethe ansatz equations. They argued that the completeness of the Bethe ansatz eigenfunctions follows from the fact that they converge to a complete set of eigenfunctions for the impenetrable bosonic particles as $k \to \infty$. By replacing the continuity argument at infinite coupling of Yang and Yang by a continuity argument at $k \searrow 0$ and using the linear independence of the Bethe ansatz eigenfunctions, Dorlas [**16**] gave a rigorous proof of the completeness of the Bethe ansatz eigenfunctions.

The techniques introduced by Dorlas [**16**] (and in particular [**16**, Theorem 3.2]) for the quantum Bose-gas on the circle with pair-wise delta-function interactions (corresponding to Σ_0 of type A) can be generalized to general root systems and allows us to give a proof of the completeness of the Bethe ansatz eigenfunctions for general root systems in this chapter.

See the next chapter for some conjectures and partial results on orthogonality and norms of the Bethe ansatz eigenfunctions.

We now proceed to formulate the main result on completeness. Before doing that let us introduce the relevant Hilbert space. For this recall the Lebesgue measure μ_E on V from Section 2.5 and the fundamental chamber C_+ (see Section 2.2).

DEFINITION 5.1.1. *Let*

$$\mathcal{H} = \{f \text{ is a } W\text{-invariant } \mu_E\text{-measurable function on } V \mid \int_{C_+} |f(v)|^2 \, d\mu_E(v) < \infty\},$$

a Hilbert space, with inner product given by

$$(f,g)_{\mathcal{H}} = \int_{C_+} f(v)\overline{g(v)} d\mu_E(v). \tag{5.1.1}$$

In this chapter it is convenient to work with several classes of multiplicity functions (see also Definition 2.2.1).

DEFINITION 5.1.2. *Let B be a set. A* multiplicity function with values in B *is a W^e-invariant function $k : \Sigma \longrightarrow B$. Let $K_{\mathbb{C}}$, K_+, $K_{>0}$ and $\overline{K_+}$ be the space of multiplicity functions with values in respectively $\mathbb{C}$, $[0,\infty)$, $(0,\infty)$ and $[0,\infty]$. On these spaces we define the following partial ordering: $k \leq k'$ if $k_a \leq k'_a$ for all $a \in \Sigma$ and $k < k'$ if $k \leq k'$ and $k \neq k'$.*

In this chapter we view $[0,\infty]$ as the one-point compactification of $[0,\infty)$.

DEFINITION 5.1.3. *Let $\mathfrak{h}$ be a complex Hilbert space.*
(i) *A set A in $\mathfrak{h}$ is called linearly independent if every finite subset of A is linearly independent.*
(ii) *A set A in $\mathfrak{h}$ is called total if the set of finite linear combinations of elements of A is a dense subspace of $\mathfrak{h}$.*

Note that the results of Chapter 3 (with the exception of the last section) holds for $k \in K_{\mathbb{C}}$. In particular the space $\mathrm{BVP}_k(\lambda)^{W_0}$ is for every $\lambda \in V_{\mathbb{C}}^*$ one-dimensional. Denote by $\phi_{\lambda,k}$ the function in $\mathrm{BVP}_k(\lambda)^{W_0}$ normalized as $\phi_{\lambda,k}(0) = 1$. Then $\phi_{\lambda,k} = T_k\psi_{\lambda,k}$ with $\psi_{\lambda,k}$ given by (3.7.12) (with $n = 1 \in \mathbb{C}_{triv}$, the trivial representation, in the terminology of Section 3.7)) with T_k the propagation operator corresponding to the scalar case (see Definition 3.5.1 or [**32**]). Since $\mathrm{BVP}_k(\lambda)^W \subseteq \mathrm{BVP}_k(\lambda)^{W_0}$, the spectrum $\mathcal{S}_k$ (see Definition 3.2.13) can be described as follows,

$$\mathcal{S}_k = \{\lambda \in V_{\mathbb{C}}^* \mid \phi_{\lambda,k} \text{ is } W\text{-invariant}\}.$$

Functions $\phi_{\lambda,k}$ ($\lambda \in \mathcal{S}_k$) are called Bethe ansatz eigenfunctions. We set $\mathcal{S}_k^+ = \mathcal{S}_k \cap i\overline{V_+^*}$. It will be shown that $\mathcal{S}_k^+$ is a complete set of representatives for $\mathcal{S}_k/W_0$ (see Corollary 5.1.5).

Recall that $\overline{C_+}$ is a fundamental domain for the reflection action (2.2.7) of W on V (see for instance [**45**, Theorem 4.8]). Therefore the map

$$C(V)^W \xrightarrow{\sim} C(\overline{C_+}) \tag{5.1.2}$$

defined by $\phi \mapsto \phi_{|\overline{C_+}}$ defines a linear bijection. Whence the vector spaces $C(V)^W$ can be seen as a subspace of $\mathcal{H}$ in an obvious way. In particular the Bethe ansatz eigenfunctions are elements of $\mathcal{H}$.

We extend the inner product $\langle \cdot, \cdot \rangle$ on V^* to a (nondegenerate) sesqui-linear form on $V^*_{\mathbb{C}}$. Then $\langle \cdot, \cdot \rangle$ defines an inner product on $V^*_{\mathbb{C}}$ and we set $\|\lambda\| := \sqrt{\langle \lambda, \lambda \rangle} \geq 0$ for $\lambda \in V^*_{\mathbb{C}}$. The following theorem is the main results of this chapter.

THEOREM 5.1.4. *Let $k \in K_+$. Then*
(i) *(Partial orthogonality) If $\phi_{\lambda,k}$, $\phi_{\mu,k}$ ($\lambda, \mu \in \mathcal{S}_k^+$) are two Bethe ansatz eigenfunctions with $\|\lambda\| \neq \|\mu\|$, then*

$$(\phi_{\lambda,k}, \phi_{\mu,k})_{\mathcal{H}} = 0.$$

(ii) *(Completeness) The Bethe ansatz eigenfunctions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k^+\}$ are linearly independent and total in $\mathcal{H}$.*

COROLLARY 5.1.5. *Let $k \in K_+$. The spectrum $\mathcal{S}_k$ is purely imaginary and $\mathcal{S}_k^+$ is a complete set of representatives of $\mathcal{S}_k / W_0$.*

The proof of this corollary be given together with the proof of Theorem 5.1.4.

We now give a section for section overview of the chapter. In Section 5.2 we generalize Propositions 4.8.1 and 4.9.1 to $k \in K_+$ and show that for a fixed $\mu \in P^{++}$ the map $K_{>0} \mapsto \overline{V^*}$ defined by $k \mapsto \widehat{\mu}_k$ extends to a continuous map $K_+ \to \overline{V^*_+}$. This allows us to give a continuous (in $k \in K_+$) parametrization of the Bethe ansatz functions. The main tool is an extension of the master function (4.5.1) S_k from Chapter 4 to $k \in K_+$.

Section 5.3 is a preparatory to the Sections 5.4-5.6. It contains general results (without proofs) on perturbation of positive self-adjoint unbounded operators.

Section 5.4 and 5.5 are devoted to construction positive self-adjoint unbounded operators H_k on $\mathcal{H}$ associated to the formal Hamiltonian given by (3.2.1) for the scalar case, i.e.

$$\mathcal{H}_k = -\Delta + \sum_{a \in \Sigma} k_a \delta(a(\cdot)). \tag{5.1.3}$$

In Section 5.6 it will be showed that the Bethe ansatz eigenfunctions $\phi_{\lambda,k}$ ($\lambda \in \mathcal{S}_k$) are in the domain of the unbounded operator H_k for $k \in K_+$, and furthermore are eigenfunctions of H_k. We then show that the H_k are non-decreasing and continuous as a function of k. The proof of Theorem 5.1.4 follows from the completeness of the Bethe ansatz eigenfunctions for $k \equiv 0$ and continuity in k. The precise conditions under which such an argument works was given by Dorlas [**16**] (see Theorem 5.3.21) and is applied in Section 5.6. After indicating why this gives another proof of the Pauli principle (Proposition 4.5.1), we end the section by generalizing Theorem 4.4.1 to all $k \in K_+$.

In Section 7 we consider completeness for general lattices X satisfying $Q^\vee \subseteq X \subseteq P^\vee$. We end the chapter by generalizing Theorem 4.6.1 to all $k \in K_+$.

5.2. Continuous parametrization of the Bethe ansatz eigenfunctions

In Chapter 4 we showed that for $k \in K_{>0}$ the Bethe ansatz eigenfunctions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k\}$ admitted a parametrization (see Theorem 4.4.1 and Proposition 4.8.1) by P^{++} via the map $\mu \mapsto \phi_{i\hat{\mu}_k,k}$. The implicit way $\hat{\mu}_k$ is defined (see (4.8.5)) allows us to apply the Implicit Function Theorem to conclude that $k \mapsto \hat{\mu}_k$ is a smooth map $K_{>0} \to V^*$. The proof of the completeness of the Bethe ansatz eigenfunctions hinges essentially on the fact that this map $K_{>0} \to V^*$ extends to a continuous map $K_+ \to V^*$.

This will be sufficient to show that $k \mapsto \phi_{i\hat{\mu}_k,k}$ is a continuous map $K_+ \to \mathcal{H}$.

For the quantum Bose-gas on the circle with pair-wise delta function interactions, i.e. when Σ_0 is of type A, this fact is mentioned and used in the physics literature. See for example [**16**, Lemma 3.3] and [**56**, Paragraph after (2.33)].

In this section we give a rigorous proof of the continuity of the above map $K_+ \to V^*$ for all affine root systems Σ. In the proof the Implicit Function Theorem will not be used.

When analyzing the case $k \in \overline{K_+}$, the following function Θ arises naturally.

DEFINITION 5.2.1. *Let Θ be the function from $\mathbb{R} \times [0,\infty]$ to $\mathbb{R}$ defined by*

$$\Theta(x,c) = \begin{cases} \int_0^x \arctan(t/c)dt & \text{if } c \in (0,\infty), \\ \frac{\pi}{2}\,|x| & \text{if } c = 0, \\ 0 & \text{if } c = \infty. \end{cases}$$

LEMMA 5.2.2. *The function Θ is continuous and $\Theta(\cdot,c)$ is convex for all $c \in [0,\infty]$.*

PROOF. Consider a point $(x_0, 0)$ ($x_0 \in \mathbb{R}$) and $c \in (0,\infty)$. Then

$$\Theta(x,c) - \Theta(x_0,0) = \int_{x_0}^{x} \arctan(t/c)dt + \left(\int_0^{x_0} \arctan(t/c)dt - \frac{\pi}{2}\,|x_0| \right).$$

The first term goes to zero if $(x,c) \to (x_0,0)$ by the uniform boundedness in c of the integrand. The second term goes to zero by Lebesgue's Dominated Convergence Theorem. Continuity of Θ in (x_0,c_0) with $c_0 \in (0,\infty]$ is completely similar (for $c = \infty$, consider the change of coordinate $c \mapsto 1/c$).

The second statement follows for $c \in (0,\infty)$ because $(\Theta(\cdot,c))''(x) = c/(x^2+c^2) > 0$ for all $x \in \mathbb{R}$. The case $c = 0$ en $c = \infty$ are obvious. □

In Chapter 4 we showed the importance of the master function S_k (see (4.5.1)) in analyzing the solutions of the Bethe ansatz equations (Proposition 4.8.1) in the case $k \in K_{>0}$. For example, in the proof of the Pauli principle (Proposition 4.5.1) it was used that $S_k(\cdot,\cdot)$ ($k \in K_{>0}$) is a strictly convex function in the second variable. We start by extending S_k to $k \in \overline{K}_+$. Although in general $S_k(\mu,\cdot)$ (with $\mu \in P$) is not a differentiable function anymore (and therefore has no Hessian), it will still be strictly convex. Together with the growth behavior in ∞, this will imply that it attains a global minimum in a unique $\widehat{\mu}_k$. By carefully analyzing (see Proposition 5.2.11) how this minimum changes when $k \in \overline{K}_+$ changes, we can prove that the minimum changes continuously.

DEFINITION 5.2.3. *For a $k \in \overline{K}_+$ we define the* master function *S_k on $P \times V^*$ by*

$$S_k(\mu,\xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle\xi,\mu\rangle + \frac{1}{2}\sum_{\alpha\in\Sigma_0} \|\alpha\|^2 \Theta(\xi(\alpha^\vee), k_\alpha).$$

We consider also $S_k' : P \times V^* \to \mathbb{R}$, defined by

$$S_k'(\mu,\xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle \xi,\mu\rangle + \frac{1}{2}\sum_{\substack{\alpha\in\Sigma_0\\ 0<k_\alpha<\infty}} \|\alpha\|^2 \int_0^{\xi(\alpha^\vee)} \arctan(t/k_\alpha)dt. \tag{5.2.1}$$

REMARK 5.2.4. Let $\mu \in P$. The function $S_k'(\mu,\cdot)$ is always differentiable, while the master function $S_k(\mu,\cdot)$ is differentiable if and only if $0 < k_\alpha \le \infty$ for all $\alpha \in \Sigma_0$.

EXAMPLE 5.2.5. We consider some extreme cases.
(a) If $k \in K_{>0}$, then we recover the master function (4.5.1).
(b) For $k \equiv 0$ we get

$$S_0(\mu,\xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle \xi,\mu\rangle + \pi \sum_{\alpha\in\Sigma_0^+} |\langle \xi,\alpha\rangle|\,,$$

and restricted to $V_+^* \ni \xi$ gives $S_0(\mu,\xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle\xi,\mu-\rho\rangle$.
(c) For $k \equiv \infty$ we have $S_\infty(\mu,\xi) = \frac{1}{2}\|\xi\|^2 - 2\pi\langle\xi,\mu\rangle$.

We will also extend Proposition 4.8.1 to all $k \in K_+$. The Bethe ansatz equations for $k \in K_+$ are the following equations in $\lambda \in iV^*$,

$$\prod_{\substack{\alpha\in\Sigma_0^+\\ k_\alpha>0}} \left(\frac{\lambda(\alpha^\vee)-k_\alpha}{\lambda(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)} = e^{\lambda(x)} \qquad \forall\, x \in Q^\vee \tag{5.2.2}$$

(if $k \equiv 0$, we interpret (5.2.2) as: $1 = e^{\lambda(x)}$). The set of solutions $\lambda \in iV^*$ of the Bethe ansatz equations (5.2.2) is denoted by BAE_k. Note that for $k \in K_{>0}$ we recover the Bethe ansatz equation (4.4.1).

For $k \in \overline{K_+}$ and $\eta \in V^*$ let

$$\widetilde{\sigma}_\eta^k = \sum_{\substack{\alpha\in\Sigma_0\\ 0<k_\alpha<\infty}} \arctan\!\left(\frac{\eta(\alpha^\vee)}{k_\alpha}\right)\alpha. \tag{5.2.3}$$

The next lemma partially generalizes Proposition 4.9.1, Corollary 4.9.3, and Lemma 4.9.2.

LEMMA 5.2.6. *Let $k \in K_+$ and $\mu \in P$.*
(i) *The function $S_k'(\mu,\cdot)$ is strictly convex and attains a unique global minimum at a $\widehat{\mu}_k' \in V^*$. Moreover, $\widehat{\mu}_k'$ is uniquely determined (cf.* (4.8.5)*) by*

$$\widehat{\mu}_k' + \tilde{\sigma}^k_{\widehat{\mu}_k'} = 2\pi\mu. \tag{5.2.4}$$

(ii) $\mu \mapsto i\hat{\mu}_k'$ *defines a W_0-equivariant bijection $P \to \mathrm{BAE}_k$.*
(iii) *For a $\mu \in P^+$ and $\beta \in \Sigma_0^+$ we have*

$$\frac{2\pi\mu(\beta^\vee)}{\left(1+\frac{h_k'}{n}\right)} \le \widehat{\mu}_k'(\beta^\vee) \le 2\pi\mu(\beta^\vee), \tag{5.2.5}$$

where $h'_k = 2\sum_{\alpha\in\Sigma_0,\, k_\alpha>0} k_\alpha^{-1}$ *(convention:* $h'_0 := 0$*). Furthermore,* $\mu \in P^+$ *if and only if* $\widehat{\mu}'_k \in \overline{V_+^*}$.

PROOF. Because $\xi \mapsto \Theta(\xi(\alpha^\vee), k_\alpha)$ is the composition of the linear function $\xi \mapsto \xi(\alpha^\vee)$ and the convex function $\Theta(\cdot, k_\alpha)$ (Lemma 5.2.2), it must be a convex function from V^* to $\mathbb{R}$. The function $\xi \mapsto \langle \xi, \xi\rangle/2$ is strictly convex. Since the set of all convex functions is closed under addition (it is actually a cone) and linear functions are also convex, the function $S'_k(\mu,\cdot)$ must be strictly convex. Since $S'_k(\mu,\cdot)$ is also dominated by the $\|\xi\|^2$-term if $|\xi| \to \infty$, it attains a unique global minimum at a $\widehat{\mu}'_k$. Since

$$S'_k(w\mu, \xi) = S'_k(\mu, w^{-1}\xi) \geq S'_k(\mu, \hat{\mu}'_k) = S'_k(w\mu, w\hat{\mu}'_k),$$

$S'_k(w\mu,\cdot)$ attains a global minimum at $w\hat{\mu}'_k$, and by uniqueness $w\hat{\mu}'_k = \widehat{(w\mu)}'_k$.

Statement **(ii)** and the fact that $\widehat{\mu}'_k$ is uniquely determined by (5.2.4) follows analogously as in the proof of Proposition 4.8.1. The proof of **(iii)** is analogous to the proof of Proposition 4.9.1. (See also the proof of Lemma 4.9.2). □

Let $\mu \in P^{++}$. Note that $S_k = S'_k$ for $k \in K_{>0}$. The map $(k,\xi) \mapsto S'_k(\mu,\xi)$ however is not a continuous function from $K_+ \times V^*$ to $\mathbb{R}$. This is the reason to consider the master function S_k instead of S'_k since $(k,\xi) \mapsto S_k(\mu,\xi)$ turns out to be a continuous map from $K_+ \times V^*$ to $\mathbb{R}$. We start with a prepatory lemma. For $k \in K_+$ let

$$\rho_{\text{short}} = \frac{1}{2}\sum_{\substack{\alpha\in\Sigma_0^+\\ \alpha \text{ short}}} \alpha, \quad \rho_{\text{long}} = \frac{1}{2}\sum_{\substack{\alpha\in\Sigma_0^+\\ \alpha \text{ long}}} \alpha, \quad \text{and} \quad \rho^k = \frac{1}{2}\sum_{\substack{\alpha\in\Sigma_0^+\\ k_\alpha=0}} \alpha. \tag{5.2.6}$$

LEMMA 5.2.7. *Let* $k \in K_+$.

(i) *We have* $\rho_{\mathit{short}}, \rho_{\mathit{long}} \in P^+$. *Furthermore*

$$\rho_{\mathit{short}} = \sum_{\substack{i\\ a_i \text{ is short}}} \omega_i \quad \textit{and} \quad \rho_{\mathit{long}} = \sum_{\substack{i\\ a_i \text{ is long}}} \omega_i. \tag{5.2.7}$$

Here $\omega_1,\ldots,\omega_n$ *denote the* fundamental weights *with respect to the simple roots* $I_0 = \{a_1,\ldots,a_n\}$*, defined as the vectors in* V^* *satisfying*

$$\omega_i(a_j^\vee) = \delta_{ij} \quad \forall i,j \in \{1,2,\ldots,n\}.$$

(ii) ρ^k *is one of the dominant weights* ρ, ρ_{short}, ρ_{long} *or zero. In particular* $P^{++} \subseteq \rho^k + P^+ \subseteq P^+$.

(iii) *Assume that* $\alpha \in \Sigma_0^+$ *and* $\alpha = \sum_i m_i a_i$*. Then* $m_i \geq 0$ *for all* i *and if* α *is long (resp. short) there is an* i_0 *such that* $m_{i_0} > 0$ *and* a_{i_0} *is long (resp. short).*

(iv) $\rho^k + P^+ = \{\mu \in P^+ | \mu(\alpha^\vee) > 0$ *for all* $\alpha \in \Sigma_0^+$ *with* $k_\alpha = 0\}$. *Let* $P^k := \{\mu \in P | \mu(\alpha^\vee) \neq 0$ *for all* $\alpha \in \Sigma_0^+$ *with* $k_\alpha = 0\}$. *Then* P^k *is* W_0*-invariant and* $\rho^k + P^+$ *is a fundamental domain for the action of* W_0 *on* P^k.

PROOF. **(i)** The set of long (resp. short) roots in Σ_0 is stable under the action of W_0. Fix a $i = 1, \ldots, n$. Since $l(s_i) = 1$ we have $s_i\Sigma_0^+ = (\Sigma_0^+\backslash\{a_i\}) \cup \{-a_i\}$ and therefore

$$s_i\rho_{\text{short}} = \begin{cases} \rho_{\text{short}} & \text{if } a_i \text{ is long} \\ \rho_{\text{short}} - a_i & \text{if } a_i \text{ is short.} \end{cases}$$

Analogously for ρ_{long}. Together with $s_i(\lambda) = \lambda - \lambda(a_i^\vee)a_i$ this implies (5.2.7).
(ii) Follows directly from **(i)**.
(iii) Assume that $m_i = 0$ if a_i is long (short) for $i = 1, \ldots, n$. Then α lies in the subsystem $\Sigma_{0,J}$ with J consisting of simple short (long) roots in I_0. Since the corresponding Dynkin diagram is simply-laced and connected, the root system $\Sigma_{0,J}$ consists of only short (long) roots. In particular α has the same length as a_i for all $a_i \in J$, and is thus short (resp. long).
(iv) The first statement follows from **(i)-(iii)**. The second statement follows form this and the fact that P^+ is a fundamental domain for P for the action of W_0 on P. □

Note that $\rho^0 = \rho$ and $\rho^k = 0$ if $k \in K_{>0}$.

LEMMA 5.2.8. *Let $k \in K_+$. Then the following holds.*
(i) *Let $\mu \in P$. Then $S_k(\mu, \cdot)$ is strictly convex and attains a unique global minimum at a $\widehat{\mu}_k \in V^*$.*
(ii) *The map $P \to V^*$ defined by $\mu \mapsto \hat{\mu}_k$ is W_0-equivariant.*
(iii) *Let $\mu \in \rho^k + P^+ \supseteq P^{++}$. Then*

$$\widehat{\mu}_k = \widehat{(\mu - \rho^k)}_k^{\prime} \tag{5.2.8}$$

(iv) *If $\mu \in \rho + P^+ = P^{++}$ then $\widehat{\mu}_0 = 2\pi(\mu - \rho)$, if $\mu \in P^+$ then $\widehat{\mu}_\infty = 2\pi\mu$.*

PROOF. The proof of **(i)** and **(ii)** is similar to that of Lemma 5.2.6**(i)-(ii)**. Assume now that $\mu \in \rho^k + P^+$. Let $g_k : V^* \to \mathbb{R}_+$ be the function

$$g_k(\xi) = \pi \sum_{\substack{\alpha \in \Sigma_0^+ \\ k_\alpha = 0}} \big(|\langle \xi, \alpha \rangle| - \langle \xi, \alpha \rangle\big). \tag{5.2.9}$$

Then $S_k(\mu, \xi) = S_k'(\mu - \rho^k, \xi) + g_k(\xi)$, $g_k(\xi) \geq 0$ and $g_{k|\overline{V_+^*}} \equiv 0$. Since $\mu - \rho^k \in P^+$ we have $\widehat{(\mu - \rho^k)}_k^{\prime} \in \overline{V_+^*}$ by Lemma 5.2.6 and therefore (5.2.8) follows, completing the proof of **(iii)**. Observe Example 5.2.5 for **(iv)**. □

Recall the subset P^k of P from Lemma 5.2.7**(iv)**. Working with the function S_k instead of the function S_k' introduces a ρ^k-shift in parametrization of BAE_k.

PROPOSITION 5.2.9. *Let $k \in K_+$.*
(i) *$\mu \mapsto i\widehat{\mu}_k$ defines a W_0-equivariant bijection $P^k \xrightarrow{\sim} \text{BAE}_k$. Furthermore for $\mu \in P^k$, we have: $\mu \in \rho^k + P^+$ iff $\widehat{\mu}_k \in \text{BAE}_k \cap i\overline{V_+^*}$.*
(ii) *The map $P^k \xrightarrow{\sim} \text{BAE}_k$ of* **(i)** *restricts to bijections*

$$P^{++} \xrightarrow{\sim} \text{BAE}_k \cap \{\lambda \in i\overline{V_+^*} | \lambda(\alpha^\vee) > 0 \ \ \forall \alpha \in \Sigma_0^+ \text{ with } k_\alpha > 0\}, \tag{5.2.10}$$

and

$$P \cap V_{reg} \xrightarrow{\sim} \mathrm{BAE}_k \cap \{\lambda \in iV^* | \lambda(\alpha^\vee) \neq 0 \;\; \forall \alpha \in \Sigma_0 \textit{ with } k_\alpha > 0\}, \qquad (5.2.11)$$

PROOF. Statement **(i)** follows by Lemma 5.2.6**(ii)** and Lemma 5.2.8**(ii)-(iii)**.

For statement **(ii)** it suffices to show (5.2.10). Now observe that (5.2.10) follows from **(i)**, Lemma 5.2.6**(iii)** and Lemma 5.2.8**(iii)**. □

REMARK 5.2.10. Let $k \in K_+$. Since $S_k(0,\cdot) \geq 0$ and $S_k(0,0) = 0$ for all $k \in K_+$, we always have $\widehat{0}_k = 0$. Similarly we get $\widehat{0}'_0 = 0$. Furthermore, by Lemma 5.2.8**(iii)** we also have $\widehat{\rho^k}_k = \widehat{0}'_0 = 0$. Now assume that $k \in K_+ \backslash K_{>0}$. Then $\rho^k \neq 0$. The map $P \to V^*$ defined by $\mu \mapsto \widehat{\mu}_k$ is in this case thus not injective.

Recall that we consider $[0,\infty]$ as the one point compactification of $[0,\infty)$. The next proposition lies at the heart of the continuous dependence of $\widehat{\mu}_k$ ($\mu \in P^{++}$ fixed) on $k \in \overline{K_+}$.

PROPOSITION 5.2.11. *Let $m \in \mathbb{N}$ and $f : \mathbb{R}^m \times [0,\infty]^r \longrightarrow \mathbb{R}$ a continuous map such that $f(\cdot,k)$ has a unique global minimum at μ_k for $k \in [0,\infty]^r$. Define the map $\mu : [0,\infty]^r \longrightarrow \mathbb{R}^m$ by $k \mapsto \mu_k$. Assume also there is a compact $D \subset \mathbb{R}^m$ such that $\mu([0,\infty]^r) \subset D$. Then μ is continuous.*

PROOF. Assume μ is not continuous in a point $\eta \in [0,\infty]^r$. Then there is a $\delta > 0$ and a sequence $k_1, k_2, \ldots$ in $[0,\infty]^r$ such that $k_j \to \eta$ and $\left|\mu_{k_j} - \mu_\eta\right| > \delta$ for all j. Defining

$$m_0 = \min_{\substack{x \in D \\ |x-\mu_\eta| \geq \delta}} \left(f(x,\eta) - f(\mu_\eta,\eta)\right),$$

we have $m_0 > 0$ by definition of μ_η and compactness. Choose an ε such that $0 < \varepsilon < m_0/2$. Since f is uniformly continuous on $D \times [0,\infty]^r$, there is a neighborhood U of η such that $|f(x,\eta) - f(x,k)| \leq \varepsilon$ for $k \in U$ and $x \in D$. Hence for $k \in U$ we have

$$f(\mu_k,\eta) - f(\mu_k,k) \leq \varepsilon. \qquad (5.2.12)$$

By definition of μ_k we have $f(\mu_k,k) \leq f(\mu_\eta,k)$, and continuity of f in k implies that there is a neighborhood U' of η such that $f(\mu_\eta,k) \leq f(\mu_\eta,\eta) + \varepsilon$ for $k \in U'$, thus $f(\mu_k,k) \leq f(\mu_\eta,\eta) + \varepsilon$ for $k \in U'$. Together with (5.2.12) this yields

$$f(\mu_k,\eta) - f(\mu_\eta,\eta) \leq 2\varepsilon < m_0$$

for $k \in U \cap U'$, which leads to a contradiction for $k = k_j$ and j large enough. □

PROPOSITION 5.2.12. *Let $\mu \in P^{++}$. Then the map $\widehat{\mu} : \overline{K}_+ \longrightarrow V^*$ defined by $k \mapsto \widehat{\mu}_k$ is continuous. In particular*

$$\lim_{k \to 0} \widehat{\mu}_k = \widehat{\mu}_0 = 2\pi(\mu - \rho) \textit{ and } \lim_{k \to \infty} \widehat{\mu}_k = \widehat{\mu}_\infty = 2\pi\mu.$$

PROOF. Let

$$D = \{\xi \in V^* \,|\, 0 \leq \xi(\beta^\vee) \leq 2\pi\mu(\beta^\vee) \,\forall \beta \in \Sigma_0^+\},$$

a compact subset of V^*, and the function $f : V^* \times \overline{K}_+ \longrightarrow \mathbb{R}$ defined by $f(\xi, k) = S_k(\mu, \xi)$, which is a continuous map, because Θ is continuous. From Lemma 5.2.8 follows that $f(\cdot, k)$ attains a unique global minimum at $\widehat{\mu}_k$ and from Lemma 5.2.6 it follows that $\widehat{\mu}_k \in D$ for all $k \in \overline{K}_+$. Now apply Proposition 5.2.11. □

Recall from the introduction that $\mathcal{S}_k = \{\lambda \in V_{\mathbb{C}}^* |\, \phi_{\lambda,k}$ is W-invariant$\}$.

In the remainder of this section we put on the linear space $C(V)$ the topology of uniform convergence on compact subsets of V.

LEMMA 5.2.13. *The map $(\lambda, k) \mapsto \phi_{\lambda,k}$ is a continuous map from $V_{\mathbb{C}}^* \times K_{\mathbb{C}}$ to $C(V)$.*

PROOF. For every fixed $w \in W_0$, $(\lambda, k) \mapsto Q_k(w)(e^\lambda)$ is a continuous map from $V_{\mathbb{C}}^* \times K_{\mathbb{C}}$ to $C(V)$ because $Q_k(w)$ is a composition of reflection-integral operators (see Section 2.5). Because of (4.3.2) (and identifying $Q_k^0(w)$ with $Q_k(w)$ for $w \in W_0$), it then follows that $(\lambda, k) \mapsto \psi_{\lambda,k}$ is a continuous map $V_{\mathbb{C}}^* \times K_{\mathbb{C}} \to C(V)$. Now observe that $(f, k) \mapsto T_k f$ is a continuous map from $C(V) \times K_{\mathbb{C}}$ to $C(V)$. □

From Chapter 4 we know that $\{i\widehat{\mu}_k | \mu \in P^{++}\} = \mathcal{S}_k^+ = \mathcal{S}_k \cap i\overline{V_+^*}$ for $k \in K_{>0}$. We shall show (see Corollary 5.6.7 and Remark 5.6.8) that this equality actually holds for all $k \in K_+$. For the moment we prove the following inclusion.

LEMMA 5.2.14. *Let $k \in K_+$. Then $\{i\widehat{\mu}_k | \mu \in P^{++}\} \subseteq \mathcal{S}_k \cap i\overline{V_+^*}$.*

PROOF. Let $k^0 \in K_+ \backslash K_{>0}$, $\mu \in P^{++}$ and put $\lambda = i\widehat{\mu}_{k^0}$.

We first show that ϕ_{λ,k^0} is W-invariant. Fix a $v \in V$ and $w \in W$. Then

$$\phi_{\lambda,k^0}(wv) = \lim_{K_{>0} \ni k \to k^0} \phi_{i\hat{\mu}_k,k}(wv) = \lim_{K_{>0} \ni k \to k^0} \phi_{i\hat{\mu}_k,k}(v) = \phi_{\lambda,k^0}(v).$$

We used in the first en last equality that for a fixed v' the map $K_+ \to \mathbb{C}$ defined by $k \mapsto \phi_{i\hat{\mu}_k,k}(v')$ is continuous by Lemma 5.2.13 and Proposition 5.2.12. The second equality used that $\phi_{i\hat{\mu}_k,k}$ is a W-invariant function for $k \in K_{>0}$. Since v, w were fixed but chosen arbitrary, it follows that ϕ_{λ,k^0} is W-invariant. Since by definition $0 \neq \phi_{\lambda,k^0} \in \mathrm{BVP}_{k^0}(\lambda)$, this shows that $\lambda \in \mathcal{S}_{k^0}$. The inclusion $\lambda \in i\overline{V_+^*}$ follows from Proposition 5.2.12 and Lemma 5.2.8**(iii)**. □

PROPOSITION 5.2.15. *Let $\mu \in P^{++}$. The map $k \mapsto \phi_{i\widehat{\mu}_k,k}$ from K_+ to $\mathcal{H}$ is continuous.*

PROOF. From Proposition 5.2.12 and Lemma 5.2.13 follows that $k \to \phi_{i\hat{\mu}_k,k}$ is a continuous map from K_+ to $C(V)$. In particular $k \to \phi_{i\hat{\mu}_k,k|\overline{C_+}}$ is continuous from K_+ to $C(\overline{C_+})$ (with the topology of uniform convergence on $C(\overline{C_+})$). Since $\phi_{i\hat{\mu}_k,k}$ is W-invariant and the uniform convergence it follows (see (5.1.1)) that $k \mapsto \phi_{i\widehat{\mu}_k,k}$ is continuous from K_+ to $\mathcal{H}$. □

5.3. Perturbations of semi-bounded self-adjoint operators and quadratic forms

In this section we denote by $\mathcal{H}$ any separable Hilbert space with scalar product $(,)$.

This section contains some general results (mostly without proofs) on perturbation theory of positive self-adjoint unbounded operators on Hilbert spaces, and is preparatory to the Sections 5.4-5.6. The main reference for this section are the books by Reed and Simon [**69**], [**67**] and Kato [**49**].

The problem we are faced with is the following. Sometimes one wants to make sense of (unbounded) operators on a Hilbert space of the form $-\Delta + V(x)$ with V a singular potential; V might not be a function, but only a distribution (see for example (3.2.1)). Often there is some heuristic reason to believe what $\beta(f,g) = (V \cdot f, g)$ should be, *if* there was such an operator "$H = -\Delta + V$". The situation then arises to find an operator H such that

$$(Hf,g) = (-\Delta f, g) + \beta(f,g)$$

for f and g in a suitable domain. This leads to the concept of a quadratic form, which we now proceed to recall.

DEFINITION 5.3.1. *A* (quadratic) form *is a map* $q : D(q) \times D(q) \longrightarrow \mathbb{C}$ *where* $D(q)$ *is a dense subspace of* $\mathcal{H}$ *called the* form domain, *such that* $q(\cdot, f)$ *is linear and* $q(f,\cdot)$ *is anti-linear for* $f \in D(q)$. *If* $q(f,g) = \overline{q(g,f)}$ *for all* f, g *we say that* q *is* symmetric. *If* $q(f,g) \geq 0$ *for all* $f \in D(q)$, q *is called* positive, *and if* $q(f,f) \geq -M\|f\|^2$ *for some* $M \geq 0$ *and for all* $f \in D(q)$, *we say that* q *is* semi-bounded *or that* q *is* bounded from below by $-M$. *We also use the following abbreviation*

$$\tilde{q}(f) = q(f,f).$$

REMARK 5.3.2. **(i)** It is assumed that all domain of definitions are dense in $\mathcal{H}$.
(ii) Let $(q, D(q))$ be a form. By the polarization identity

$$q(f,g) = \frac{\tilde{q}(f+g) - \tilde{q}(f-g) + i\tilde{q}(f+ig) - i\tilde{q}(f-ig)}{4} \quad (f, g \in D(q)), \tag{5.3.1}$$

it follows that any semi-bounded form is automatically symmetric. The polarization identity also shows that q is completely determined by $\tilde{q}$. This fact allows us to define forms by only giving $\tilde{q}$.

Let $(q, D(q))$ be a semi-bounded form. A sequence $\{h_n\} \subseteq D(q)$ of vectors will be said to be q-convergent to $h \in \mathcal{H}$ (although strictly speaking, we should say $(q, D(q))$-convergent), denoted by $h_n \underset{q}{\rightarrow} h$, if $h_n \rightarrow h$ with respect to the topology of $\mathcal{H}$ and $\tilde{q}(h_n - h_m) \rightarrow 0$ as $n, m \rightarrow \infty$. The semi-bounded form $(q, D(q))$ is said to be *closed* if $h_n \underset{q}{\rightarrow} h$ implies $h \in D(q)$. The following theorem gives an alternative description of closed semi-bounded forms (see [**49**, Theorem VI.1.1] for a proof).

LEMMA 5.3.3. *Let* $(q, D(q))$ *be a form bounded below by* $-M$. *Then* $D(q)$ *becomes a pre-Hilbert space with respect to the following inner product,*

$$(f,g)_q = q(f,g) + (M+1)(f,g). \tag{5.3.2}$$

The form q is closed iff $D(q)$ is a Hilbert-space with respect to $(\cdot,\cdot)_q$.

REMARK 5.3.4. It is not difficult to show that the norms

$$\|f\|_q = \sqrt{(f,f)_q} \tag{5.3.3}$$

corresponding to different lower bounds are equivalent.

DEFINITION 5.3.5. *Let $(q, D(q))$ and $(r, D(r))$ be two forms on $\mathcal{H}$. We say that r is an* extension *of q or that q is a* restriction *of r (in short: $r \supset q$ or $q \subset r$) if $D(q) \subset D(r)$ and $q(f,g) = r(f,g)$ for all $f, g \in D(q)$.*

A semi-bounded form $(q, D(q))$ is said to be *closable* if it has a closed extension. The following theorem gives an alternative description of closable semi-bounded forms (see [**49**, Theorem VI.1.17] for a proof).

THEOREM 5.3.6. *A semi-bounded form $(q, D(q))$ is closable iff $h_n \underset{q}{\to} 0$ implies $\tilde{q}(h_n) \to 0$.*

When this condition is satisfied we define the closure $(\overline{q}, D(\overline{q}))$ as follows. $D(\overline{q})$ is the subspace of $\mathcal{H}$ of all $h \in \mathcal{H}$ such that there exist a sequence $\{h_n\} \subset D(q)$ with $h_n \underset{q}{\to} h$, and

$$\overline{q}(f,g) = \lim_{n\to\infty} q(f_n, g_n) \quad \text{for any } f_n \underset{q}{\to} f,\ g_n \underset{q}{\to} g.$$

Then $(\overline{q}, D(\overline{q}))$ is a closed semi-bounded form with the same lower bound as $(q, D(q))$, and it is the smallest possible semi-bounded closed extension of $(q, D(q))$.

REMARK 5.3.7. If $(q, D(q))$ is a form on $\mathcal{H}$ bounded form below by $-M$ for a $M \geq 0$, then one can take the abstract closure of the pre-Hilbert space $D(q)$ with inner product $(\cdot,\cdot)_q$ given by (5.3.2). A a set this Hilbert space consists of equivalence classes of Cauchy sequences $[(h_n)_n]$ with respect to the norm $\|\cdot\|_q$. There is a well-defined map from this Hilbert space to $\mathcal{H}$, defined by sending a $[(h_n)_n]$ to the limit $\lim_n h_n$ in $\mathcal{H}$. This map is in general not injective. Theorem 5.3.6 shows that this map is injective in the case that $(q, D(q))$ is closable, with image $D(\overline{q}) \subseteq \mathcal{H}$.

When $(q, D(q))$ is a closed semi-bounded form, a linear subspace D of $D(q)$ is called a *form core* of $(q, D(q))$ if the closure of the restriction (q, D) of $(q, D(q))$ is equal to $(q, D(q))$. The following (see [**49**, Theorem VI.1.21] for a proof) gives an alternative description of a form core.

LEMMA 5.3.8. *Let $(q, D(q))$ be a semi-bounded closed form. A linear subspace D of $D(q)$ is a form core iff D is dense in $D(q)$ with respect to the norm $\|\cdot\|_q$.*

DEFINITION 5.3.9. *We say that an unbounded operator $(A, D(A))$ on $\mathcal{H}$ (the domain of definition is always assumed to be dense in $\mathcal{H}$) is* closed *if its graph $\Gamma(A) = \{(Af, f) | f \in D(A)\}$ is closed in $\mathcal{H} \oplus \mathcal{H}$. An unbounded operator $(A, D(A))$ is* closable *if there exists a closed extension $(B, D(B))$ of $(A, D(A))$, i.e.*
(i) *$(B, D(B))$ is a closed unbounded operator on $\mathcal{H}$,*
(ii) *$D(A) \subset D(B)$ and $B_{|D(A)} = A$ for all $f \in D(A)$.*

A closable unbounded operator $(A, D(A))$ on $\mathcal{H}$ has a smallest closed extension $(\overline{A}, D(\overline{A}))$, called the *closure* of A. It is known that a symmetric operator is closable, and a self-adjoint operator closed.

DEFINITION 5.3.10. *Let $(A, D(A))$ be a self-adjoint operator. Consider the symmetric form $(q_A, D(A))$, with $q_A(f,g) = (Af, g)$.*
(i) *$(A, D(A))$ is said to be* bounded from below by $-M$ *for some $M \geq 0$ if the form $(q_A, D(A))$ is bounded from below by $-M$. $(A, D(A))$ is said to be* semi-bounded *if $(q_A, D(A))$ is bounded from below by $-M$ for some $M \geq 0$. If $M = 0$ can be taken, then $(A, D(A))$ is called* positive.
(ii) *Let $D \subset D(A)$ be a dense subspace of $\mathcal{H}$. We say that D is a* operator core *for A if $(A, D(A))$ is the closure of the symmetric operator $(A_{|D}, D)$.*

It is possible to associate to any semi-bounded self-adjoint operator a closed semi-bounded form.

THEOREM 5.3.11. *Let $(A, D(A))$ be a self-adjoint operator that is bounded from below by $-M$ for a $M \geq 0$. Then the form $(q_A, D(A))$ is closable. We denote (by abuse of notation) the closure (see Theorem 5.3.6) $(\overline{q_A}, D(\overline{q_A}))$ of $(q_A, D(A))$ by $(q_A, Q(A))$. Let $D \subset D(A)$ be an operator core for $(A, D(A))$. Then D is a form core for $(q_A, Q(A))$ and*

$$Q(A) = \{f \in \mathcal{H} | \exists f_n \in D, f_n \to f \text{ in } \mathcal{H} \\ \text{and } (A(f_n - f_m), f_n - f_m) \to 0 \text{ for } n, m \to \infty\}. \tag{5.3.4}$$

Furthermore,

$$q_A(f,g) = (Af, g) \quad f \in D(A), g \in Q(A). \tag{5.3.5}$$

For a proof of Theorem 5.3.11 see [**69**, Example 2 in VIII.6 and Problem 16 of VIII] or [**49**, Corollary 1.28].

The following proposition gives a description of the domain of $D(A)$ of a semi-bounded closed self-adjoint operator in terms of a from core D of $(q_A, Q(A))$.

PROPOSITION 5.3.12. *Let $(A, D(A))$ be a semi-bounded self-adjoint operator on $\mathcal{H}$, and $D \subset Q(A)$ a form core (for example, one can take here an operator core $D \subset D(A)$). If $f \in Q(A)$, $h \in \mathcal{H}$ and $q_A(f,g) = (h,g)$ $\forall g \in D$, then $f \in D(A)$ and $Af = h$. Furthermore,*

$$D(A) = \{f \in Q(A) | \, \exists \, h \in \mathcal{H} \text{ such that } q_A(f,g) = (h,g) \text{ for all } g \in D\}. \tag{5.3.6}$$

For a proof of this theorem see [**49**, Theorem VI.2.1].

We have the following reverse to Theorem 5.3.11, sometimes called *the first representation theorem*.

THEOREM 5.3.13. *Let $(q, D(q))$ be a closed semi-bounded form. There exists an unique semi-bounded self-adjoint operator $(A, D(A))$ such that $(q, D(q)) = (q_A, Q(A))$.*

For a proof of this theorem see [**69**, VIII.15] and [**49**, Theorem VI.2.1].

Although the following theorem about positive self-adjoint operators is not used in this thesis we state it here nonetheless, since it is illuminating (and all self-adjoint operators which we will encounter in this chapter are positive).

THEOREM 5.3.14 (Second representation theorem [**49**, Theorem VI.2.23]). *Let $(A, D(A))$ be a positive self-adjoint operator. Denote by $(A^{1/2}, D(A^{1/2}))$ the square root of A. Then we have $Q(A) = D(A^{1/2})$ and*

$$q_A(f, g) = (A^{1/2} f, A^{1/2} g), \quad f, g \in Q(A).$$

A subset D of $Q(A)$ is a form core of $(q_A, Q(A))$ iff D is a operator core of $(A^{1/2}, D(A^{1/2}))$.

An important result in the theory of perturbation theory of self-adjoint operators and forms is the so-called KLMN theorem [1].

THEOREM 5.3.15 (the KLMN theorem). *Let $(A, D(A))$ be a positive self-adjoint operator on $\mathcal{H}$ and suppose that $(\beta, Q(A))$ is a symmetric form such that*

$$|\beta(f, f)| \le a(Af, f) + b(f, f) \quad \forall f \in D(A) \tag{5.3.7}$$

for some $a < 1$ and $b \in \mathbb{R}$. Then there exists a unique semi-bounded self-adjoint operator $(C, D(C))$ such that the corresponding form $(q_C, Q(C))$ has form domain $Q(C) = Q(A)$ and

$$q_C(f, g) = q_A(f, g) + \beta(f, g) \quad \forall f, g \in Q(C) = Q(A). \tag{5.3.8}$$

C is bounded below by $-b$ and any operator core D for $(A, D(A))$ is a form core for $(q_C, Q(C))$.

The proof of the KLMN theorem uses Theorem 5.3.13 (see [**67**, Theorem X.17] and [**49**, Theorem VI.3.9] for details). Theorem 5.3.15 can be proved for much wider classes of forms (see for instance [**49**, Theorem VI.3.4]).

We turn our attention to the question of convergence of unbounded operators. Our basic reference is [**69**, Section VIII.7 and Supplement to VIII.7]. Let $(A, D(A))$ be a closed operator on $\mathcal{H}$. A $\lambda \in \mathbb{C}$ is in the *resolvent set* $\rho(A)$ of A if $\lambda - A$ is a bijection of $D(A)$ onto $\mathcal{H}$ with a bounded inverse. If $\lambda \in \rho(A)$, $R_\lambda(A) = (\lambda - A)^{-1}$ is called the *resolvent* of A at λ. Note that $\rho(A) = \mathbb{C}\backslash\sigma(A)$, with $\sigma(A)$ the spectrum of A. In particular for self-adjoint A we have $\rho(A) \supset \{\lambda \in \mathbb{C} | \operatorname{Im}(\lambda) \neq 0\}$.

DEFINITION 5.3.16. *Let $(A_n, D(A_n))$ ($n = 1, 2, \dots$) and $(A, D(A))$ be self-adjoint operators on $\mathcal{H}$. Then A_n is said to converge to A in the* strong-resolvent sense *if $R_\lambda(A_n) \to R_\lambda(A)$ in the strong operator topology for all λ with $\operatorname{Im}(\lambda) \neq 0$.*

DEFINITION 5.3.17. *Let $(q, D(q))$ and $(r, D(r))$ be two symmetric forms. We say that $q \le r$ iff $D(q) \supseteq D(r)$ and $q(f, f) \le r(f, f)$ for all $f \in D(r)$.*

[1] The theorem is due to Kato, Lions, Lax and Milgram, and Nelson; see [**67**, Notes to Chapter X] for some historical comments.

Note that if $(q, D(q)) \subset (r, D(r))$, then $(r, D(r)) \leq (q, D(q))$.

Although it is troublesome to define a partial ordering on the set of unbounded operators on $\mathcal{H}$ due to possibly non-overlapping domains, there is a natural definition for the class of semi-bounded self-adjoint operators (see also Theorem 5.3.11).

DEFINITION 5.3.18. *If A and B are semi-bounded self-adjoint operators, then we say that A is smaller than B or that B is greater than A (in short: $A \leq B$ or $B \geq A$) whenever $q_A \leq q_B$.*

THEOREM 5.3.19 ([**69**, Theorem S.15]). *Let q be any positive form. Then there exists a largest closable form q_r that is smaller than q, i.e: If t is a closable positive form and $t \leq q$ then $t \leq q_r$.*

We define 0 to be the form with form domain $\mathcal{H}$ and form $(f, g) \mapsto 0$.

THEOREM 5.3.20 ([**69**, Theorem S.16]). *Let $q_1 \geq q_2 \geq q_3 \geq \cdots \geq 0$ be a sequence of closed positive forms. Consider $D_\infty = \cup_n D(q_n)$. Then the pair (q_∞, D_∞) is a positive (but in general not closed) form, where*

$$q_\infty(f, g) = \lim_{n \to \infty} q_n(f, g) = \inf_{n \in \mathbb{N}} q_n(f, g)$$

for $f, g \in D_\infty$. Let A_n be the positive self-adjoint operator corresponding (in the sense of Theorem 5.3.13) to q_n and B be the positive self-adjoint operators corresponding to the closure of $(q_\infty)_r$. Then $A_n \to B$ in the strong-resolvent sense.

The proof of the completeness of the Bethe ansatz eigenfunctions (Theorem 5.1.4) is based on the following theorem of Dorlas. We say that an unbounded operator $(A, D(A))$ on $\mathcal{H}$ has compact resolvent if $(A - \mu)^{-1}$ is a compact operator for all $\mu \in \rho(A)$.

THEOREM 5.3.21. [**16**, Theorem 3.2] *Let $\{H_t\}_{t \geq 0}$ be a one-parameter family of positive self-adjoint operators acting on a Hilbert space $\mathcal{H}$. Assume that $t \mapsto H_t$ is monotonically nondecreasing (in the sense of Definition 5.3.18) and right-continuous in the strong-resolvent sense and that H_0 has compact resolvent. Suppose that there exists a set of linearly independent eigenfunctions $\{\phi_n^t\}_{n \in N}$ for H_t, parameterized by a discrete set N, which depends continuously on t, and which is complete at $t = 0$. Then $\{\phi_n^t\}_{n \in N}$ is a total set in $\mathcal{H}$ for all $t \geq 0$.*

REMARK 5.3.22. In [**16**, Theorem 3.2] it is assumed that $t \mapsto H_t$ should be continuous in the strong-resolvent sense. From the proof follows that right-continuity in strong-resolvent sense is sufficient.

For later reference we state some other theorems that will be used in the following sections.

THEOREM 5.3.23. *Let A be a semi-bounded self-adjoint operator on $\mathcal{H}$. Then the following are equivalent:*
(a) *A has compact resolvent.*
(b) *There exists a complete orthonormal basis $\{f_n\}_{n=1}^\infty$ of $\mathcal{H}$ in $D(A)$ so that $Af_n =$*

$\lambda_n f_n$, $\lambda_1 \leq \lambda_2 \leq \ldots$ *and* $\lambda_n \to \infty$.

For a proof, see [**68**, Theorem XIII.64].

The spectrum $\sigma(A)$ of a self-adjoint operator $(A, D(A))$ decomposes into two disjoint parts, the *essential spectrum* $\sigma_{ess}(A)$ and the *discrete spectrum*

$$\sigma_{disc}(A) = \{\lambda \in \sigma(A) |\, \lambda \text{ is an isolated point of } \sigma(A) \text{ with finite multiplicity}\}.$$

THEOREM 5.3.24. *Let* $(A, D(A))$ *be a positive self-adjoint operator on* $\mathcal{H}$, $\phi_1, \phi_2, \ldots$ *a total set of linear independent eigenfunctions,* $A\phi_j = m_j \phi_j$ *such that* $m_j \to \infty$. *Then there is only discrete spectrum, i.e.* $\sigma(A) = \sigma_{disc}(A)$. *Moreover* $\sigma(A) = \{m_1, m_2, \ldots\} \subset [0, \infty)$. *For* $m \in \sigma(A)$ *the eigenvalue space* $\mathcal{H}_m = \{\phi \in D(A) |\, A\phi = m\phi\}$ *is finite dimensional and is spanned by* $\{\phi_i |\, m_i = m\}$. *Also* $\mathcal{H} = \bigoplus_{m \in \sigma(A)} \mathcal{H}_m$.

The only non-trivial part of Theorem 5.3.24 is to show that the essential spectrum is empty. This follows immediately from the operator form of the min-max theorem (see [**68**, Theorem XIII.1]). Note that by Theorem 5.3.23 a positive self-adjoint operator $(A, D(A))$ satisfies the condition of Theorem 5.3.24 iff A has compact resolvent.

DEFINITION 5.3.25. *Let* A *be a operator on a Hilbert space* $\mathcal{H}$. *A vector* $f \in D(A)$ *satisfying* $Af \in D(A)$, $A^2 f = A(Af) \in D(A)$, $A^3 f = A(A^2 f) \in D(A)$ *etc. is called a* smooth vector for A. *A smooth vector* f *for* A *is called an* analytic vector for A *if* $\sum_{n=1}^{\infty} \|A^n f\| t^n / n! < \infty$ *for some* $t > 0$. *In particular, any eigenvector of* A *is an analytic vector for* A.

THEOREM 5.3.26 (Nelson's analytic vector theorem). *Let* A *be a symmetric operator on a Hilbert space* $\mathcal{H}$. *If* $D(A)$ *contains a total set of analytic vectors, then* A *is essentially self-adjoint.*

For a proof, see [**67**, Theorem X.39].

5.4. Definition of the Hamiltonians: the free system

We will apply Dorlas' Theorem 5.3.21 to the Bethe ansatz eigenfunctions $\{\phi_{i\widehat{\mu}_k,k} | \mu \in P^{++}\}$ and the self-adjoint operators obtained from the form interpretations of of the formal Hamiltonian given by (5.1.3). Theorem 5.3.21 reduces the completeness for general $k \in K_+$ to the completeness for the uncoupled case $k \equiv 0$. We therefore have to investigate this case first.

In Example 3.2.8 we have seen that Bethe ansatz eigenfunctions

$$\phi_{\lambda,0} = \frac{1}{\# W_0} \sum_{w \in W_0} e^{w\lambda}$$

are parameterized by $\mathcal{S}_0^+ = 2\pi i P^+$. Our aim is to define a self-adjoint operator H_0 on $\mathcal{H}$ that is related the Laplacian $-\Delta$ (see (5.1.3) with $k \equiv 0$) with compact resolvent such that the Bethe ansatz eigenfunctions corresponding to $k \equiv 0$ are a complete set of orthogonal eigenfunctions of H_0.

It is sometimes more convenient to work with other Hilbert spaces that are isomorphic to $\mathcal{H}$ via a unitary transformation. The measure μ_E on V induces a measure on the compact torus $T = V/Q^\vee$, denoted by μ_T. It is the unique Haar measure on the abelian group T normalized by $\mu_T(T) = \#W_0 \cdot \mu_E(C_+)$. Since W_0 leaves $Q^\vee$ stable, the following defines an action of W_0 on T

$$w(v + Q^\vee) = wv + Q^\vee.$$

Consider the map $\imath$ from W-invariant functions on V to W_0-invariant functions on T, defined by

$$f \mapsto (v + Q^\vee \mapsto f(v)).$$

On the level of Hilbert spaces it defines a unitary isomorphism,

$$\imath : \mathcal{H} \xrightarrow{\sim} \mathcal{H}_T = L^2\left(T, \frac{d\mu_T}{\#W_0}\right)^{W_0},$$

with the inner product $(\cdot,\cdot)_T$ on $\mathcal{H}_T$ given by

$$(f,g)_{\mathcal{H}_T} = \frac{1}{\#W_0}\int_T f(t)\overline{g(t)}d\mu_T(t).$$

The torus T has a smooth manifold structure. The map $\imath$ defines an isomorphism from the space of smooth W-invariant functions $C^\infty(V)^W$ on V to the space of W_0-invariant smooth functions $C^\infty(T)^{W_0}$ on T. There is a Laplacian Δ_T on $C^\infty(T)^{W_0}$, defined by $\Delta_T \circ \imath = \imath \circ \Delta$.

We define first H_0 as an operator on $\mathcal{H}_T$.

THEOREM 5.4.1. *Consider the positive symmetric operator* $(-\Delta_T, C^\infty(T)^{W_0})$ *on* $\mathcal{H}_T$.

(a) *The operator* $(-\Delta_T, C^\infty(T)^{W_0})$ *is essentially self-adjoint. Its unique self-adjoint extension* $(H_0, D(H_0))$ *is positive and* $C^\infty(T)^{W_0}$ *is an operator core for* $(H_0, D(H_0))$.

(b) *The set* $\{\phi_{\lambda,0} |\, \lambda \in 2\pi i P^+\} \subset C^\infty(T)^{W_0} \subset D(H_0)$ *is a complete set of orthogonal functions in* $\mathcal{H}_T$*, that are also eigenfunctions of* H_0,

$$H_0\phi_{\lambda,0} = \|\lambda\|^2\phi_{\lambda,0} \qquad (5.4.1)$$

for all $\lambda \in 2\pi i P^+$.

(c) H_0 *has compact resolvent.*

PROOF. The symmetry and positivity of $(-\Delta_T, C^\infty(T)^{W_0})$ is obvious (see also Lemma 6.2.6**(i)** and (5.5.7)).

Assertion (c) follows from (b) and Theorem 5.3.23. The orthogonality of the functions in (b) is shown by a simple calculation. Consider the unital algebra A of functions in $C(T)$ generated by $\{e^\lambda |\, \lambda \in 2\pi i P\}$. We claim that A is dense in $C(T)$ in the topology of uniform convergence on T. This follows from the Stone-Weierstrass theorem. It is obvious that A contains the constants and is closed under conjugation. That it separates points is equivalent with: for every $v + Q^\vee \in T$ with $v \notin Q^\vee$, there is a $\lambda \in P$ such that

$\lambda(v) \notin 2\pi i\mathbb{Z}$. This is the case since P and $Q^\vee$ are lattices which by definition are in perfect duality. Whence by Stone-Weierstrass A is dense in $C(T)$ and elementary L^2-theory tells us that therefore A must be dense in $L^2\left(T, d\mu_T/\#W_0\right)$.

Consider the bounded symmetrization operator Sym_{W_0} from $L^2\left(T, d\mu_T/\#W_0\right)$ onto $\mathcal{H}_T$. On the dense subspace $C(T)$ it is defined by

$$\mathrm{Sym}_{W_0}(\phi) = \frac{1}{\#W_0}\sum_{w\in W_0} w\phi.$$

The image of A under Sym_{W_0} is dense by the continuity of Sym_{W_0}. But $\mathrm{Sym}_{W_0}(A)$ is equal to the $\mathbb{C}$-linear span of $\{\phi_{\lambda,0} |\, \lambda \in 2\pi i P^+\}$. Noting that $-\Delta_T\phi_{\lambda,0} = \|\lambda\|^2\phi_{\lambda,0}$ holds for all $\lambda \in 2\pi i P^+$, we get (b). Since eigenfunctions of H_0 are analytic vectors for H_0 (see Definition 5.3.25), the essential self-adjointness of $(-\Delta_T, C^\infty(T)^{W_0})$ follows from (b) and Nelson's analytic vector theorem (Theorem 5.3.26). □

Let $(h_0, Q_0) = (q_{H_0}, Q(H_0))$ be the closed positive form corresponding to the positive self-adjoint operator $(H_0, D(H_0))$ (see Theorem 5.3.11). Since $C^\infty(T)^{W_0}$ is an operator core of $(H_0, D(H_0))$, it is also a form core for (h_0, Q_0) by Theorem 5.3.11. In the next section we show that Q_0 is the Sobolov space of once differentiable function functions on $\overline{C_+}$ (see Proposition 5.5.3).

5.5. Definition of the Hamiltonians: the coupled system

For technical reasons it is more convenient to work with the following Hilbert space, which is unitary isomorphic to $\mathcal{H}$,

$$\mathcal{H}' = L^2(\overline{C_+}, d\mu_E),$$

with the inner product the obvious one (see also (5.1.2)). A domain of self-adjointness for H_0 on $\mathcal{H}'$ is then

$$D_0 = \{\phi_{|\overline{C_+}} | \phi \in C^\infty(V)^W\}$$

by Theorem 5.4.1(b).

For a bounded domain Ω in V and $m \in \mathbb{Z}_{\geq 1}$ or $m = \infty$ we let

$$C^m(\overline{\Omega}) = \{f_{|\overline{\Omega}} |\, f \in C^m(V)\}.$$

Denote the wall of the fundamental chamber C_+ corresponding to the simple root a_i (with $i = 0, 1, 2, \ldots, n$) by L_i,

$$L_i = \overline{C_+} \cap V_{a_i} = \{v \in V |\, a_i(v) = 0, a_j(v) \geq 0\, \forall j \neq i\}. \tag{5.5.1}$$

Since smooth functions symmetric with respect to orthogonal reflection in an affine hyperplane have zero normal derivative jumps, we have

$$D_0 \subseteq \{\phi \in C^\infty(\overline{C_+}) |\, \partial_{Da_i^\vee}\phi(x+0Da_i^\vee) = 0 \quad \text{for semi-regular } x \in L_i \quad \forall\, i\}. \tag{5.5.2}$$

In this section we construct positive self-adjoint unbounded operators H_k on $\mathcal{H}'$ ($k \in K_+$) associated to the formal Hamiltonian (5.1.3). The Bethe ansatz eigenfunctions will be contained in the domain of H_k. Moreover, they will be eigenfunctions. The operators

H_k will be constructed by applying the KLMN theorem (Theorem 5.3.15). The most natural way of defining the relevant forms is by using Sobolev theory and we therefore recall some basic facts, referring for more details to the textbooks [**79**], [**2**].

Let Ω be a domain in V. Denote the space of L^1-locally integrable function with respect to $d\mu_E$ on Ω by $L^1_{loc}(\Omega, d\mu_E)$. Every $g \in L^1_{loc}(\Omega, d\mu_E)$ defines a distribution $T_g \in C_c^\infty(\Omega)'$ on Ω as follows: It sends a test function ϕ in in the space of compactly supported smooth functions $C_c^\infty(\Omega)$ on Ω to $T_g(\phi) = \int_\Omega g\phi d\mu_E$.

For a homogeneous polynomial p of degree r and a distribution S on Ω, the distributional derivative $\phi \mapsto (-1)^r S(p(\partial)\phi)$ ($\phi \in C_c^\infty(\Omega)$) is denoted by $p(\partial)(S)$. We extend this by linearity to all polynomials. If $g \in L^1_{loc}(\Omega, d\mu_E)$ and $p(\partial)(T_g) = T_h$ for a $h \in L^1_{loc}(\Omega, d\mu_E)$, then h is unique and is called the *weak partial derivate* of g. Thus "$p(\partial)g = h$" in the weak sense for a homogeneous polynomial p of degree r, provided $h \in L^1_{loc}(\Omega, d\mu_E)$ and $\int_\Omega gp(\partial)\phi d\mu_E = (-1)^r \int_\Omega h\phi d\mu_E$ for every test function $\phi \in C_c^\infty(\Omega)$. If g is sufficiently smooth (say $g \in C^\infty(\Omega)$), then the point-wise partial derivative $p(\partial)g$ is the weak derivative of g. For a $g \in L^1_{loc}(\Omega, d\mu_E)$ we write $p(\partial)g \in L^2(\Omega, d\mu_E)$ whenever $p(\partial)g = h \in L^2(\Omega, d\mu_E)$.

Let $\{\eta_1, \ldots, \eta_n\}$ be an orthonormal bases (in short, ONB) of V. Assume $f, g \in L^1_{loc}(\Omega, d\mu_E)$ such that $\frac{\partial f}{\partial \eta_j} \in L^2(\Omega, d\mu_E)$ and $\frac{\partial g}{\partial \eta_j} \in L^2(\Omega, d\mu_E)$. Then it is easily checked that

$$\sum_{j=1}^n \int_\Omega \frac{\partial f}{\partial \eta_j} \overline{\frac{\partial g}{\partial \eta_j}} d\mu_E \tag{5.5.3}$$

is independent of the choice of ONB.

DEFINITION 5.5.1. *The space*

$$W^{1,2}(\Omega, d\mu_E) = \{f \in L^2(\Omega, d\mu_E) | \frac{\partial f}{\partial \eta_j} \in L^2(\Omega, d\mu_E) \textit{ for all } j = 1, 2, \ldots, n\}$$

is called the Sobolev space of once weak differentiable functions on Ω with respect to the measure μ_E.

$W^{1,2}(\Omega, d\mu_E)$ is a Hilbert space with respect to the inner product

$$(f, g)_{W^{1,2}(\Omega, d\mu_E)} = \int_\Omega f\bar{g} d\mu_E + \sum_{j=1}^n \int_\Omega \frac{\partial f}{\partial \eta_i} \frac{\partial \bar{g}}{\partial \eta_i} d\mu_E, \tag{5.5.4}$$

with $\{\eta_1, \ldots, \eta_n\}$ an ONB for V. The inner product is well-defined since by (5.5.3) it is independent of the choice of orthonormal basis.

Let $W^{1,2} = W^{1,2}(C_+, d\mu_E)$ the Sobolev space of once weak-differentiable functions on C_+.

Denote the measure on ∂C_+ induced from $d\mu_E$ on $\overline{C_+}$ by $d\sigma_E$.

THEOREM 5.5.2 (Trace operator). *There exist a linear operator*

$$\mathcal{B} : W^{1,2} \longrightarrow L^2(\partial C_+, d\sigma_E) \tag{5.5.5}$$

such that

(i) $\mathcal{B}\phi = \phi_{|\partial C_+}$ *if* $\phi \in W^{1,2} \cap C^1(\overline{C_+})$,

(ii) $\mathcal{B}$ *is continuous, i.e., there is a constant* $C > 0$ *such that*

$$\|\mathcal{B}\phi\|_{L^2(\partial C_+, d\sigma_E)} \leq C \cdot \|\phi\|_{W^{1,2}} \quad \forall \phi \in W^{1,2}.$$

PROOF. This is an immediate consequence of [**79**, Theorem I.8.7]. □

For the rest of this section we fix an ONB $\{\eta_1, \dots, \eta_n\}$.

Recall the closed positive form (h_0, Q_0) corresponding to the positive self-adjoint operator $(H_0, D(H_0))$. Note in particular that $M = 0$ is a lower bound for H_0. The form $(\cdot,\cdot)_{h_0}$ on Q_0 given by (5.3.2) has the following expression on $D_0 \subset Q_0$,

$$(\phi,\psi)_{h_0} = (-\Delta_T \phi, \psi)_{\mathcal{H}'} + (\phi,\psi)_{\mathcal{H}'} = (\phi,\psi)_{\mathcal{H}'} + \sum_{j=1}^{n} (\frac{\partial\phi}{\partial\eta_j}, \frac{\partial\psi}{\partial\eta_j})_{\mathcal{H}'} = (\phi,\psi)_{W^{1,2}}. \tag{5.5.6}$$

The second equality follows by Stokes' theorem,

$$\sum_{i=1}^{n} \left(\frac{\partial\phi}{\partial\eta_i}, \frac{\partial\psi}{\partial\eta_i} \right)_{\mathcal{H}'} = (-\Delta_T \phi, \psi)_{\mathcal{H}'} + \int_{\partial C_+} \frac{\partial\phi}{\partial\nu} \overline{\psi} d\sigma_E, \tag{5.5.7}$$

with ν the inner normal to C_+, and by observing that the second term on the right hand side is zero because of (5.5.2).

Since D_0 is an operator core for $(H_0, D(H_0))$ (by Theorem 5.4.1), it follows by (5.3.4) and (5.5.6) that Q_0 is precisely the closure of D_0 in $\mathcal{H}'$ with respect to the Sobolev norm $\|\cdot\|_{W^{1,2}}$. Since $W^{1,2}$ is a Hilbert space, Q_0 is the closure of D_0 in $W^{1,2}$, i.e.

$$Q_0 = \{\phi \in W^{1,2} | \, \exists\, \phi_n \in D_0 \text{ such that } \|\phi_n - \phi\|_{W^{1,2}} \to 0 \text{ as } n \to \infty\}. \tag{5.5.8}$$

The next proposition shows that in fact $D_0 \subseteq W^{1,2}$ is dense, hence $Q_0 = W^{1,2}$.

PROPOSITION 5.5.3. *The form domain* Q_0 *of* h_0 *is equal to the Sobolev space of once differentiable functions on* C_+*, i.e.* $Q_0 = W^{1,2}$*. Furthermore,*

$$h_0(\phi,\psi) = \sum_{i=1}^{n} \left(\frac{\partial\phi}{\partial\eta_i}, \frac{\partial\psi}{\partial\eta_i} \right)_{\mathcal{H}'} \quad \forall \phi,\psi \in Q_0 = W^{1,2}. \tag{5.5.9}$$

We introduce a regularization procedure that will be used in the proof of Proposition 5.5.3. For a $r > 0$ and $x \in V$ let $\bar{B}(x,r) = \{y \in V | \|y - x\| \leq r\}$. Choose a $j \in C_c^\infty(V)^{W_0}$ with $j \geq 0$, $\mathrm{Supp}(j) \subseteq \bar{B}(0,1)$ and $\int_V j(x) d\mu_E(x) = 1$. For $\varepsilon > 0$ consider the *mollifier* $j_\varepsilon \in C_c^\infty(V)^{W_0}$ defined by $j_\varepsilon(x) = j(x/\varepsilon)/\varepsilon^n$. For any $u \in L^1_{loc}(V, d\mu_E)$ let $u_\varepsilon \in C^\infty(V)$ be the regularization $j_\varepsilon \star u$ of u by j_ε, i.e.

$$u_\varepsilon(x) = \int_V j_\varepsilon(x-y) u(y) d\mu_E(y) = \int_V j_\varepsilon(y) u(x-y) d\mu_E(y).$$

In the next lemma we collect some elementary properties of u_ε.

LEMMA 5.5.4. *Let* $u \in L^1_{loc}(V, d\mu_E)$.

(i) *Let* Ω *be an open subset of* V *and* $u_{|\Omega} \in C(\Omega)$*. If* G *is a compact subset of* Ω*, then* $u_\varepsilon \to u$ *uniformly on* G.

(ii) *Let* $u \in L^1_{loc}(V, d\mu_E)^W$. *Then* $u_\varepsilon \in C^\infty(V)^W$ *and therefore* $u_{|\overline{C_+}} \in D_0$.

PROOF. **(i)** The method of proof is standard in L^p-theory (cf. proof of [**2**, Lemma 2.18(d)]). Since G is compact, there is a compact G' and $\varepsilon_0 > 0$ such that $G \subset G' \subset \Omega$ and $\bar{B}(x, \varepsilon_0) \subset G'$ for all $x \in G$. The statement is now an immediate consequence of

$$|u_\varepsilon(x) - u(x)| = \left| \int_V j_\varepsilon(x-y)(u(y) - u(x)) d\mu_E(y) \right|$$
$$\leq \sup_{y \in \bar{B}(x,\varepsilon)} |u(y) - u(x)| \leq \sup_{y,z \in G', \|y-z\| \leq \varepsilon} |u(y) - u(z)|$$

for all $x \in G$ and $0 < \varepsilon \leq \varepsilon_0$ and the fact that u is uniformly continuous on G'.
(ii) Because $W = W_0 \ltimes Q^\vee$ it suffices to show that u_ε is W_0-invariant and $\tau(Q^\vee)$-invariant. Let $x \in V$, $w \in W_0$ and $a \in Q^\vee$. Then

$$u_\varepsilon(x+a) = \int_V j_\varepsilon(y) u(x+a-y) d\mu_E(y) = \int_V j_\varepsilon(y) u(x-y) d\mu_E(y) = u_\varepsilon(x).$$

Also

$$u_\varepsilon(wx) = \int_V j_\varepsilon(y) u(wx-y) d\mu_E(y) = \int_V j_\varepsilon(wy') u(wx - wy') d\mu_E(y')$$
$$= \int_V j_\varepsilon(y) u(x-y) d\mu_E(y) = u_\varepsilon(x),$$

where we in the third equality used that j is itself W_0-invariant. □

PROOF OF PROPOSITION 5.5.3. Because of (5.5.8) it suffices to show that D_0 is a dense subspace of $W^{1,2}$ (in the Sobolev topology) for the first statement. The second statement then follows immediately from (5.5.6) and the fact that D_0 is a form core for $(h_0, Q_0) = (h_0, W^{1,2})$.

A standard result in Sobolov theory states that $C^\infty(\overline{C_+})$ is dense in $W^{1,2}$ (see for example [**79**, Theorem I.3.6]). To prove the proposition it therefore suffices to show that every $\phi \in C^\infty(\overline{C_+})$ can be approximated in $W^{1,2}$ by elements from D_0.

Assume that a $\phi \in C^\infty(\overline{C_+})$ is fixed. By (5.1.2) there is a unique continuous W-invariant function u on V extending ϕ. By Lemma 5.5.4**(ii)** $u_{\varepsilon|\overline{C_+}} \in D_0$. We shall show that $u_{\varepsilon|\overline{C_+}} \to \phi$ in the Sobolev norm as $\varepsilon \searrow 0$.

Recall that we had fixed an ONB $\{\eta_1, \ldots, \eta_n\}$ of V. For $j = 1, 2, \ldots n$, consider the function in $v^j \in L^1_{loc}(V, d\mu_E)$ defined by on V_{reg} by $\frac{\partial u}{\partial \eta_j}$. We claim that $\frac{\partial u_\varepsilon}{\partial \eta_j} = (v^j)_\varepsilon$ for all $\varepsilon > 0$.

For $t > 0$ and $j = 1, 2, \ldots, n$ consider the following function in $C(V)^{\tau(Q^\vee)}$,

$$g_{t,j}(y) = (u(y + t\eta_j) - u(y))/t \quad (y \in V).$$

Because of the geometry of the hyperplane arrangement associated to the affine root system Σ, it is easily seen that there is a $t_0 > 0$ such that for every $y \in V$ and $0 < t < t_0$, the number of times the line segment between y and $y + t\eta_j$ changes from chamber is at most $\#W_0$-times. Observe that $u_{|\overline{C}} \in C^\infty(\overline{C})$, and therefore also $\left(\frac{\partial u}{\partial \eta_j}\right)_{|C} \in C^\infty(\overline{C})$, for every alcove C. A simple application of the mean value theorem (possibly multiple times, but at most $\#W_0$-times) together with the W-invariance and continuity of u, then shows

$$|g_{t,j}(v)| \leq \#W_0 \sup_{C_+} \left|\frac{\partial u}{\partial \eta_j}\right| =: \#W_0 M_j$$

for all $t \in (0, t_0)$ and $v \in V$. Fix a $x \in V_{reg}$. Then $\mathrm{Supp}(j_\varepsilon g_{t,j}(x - \cdot)) \subset \bar{B}(0, \varepsilon)$, and $|j_\varepsilon g_{t,j}(x - \cdot)|$ is dominated by $(\#W_0 M_j) \sup_V j/(\varepsilon^n)$ and $\lim_{t\to 0} g_{t,j}(y) = \frac{\partial u}{\partial \eta_j}(y)$ for every $y \in V_{reg}$ and j. Observing that the complement of $x - V_{reg}$ in V has measure zero and applying Lebesgue's dominated convergence theorem gives

$$\begin{aligned}
\frac{\partial u_\varepsilon}{\partial \eta_j}(x) &= \lim_{t\to 0} \int_{x - V_{reg}} j_\varepsilon(y) g_{t,j}(x-y) d\mu_E(y) \\
&= \int_{x - V_{reg}} \lim_{t\to 0} (j_\varepsilon(y) g_{t,j}(x-y)) d\mu_E(y)) \\
&= \int_{x - V_{reg}} j_\varepsilon(y) v^j(x-y) d\mu_E(y) = (v^j)_\varepsilon(x).
\end{aligned}$$

Since $x \in V_{reg}$ was arbitrary and V_{reg} is a dense subset of V and continuity, it follows indeed that $\frac{\partial u_\varepsilon}{\partial \eta_j} = (v^j)_\varepsilon$ for all $\varepsilon > 0$. Together with $v^j{}_{|C_+} \in C(C_+)$ and Lemma 5.5.4**(i)** this shows that $(v^j)_\varepsilon \to v^j$, and therefore $\frac{\partial u_\varepsilon}{\partial \eta_j} \to \frac{\partial u}{\partial \eta_j}$, uniformly on compact subsets of C_+, and thus in particularly point-wise on C_+. If a locally integrable function f is bounded by M, then so does the regularization f_ε. Applying this observation to $f = v^j$ and using $\frac{\partial u_\varepsilon}{\partial \eta_j} = (v^j)_\varepsilon$ gives

$$\left|\frac{\partial(u_\varepsilon - u)}{\partial \eta_j}\right|^2 = \left|\frac{\partial u_\varepsilon}{\partial \eta_j} - \frac{\partial u}{\partial \eta_j}\right|^2 \leq 4M_j^2,$$

uniformly on C_+. Lebesgue's dominated convergence theorem now gives

$$\lim_{\varepsilon\searrow 0} \int_{C_+} \left|\frac{\partial(u_\varepsilon - u)}{\partial \eta_j}\right|^2 d\mu_E = 0.$$

Because $u \in C(V)$, it follows by Lemma 5.5.4**(i)** that $u_\varepsilon \to u$ uniformly on $\overline{C_+}$. Together with (5.5.4) applied to $\Omega = C_+$ and $f = g = (u - u_\varepsilon)_{|\overline{C_+}} \in C^\infty(\overline{C_+})$, we see that $u_{\varepsilon|C_+} \to u_{|C_+} = \phi$ in $W^{1,2}$, and this concludes the proof because of the remarks made at the beginning of the proof.

□

We are now able to define the forms used to make sense of the formal Hamiltonian (5.1.3) as positive self-adjoint operators.

DEFINITION 5.5.5. *Let $k \in K_+$. We denote by δ_k the form with form domain $Q_0 = W^{1,2}$ and*

$$\delta_k(\phi, \psi) = \sum_{i=0}^{n} \frac{k_i}{\|Da_i^\vee\|} \int_{L_i} \mathcal{B}\phi(v)\overline{\mathcal{B}\psi(v)}\ d\sigma_E(v) \quad (\phi, \psi \in Q_0),$$

with $\mathcal{B}$ the trace operator from Theorem 5.5.2. Define also the form $h_k = h_0 + \delta_k$ (with Q_0 as form domain).

REMARK 5.5.6. If $\phi, \psi \in D_0$, then by Theorem 5.5.2**(i)**

$$\delta_k(\psi, \phi) = \sum_{i=0}^{n} \frac{k_i}{\|Da_i^\vee\|} \int_{L_i} \psi\bar{\phi} d\sigma_E,$$

and

$$h_k(\phi, \psi) = (-\Delta_T \phi, \psi) + \sum_{i=0}^{n} \frac{k_i}{\|Da_i^\vee\|} \int_{L_i} \psi\bar{\phi} d\sigma_E.$$

To apply the KLMN theorem, we need to check the inequality (5.3.7). This will be a consequence of the following lemma.

LEMMA 5.5.7. *There is a constant $c > 0$ (only dependent on Σ) such that*

$$\delta_1(\phi, \phi) \leq c\,(h_0(\phi, \phi) + (\phi, \phi)) \tag{5.5.10}$$

for all $\phi \in Q_0$.

This follows from Theorem 5.5.2 and Proposition5.5.3

For multiplicity functions k and k' in K_+ such that $k \geq k'$ we denote by $k - k' \in K_+$ the following multiplicity function: $(k - k')_a = k_a - k'_a$. For $k \in K_+$ let $k_\infty = \max_{a\in\Sigma} k_a$.

PROPOSITION 5.5.8. *For all $k \in K_+$, there is an unique positive self-adjoint operator $(H_k, D(H_k))$ on $\mathcal{H}'$ with form domain $Q(H_k) = Q_0$ such that $q_{H_k} = h_k$. The operators H_k are increasing in k:*

$$H_k \leq H_{k'} \quad \text{if } k \leq k'. \tag{5.5.11}$$

PROOF. The uniqueness follows because a self-adjoint operator is uniquely defined by its form (see Theorem 5.3.13). Equation (5.5.11) follows directly from the fact that $h_k \leq h_{k'}$ (see Definition 5.5.5) whenever $k \leq k'$, and the first part of the Proposition.

To prove the existence we use induction on k_∞. Assume that H_{k^0} is defined for some $k^0 \geq 0$ with the properties described in the statement of the proposition. Let β be the form δ_{k-k^0}. Then h_k is a perturbation of h_{k^0} by β since $h_k = h_{k^0} + \beta$. The following estimation

$$\beta(\phi, \phi) = \delta_{k-k^0}(\phi, \phi) \leq (k - k^0)_\infty \delta_1(\phi, \phi) \leq c(k - k^0)_\infty \bigg(h_{k^0}(\phi, \phi) + (\phi, \phi)\bigg)$$

is true for all $\phi \in Q_0$ since the forms h_k are nondecreasing and inequality (5.5.10). The KLMN theorem (Theorem 5.3.15) tells us that there is a *unique* self-adjoint operator

$(H_k, D(H_k))$ with associated form (h_k, Q_0) when $c(k - k^0)_\infty < 1$, i.e. for all k with $k^0 \leq k < k^0 + 1/c$. Apply induction to conclude the proof of the proposition. □

REMARK 5.5.9. Although D_0 is a form core for the form (h_k, Q_0) associated to the operator $(H_k, D(H_k))$ it is in general not true that $D_0 \subset D(H_k)$ for $k \neq 0$.

5.6. Proof of the completeness

In this section we shall prove Theorem 5.1.4. We finish the section with some corollaries to Theorem 5.1.4.

Now we have defined positive self-adjoint operators H_k on $\mathcal{H}'$, we turn our attention to the domain of definition $D(H_k)$. As observed earlier on (see statement Theorem 5.3.21) the key step will be to show that the Bethe ansatz functions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k\}$ are in the domain of definition $D(H_k)$. We first define a convenient space D_k containing the Bethe ansatz eigenfunctions. For this recall the space $C^{\omega,(k)}(V)$ from Chapter 3 (see Section 3.5). For $k \in K_+$, $k \neq 0$, let

$$D_k = \{ \ \phi_{|\overline{C_+}} | \ \phi \in C^{\omega,(k)}(V)^W \}$$

(Note that for $k \equiv 0$, the above would not be equal to the space D_0 defined at the beginning of Section 5.5, but would only be contained in it). Alternatively,

$$D_k = \{\phi \in C^\omega(\overline{C_+}) | \, \partial_{Da_i^\vee} \phi(x + 0Da_i^\vee) = k_i \phi(x) \quad \text{for semi-regular } x \in L_i, \\ i = 0, 1, \ldots, n\}. \quad (5.6.1)$$

For a $\phi \in C^\infty(\overline{C_+})$ we denote by $\Delta\phi$ the unique function in $C^\infty(\overline{C_+})$ that on C_+ is given by $\Delta\phi$. It extends the operator Δ on $D_0 \subset C^\infty(\overline{C_+})$ from Section 4.

PROPOSITION 5.6.1. *Let $k \in K_+$.*

- **(1)** *The Bethe ansatz eigenfunctions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k\}$ are in D_k.*
- **(2)** $D_k \subset D(H_k)$.
- **(3)** *$H_k \phi = -\Delta\phi$ in $\mathcal{H}'$ for all $\phi \in D_k$. In particular $H_k \phi_{\lambda,k} = \|\lambda\|^2 \phi_{\lambda,k}$ for all $\lambda \in \mathcal{S}_k$.*

PROOF. **(1)** follows from Corollary 3.6.2 and $\phi_{\lambda,k} \in \mathrm{BVP}_k(\lambda)^W$ for $\lambda \in \mathcal{S}_k$ (note that all the results in Chapter 3, with the exception of the last section, holds for complex valued multiplicity functions k).

Observe that $D_k \subset W^{1,2} = Q_0$, the inclusion coming from $D_k \subset C^\infty(\overline{C_+}) \subset W^{1,2}$ and the equality being Proposition 5.5.3. To prove **(2)** and **(3)** it suffices by Proposition 5.3.12 to show that for all $\phi \in D_k \subset Q_0$, $h_k(\phi, \psi) = (-\Delta\phi, \psi)$ holds for all $\psi \in D_0$. By symmetry of the forms, this is equivalent to: $h_k(\psi, \phi) = (\psi, -\Delta\phi)$ holds for all $\psi \in D_0$, and this is equivalent to

$$-(\Delta\psi, \phi)_{\mathcal{H}'} + \delta_k(\psi, \phi) = (\psi, -\Delta\phi)_{\mathcal{H}'}, \quad \forall \psi \in D_0, \qquad (5.6.2)$$

by the definition of h_k (see Definition 5.5.5) and (5.3.5). By Stokes' theorem:

$$(\Delta\psi,\phi)_{\mathcal{H}'} = (\psi,\Delta\phi)_{\mathcal{H}'} + \int_{\partial C_+}\left(\psi\overline{\frac{\partial\phi}{\partial\eta}} - \frac{\partial\psi}{\partial\eta}\overline{\phi}\right)d\sigma_E,$$

with η denoting the inner normal vector to ∂C_+. The second term of the integrand is zero because $\psi \in D_0$ (see (5.5.2)). The remaining term gives

$$\sum_{i=0}^{n}\frac{1}{\|Da_i^\vee\|}\int_{L_i}\psi\overline{\partial_{Da_i^\vee}\phi}d\sigma_E.$$

Now use that $\phi \in D_k$ (see (5.6.1)) to conclude (5.6.2). □

We turn now our attention the right-continuity of the map defined by $k \mapsto H_k$.

LEMMA 5.6.2. *Let $k^0 \in K_+$. Then $h_{k^0}(\phi,\phi) = \inf_{k>k^0} h_k(\phi,\phi)$ for all $\phi \in D_0$.*

PROOF. This follows from

$$0 \leq h_k(\phi,\phi) - h_{k^0}(\phi,\phi) = \delta_{k-k^0}(\phi,\phi) \leq (k-k^0)_\infty\delta_1(\phi,\phi).$$

□

COROLLARY 5.6.3. *The map defined by $k \mapsto H_k$ ($k \in K_+$) is right-continuous in the strong-resolvent sense.*

PROOF. Let $k_1 \geq k_2 \geq \cdots \geq k^0$ be a sequence in $\{k \in K_+ | k > k^0\}$ s.t. $k_j \to k^0$. Then $h_{k_1} \geq h_{k_2} \geq \cdots \geq 0$ as forms on Q_0, and $h_{k^0} = \lim_{j\to\infty} h_{k_j} = \inf_{j\in\mathbb{N}} h_{k_j}$ by the first part of Theorem 5.3.20 and Lemma 5.6.2. By the second part of Theorem 5.3.20, $H_{k_j} \to H_{k^0}$ as $j \to \infty$ in the strong resolvent sense. □

To apply Dorlas' theorem (see Theorem 5.3.21) to our operators H_k, we still need to verify the linear independence of the Bethe ansatz eigenfunctions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k\}$. Recall the Dunkl-type operators $p(\mathcal{D}^k)$ ($p \in S(V)_\mathbb{C}$) from Section 3.4 (which were defined for all complex valued multiplicity functions k). By Theorem 3.6.4**(i)** we have for any $\lambda \in V_\mathbb{C}^*$ and $p \in S(V)_\mathbb{C}^{W_0}$,

$$p(\mathcal{D}^k)\phi_{\lambda,k} = p(\lambda)\phi_{\lambda,k}. \tag{5.6.3}$$

LEMMA 5.6.4. *Let $k \in K_+$. The Bethe ansatz eigenfunctions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k/W_0\}$ are linearly independent in $\mathcal{H}$.*

PROOF. Let $W_0\lambda_1, W_0\lambda_2, \ldots W_0\lambda_m \in \mathcal{S}_k/W_0$ be pairwise different and $a_1,\ldots,a_m$ elements in $\mathbb{C}$ such that $\phi = \sum_{i=1}^m a_i\phi_{\lambda_i,k} = 0$ in $\mathcal{H}$. We have to show that $a_1 = \cdots = a_m = 0$. Since ψ is μ_E-almost everywhere zero on V, it must be zero as a continuous function on V.

Since $S(V)_\mathbb{C}^{W_0}$ separates the orbits $V_\mathbb{C}^*/W_0$ there are $p_1, p_2, \ldots, p_m \in S(V)_\mathbb{C}^{W_0}$ such that $p_i(\lambda_j) = \delta_{ij}$. For any $1 \leq i \leq m$ we therefore have

$$a_i\phi_{\lambda_i,k} = \sum_j a_j p_i(\lambda_j)\phi_{\lambda_j,k} = p_i(\mathcal{D}^k)\left(\sum_j a_j\phi_{\lambda_j,k}\right) = p_i(\mathcal{D}^k)(0) = 0,$$

(using (5.6.3) in the second equality). Whence $a_i = 0$ for $1 \le i \le m$, as desired. $\square$

THEOREM 5.6.5. *Let $k \in K_+$. The functions $\{\phi_{i\widehat{\mu}_k,k} | \mu \in P^{++}\}$ form a total set in $\mathcal{H}$.*

PROOF. By Theorem 5.4.1(2) we may assume that $k \neq 0$. All the conditions of Theorem 5.3.21 are satisfied for the operators H_{tk} and the Bethe ansatz eigenfunctions $\{\phi_{i\widehat{\mu}_{tk},tk} | \mu \in P^{++}\}$ ($t \ge 0$) by Lemma 5.6.4, Theorem 5.4.1, Proposition 5.6.1, Proposition 5.2.15 and Corollary 5.6.3. Hence $\{\phi_{i\widehat{\mu}_{tk},tk} | \mu \in P^{++}\}$ is a total set in $\mathcal{H}$ for all $t \ge 0$. Now take $t = 1$ to conclude the proof. $\square$

COROLLARY 5.6.6. *Let $k \in K_+$.*
(a) *The spectrum $\sigma(H_k)$ of H_k is given by $\{\|\widehat{\mu}_k\|^2 | \, \mu \in P^{++}\}$.*
(b) *For a $m \in \sigma(A)$ the set of functions $\{\phi_{\lambda,k} | \, \lambda = i\widehat{\mu}_k, \, \mu \in P^{++}, \, \|\lambda\|^2 = m\}$ is a basis for the corresponding eigenspace $\mathcal{H}_m$ (and thus $\mathcal{H}_m$ is in particular finite dimensional). Also $\mathcal{H} = \oplus_{m \in \sigma(A)} \mathcal{H}_m$.*

PROOF. Follows from Theorem 5.3.24, Proposition 5.6.1, Lemma 5.6.4 and Theorem 5.6.5. $\square$

PROOF OF THEOREM 5.1.4 AND COROLLARY 5.1.5. Because of the above corollary it suffices to show that any $\lambda \in \mathcal{S}_k$ is W_0-conjugate to an element from $\{i\widehat{\mu}_k | \, \mu \in P^{++}\}$. Assume that this is not the case for a $\lambda \in \mathcal{S}_k$. The eigenfunction $\phi_{\lambda,k}$ of H_k must be in $\mathcal{H}_m$ for a $m = \|\widehat{\mu}_k\|^2$ and $\mu \in P^{++}$ by the above corolllary. But this is in contradiction with Lemma 5.6.4. $\square$

The above proof gives immediately the following.

COROLLARY 5.6.7. *For $k \in K_+$ the spectrum $\mathcal{S}_k^+$ is given by $\{i\widehat{\mu} | \, \mu \in P^{++}\}$ (cf. Corollary 5.1.5).*

REMARK 5.6.8. Note that in the proof of Theorem 5.6.5 only the inclusion $\{i\widehat{\mu}_k | \mu \in P^{++}\} \subseteq \mathcal{S}_k$ is needed. For $k \in K_{>0}$ the inclusion $\{i\widehat{\mu}_k | \mu \in P^{++}\} \subseteq \mathcal{S}_k$ follows (see in particular Corollary 4.4.8, Proposition 4.8.1 and Proposition 4.9.1) without using the Pauli principle (Proposition 4.5.1). By continuity (see Lemma 5.2.14) this inclusion holds for all $k \in K_+$. The arguments in this chapter thus give an independent proof of the Pauli principle.

LEMMA 5.6.9. *Let $\mu \in P^{++}$ and $k^0 \in K_+$. Then for all $\alpha \in \Sigma_0^+$ with $k_\alpha^0 = 0$ we have*

$$\lim_{K_{>0} \ni k \to k^0} \widehat{\mu}_k(\alpha^\vee)/k_\alpha = +\infty. \tag{5.6.4}$$

PROOF. By Lemma 5.2.6 and 5.2.8 we have for $k \in K_{>0}$,

$$\widehat{\mu}_k + \sigma_{\widehat{\mu}_k}^k = 2\pi\mu = \widehat{\mu}_{k^0} + \widetilde{\sigma}_{\widehat{\mu}_{k^0}}^{k^0} + 2\pi\rho^{k^0},$$

with $\widetilde{\sigma}^{k^0}_{\widehat{\mu}_{k^0}}$ and $\sigma^{k}_{\widehat{\mu}_k}$ given by (5.2.3) and the formula after (4.8.5). Proposition 5.2.12 thus gives

$$2 \lim_{K_{>0}\ni k\to k^0} \sum_{\alpha\in\Sigma_0^+,\, k^0_\alpha=0} \arctan\left(\frac{\widehat{\mu}_k(\alpha^\vee)}{k_\alpha}\right)\alpha = \sigma^{k}_{\widehat{\mu}_k} - \widetilde{\sigma}^{k^0}_{\widehat{\mu}_{k^0}} = 2\pi\rho^{k^0}.$$

In particular

$$2 \lim_{K_{>0}\ni k\to k^0} \sum_{\alpha\in\Sigma_0^+,\, k^0_\alpha=0} \arctan\left(\frac{\widehat{\mu}_k(\alpha^\vee)}{k_\alpha}\right)\alpha(v) = \pi \sum_{\alpha\in\Sigma_0^+,\, k^0_\alpha=0} \alpha(v)$$

holds for all $v \in V$. Applying this to a $v \in V_+ = \{v \in V | \alpha(v) > 0 \,\forall \alpha \in \Sigma_0^+\} \supset C_+$ and observing that $0 \leq 2\arctan(t) \leq \pi$ for $t \geq 0$, we get

$$2 \lim_{K_{>0}\ni k\to k^0} \arctan\left(\frac{\widehat{\mu}_k(\alpha^\vee)}{k_\alpha}\right) = \pi,$$

which leads to (5.6.4). □

Recall the c-function $\widetilde{c}_k$, for every $k \in K_{\mathbb{C}}$ a rational function on $V_{\mathbb{C}}^*$ given by (3.7.7). Note that for example $\widetilde{c}_0 \equiv 1$. Lemma 5.6.9 leads directly to the following.

LEMMA 5.6.10. *Let $\mu \in P^{++}$. Then $k \mapsto \tilde{c}_k(i\widehat{\mu}_k)$ is a continuous function from K_+ to $\mathbb{C}$.*

We now can prove the following extension of Theorem 4.4.1 to $k \in K_+$.

THEOREM 5.6.11. *Let $\lambda \in V_{\mathbb{C}}^*$ and $k \in K_+$. The space $\mathrm{BVP}_k(\lambda)^W$ of W-invariant solutions to the boundary value problem is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is purely imaginary solution to the Bethe ansatz equations* (5.2.2) *satisfying $\lambda(\alpha^\vee) \neq 0$ for all $\alpha \in \Sigma_0$ with $k_\alpha > 0$. If $\mathrm{BVP}_k(\lambda)^W$ is one-dimensional, then there exists a unique $\phi_{\lambda,k} \in \mathrm{BVP}_k(\lambda)^W$ normalized by $\phi_{\lambda,k}(0) = 1$. The solution $\phi_{\lambda,k}$ is the unique W-invariant function satisfying*

$$\phi_{\lambda,k}(v) = \frac{1}{\#W_0} \sum_{w\in W_0} \widetilde{c}_k(w\lambda) e^{w\lambda(v)}, \qquad v \in \overline{C_+}.$$

PROOF. This follows from Corollary 5.1.5, Lemma 5.2.9**(ii)**, Theorem 4.4.1, Lemma 5.6.10 and Proposition 5.2.15. □

5.7. General lattices

In this section we consider Hilbert space completeness for general lattices X in V with $Q^\vee \subseteq X \subset P^\vee$.

Formula (2.2.7) defines (cf. Definition 3.2.10) an action of W_X on $\mathcal{H}$. Note that $W \subset W_X$ acts trivially on $\mathcal{H}$.

LEMMA 5.7.1. *The action of W_X on $\mathcal{H}$ is unitary.*

PROOF. If $w_1 \in W$, $\omega \in \Omega_X$ and $f, g \in \mathcal{H}$ we have

$$(w_1\omega f, w_1\omega g)_{\mathcal{H}} = (\omega f, \omega g)_{\mathcal{H}} = \int_{C_+} f(\omega^{-1}v)\overline{g(\omega^{-1}v)}d\mu_E(v) = \int_{C_+} f(v)\overline{g(v)}d\mu_E(v).$$

The last equality follows after the change of coordinates $v \mapsto \omega v$ and the fact that Jacobian of this coordinate transform is equal to 1 since it equals $(-1)^{l(\omega)}$ and ω has length zero. □

LEMMA 5.7.2. *The Hilbert space $\mathcal{H}$ admits the following orthogonal decomposition*

$$\mathcal{H} = \bigoplus_{\chi \in \widehat{X/Q^\vee}} \mathcal{H}_\chi, \tag{5.7.1}$$

where

$$\mathcal{H}_\chi = \{f \in \mathcal{H} | w\tau_x f = \chi(x + Q^\vee) f \ \forall w \in W_0, x \in X\}.$$

PROOF. The decomposition (5.7.1) as direct sum follows from general representation theoretic considerations (cf. Lemma 3.2.11). That closed subspaces $\mathcal{H}_\chi$ ($\chi \in \widehat{X/Q^\vee}$) are pair-wise orthogonal follows directly from Lemma 5.7.1. □

When the closed subspace $\mathcal{H}_\chi$ ($\chi \in \widehat{X/Q^\vee}$) of $\mathcal{H}$ is considered as a Hilbert space on itself, the inner product on $\mathcal{H}_\chi$ is denoted by $(,)_{\mathcal{H}_\chi}$. Thus $(f,g)_{\mathcal{H}} = (f,g)_{\mathcal{H}_\chi}$ for $f, g \in \mathcal{H}_\chi$.

Recall (see Definition 3.2.13) the spectrum $\mathcal{S}_k(\chi)$ associated to any $\chi \in \widehat{X/Q^\vee}$.

THEOREM 5.7.3. *Let $k \in K_+$. Then for all $\chi \in \widehat{X/Q^\vee}$ the spectrum $\mathcal{S}_k(\chi)$ is purely imaginary and $\mathcal{S}_k^+(\chi) = \mathcal{S}_k(\chi) \cap i\overline{V_+^*}$ forms a complete set of representatives for $\mathcal{S}_k(\chi)/W_0$. Furthermore the following holds.*

(i) *If $\lambda \in \mathcal{S}_k^+(\chi)$ and $\mu \in \mathcal{S}_k^+(\chi')$ with χ and χ' different characters in $\widehat{X/Q^\vee}$, then $(\phi_{\lambda,k}, \phi_{\mu,k})_{\mathcal{H}} = 0$.*

Let $\chi \in \widehat{X/Q^\vee}$. Then the following holds.

(ii) *(Partial orthogonality) If $\phi_{\lambda,k}$, $\phi_{\mu,k}$ ($\lambda, \mu \in \mathcal{S}_k^+(\chi)$) are two Bethe ansatz eigenfunctions with $\|\lambda\| \neq \|\mu\|$, then $(\phi_{\lambda,k}, \phi_{\mu,k})_{\mathcal{H}_\chi} = 0$.*

(iii) *(Completeness) The Bethe ansatz eigenfunctions $\{\phi_{\lambda,k} | \lambda \in \mathcal{S}_k^+(\chi)\}$ are algebraically linearly independent and total in $\mathcal{H}_\chi$.*

PROOF. Since $\mathcal{S}_k(\chi) \subset \mathcal{S}_k$, the first statement follows from the first statement of Theorem 5.1.4.

Observe that $\mathrm{BVP}_k(\lambda)^{W,\chi} \subseteq \mathcal{H}_\chi$ for all $\lambda \in \mathcal{S}_k^+(\chi)$. Therefore statement **(i)** is an immediate consequence of Lemma 5.7.2. Statement **(ii)** follows directly from Theorem 5.1.4**(i)**. Statement **(iii)** follows from Lemma 5.7.2 and Theorem 5.1.4**(ii)**. □

We now extend Theorem 5.7.6 to all $k \in K_+$. The Bethe ansatz equations for $k \in K_+$ and $\chi \in \widehat{X/Q^\vee}$ are the following equations in $\lambda \in iV^*$,

$$\prod_{\substack{\alpha\in\Sigma_0^+\\ k_\alpha>0}} \left(\frac{\lambda(\alpha^\vee)-k_\alpha}{\lambda(\alpha^\vee)+k_\alpha}\right)^{\alpha(x)} = \chi(x+Q^\vee)e^{\lambda(x)} \qquad \forall\, x\in X \tag{5.7.2}$$

(if $k \equiv 0$, we interpret these equations as: $1 = \chi(x+Q^\vee)e^{\lambda(x)}$). The set of solutions $\lambda \in iV^*$ of the Bethe ansatz equations (5.7.2) is denoted by BAE_k^χ. For $k \in K_{>0}$ we recover the Bethe ansatz equation (4.6.1).

Recall from Section 4.7 the Y-cosets Y^χ in P. We have the following corollary.

COROLLARY 5.7.4. *Let $k \in K_+$ and $\chi \in \widehat{X/Q^\vee}$. Then $\mathcal{S}_k \subset iV^*$ and $\mathcal{S}_k^+(\chi) = \{i\widehat{\mu}_k | \mu \in (\rho + Y^\chi) \cap P^{++}\}$.*

PROOF. Proceeding as in the proof of Lemma 5.2.14 we conclude that $\mathcal{S}_k^+(\chi) \subset \{i\widehat{\mu}_k | \mu \in (\rho + Y^\chi) \cap P^{++}\}$. The corollary now follows from (3.2.8), Corollary 5.1.5 and (4.7.4). □

Recall the subset P^k of P from Lemma 5.2.7**(iv)**. We have the following extension of Proposition 4.8.3 to all $k \in K_+$.

PROPOSITION 5.7.5. *Let $k \in K_+$ and $\chi \in \widehat{X/Q^\vee}$. The assignment $\mu \mapsto i\widehat{\mu}_k$ defines a W_0-equivariant bijection $(\rho + Y^\chi) \cap P^k \xrightarrow{\sim} \mathrm{BAE}_k^\chi$.*

PROOF. This is analogous to the proof of Proposition 4.8.3 (and using Proposition 5.2.9 instead of Proposition 4.8.1). □

THEOREM 5.7.6. *Let $k \in K_+$, $\lambda \in V_{\mathbb{C}}^*$ and $\chi \in \widehat{X/Q^\vee}$. The space $\mathrm{BVP}_k(\lambda)^{W,\chi}$ of (W,χ)-invariant solutions to the boundary value problem is one-dimensional or zero-dimensional. It is one-dimensional if and only if the spectral value λ is purely imaginary solution to the Bethe ansatz equations (5.7.2) satisfying $\lambda(\alpha^\vee) \neq 0$ for all $\alpha \in \Sigma_0$ with $k_\alpha > 0$. If $\mathrm{BVP}_k(\lambda)^{W,\chi}$ is one-dimensional, then there exists a unique $\phi_{\lambda,k} \in \mathrm{BVP}_k(\lambda)^{W,\chi}$ normalized by $\phi_{\lambda,k}(0) = 1$. The solution $\phi_{\lambda,k}$ is the unique (W,χ)-invariant function satisfying*

$$\phi_{\lambda,k}(v) = \frac{1}{\#W_0}\sum_{w\in W_0} \widetilde{c}_k(w\lambda)e^{w\lambda(v)}, \qquad v \in \overline{C_+}. \tag{5.7.3}$$

PROOF. This follows from Proposition 5.7.5, Corollary 5.7.4 and Theorem 5.6.11. □

CHAPTER 6

Conjectures on orthogonality and norms

6.1. Introduction

In Chapter 5 we showed that the Bethe ansatz eigenfunctions of the quantum integrable systems with delta-potentials associated to affine root systems studied in the Chapter 3 and 4 are complete in the Hilbert space of square-integrable functions on the fundamental domain for the reflection representation of the affine Weyl group. In this chapter we conjecture (Conjecture 6.2.4) that the Bethe ansatz eigenfunctions are orthogonal and that their quadratic norms are expressible in terms of the determinant of the Hessian of the master function.

Similar conjecture about quadratic norms were made by Gaudin [**27**] for the quantum Bose-gas with repulsive delta-function potential. An important feature of the quantum Bose-gas on the circle with pair-wise delta-function interactions is its realization as the restriction to a fixed particle sector of the quantum integrable field theory in $1 + 1$ dimensions governed by the quantum nonlinear Schrödinger equation (see for example Section 1.1). Korepin [**54**] managed to overcome the intimidating combinatorial difficulties and proved the conjecture of Gaudin using quantum inverse scattering techniques. However not much progress has been achieved in the conceptual understanding of this quadratic norm formula since then.

Izergin and Korepin [**46**] introduced a lattice version a quantum integrable field theory in $1+1$ dimensions governed by the quantum nonlinear Schrödinger equation. By applying quantum inverse scattering techniques to this discrete model and letting the lattice constant tend to zero, Dorlas [**16**] showed the orthogonality of the Bethe ansatz eigenfunctions. Using a continuity argument at zero coupling he also showed these eigenfunctions are complete (see also Chapter 5).

Another interesting approach is taken by van Diejen [**15**], where an integrable lattice discretization of the quantum Bose-gas with delta-function potential is introduced. The corresponding spectral problem of the integrable lattice model is solved by means of the Bethe ansatz method. In the continuum limit the results of Lieb and Liniger [**56**] are recovered and the orthogonality results of Dorlas [**16**]. Many constructions in [**15**] can be carried over to the root system generalizations of the quantum Bose gas on the circle with repulsive delta-potentials. This looks a promising path for proving the full orthogonality for all root systems.

In the next section we give the precise conjectures about orthogonality and the quadratic norm formulas. The free case and the rank one case are calculated explicitly.

In the last section we show that the conjecture holds when Σ_0 is of type **A**. This is done by showing that it is essentially equivalent to the similar statement for the Bethe ansatz eigenfunctions for the quantum Bose gas on the circle with pair-wise repulsive delta-function potential.

6.2. The conjectures

Before stating our conjecture on orthogonality and norms of the Bethe ansatz eigenfunctions, we consider the free case $k \equiv 0$ and the the rank one case for general $k > 0$.

To calculate the inner product of the Bethe ansatz eigenfunctions for $k \equiv 0$ it will be useful to start with the following simple lemma (which is well-known from harmonic analysis of compact groups, see e.g. [**42**]).

LEMMA 6.2.1. *Let G be a compact abelian group and μ a Haar measure on G and $\gamma \in \hat{G} = \mathrm{Hom}_{\mathbb{Z}}(G, S^1)$. Then*

$$\int_G \gamma(g) d\mu(g) = \begin{cases} \mu(G) & \textit{if } \gamma = 1 \textit{ and,} \\ 0 & \textit{if } \gamma \neq 1. \end{cases}$$

PROOF. If $g' \in G$, then applying the change of coordinate $g \mapsto g'g$ gives

$$\int_G \gamma(g) d\mu(g) = \int_G \gamma(g'g) d\mu(g'g) = \gamma(g') \int_G \gamma(g) d\mu(g).$$

The lemma follows easily from this equality. □

Recall the measure μ_T on the compact torus $T = V/Q^\vee$ (cf. Section 5.4). For a $\lambda \in 2\pi i P$ we define $e^\lambda \in \widehat{T}$ by

$$e^\lambda(v + Q^\vee) = e^{\lambda(v)}, \quad v \in V.$$

Then $\lambda \mapsto e^\lambda$ defines an isomorphism $2\pi i P \xrightarrow{\sim} \widehat{T}$ of groups. Applying Lemma 6.2.1 to T gives

$$\int_T e^\lambda(t) d\mu_E(t) = \#W_0 \mu_E(C_+) \delta_{\lambda,0} \qquad (\lambda \in 2\pi i P). \tag{6.2.1}$$

Let λ and λ' be in $2\pi i P^+$. The following follows easily from (6.2.1) (see also Example 3.2.8)

$$\int_T \phi_{\lambda,0}(t) \overline{\phi_{\lambda',0}(t)} d\mu_T = \delta_{\lambda,\lambda'} \mu_E(C_+) \#W_{0,\lambda},$$

with $W_{0,\lambda} = \{w \in W_0 |\, w\lambda = \lambda\}$ the isotropy subgroup of λ in W_0. Reformulated in terms of the Hilbert space $\mathcal{H}' = L^2(\overline{C_+}, dx)$ (see Section 5.5) gives the following.

LEMMA 6.2.2 (Free case $k \equiv 0$). *Let λ and λ' be in $2\pi i P^+$ with $\lambda \neq \lambda'$. Then the following holds.*

$$(\phi_{\lambda,0}, \phi_{\lambda',0})_{\mathcal{H}'} = 0 \qquad \text{(Orthogonality)}$$

$$(\phi_{\lambda,0},\phi_{\lambda,0})_{\mathcal{H}'}=\frac{\mu_E(C_+)\#W_{0,\lambda}}{\#W_0} \qquad \text{(Norms)}$$

Recall that we denoted the set of strictly positive multiplicity functions by $K_{>0}$ (see Definition 5.1.2).

For Σ_0 of type A_1 the inner products of the Bethe ansatz eigenfunctions in $\mathcal{H}'$ can be easily calculated.

PROPOSITION 6.2.3 (Rank one). *Let $k\in K_{>0}$, $\lambda,\lambda'\in\mathcal{S}_k^+$ and $\lambda'\neq\lambda$. Then the following holds.*

$$(\phi_{\lambda,k},\phi_{\lambda',k})_{\mathcal{H}'}=0 \qquad \text{(Orthogonality)}$$

$$(\phi_{\lambda,k},\phi_{\lambda,k})_{\mathcal{H}'}=\frac{\mu_E(C_+)\left|\widetilde{c}_k(\lambda)\right|^2}{\#W_0}\frac{d^2S_k(0,\xi)}{d\xi^2}(-i\lambda), \qquad \text{(Norms)}$$

PROOF. We only give the details for the norms, since orthogonality follows similarly.

We identify V^* with $V=\mathbb{R}$ and set $\Sigma_0=\{\alpha,-\alpha\}$, with $\alpha=\sqrt{2}$. Then $C_+=(0,\omega)$ with $\omega=\sqrt{2}/2$. Let $\lambda\in\mathcal{S}_k^+\subset i\mathbb{R}$. Note that $\lambda\neq 0$ because $k>0$ (By the Pauli principle, Proposition 4.5.1). Whence

$$\begin{aligned}4(\phi_{\lambda,k},\phi_{\lambda,k})_{\mathcal{H}'}&=\int_0^\omega\left(2\left|\widetilde{c}_k(\lambda)\right|^2+\widetilde{c}_k(\lambda)^2e^{2\lambda x}+\widetilde{c}_k(-\lambda)^2e^{-2\lambda x}\right)dx\\&=\omega\left(2\left|\widetilde{c}_k(\lambda)\right|^2+\widetilde{c}_k(\lambda)^2\frac{e^{2\lambda\omega}-1}{2\lambda}+\widetilde{c}_k(-\lambda)^2\frac{e^{-2\lambda\omega}-1}{-2\lambda}\right),\end{aligned}$$

with $\widetilde{c}_k(\lambda)=(\lambda\sqrt{2}+k)/(\lambda\sqrt{2})$. Furthermore, since $\lambda\in\mathcal{S}_k^+\subset\mathrm{BAE}_k$ (see (4.4.1)) we also have

$$e^{2\lambda\omega}=\left(\frac{\lambda\sqrt{2}-k}{\lambda\sqrt{2}+k}\right)^2=\frac{\widetilde{c}_k(-\lambda)^2}{\widetilde{c}_k(\lambda)^2}.$$

A small calculation gives

$$4(\phi_{\lambda,k},\phi_{\lambda,k})_{\mathcal{H}'}=2\omega\left|\widetilde{c}_k(\lambda)\right|^2\left(1-\frac{4k}{\left|\lambda\sqrt{2}+k\right|^2}\right).$$

The derivative of the master function $S_k(0,\xi)=\xi^2/2+2\int_0^{\sqrt{2}\xi}\arctan(t/k)dt$ is easily calculated,

$$\frac{d^2S_k(0,\xi)}{d\xi^2}(-i\lambda)=\frac{4k}{\left|\lambda\sqrt{2}+k\right|^2}=\frac{4k}{2(-i\lambda)^2+k^2}$$

Now use that $\mu_E(C_+)=\omega$ and $\#W_0=2$ to conclude the proof. □

Observe that the Hessian of $S_k(\mu,\cdot)$ is independent of μ (see (4.5.2) for an explicit expression of the Hessian).

After these simple examples we now state the main conjecture of the chapter.

CONJECTURE 6.2.4. *Let $k \in K_{>0}$. Let $\lambda, \lambda' \in \mathcal{S}_k^+$ and $\lambda' \neq \lambda$. Then the following holds.*

$$(\phi_{\lambda,k}, \phi_{\lambda',k})_{\mathcal{H}'} = 0 \qquad \text{(Orthogonality)}$$

$$(\phi_{\lambda,k}, \phi_{\lambda,k})_{\mathcal{H}'} = \frac{\mu_E(C_+)\,|\widetilde{c}_k(\lambda)|^2}{\#W_0} \det(\operatorname{Hess} S_k(0,\cdot)(-i\lambda)), \qquad \text{(Norms)}$$

REMARK 6.2.5. Note that when the Bethe ansatz eigenfunctions $\phi_{\lambda,k}$ are considered as functions in $C(T)^{W_0}$ (with $T = V/Q^\vee$ and dt is the unique normalized Haar measure (thus $dt = d\mu_T/(\#W_0\mu_E(C_+))$ (see Section 5.4), then the norm formula of Conjecture 6.2.4 is equivalent to

$$\int_T |\phi_{\lambda,k}|^2\, dt = \frac{|\widetilde{c}_k(\lambda)|^2 \det(\operatorname{Hess} S_k(0,\cdot)(-i\lambda))}{\#W_0}.$$

Recall the space $C^{1,(k)}(V)^W$ from Chapter 3 (see Definition 3.2.3) that contains the Bethe ansatz eigenfunctions $\phi_{\lambda,k}$ ($\lambda \in \mathcal{S}_k^+$).

LEMMA 6.2.6. *Let $k \in K_{>0}$.*
(i) *For $f, g \in C^{1,(k)}(V)^W$ we have*

$$(\Delta f, g)_{H'} = (f, \Delta g)_{H'}.$$

(ii) *(Partial orthogonality) For $\lambda, \mu \in \mathcal{S}_k^+$ and $\|\lambda\| \neq \|\mu\|$ we have*

$$(\phi_{\lambda,k}, \phi_{\mu,k})_{\mathcal{H}'} = 0.$$

PROOF. Note that **(ii)** follows immediately from **(i)** and $\Delta\phi_{\lambda,k}{}_{|C_+} = -\|\lambda\|^2\phi_{\lambda,k}{}_{|C_+}$. As for **(i)**, observe that by Stokes' theorem we have

$$(\Delta f, g)_{H'} - (f, \Delta g)_{H'} = \sum_{j=0}^{n} \frac{1}{\|Da_j^\vee\|} \int_{L_j} \left(f\overline{\partial_{Da_i^\vee} g} - (\partial_{Da_i^\vee} f)\overline{g} \right) d\sigma_E, \qquad (6.2.2)$$

with L_j ($j = 0, 1, \ldots, n$) denoting the walls of C_+ (see (5.5.1)) and $d\sigma_E$ denotes the measure on ∂C_+ induced from $d\mu_E$ on $\overline{C_+}$. Furthermore,

$$(\partial_{Da_i^\vee} f)(v) = k_j f(v), \quad v \in L_j,$$

and similarly for g, because of (3.2.3) and W-invariance of f, g. Therefore the right hand side of (6.2.2) vanishes. ☐

Note that Lemma 6.2.6**(ii)** is actually Theorem 5.1.4**(i)**. The proof above however is elementary compared the proof in Chapter 5.

REMARK 6.2.7. Proposition 6.2.3 tells us that Conjecture 6.2.4 holds when Σ_0 is of type A_1. Lemma 6.2.6 states that we have partial orthogonality for all root systems (for Σ_0 of type A_1 this is full orthogonality). In the next section we show that Conjecture 6.2.4 holds for Σ_0 of type A_{n-1} for all $n \geq 2$.

COROLLARY 6.2.8. *Assume that Conjecture 6.2.4 is true. For $\mu \in P^{++}$ we have*

$$\lim_{k\searrow 0} \det(\operatorname{Hess} S_k(0,\cdot)(\widehat{\mu}_k)) = \# W_{0,\mu-\rho}. \tag{6.2.3}$$

In particular

$$\lim_{k\searrow 0} \det(\operatorname{Hess} S_k(0,\cdot)(\widehat{\rho}_k)) = \# W_0. \tag{6.2.4}$$

PROOF. This follows from Conjecture 6.2.4, Proposition 5.2.15 and Lemma 5.6.10. □

REMARK 6.2.9. **(i)** The formula (6.2.4) shows similarity to the limit formula [**41**, (3.5.14)] for the root system version of the Jacobi polynomials introduced by Heckman and Opdam.
(ii) Compare (6.2.3) also with the fact that for a fixed $\xi \in V^*$ we have

$$\lim_{k\searrow 0} \operatorname{Hess} S_k(0,\cdot)(\xi) = 1.$$

Thus (6.2.3) shows the non-trivial nature of the limit $k \searrow 0$ for the quantum system with delta-potentials we are considering.

6.3. The quantum Bose-gas on the circle with delta-potential revisited

In this section we show that Conjecture 6.2.4 holds for Σ_0 of type A_{n-1} (see Theorem 6.3.14). This is done by showing that the quantum integrable system corresponding to this case (see Example 1.5.1 and the paragraph following Example 3.2.12) is essentially the same as the system of n quantum bosons on the circle with pair-wise delta-function potential and using the results of Dorlas [**16**] (orthogonality) and Korepin [**54**] (norms).

To show this relation we introduce some notations related to the root systems Σ_0 of type A (we advise the reader to recall Section 2.2). Consider the Euclidean vector space $\mathbb{R}^n$ with the usual inner product $\langle\cdot,\cdot\rangle$ and denote by $e_1,\ldots,e_n$ the standard orthonormal basis of $\mathbb{R}^n$. We denote by $\mathbb{R}^{n*} = (\mathbb{R}^n)^*$ the dual vector space of $\mathbb{R}^n$. The dual basis $\varepsilon_1,\ldots,\varepsilon_n$ of $e_1,\ldots,e_n$, defined by $\varepsilon_i(e_j) = \delta_{ij}$, is a basis for $\mathbb{R}^{n*}$. We set $e = e_1 + \cdots + e_n$ and $\varepsilon = \varepsilon_1 + \cdots + \varepsilon_n$. Denote by V the orthogonal complement of $\mathbb{R}e$ in $\mathbb{R}^n$, i.e.

$$V = \{x = \sum_{i=1}^n x_i e_i \in \mathbb{R}^n |\, \langle x, e\rangle = x_1 + \cdots + x_n = 0\},$$

considered as Euclidean vector spaces with the restricted inner product $\langle,\rangle$. The dual space of V can be identified in a natural way with the following subset of $\mathbb{R}^{n*}$,

$$V^* = \{\lambda = \sum_{i=}^n \lambda_i\varepsilon_i \in \mathbb{R}^{n*} | \lambda(e) = \langle\lambda,\varepsilon\rangle = \lambda_1 + \cdots + \lambda_n = 0\} \tag{6.3.1}$$

With this identification we have $\mathbb{R}^{n*} = V^* \oplus \mathbb{R}\varepsilon$. Let $\Sigma_0 = \{\varepsilon_i - \varepsilon_j | n \geq i \neq j \geq 1\} \subset V^*$, and $I_0 = \{a_1,\ldots,a_{n-1}\}$ with $a_i = \varepsilon_{i+1} - \varepsilon_i$. Then $\Sigma_0 \subset V^*$ is a finite integral irreducible root system of type A_{n-1} with a basis of simple roots I_0. We define the usual lattices Q and P in V^* and $Q^\vee$ and $P^\vee$ in V. Let also $\Sigma = \Sigma_0 + \mathbb{Z} \subset \widehat{V}$,

$I = \{a_0, a_1, \ldots, a_{n-1}\}$ with $a_0 = -(\varepsilon_n - \varepsilon_1) + 1 \in \widehat{V}$. Then Σ is an affine irreducible root system of type $\widehat{\mathrm{A}}_{n-1}$ with basis of simple affine roots I. The fundamental chamber corresponding to this choice of Σ and I is given by

$$C_+ = \{v = v_1e_1 + \ldots v_ne_n \in V \,|\, v_1 < v_2 < \cdots < v_n < v_1 + 1\}.$$

The standard action of the Weyl group $W = S_n \ltimes Q^\vee$ of Σ on V extends in a natural way to an action on $\mathbb{R}^n$ (it acts trivial on the subspace $\mathbb{R}e$). The finite Weyl group S_n also acts on $\mathbb{R}^{n*}$ via the adjoint of the action of S_n on $\mathbb{R}^n$.

An element $\lambda = \lambda_1\varepsilon_1 + \ldots \lambda_n\varepsilon_n \in \mathbb{C}^{n*} = (\mathbb{C}^n)^* = \mathbb{R}^{n*} \otimes \mathbb{C}$ is called *regular* if $\lambda_i \neq \lambda_j$ for all $n \geq i \neq j \geq 1$. The regular elements in $\mathbb{R}^{n*} \subset \mathbb{C}^{n*}$ are denoted by $\mathbb{R}^{n*}_{reg}$. A fundamental domain for the action of S_n on $\mathbb{R}^{n*}$ is

$$\mathbb{R}^{n*}_+ = \{\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n \in R^{n*} |\, \lambda_1 < \lambda_2 < \cdots < \lambda_n\}.$$

We denote by X the standard full lattice $\mathbb{Z}^n$ in $\mathbb{R}$ with $\mathbb{Z}$-basis $e_1, \ldots, e_n$. Denote by Y the (full) lattice dual to X in $\mathbb{R}^{n*}$. Then $\varepsilon_1, \ldots, \varepsilon_n$ is a $\mathbb{Z}$-basis of Y. Since X is invariant under the action of S_n we can form the semi-direct product $G = S_n \ltimes X$, with X acting by translations on $\mathbb{R}^n$. The following subset of $\mathbb{R}^n$,

$$D = \{x = \sum_i x_ie_i \in \mathbb{R}^n | 1 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 0\}.$$

is a fundamental domain for the action of G on $\mathbb{R}^n$.

Since all the hyperplanes V_a $(a \in \Sigma_0)$ for Σ_0 of type A are conjugate, multiplicity function $k \in K_{>0}$ must be a constant function. We fix such a multiplicity function for the rest of the chapter and identify it with its constant value.

The system of n quantum bosons on the circle with pair-wise repulsive δ-function interactions is described by the following formal Hamiltonian on $[0, 1]^n$ (cf. (1.1.1)),

$$H^n_k = -\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} + k \sum_{n \geq i \neq j \geq 1} \delta(x_i - x_j). \tag{6.3.2}$$

Recall the concept of a sub-regular point of V from see Subsection 2.2.1. For an $a \in \Sigma$, let $\mathbb{R}^n_a = \{x \in \mathbb{R}^n : a(x) = 0\} = V_a + \mathbb{R}e$, with $V_a = \{v \in V |\, a(v) = 0\}$. We call a $x \in \mathbb{R}^n_a$ $(a \in \Sigma)$ *sub-regular* if it does not lie on any other hyperplane $\mathbb{R}^n_b$. Alternatively, an $x \in \mathbb{R}^n_a$ $(a \in \Sigma)$ is regular if and only if the orthogonal projection of x in V_a is sub-regular in V.

Let f be a continuous S_n-invariant function f on $[0, 1]^n$ satisfying periodic boundary conditions. Because of the S_n-invariance, the periodic boundary conditions are equivalent with the single condition

$$f(0, x_1, \ldots, x_{n-1}) = f(x_1, \ldots, x_{n-1}, 1).$$

In the physics literature (cf. Lieb and Liniger [**56**]) such a function f is considered an eigenvector with eigenvalue E of the formal Hamiltonian H^n_k if:

(a) f satisfies the following boundary jump condition. For $x = (x_1, \ldots, x_n) \in D$,

$$\partial_{x_1} f(0, x_1, \ldots, x_{n-1}) = \partial_{x_n} f(x_1, \ldots, x_{n-1}, 1).$$

(b) f is an eigenvalue of $-\Delta$ on the interior of D with eigenvalue E.
(c) f satisfy the following boundary value condition,

$$\big(\partial_{e_{i+1}-e_i} f\big)(x + 0(e_{i+1} - e_i)) - \big(\partial_{e_{i+1}-e_i} f\big)(x - 0(e_{i+1} - e_i)) = 2kf(x) \quad (6.3.3)$$

for a subregular $x \in D \cap \mathbb{R}^n_{\varepsilon_{i+1}-\varepsilon_i}$, $i = 1, 2, \ldots, n-1$ (cf. this with Proposition 3.2.5).

REMARK 6.3.1. If $f : [0,1]^n \to \mathbb{C}$ is a S_n-invariant eigenfunction of H^n_k, we denote the unique continuous G-invariant extension of f to $\mathbb{R}^n$ also by f. It satisfies jump conditions analogous to (6.3.3) on all sub-regular points of $\mathbb{R}^n$. This is because G maps maps the union of the $\{x = (x_1, \ldots, x_n) \in D |\, x_i = x_{i+1}\}$, $i = 1, 2, \ldots, n-1$, *onto* the union of the affine hyperplanes $\mathbb{R}^n_a$ ($a \in \Sigma$). Furthermore, away from these affine hyperplanes $\mathbb{R}^n_a$ ($a \in \Sigma$) the function f is differentiable because of (a). It is these facts that allows us to relate (see e.g. Lemma 6.3.12) the system of n quantum bosons with delta-potentials on the circle to the quantum systems defined in the previous chapters for the root system Σ (see also Example 1.5.1 and the paragraph following Example 3.2.12).

Let $\lambda \in i\mathbb{R}^{n*}_{reg}$. A G-invariant eigenfunction of H^n_k on $\mathbb{R}^n$ that has a plane wave decomposition

$$\sum_{w \in S_n} a_w e^{w\lambda(x)}, \quad x \in gD,$$

on every chamber gD ($g \in G$) for certain constants $a_w \in \mathbb{C}$ (depending on $g \in G$) is called a *Bethe ansatz eigenfunction with spectral parameter λ (more precisely, $S_n\lambda$).*

We extend the c-function (3.7.7) on $V^*_{\mathbb{C}}$ to a function on $\mathbb{C}^{n*}$,

$$\tilde{c}_k(\mu) = \prod_{a \in \Sigma_0^+} \frac{\mu(\alpha^\vee) + k}{\mu(\alpha^\vee)} = \prod_{n \geq i > j \geq 1} \frac{\mu_i - \mu_j + k}{\mu_i - \mu_j}, \quad (6.3.4)$$

which we consider as a rational function of $\mu = \mu_1\varepsilon_1 + \cdots + \mu_n\varepsilon_n \in \mathbb{C}^{n*}$.

THEOREM 6.3.2 (Lieb-Liniger [**56**]). *Let $\lambda \in i\mathbb{R}^{n*}_+$. Then there is a Bethe ansatz eigenfunction with spectral parameter λ if and only if λ is a solution to the* Bethe ansatz equations

$$e^{\lambda_j} = \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{\lambda_j - \lambda_k - k}{\lambda_j - \lambda_k + k} \qquad (j = 1, 2, \ldots, n). \quad (6.3.5)$$

Let $\lambda \in i\mathbb{R}^{n*}_+$ be a solution to (6.3.5). *Every Bethe ansatz eigenfunction with spectral parameter λ is a scalar multiple of the following function*[1] $\phi^{LL}_{\lambda,k}$, *uniquely defined by*

$$\phi^{LL}_{\lambda,k}(x) = \frac{1}{n!} \sum_{w \in S_n} \widetilde{c}(w\lambda) e^{w\lambda(x)}, \quad x \in D. \quad (6.3.6)$$

[1]The LL in $\phi^{LL}_{\lambda,k}$ stands for Lieb and Liniger. This is, however, not a standard notation

The set of all solutions $\lambda \in i\mathbb{R}^{n*}$ to the Bethe ansatz equations (6.3.5) is denoted by cBAE_k (the prefix "c", standing for classical, is added to distinguish it with the set BAE_k of solutions $\lambda \in iV^*$ of the Bethe ansatz equations (4.4.1) corresponding to Σ). We shall now discuss the relation of cBAE_k to BAE_k.

DEFINITION 6.3.3. *The* Yang-Yang action *(also called the master function)* B_k : $(P \oplus \mathbb{R}\varepsilon) \times \mathbb{R}^{n*} \to \mathbb{R}$ *is defined by*

$$B_k(\mu, \xi) = \frac{1}{2}\langle \xi, \xi\rangle - 2\pi\langle \mu, \xi\rangle + \sum_{\alpha\in\Sigma_0} \int_0^{\xi(\alpha^\vee)} \arctan(t/k)dt. \tag{6.3.7}$$

Let $pr_{V^*_{\mathbb{C}}}$ be orthogonal projection of $\mathbb{C}^{n*}$ onto $V^*_{\mathbb{C}}$, defined by

$$pr_{V^*_{\mathbb{C}}}(\lambda) = \lambda - \lambda(e)\varepsilon/n \in V^*_{\mathbb{C}}, \quad \lambda \in \mathbb{C}^{n*}.$$

Recall that the master function (4.5.1) $S_k(\cdot,\cdot)$ from Chapter 4. For a $\mu \in P$ it attains a unique global minimum at a $\widehat{\mu}_k$. The following theorem follows similarly as the results in Section 4.8 (see in particular the proof of Proposition 4.8.1).

THEOREM 6.3.4. *Let $\mu \in P \oplus \mathbb{R}\varepsilon$. The Yang-Yang action $B_k(\mu,\cdot)$ is strictly convex and attains a unique global minimum at a $\widehat{\mu}^c_k$. The map $\mu \mapsto i\widehat{\mu}^c_k$ defines a S_n-equivariant bijection $P \oplus \mathbb{R}\varepsilon \xrightarrow{\sim} \mathrm{BAE}_k \oplus i\mathbb{R}\varepsilon$. Furthermore, $\widehat{\mu}^c_k$ is uniquely determined by*

$$\widehat{\mu}^c_k + \sum_{\alpha\in\Sigma_0} \arctan\left(\frac{\widehat{\mu}^c_k(\alpha^\vee)}{k}\right)\alpha = 2\pi\mu. \tag{6.3.8}$$

Moreover, μ is regular iff $\widehat{\mu}^c_k$ is regular. Also $\widehat{\mu}^c_k = \widehat{\tilde{\mu}}_k + 2\pi\mu(e)\varepsilon/n$, with $\tilde{\mu} = pr_{V^*_{\mathbb{C}}}(\mu)$.

Note that a $\lambda \in i\mathbb{R}^{n*}$ is in $\mathrm{BAE}_k \oplus i\mathbb{R}\varepsilon$ if and only if λ is a solution of

$$e^{\lambda(q)} = \prod_{\alpha\in\Sigma_0^+} \left(\frac{\lambda(\alpha^\vee) - k}{\lambda(\alpha^\vee) + k}\right)^{\alpha(q)} \qquad \forall q \in Q^\vee.$$

The system of n equations (6.3.5) Bethe ansatz equations can be reformulated in the following "invariant" form.

LEMMA 6.3.5. *The equations* (6.3.5) *are equivalent with the following set of equations in $\lambda \in i\mathbb{R}^{n*}$:*

$$e^{\lambda(m)} = \prod_{\alpha\in\Sigma_0^+} \left(\frac{\lambda(\alpha^\vee) - k}{\lambda(\alpha^\vee) + k}\right)^{\alpha(m)} \qquad \forall m \in X. \tag{6.3.9}$$

PROOF. First note that (6.3.5) is (6.3.9) with $m = e_1, \ldots, e_n$. If (6.3.9) is true for a m and m', then it easily seen that is also true for $-m$ and $m+m'$. Now use that $e_1, \ldots, e_n$ is a $\mathbb{Z}$-basis for X to conclude that (6.3.5) implies (6.3.9). □

Recall that

$$2\rho = \sum_{n\geq i>j\geq 1} \varepsilon_i - \varepsilon_j = (n-1)\varepsilon_n + (n-3)\varepsilon_{n-1} + \cdots - (n-3)\varepsilon_2 - (n-1)\varepsilon_1,$$

that $\rho \in P^+$ and that $Y \subset \mathbb{R}^{n*}$ by definition the lattice dual to $X = \mathbb{Z}^n$ is. Although the following lemma is trivial, we state it for later reference.

LEMMA 6.3.6. **(i)** *The projection $pr_{V_{\mathbb{C}}^*} : \mathbb{C}^{n*} \to V_{\mathbb{C}}^*$ maps Y onto P and has kernel $n\mathbb{Z}e + in\mathbb{Z}e$.*
(ii) *$P \oplus \mathbb{R}\varepsilon = Y + \mathbb{R}\varepsilon$.*
(iii) *$Y + \rho = Y$ for n odd and $Y + \rho = Y + \mathbb{Z}\varepsilon/2$ for n even.*

Using Proposition 4.8.1, Lemma 6.3.5,Theorem 6.3.4 and Lemma 6.3.6 gives the following.

THEOREM 6.3.7 (Yang-Yang [**81**]). *The restriction of the map $\mu \mapsto i\widehat{\mu}_k^c$ defines a S_n-equivariant bijection $\rho + Y \xrightarrow{\sim} \mathrm{cBAE}_k$. Furthermore, $\widehat{\mu}_k^c$ is uniquely determined by*

$$\widehat{\mu}_k^c + \sum_{\alpha \in \Sigma_0} \arctan\left(\frac{\widehat{\mu}_k^c(\alpha^\vee)}{k}\right)\alpha = 2\pi\mu. \tag{6.3.10}$$

Moreover, $\mu \in \rho + Y$ is regular if and only if $\widehat{\mu}_k^c$ is regular. Also $\widehat{\mu}_k^c = \widehat{\tilde{\mu}}_k + 2\pi\mu(e)\varepsilon/n$, with $\tilde{\mu} = pr_{V_{\mathbb{C}}^*}(\mu)$.

Recall the spectrum $\mathcal{S}_k$ (see for example Theorem 4.4.1 and the paragraph following Definition 5.1.3) and

$$\mathcal{S}_k^+ = \mathcal{S}_k \cap iV_+^* = \{\lambda \in iV_+^* |\, \phi_{\lambda,k} \text{ is } W\text{-invariant}\}.$$

COROLLARY 6.3.8. *The assignment $\lambda \mapsto pr_{V_{\mathbb{C}}^*}(\lambda)$ maps $\mathrm{cBAE}_k \cap i\mathbb{R}_+^{n*}$ onto $\mathrm{BAE}_k \cap iV_+^* = \mathcal{S}_k^+$ (cf. Theorem 4.4.1).*

We now consider inner product of the Bethe ansatz eigenfunctions $\phi_{\lambda,k}^{LL}$. Denote by dx the usual Lebesgue measure on $\mathbb{R}^n$. Then the measure $d\mu_E$ on V is precisely the restriction of dx to V. The relevant Hilbert space is the space of square-integrable functions on D with respect to Lebesgue measure, i.e.

$$\mathcal{H}_c = L^2(D, dx)$$

THEOREM 6.3.9 (Dorlas [**16**]). *The functions $\phi_{\lambda,k}^{LL}$ ($\lambda = i\widehat{\mu}_k^c$, $\mu \in (Y + \rho) \cap \mathbb{R}_+^{n*}$) are pair-wise orthogonal in $\mathcal{H}_c$.*

THEOREM 6.3.10 (Korepin [**54**]). *The quadratic norm of a Bethe ansatz eigenfunctions $\phi_{\lambda,k}^{LL}$ ($\lambda \in \mathrm{cBAE}_k \cap i\mathbb{R}_+^{n*}$) is given by*

$$(\phi_{\lambda,k}^{LL}, \phi_{\lambda,k}^{LL})_{\mathcal{H}_c} = \frac{|\widetilde{c}_k(\lambda)|^2 \det(\mathrm{Hess}\, B_k(0,\cdot)(-i\lambda))}{(n!)^2}. \tag{6.3.11}$$

REMARK 6.3.11. Theorem 6.3.10 was conjectured by Gaudin [**26**] and proved by Korepin [**54**] using quantum inverse scattering method. Theorem 6.3.9 was proved by Dorlas using a lattice version of the quantum inverse scattering method.

Recall the Bethe ansatz functions $\phi_{\tilde{\lambda},k} \in \mathrm{BVP}_k(\tilde{\lambda})$ (with $\lambda \in \mathcal{S}_k^+$) corresponding to the root system $\Sigma \subset \widehat{V}$ (see Theorem 4.4.1) . For $\tilde{\lambda} \in \mathcal{S}_k^+$, the W-invariant function $\phi_{\tilde{\lambda},k}$ is uniquely determined by

$$\phi_{\tilde{\lambda},k} = \frac{1}{n!} \sum_{w \in S_n} \widetilde{c}(w\tilde{\lambda}) e^{w\lambda(v)}, \quad v \in C_+. \tag{6.3.12}$$

LEMMA 6.3.12. *Let $\lambda \in \mathrm{cBAE}_k \cap i\mathbb{R}^{n*}_{reg}$. Consider the decomposition $\lambda = \tilde{\lambda} + ic\varepsilon$ with respect to the orthogonal decomposition $i\mathbb{R}^{n*} = iV^* + i\mathbb{R}\varepsilon$. Then $\tilde{\lambda} = pr_{V^*_{\mathbb{C}}}(\lambda) \in \mathcal{S}_k^+$. Furthermore,*

$$\phi^{LL}_{\lambda,k}(v + te) = e^{itnc}\phi_{\tilde{\lambda},k}(v) \quad (v \in V, t \in \mathbb{R}). \tag{6.3.13}$$

In particular, $c \in 2\pi\mathbb{Z}/n$.

PROOF. Since $\phi^{LL}_{\lambda,k} \in C(\mathbb{R}^n)^G$ we have $g = \phi^{LL}_{\lambda,k|V} \in C(V)^W$. Because of the plane wave form of $\phi^{LL}_{\lambda,k}$ on all chambers gD ($g \in G$) and Remark 6.3.1, it follows that $g \in \mathrm{BVP}_k(\tilde{\lambda})$. Because of $g(0) = \phi^{LL}_{\lambda,k}(0) = 1$, Theorem 4.4.1 gives $g = \phi_{\tilde{\lambda},k}$. Now (6.3.13) follows from the G-invariance of $\phi^{LL}_{\lambda,k}$, the W-invariance of g, (6.3.6) and (6.3.12). □

LEMMA 6.3.13. *Let $f_j \in C(V)^G$ ($j = 1, 2$) such that $f_j(te + v) = e^{tnc_j} f_j(v)$ for a $c_j \in 2\pi i\mathbb{Z}/n$ and all $t \in \mathbb{R}$, $v \in V$. Put $g_j = f_{j|V}$. Then $g_j \in C(V)^W$ and*

$$(f_1, f_2)_{\mathcal{H}_c} = \delta_{c_1,c_2} \frac{(g_1, g_2)_{\mathcal{H}'}}{\sqrt{n}}.$$

PROOF. We consider the following auxiliary Hilbert space $\mathcal{H}^0_c := L^2(\overline{C_+} + [0, 1]e, dx)$ and full lattice $X_0 = Q^\vee + \mathbb{Z}e \subseteq X$ in $\mathbb{R}^n$. Consider the following S_n-stable fundamental domain for the action (by translations) of $Q^\vee$ on V,

$$L_0 = \bigcup_{w \in S_n} w\overline{C_+}.$$

Then $L_0 + [0, 1]e$ is a S_n-stable fundamental domain for the action (by translations) of X_0 on V. Using the G-invariance of f_1, f_2 and the W-invariance of g_1, g_2 gives

$$\begin{aligned}
(f_1, f_2)_{\mathcal{H}^0_c} &= \frac{1}{n!} \int_{[0,1]e + L_0} f_1(x)\overline{f_2(x)} dx \\
&= \frac{1}{n!} \int_{t=0}^{\sqrt{n}} \int_{v \in L_0} f_1(te/\sqrt{n} + v)\overline{f_2(te/\sqrt{n} + v)} dt dv \\
&= \frac{1}{n!} \int_{t=0}^{\sqrt{n}} \int_{v \in L_0} e^{t\sqrt{n}(c_1 - c_2)} f_1(v)\overline{f_2(v)} dt dv \\
&= \delta_{c_1,c_2} \sqrt{n} (g_1, g_2)_{\mathcal{H}'}.
\end{aligned}$$

Whence to conclude the lemma, it suffices to show that the following holds,

$$(f_1, f_2)_{\mathcal{H}_c} = \frac{(f_1, f_2)_{\mathcal{H}_c^0}}{n}. \tag{6.3.14}$$

Note that since $X_0 \subset X$, this will follow from $\#X/X_0 = n$. We actually claim that X/X_0 is isomomorphic to $\mathbb{Z}/n\mathbb{Z}$ as abelian groups. For this it suffices to show that $0, e_1, 2e_1, \ldots, (n-1)e_1$ are a complete set of representatives for the quotient X/X_0. We start with the observation that $ce_1 \in X_0$ if and only if $c \in n\mathbb{Z}$. This is so because $ce_1 = (ce_1 - c/ne) + (c/n)e$ in the orthogonal decomposition $V \oplus \mathbb{R}e$ and

$$ne_1 - e = (e_1 - e_1) + (e_1 - e_2) + (e_1 - e_3) + \ldots (e_1 - e_n) \in Q^\vee.$$

Whence the cosets $X_0, e_1 + X_0 \ldots, (n-1)e_1 + X_0$ are pairwise different cosets. For the claim, and whence the lemma, it remains to be shown that these are the only cosets. This follows from the following relation for $m = m_1e_1 + \cdots + m_ne_n \in X$,

$$m + X_0 = \sum_i (m_i(e_i - e_1) + m_ie_1) + X_0 = (\sum_i m_i)e_1 + X_0.$$

□

THEOREM 6.3.14. *Theorem 6.3.9 and Theorem 6.3.10 are equivalent with Conjecture 6.2.4 for Σ_0 of type A_{n-1}.*

PROOF. If $\lambda, \mu \in \mathrm{cBAE}_k \cap i\mathbb{R}_+^{n*}$, then by Lemma 6.3.12 and Lemma 6.3.13 follows immediately that

$$(\phi_{\lambda,k}^{LL}, \phi_{\mu,k}^{LL})_{\mathcal{H}_c} = \delta_{c_1,c_2} \frac{(\phi_{\tilde{\lambda},k}, \phi_{\tilde{\mu},k})_{\mathcal{H}'}}{\sqrt{n}}, \tag{6.3.15}$$

where $\lambda = \tilde{\lambda} + ic_1e$, $\mu = \tilde{\mu} + ic_2e$ are orthogonal decomposition with respect to $i\mathbb{R}^{n*} = iV^* \oplus i\mathbb{R}\varepsilon$. Note that $\tilde{\lambda}, \tilde{\mu} \in \mathcal{S}_k^+$. To compare the norms in the different Hilbert spaces we relate the Yang-Yang action B_k to the master function S_k. Let $\eta_1, \ldots, \eta_{n-1}$ be an orthonormal basis of V^*. Then $\eta_1, \ldots, \eta_{n-1}, \varepsilon/\sqrt{n}$ is an orthonormal basis of $\mathbb{R}^{n*}$. Let $\xi \in \mathbb{R}^{n*}$ and put $\tilde{\xi} = pr_{V_{\mathbb{C}}^*}(\xi)$. With respect to these ordered bases' the Hessian's of B_k and S_k are related as follows,

$$\mathrm{Hess}\, B_k(0,\cdot)(\xi) = \begin{pmatrix} \mathrm{Hess}\, S_k(0,\cdot)(\tilde{\xi}) & 0 \\ 0 & 1 \end{pmatrix},$$

with the latter matrix a block diagonal matrix. Therefore

$$\det \mathrm{Hess}\, B_k(0,\cdot)(\xi) = \det \mathrm{Hess}\, S_k(0,\cdot)(\tilde{\xi}).$$

The choice of $\Sigma \subset \widehat{V}$ taken in this section gives $\mu_E(C_+) = \sqrt{n}/n!$, which follows by applying Lemma 6.3.13 to $f_1 = f_2 = 1$ and observing that the volume of D equals $1/n!$. The theorem now follows from Theorem 6.3.4, Theorem 6.3.7 and Corollary 6.3.8. □

COROLLARY 6.3.15. *Conjecture 6.2.4 holds for Σ_0 of type* **A**.

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Samenvatting

Dit proefschrift is het beslag van onderzoek aan wiskundige generalisaties van het quantummechanische systeem van een vast aantal deeltjes zonder spin dat paarsgewijs via een deltafunctiepotentiaal (dat we afkorten als *deltapotentiaal*) wisselwerkt. Informeel gezegd betekent de deltapotentiaal: er is alleen een wisselwerking bij contact tussen de deeltjes.

Zoals de titel van dit proefschrift al aangeeft heeft het onderzoek zich gericht op *integreerbare systemen*. Een integreerbare systeem is ten eerste een dynamische systeem. Met dynamische systemen kan men allerlei belangrijke en nuttige (maar ook nare) zaken die in tijd veranderen beschrijven. In ieder geval houden we ons in dit proefschrift niet bezig met (praktische) toepassingen. De toestand van een geïsoleerd systeem van N deeltjes dat evolueert met de tijd kan vaak gekarakteriseerd worden door de waarden van bepaalde (in tijd) behouden grootheden. Voorbeelden hiervan zijn de totale energie of het totale impulsmoment van systeem. Het is uitzonderlijk als een systeem meer onafhankelijke behouden grootheden heeft dan deze. Dit heeft te maken het feit dat de meeste systemen chaotische gedrag vertonen. Hier tegenover staat de categorie van zogenaamde (volledig) integreerbare systemen die N onafhankelijke commuterende behouden grootheden bezitten.

Het blijkt dat de quantummechanische systeem op een ring van wisselwerkende deeltjes via een deltapotentiaal generalisaties toelaat in de categorie van quantumsystemen in de context van zogenaamde *affiene Weylgroepen*. Een van de resultaten uit dit proefschrift is dat deze generalisaties integreerbare systemen zijn. Behalve affiene Weylgroepen zijn er ook eindige Weylgroepen met eindig veel elementen(in tegenstelling tot de affiene versies). Affiene en eindige Weylgroepen vormen een bijzondere deelverzameling van de verzameling van groepen voortgebracht door spiegelingen in (affiene) hypervlakken van een Euclidische vectorruimte. De permutatiegroep op N symbolen is wellicht het bekendste voorbeeld van een eindige Weylgroep. De dynamica van het quantumsysteem wordt volledig bepaald door de quantum Hamiltoniaan. In ons geval bestaat de potentiaalterm uit een gewogen som (met gewicht bepaald door koppelingsconstanten) van deltafuncties over de affiene hypervlakken geassocieerd met de affiene Weylgroep. Men kan ook integreerbare systemen met deltapotentialen beschouwen die corresponderen met eindige Weylgroepen. Het quantumsysteem geassocieerd met de permutatiegroep op N symbolen beschrijft bijvoorbeeld N deeltjes zonder spin op een lijn die paarsgewijs via een deltapotentiaal wisselwerken. In dit proefschrift bestuderen we deze quantumsystemen voor alle affiene Weylgroepen tegelijkertijd. Hoewel niet alle zulke quantumsystemen wisselwerkende deeltjes beschrijven, is het vanuit wiskundige oogpunt vanzelfsprekend om ze voor alle affiene Weylgroepen te bestuderen.

In het geval van quantumsystemen corresponderend met eindige Weylgroepen is veel bekend. De spectraaldecompositie (i.e. de Plancherelformule) voor de symmetrische golffuncties (dit zijn golffuncties invariant onder de Weylgroep) is bijvoorbeeld helemaal expliciet gemaakt door Eric Opdam en Gert Heckman. Een fundamenteel inzicht van hen was ook dat de symmetrie van het quantumsysteem gerelateerd is aan representaties van de gedegenereerde affiene Hecke algebra via reflectie-integraal operatoren. Een belangrijk doel van dit proefschrift was om deze resultaten uit te breiden tot de quantumsystemen geassocieerd met affiene Weylgroepen.

We laten zien dat voor quantumsystemen geassocieerd met een affiene Weylgroep de relevante algebraïsche structuur Cherednik's (geschikt gefilterde) gedegeneerde dubbele affiene Hecke algebra is. Na een inleidend hoofdstuk bestuderen we in het tweede hoofdstuk enige representaties van deze algebra. Het belangrijkste resultaat uit dit hoofdstuk is de constructie van representaties in termen van vectorwaardige reflectie-integraal operatoren en differentiaal operatoren. Met vector-waardig bedoelen we dat deze operatoren werken op golffuncties die waarden aannemen in een vectorruimte die tevens een representatie is van de affiene Weylgroep. Deze representaties zijn fundamenteel voor de hele theorie.

Het derde hoofdstuk is de kern van het proefschrift. Hierin worden vectorwaardige quantumsystemen met deltapotentialen bestudeerd. Voor affiene Weylgroepen van klassieke types en bepaalde representaties beschrijven dit wisselwerkende quantumsystemen van deeltjes met interne structuur, i.e. spin. We laten zien dat er een tweede representatie is van Cherednik's (geschikt gefilterde) gedegeneerde dubbele affiene Hecke algebra in termen van vectorwaardige commuterende reflectie-differentiaal operatoren. Ook laten we zien dat deze quantumsystemen natuurlijk passen in de klasse van quantum Calogero-Moser integreerbare systemen door voldoende commuterende onafhankelijk behouden grootheden, ook wel de hogere Hamiltonianen genoemd, te construeren. Dan laten we zien dat de spectrum van het quantumsysteem volledig bepaald wordt door een stelsel van vectorwaardige transcendente vergelijkingen en die we de naam Bethe ansatz vergelijkingen geven. Voor positieve koppelingsconstanten (voor quantumsystemen die wisselwerkende deeltjes beschrijven komt dit neer op een afstotende wisselwerking) laten we zien dat de spectrum van het quantumsysteem volledig imaginair is. Dit impliceert onder andere dat de energiespectrum van het quantumsysteem reëel is.

In de rest van het proefschrift beschouwen we alleen het scalaire geval, i.e. we beschouwen golffuncties met waarden in $\mathbb{C}$: de triviale representaties van de affiene Weylgroep.

In het vierde hoofdstuk beschouwen we alleen strikt positieve koppelingconstanten. In dit geval zijn we in staat alle oplossingen van de Bethe ansatz vergelijkingen uit te drukken als de kritieke punten van een zogenaamde meesterfunctie (*master function* in het Engels). Deze meesterfunctie blijkt een belangrijke rol te spelen in de hele theorie.

Door bijvoorbeeld gebruik te maken van de convexiteit van de meesterfunctie tonen we het bestaan aan van een affiene Weylgroep versie van Pauli's uitsluitingsprincipe.

In het vijfde hoofdstuk behandelen we functionaalanalytische aspecten. De hoofdresultaat luidt dat de simultane eigenfuncties van alle hogere Hamiltoniaan, die we de naam Bethe ansatz eigenfuncties geven, een dichte deelruimte opspannen in de Hilbertruimte van symmetrische functies. Dit bewijzen we met een continuïteitsargument in de koppelingsconstanten en door gebruik te maken van methoden uit de theorie van perturbaties van onbegrensde, zelfgeadjungeerde operatoren en kwadratische vormen op Hilbertruimtes.

In het laatste hoofdstuk formuleren we het vermoeden dat de Bethe ansatz eigenfuncties orthogonaal zijn en dat de normen hiervan uitgedrukt kunnen worden als de determinant van de Hessiaan van de meesterfunctie.

Dankwoord

Allereerst wil ik bij deze gelegenheid mijn promotor Eric Opdam bedanken voor zijn begeleiding gedurende het onderzoek van de afgelopen jaren. Zonder Eric zou ik geen promotieonderzoek hebben gedaan aan de intrigerende integreerbare systemen met deltapotentialen. Mijn copromotor Jasper Stokman wil ik bedanken voor zijn actieve begeleiding, zeker toen het echt nodig was. Samen hebben Eric en Jasper met hun geduld en vele suggesties bijgedragen aan de totstandkoming van dit proefschrift.

Mijn dank gaat ook naar de leden van de leescommissie voor het nauwkeurig doorlezen van het manuscript.

Mijn collega's van het Korteweg-de Vries Instituut voor Wiskunde wil ik bedanken voor hun belangstelling en prettige werksfeer. Mijn kamergenoten Elena, Nabila, Nienke, Rogier, Fokko, Ali, Philippe, Khanh, Ruud en Luca worden in het bijzonder bedankt voor de prettige kamersfeer. René wordt bedankt voor de gezelligheid en zijn praktische hulp bij veel zaken (ook buiten Euclid). Sjoerd ben ik dankbaar voor zijn hulp in de bibliotheek. Margit Rösler: It was a pleasure to chat with you about mathematics and other things. Catarina, Klaske, Henk, Niels, Michael, Uri en Klaas: bedankt voor de (soms te) gezellige afleiding op het instituut en daar buiten. Piet Molenaar: bedankt dat je (na een valse start) je enthousiasme voor wiskunde met mij wilde delen. Mijn broer Nevzat en Olav ben ik (onder andere) dankbaar omdat ze bereid zijn als mijn paranimfen op te treden. Els Bokstein en Dušica: bedankt voor al jullie aanmoedigingen.

Een bijzondere dank verdient mijn familie. Zonder hun morele steun zou dit proefschrift nooit mogelijk geweest zijn.

Curriculum vitae

De auteur is geboren op 3 maart 1974 te Alagöz, Kiğı, Turkije uit Koerdische ouders. Op zesjarige leeftijd is hij verhuisd naar 's-Gravenhage met zijn familie. Na de basisschool, de MAVO, de MTS en een propedeutisch diploma Technische Natuurkunde van de HTS liet hij zich in augustus 1995 inschrijven aan de (toen nog) Rijksuniversiteit Leiden om daar wiskunde en natuurkunde te studeren. In juni 1996 behaalde hij de propedeutisch diploma's voor natuurkunde en wiskunde. Zijn derde studiejaar bracht hij als uitwisselingstudent door op de Cornell University in de staat New York van de Verenigde Staten van Amerika. Verder heeft hij tijdens zijn studie geroeid bij roeivereniging Asopos de Vliet. Eind juni 2001 is hij afgestudeerd in wiskunde in de richting complexe analyse van meer variabelen bij Prof. dr. Jan Wiegerinck met de scriptie "Een nieuwe constructie van plurisubharmonische functies en enkele toepassingen".

Op 2 juli 2001 begon hij aan een promotie onderzoek bij het Korteweg-de Vries Instituut voor Wiskunde (KdVI) onder de begeleiding van Prof. dr. Eric Opdam en dr. Jasper Stokman. Tijdens zijn promotie gaf hij zeven werkcolleges aan studenten wiskunde, natuurkunde en kunstmatige intelligentie. Verder was hij een tijd mede-webmaster van het KdVI. Het onderzoek dat hij verrichte vond zijn weerslag in dit proefschrift.

Na zijn promotie zal hij voor een jaar werkzaam zijn als postdoc aan de Universidad de Talca (Chili) in de groep van Prof. dr. Jan Felipe van Diejen.