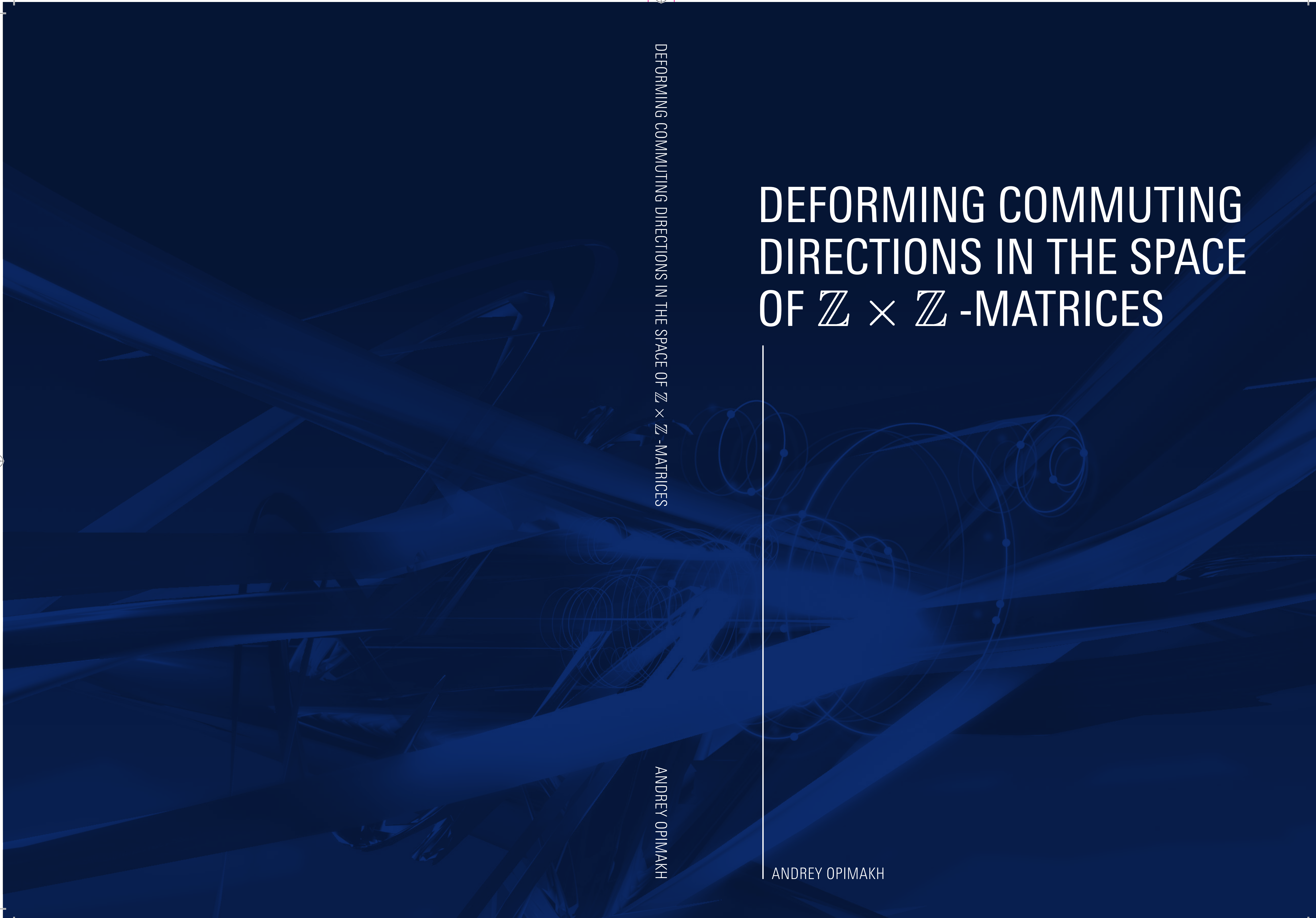

DEFORMING COMMUTING DIRECTIONS IN THE SPACE OF $\mathbb{Z} \times \mathbb{Z}$ -MATRICES

ANDREY OPIMAKH

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Deforming commuting directions in the space of $\mathbb{Z} \times \mathbb{Z}$-matrices

The research carried out in this thesis was done at the University of Amsterdam, Amsterdam, The Netherlands. Financial support was provided by the NWO–RFBR project "Geometric Aspects of Quantum Theory and Integrable Systems", nr. 047.017.015.

Netherlands Organisation for Scientific Research

Printed by Xerox Service, Enschede, The Netherlands

Cover design: Olaf Stokkers, Communication Department, University of Twente, Enschede, The Netherlands

ISBN: 9789057762239

DEFORMING COMMUTING DIRECTIONS IN THE SPACE OF $\mathbb{Z} \times \mathbb{Z}$-MATRICES

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
ten overstaan van een door het college van promoties
ingestelde commissie,
in het openbaar te verdedigen in de Agnietenkapel
op woensdag 23 februari 2011, te 14.00 uur

door

ANDREY VLADIMIROVICH OPIMAKH

geboren te Orenburg, Rusland

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Contents

1 Introduction

1.1 Historic background

The present overview of what brought people to the study of systems like the ones that form the main topic of this thesis, is based on [1], [4], [19] and [22]. More details can be found there.

The description of various types of water waves goes back to the nineteenth century. We mention two antagonists in this respect: J. Scott Russell who devoted much of his attention to solitary waves and Airy, who published in 1845 in his *Tides and Waves* a theory of long waves of small but finite amplitude and concluded that solitary waves of permanent form do not exist. This dispute lasted for 50 years till in 1895 Korteweg and his PhD-student de Vries, inspired by the work of Boussinesq, derived the following partial differential equation

$$u_t := \frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}u\frac{\partial u}{\partial x} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x,$$

to describe the propagation of shallow water waves in a narrow channel. Here the dependent variable $u = u(x,t)$ describes the height of waves in the channel as a function of a space coordinate x along the channel and a time coordinate t. Since then it carries their name and is abbreviated as KdV equation. This equation possesses a family of permanent wave solutions

$$u(x,t) = k^2 \cosh^{-2}(kx + k^3 t + c), \tag{1}$$

where k and c are constants. They are examples of so-called soliton solutions of the KdV equation.

For a long time this equation remained obscure. It was another physical problem that triggered interest in this equation: Fermi, Pasta and Ulam performed numerical experiments with an anharmonic one-dimensional lattice and observed to their surprise not a steady continuous flow of energy from the first mode to the higher ones, but more an exchange of energy, essentially, among only a certain few. To understand this phenomenon Kruskal and Zabrusky made a continuous model, where the equations of motions in the approximation up to the second order in the distance between the springs reduced to the KdV equation. They observed that the interaction of two solutions of the form (1) was inelastic: starting at $t = 0$ with two well-separated solutions with the highest(= fastest) one at the left, the fastest one starts overhauling the slower one, they interact for some time and continue to move on, whereby they regain asymptotically for $t \mapsto \infty$ their original shape and only a phase shift has taken place. These observations triggered a lot research. In particular, it led to the development by Gardner, Greene, Kruskal and Miura, see [11] and [12], of the inverse scattering transform, roughly speaking a method to reconstruct fast decaying solutions $u(x,t)$ of the KdV equation out of $u(x,0)$ and its scattering data.

Inspired by the outcome of the numerical work of Fermi and coworkers, M. Toda started the search for nonlinear lattices that are susceptible for analysis. He considered systems of one-dimensional particles where the equations of motion of each particle are determined by a potential depending only of the distance to the particle on the left and the distance to the particle on the right, see [51] and [52]. From an integrable perspective such systems were treated by J. Moser, [36]. To give a concrete idea of these systems we present following [41] an example, namely that of the so-called *finite open Toda lattice*. Consider n particles on a line and let $q_1(t), \cdots, q_n(t)$ denote their positions and assume for simplicity that each particle has mass equal to 1. The momentum p_i of the i-th particle is then $p_i = \frac{d}{dt} q_i$. Further we assume that the center of gravity remains in the origin

$$\sum_{j=1}^{n} q_j = \sum_{j=1}^{n} p_j = 0.$$

The Hamiltonian function H for this model is given by

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} e^{2(q_j - q_{j+1})}.$$

The corresponding Hamiltonian system consists then of the equations

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \text{ and } \dot{p}_j = -\frac{\partial H}{\partial q_j}.$$

In terms of the q_j and the p_j this amounts to the equations

$$\begin{aligned}
\dot{q}_j &= p_j, j = 1, \cdots, n \\
\dot{p}_1 &= -2e^{2(q_1-q_2)} \\
\dot{p}_j &= 2e^{2(q_{j-1}-q_j} - 2e^{2(q_j-q_{j+1}}, 2 \leq j \leq n-1, \\
\dot{p}_n &= 2e^{2(q_{n-1}-q_n)}.
\end{aligned}$$

This finite-dimensional Hamiltonian system is a special case of a series of Hamiltonian systems connected with Dynkin diagrams, see [39]. The representation theoretic and symplectic aspects of these systems and the most explicit integration are treated extensively in [31], see also [50] for a more compact and slightly different view on them. Many of the techniques used in these papers could be adapted to integrable systems connected to affine Lie algebras, see [43], [44] and [8].

Another important discovery was made by P.D. Lax [33]. He considered certain deformations of constant coefficient differential operators and showed that the KdV equation follows as the compatibility condition of an appropriate linearization and as such the KdV equation can be written as an identity between differential operators in $\partial := \frac{\partial}{\partial x}$ with coefficients from some ring of functions R in the variables x and t. For simplicity we assume here that $R = C^\infty(\mathbb{R}^2)$.

If one associates to any function $v \in R$ the differential operators

$$\mathcal{L} = \partial^2 + v \text{ and } P = \partial^3 + \frac{3}{2}v\partial + \frac{3}{4}\partial(v), \tag{2}$$

then a direct computation shows that $[P, \mathcal{L}] = P \circ \mathcal{L} - \mathcal{L} \circ P$ is equal to the zero-th-order operator in ∂

$$[P, \mathcal{L}] = \frac{1}{4}\partial^3(v) + \frac{3}{2}v\partial(v).$$

Now one considers $\partial^2 + v(x,t)$ as a flow of Schrödinger operators and in that light one lets $\frac{\partial}{\partial t} =: \partial_t$ act on $\mathcal{L}$ by differentiating its coefficients w.r.t. t. Thus we see that the fact "v is a solution of the KdV-equation" is equivalent to the following identity for $\mathcal{L}$ and P

$$\frac{\partial}{\partial t}(\mathcal{L}) = 0 \cdot \frac{\partial^2}{\partial x^2} + \frac{\partial v}{\partial t} = [P, \mathcal{L}]. \tag{3}$$

This explains the name *Lax form* of the KdV equation for the equation (3).

Also in the context of the Toda lattices Lax forms occur. We illustrate this at the finite open Toda lattice. Define for each i and $j \in \{1, \cdots, n\}$, the function $q_{i,j}$ by $e^{q_i - q_j}$. Then one introduces the matrices

$$L = \begin{pmatrix} p_1 & q_{1,2} & 0 & 0 & \cdots & \cdots & 0 \\ q_{1,2} & p_2 & q_{2,3} & 0 & \cdots & \cdots & 0 \\ 0 & q_{2,3} & p_3 & q_{3,4} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & p_{n-2} & q_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \ddots & q_{n-2,n-1} & p_{n-1} & q_{n-1,n} \\ 0 & \dots & \dots & \dots & 0 & q_{n-1,n} & p_n \end{pmatrix}$$

and the skew symmetric matrix

$$M = \begin{pmatrix} 0 & q_{1,2} & 0 & 0 & \cdots & \cdots & 0 \\ -q_{1,2} & 0 & q_{2,3} & 0 & \cdots & \cdots & 0 \\ 0 & -q_{2,3} & 0 & q_{3,4} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 & q_{n-2,n-1} & 0 \\ 0 & 0 & 0 & \ddots & -q_{n-2,n-1} & 0 & q_{n-1,n} \\ 0 & \dots & \dots & \dots & 0 & -q_{n-1,n} & 0 \end{pmatrix}$$

A direct computation shows then that the Hamilton equations are equivalent to

$$\dot{L} = [L, M] \tag{4}$$

This form was found by H. Flaschka in [9] and used in [10] to show that the inverse scattering transform works also in this setting.

In the Lie algebraic picture one has a description of a general solution of systems like (4). Note that, as the center of gravity is preserved, both L and M are maps from $\mathbb{R} \to \mathrm{sl}_n(\mathbb{R})$, the Lie algebra of real $n \times n$-matrices with trace zero. By the Iwasawa decomposition

$$\mathrm{SL}_n(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R}) P_-, \tag{5}$$

where $\mathrm{SO}_n(\mathbb{R}) \cap P_- = \mathrm{Id}$ and $P_- = AU_-(\mathbb{R})$ with A the positive real diagonal matrices and $U_-(\mathbb{R})$ the unipotent real lower triangular matrices. If one denotes by $g = g_1 g_2$ the decomposition of each $g \in \mathrm{SL}_n(\mathbb{R})$ w.r.t. the decomposition (5), then there holds, see [19] and [40]

$$L(t) = \exp(-tL(0))_1 L(0) \exp(tL(0))_1. \tag{6}$$

The phenomenon that specific parts of certain decompositions in a Lie group are at the basis of the construction of solutions of your system is also met at the hierarchies that are being dealt with in this thesis.

In 1970 Kadomtsev and Petviashvili proposed a two dimensional version of the KdV equation to study the transversal stability of soliton solutions of the KdV equation. In this case we have a function $u = u(x, y, t)$ and it satisfies

$$\frac{3}{4}u_{yy} = (u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x)_x \tag{7}$$

In particular, you see that if you have a solution of the KdV equation and you extend it to a function in three variables by making the y-dependence trivial, you have a solution of the KP equation. Also the KP equation results from an identity between differential operators in $\partial = \frac{\partial}{\partial x}$, but now the coefficients depend in a differentiable way of three variables, namely x, y and t. Assume for simplicity that $u = u(x, y, t)$ and $v = v(x, y, t) \in C^\infty(\mathbb{R}^3)$ and consider the operators

$$B_2 = \partial^2 + u \text{ and } B_3 = \partial^3 + \frac{3}{2}u\partial + v.$$

One relates to these operators the linear system for the function $\Psi = \Psi(x, y, t)$

$$\begin{cases} \frac{\partial \Psi}{\partial y} = B_2 \Psi \\ \frac{\partial \Psi}{\partial t} = B_3 \Psi \end{cases} \tag{8}$$

The compatibility conditions of this system are the equations

$$\frac{\partial}{\partial t}(\frac{\partial \Psi}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial \Psi}{\partial t})$$

The left hand side of this equation equals

$$\frac{\partial}{\partial t}(\frac{\partial \Psi}{\partial y}) = \{\frac{\partial}{\partial t}(B_2) + B_2 B_3\}\Psi = \{u_t + B_2 B_3\}\Psi$$

and the right hand one

$$\frac{\partial}{\partial y}(\frac{\partial \Psi}{\partial t}) = \{\frac{\partial}{\partial y}(B_3) + B_3 B_2\}\Psi = \{\frac{3}{2}u_y\partial + v_y + B_3 B_2\}\Psi.$$

The linear system (8) can be seen as a Cauchy-problem in the two variables y and t for some initial value $x \mapsto \Psi(x, y_0, t_0)$. If one wants to choose this initial value as generic as possible, then it is necessary that the two operators acting on Ψ in the compatibility conditions have to be equal. Thus one arrives at the relation

$$\frac{\partial}{\partial t}(B_2) - \frac{\partial}{\partial y}(B_3) - [B_3, B_2] = 0 \Leftrightarrow u_t - \frac{3}{2}u_y\partial - v_y - [B_3, B_2] = 0 \qquad (9)$$

A direct computation shows that the commutator $[B_3, B_2]$ is equal to the first order operator

$$\{-\frac{3}{2}u_{xx} + 2v_x\}\partial + (u_{xxx} - \frac{3}{2}uu_x + v_{xx}).$$

Thus the equations (9) boil down to two equations

$$u_t = v_y + u_{xxx} + \frac{3}{2}uu_x - v_{xx} \qquad (10)$$

$$0 = \frac{3}{2}u_y + (\frac{3}{2}u_{xx} - 2v_x) \qquad (11)$$

Now solving v_x from equation (11), i.e. $v_x = \frac{3}{4}u_y + \frac{3}{4}u_{xx}$, and substituting this identity in equation (10) result in

$$u_t = v_y + \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x - \frac{3}{4}u_{yx}.$$

This equation one differentiates once w.r.t. x and substituting

$$v_{yx} = \frac{3}{4}u_{yy} + \frac{3}{4}u_{xxy}$$

yields the KP equation for u.

There exist also two-dimensional variants of the Toda lattice. We mention the so-called *elliptic periodic two-dimensional Toda lattice* from [34]. In this model one considers functions $w_i : \mathbb{C} \to \mathbb{R}, i \in \mathbb{Z}$, satisfying the periodicity condition

$$w_{i+n+1} = w_i, \text{ for all } i,$$

and the center of gravity condition

$$\sum_{i=0}^{n} w_i = 0.$$

These functions should satisfy

$$2\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}}(w_i) =: 2(w_i)_{z\bar{z}} = e^{2(w_{i+1}-w_i)} - e^{2(w_i - w_{i+1})}, \tag{12}$$

and are of a geometric importance as solutions for $n = 1$ describe surfaces of constant mean curvature. Define as before for each i and $j \in \{1, \cdots, n\}$, the function $w_{i,j}$ by $e^{w_i - w_j}$ and introduce the matrices

$$W = \begin{pmatrix} (w_0)_z & 0 & 0 & 0 & \cdots & \cdots & w_{0,n} \\ w_{1,0} & (w_1)_z & 0 & 0 & \cdots & \cdots & 0 \\ 0 & w_{2,1} & (w_3)_z & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & (w_{n-2})_z & 0 & 0 \\ 0 & 0 & 0 & \ddots & w_{n-1,n-2} & (w_{n-1})_z & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & w_{n,n-1} & (w_n)_z \end{pmatrix}$$

and $V = -W^*$. Then a direct verification shows that the equations (12) are equivalent to the matrix identity

$$\frac{\partial}{\partial \bar{z}}(W) - \frac{\partial}{\partial z}(V) = [W, V] \tag{13}$$

Also the generalized Toda lattices have a two-dimensional version, see [35].

Gelfand and Dickey determined the Hamiltonian structure of many equations similar to the KdV equation, see [13] and also [6]. From their work and that of Wilson [55], it became clear that it made sense to group certain Lax equations or zero curvature equations together as they correspond to commuting flows on the same object. It was Sato and his school, see [45], [5] and their successors , who took up the study of these combined systems, so-called hierarchies, systematically and showed their relation to quantum field theory and infinite dimensional Lie algebras. V. Kac , see [26] , used the representation theory of these algebras to describe rational solutions of the bilinear form of these hierarchies.

To illustrate this notion of hierarchy, we present here the description of the KdV hierarchy. The key to the other Lax equations in this hierarchy lies in the relation between the Schrödinger operator $\mathcal{L} = \partial^2 + v$ and the third order operator P in ∂ from equation (2). In general, $\mathcal{L}$ is not the square of some first order differential operator. However, if one extends the ring of differential operators in ∂ with coefficients from the ring R to that of the pseudodifferential operators in ∂ with coefficients from R, then one can define a square root $\mathcal{L}^{\frac{1}{2}}$ of $\mathcal{L}$ in this extended ring. It has the form

$$\mathcal{L}^{\frac{1}{2}} = \partial + \sum_{i=0} l_i \partial^{-i},$$

where all the coefficients l_i of the formal integral operators ∂^{-i} are polynomial expressions in the potential v and all its derivatives w.r.t. ∂ because of the relation $(\mathcal{L}^{\frac{1}{2}})^2 = \mathcal{L}$. If one denotes the differential operator part $\sum_{j\geq 0}^{N} p_j \partial^j$ of a pseudodifferential operator $\mathcal{P} = \sum_{j=-\infty}^{N} p_j \partial^j$ simply by $\mathcal{P}_+$, then the the third order operator P is equal to $((\mathcal{L}^{\frac{1}{2}})^3)_+$. The relation

$$[P, \mathcal{L}] = -[((\mathcal{L}^{\frac{1}{2}})^3) - ((\mathcal{L}^{\frac{1}{2}})^3)_+, \mathcal{L}]$$

shows that the commutator $[P, \mathcal{L}]$ is a zero-th order operator in ∂. By replacing three by any odd k, the same reasoning yields that it makes sense to consider the Lax equations

$$\frac{\partial}{\partial t_k}(\mathcal{L}) = [P_k, \mathcal{L}], \text{ where } P_k = ((\mathcal{L}^{\frac{1}{2}})^k)_+, \tag{14}$$

where the potential v depends now of all the parameters $\{t_k \mid k \text{ odd }\}$. Note that for even k the commutator in the right hand side of (14) is zero so that the Lax equation (14) in that case means that the potential is independent of this parameter. For $k = 1$, the equation (14) is equal to $\frac{\partial}{\partial t_1}(v) = \partial(v)$. The equations (14) are the equations of the KdV hierarchy. Also the KP equation is the lowest nontrivial member of a series of Lax equations similar to (14) for a first order pseudodifferential operator, see e.g. [5].

The Toda equations that have their origin in the description of the equations of motion for chains of particles with nearest-neighbour interaction, can also be embedded into hierarchies that constitute towers of nonlinear differential difference equations, see e.g. [2] . By now, it has become clear that their importance goes far beyond the description of these particle systems as they play a role in topics ranging from quantum crystals, see [38] and [37], matrix models [29], quantum gravity, see [7], string theory, see [14] and [54], special functions, see [3] and [20], harmonic maps, see [19], to mirror symmetry, see [15].

1.2 Examples

In this subsection we present two examples of the form of the nonlinear equations that we will meet in this thesis.

The first has the key equations in the Lax form. Following [52], we consider the infinite Toda-chain consisting of an infinite number of particles on a straight line with the same mass that are labeled by $\mathbb{Z}$ and whose equations of motion in dimensionless form are described by

$$\frac{dq_n}{dt} = p_n \ \text{ and } \ \frac{dp_n}{dt} = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1}-q_n)}, \quad n \in \mathbb{Z}. \tag{15}$$

Here q_n is the displacement of the n-th particle and the two exponential factors in equation (15) describe the forces exerted on the n-th particle by each of its neighbours. These equations can be rewritten as an equality between infinite matrices by putting

$$a_n := \frac{1}{2}e^{-(q_n - q_{n-1})} \ \text{ and } \ b_n := \frac{1}{2}p_n.$$

The equations (15) get then the following form

$$\frac{da_n}{dt} = a_n(b_n - b_{n-1}) \ \text{ and } \ \frac{db_n}{dt} = 2(a_{n-1}^2 - a_n^2), \quad n \in \mathbb{Z}. \tag{16}$$

Introduce the $\mathbb{Z} \times \mathbb{Z}$-matrices L and B by

$$L = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{b_{n-1}} & a_n & 0 & \ddots \\ \ddots & a_n & \mathbf{b_n} & a_{n+1} & \ddots \\ & 0 & a_{n+1} & \mathbf{b_{n+1}} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix} \ \text{ and } \ B = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{0} & -a_n & 0 & \ddots \\ \ddots & a_n & \mathbf{0} & -a_{n+1} & \ddots \\ & 0 & a_{n+1} & \mathbf{0} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix},$$

where the central diagonals as a reference point have been printed in boldface. Then a direct computation shows that the equations (16) amount to the matrix equation

$$\frac{dL}{dt} = BL - LB = [B, L]. \tag{17}$$

This is a $\mathbb{Z} \times \mathbb{Z}$-version of the so-called *Lax equation* and the solution (6) of the finite open Toda lattice suggests that the matrix L can be obtained by conjugating a matrix that does not depend of t with a suitable t-dependent one. This turns out to be true for the solutions that we construct.

The second example is a two-dimensional variant similar to the elliptic periodic two-dimensional Toda lattice and yields the main equations in zero curvature form. Following [53] the two-dimensional infinite Toda lattice is a set of nonlinear wave equations for an infinite chain of particles with only nearest neighbour interaction

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}u_n = e^{u_n - u_{n-1}} - e^{u_{n+1} - u_n}, n \in \mathbb{Z}. \tag{18}$$

It can also be formulated as a relation between $\mathbb{Z} \times \mathbb{Z}$-matrices, but of a slightly different form. Consider namely two $\mathbb{Z} \times \mathbb{Z}$-matrices B and C of the form

$$B = \Lambda + b = \Lambda + b\,\mathrm{Id} = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{b_{n-1}} & 1 & 0 & \ddots \\ \ddots & 0 & \mathbf{b_n} & 1 & \ddots \\ & 0 & 0 & \mathbf{b_{n+1}} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix},$$

resp.

$$C = c\Lambda^{-1} = \begin{pmatrix} \ddots & \ddots & \ddots & & 0 \\ \ddots & \mathbf{0} & 0 & 0 & \ddots \\ \ddots & c_n & \mathbf{0} & 0 & \ddots \\ & 0 & c_{n+1} & \mathbf{0} & \ddots \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix}, \text{ with } c_n \neq 0, \text{ for all } n \in \mathbb{Z}.$$

Here Λ is the upper triangular matrix corresponding to the basis shift $e_i \mapsto e_{i-1}$ and the entries $\{c_n \mid n \in \mathbb{Z}\}$, $\{b_n \mid n \in \mathbb{Z}\}$ of both diagonal $\mathbb{Z} \times \mathbb{Z}$-matrices c and b are assumed to be differentiable functions in the variables t and s. Analogously to the finite dimensional terminology, the equation

$$\frac{\partial}{\partial s}(B) - \frac{\partial}{\partial t}(C) = [C, B] \tag{19}$$

is called a zero curvature relation for B and C. A direct computation yields that this amounts to the equations

$$\frac{\partial}{\partial s}(b) = c - \Lambda c \Lambda^{-1} \text{ and } c^{-1}\frac{\partial}{\partial t}(c) = b - \Lambda^{-1} b \Lambda.$$

Hence, if one chooses for all $n \in \mathbb{Z}$

$$b_n = \frac{\partial}{\partial t}(u_n) \text{ and } c_n = e^{u_n - u_{n-1}},$$

then the zero curvature relation is equivalent to the nonlinear wave equations of the two-dimensional infinite Toda lattice. Also the nonlinear equations of the hierarchies considered in this paper can be presented in a a zero curvature form like the equations (19).

1.3 Topics in the thesis

In this thesis we consider certain basic sets of commuting $\mathbb{Z} \times \mathbb{Z}$-matrices. We are interested in various deformations of these directions depending of the commuting flows associated with these basic directions and in particular in the nonlinear evolution equations satisfied by these matrices. It will turn out that it makes sense to group evolution equations of a similar form together into systems of compatible equations, the so-called hierarchies. We will consider three types of deformations: the first takes place in the lower triangular matrices and preserves the leading term. The second one that will be analyzed is in the upper triangular matrices and allows also a distortion of the leading term. Finally one considers a combination of the two foregoing types, where, roughly said, half of them is deformed in the lower triangular matrices and the other half in the upper triangular ones.

The equations of each hierarchy may appear in various forms like the Lax form, the zero curvature form or the bilinear form. So at every type the first goal will be to analyze the underlying algebraic structure of the equations and to show the equivalence of the different forms that are used. The next step will be to give a geometric construction of solutions of these systems of compatible equations.

This work is built up as follows: we start after the introduction with giving examples of $\mathbb{Z} \times \mathbb{Z}$-matrices that will be widely used in the text, we introduce notations for $\mathbb{Z} \times \mathbb{Z}$-matrices that will occur frequently and we prove various properties for this class of matrices like relevant decompositions.

The equations of the hierarchies are intimately linked with infinite dimensional Cauchy problems. This link also justifies the use of the terminology zero curvature form. Therefore a separate section is devoted to a discussion of their formal solvability.

After these preparations we are ready to discuss the three types of deformations, starting with the lower triangular ones. Chapter 4 is devoted to the study of these hierarchies. One starts with a discussion of the Lax form of the hierarchies. Then a minimal realization of the hierarchy is presented, where minimal refers to the number of relations between the deformed generators of the basic directions. The next step is taken in the subsequent subsection, where the discussion of the zero curvature form can be found and the proof of the equivalence with the Lax form. In the finite dimensional case zero curvature relations occur naturally as the compatibility equations for the horizontal sections of an integrable connection. To show that the nonlinear equations of the hierarchy can be seen as the compatibility equations of a linear system, we discuss in the fifth subsection of this chapter the so-called linearization of the hierarchy. Here one can also find the description of the link with the Cauchy problems. The final two subsections describe the infinite dimensional flag variety from which one can construct solutions of the hierarchy.

The second class of deformations form the topic of the next chapter. It is built up in a similar way as the one with the lower triangular hierarchies with delicate adjustments of the algebraic structure to include now also perturbations of the leading term. The algebraic description for both type of deformations of the equivalence of the Lax form and the zero curvature form forms the contents of [23]. Apart from that, the main difference is the infinite dimensional geometry that yields the solution of the hierarchy: this time it is a relative frame bundle over a flag variety similar to the one occurring in the lower triangular case that does the job. As we know from [46] and [49] infinite dimensional Grassmann varieties are basic for the construction of solutions for KP-type hierarchies and finite chains of such planes determine Darboux transformations for those hierarchies, see [28]. The present two constructions furnish two new illustrations of the important role various infinite dimensional varieties play for integrable hierarchies and are treated in [24].

In the last chapter we discuss what happens if you deform half of your basic directions into the lower triangular matrices in the way as described in chapter 4, the other half into the upper triangular matrices in a manner as treated in chapter 5 and you consider the evolution of all generators w.r.t. both sets of parameters. The hierarchy that is obtained in this way is a multi component version of the two-dimensional Toda hierarchy, where the terminology is analogous to one used for a similar hierarchy of matrix pseudo differential operators. Also with this type of deformation we have such a minimal set-up, we show the equivalence of the Lax form and the zero curvature form and we present the appropriate linearization. Besides the two forms just mentioned, there is still another way to describe the system, namely the bilinear form, and we present also this one. One concludes with a geometric construction of solutions of the multi component Toda hierarchies. Paper [25] is based on chapter 6.

2 The space $M_{\mathbb{Z}}(R)$

Since the nonlinear systems of differential and difference equations that are the central topic of this paper are formulated in terms of relations for $\mathbb{Z}\times\mathbb{Z}$-matrices, one will first discuss the necessary ingredients from that space.

Let R be a commutative ring. Then one writes $M_{\mathbb{Z}}(R)$ for the R-module of $\mathbb{Z}\times\mathbb{Z}$-matrices with coefficients from R. On this space one uses the ordering of columns and rows that is compatible with the finite dimensional case, i.e. any matrix $A=(\alpha_{ij})$ is denoted by

$$A=\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \boldsymbol{\alpha_{n-1\,n-1}} & \alpha_{n-1\,n} & \alpha_{n-1\,n+1} & \ddots \\ \ddots & \alpha_{n\,n-1} & \boldsymbol{\alpha_{n\,n}} & \alpha_{n\,n+1} & \ddots \\ \ddots & \alpha_{n+1\,n-1} & \alpha_{n+1\,n} & \boldsymbol{\alpha_{n+1\,n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

There is a number of special elements in $M_{\mathbb{Z}}(R)$ that will be used frequently. First of all, there are the basic matrices $E_{(i,j)}$, i and $j\in\mathbb{Z}$, given by

$$(E_{(i,j)})_{mn}=\delta_{im}\delta_{jn}, \tag{20}$$

where δ_{ts}, for t and $s\in\mathbb{Z}$, denotes the Kronecker symbol. Thus one can describe every $A=(\alpha_{ij})\in M_{\mathbb{Z}}(R)$ as a formal linear combination of the basic matrices

$$A=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\alpha_{ij}E_{(i,j)}=\sum_{i\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\alpha_{(i,j)}E_{(i,j)}. \tag{21}$$

The notation $\alpha_{(i,j)}$ for the matrixcoefficient α_{ij} of A will be used only if confusion in the labeling might occur. This will be done without further mentioning. Any map $\Delta : R\to R$ extends in a natural way to a map $\Delta : M_{\mathbb{Z}}(R)\to M_{\mathbb{Z}}(R)$ by putting

$$\Delta(A)_{ij}=\Delta(\alpha_{ij}),\ \text{if } A=(\alpha_{ij}).$$

An important role is played by the shift matrix Λ and its powers $\{\Lambda^m \mid m\in\mathbb{Z}\}$ defined by

$$\Lambda^m=\sum_{i\in\mathbb{Z}}E_{(i-m,i)}, m\in\mathbb{Z}.$$

It permits you to decompose each matrix $A \in M_{\mathbb{Z}}(R)$ in diagonals that are handy at explicit computations and enable a simple description in the present context of the notions lower and upper triangular matrix.

For each nonzero $k \in \mathbb{N}$, one denotes the ring of $k \times k$-matrices with coefficients from the ring R by $M_k(R)$. Assume one has a collection of $k \times k$-matrices $\{d(s)|s \in \mathbb{Z}\}$ in $M_k(R)$. To such a collection one associates a diagonal of $k \times k$-blocks $\mathrm{diag}(d(s))$ in $M_{\mathbb{Z}}(R)$ given by

$$\mathrm{diag}(d(s)) := \sum_{s \in \mathbb{Z}} \sum_{\alpha=1}^{k} \sum_{\beta=1}^{k} d(s)_{\alpha\beta} E_{(sk+\alpha-1, sk+\beta-1)}. \tag{22}$$

The form of this matrix is as follows

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{d(n-1)} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{d(n)} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{d(n+1)} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} \tag{23}$$

and justifies the terminology used. For any such $k \geq 1$ one denotes the ring of $k \times k$-block diagonal matrices in $M_{\mathbb{Z}}(R)$ by

$$\mathcal{D}_k(R) = \{d = \mathrm{diag}(d(s))|d(s) \in M_k(R) \text{ for all } s \in \mathbb{Z}\}.$$

One has a diagonal embedding i_k from $M_k(R)$ into $\mathcal{D}_k(R)$ by taking for any $A \in M_k(R)$ all diagonal blocks of $i_k(A)$ equal to A. The elements Λ^{km}, $m \in \mathbb{Z}$, act on $\mathcal{D}_k(R)$ according to the formula

$$\Lambda^{km} \mathrm{diag}(d(s)) \Lambda^{-km} = \mathrm{diag}(d(s+m)). \tag{24}$$

It implies e.g. that the image of i_k consists of all matrices in $\mathcal{D}_k(R)$ that commute with Λ^k. This brings one to the decomposition in diagonals mentioned above. For, if $A = (\alpha_{ij}) \in M_{\mathbb{Z}}(R)$, then one puts

Definition 1. The *j-th $k \times k$-block diagonal* of any matrix A, $j \in \mathbb{Z}$, is the matrix

$$\sum_{i \in \mathbb{Z}} \sum_{\gamma=1}^{k} \sum_{\beta=1}^{k} \alpha_{(ki-kj+\gamma-1, ki+\beta-1)} E_{(ki-kj+\gamma-1, ki+\beta-1)}.$$

From equation (24) it is clear that the j-th $k \times k$-block diagonal of a $\mathbb{Z} \times \mathbb{Z}$-matrix A can uniquely be written in the form $\mathrm{diag}(d(s))\Lambda^{kj}$ or $\Lambda^{kj}\mathrm{diag}(c(s))$ where both $\mathrm{diag}(d(s))$ and $\mathrm{diag}(c(s))$ belong to $\mathcal{D}_k(R)$. Thus each matrix $A = (\alpha_{(i,j)}) \in M_{\mathbb{Z}}(R)$ can uniquely be written as a formal infinite sum

$$A = \sum_{j \in \mathbb{Z}} d_j \Lambda^{kj} \;\text{ or }\; A = \sum_{j \in \mathbb{Z}} \Lambda^{kj} c_j, \tag{25}$$

with all the d_j and c_j in $\mathcal{D}_k(R)$. Thanks to this decomposition and the equation (24) it is clear now that multiplying any matrix A from the left or right with a power of Λ is well-defined and easily computable. In particular one sees that any matrix that commutes with Λ^k has the form (25) with d_j and c_j in the image of i_k. The elements of $\mathcal{D}_k(R)$ were given in terms of $k \times k$-blocks. It will be convenient to work with a similar, compatible decomposition for general matrices $A \in M_{\mathbb{Z}}(R)$. One has then

$$A = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{A_{n-1\,n-1}} & A_{n-1\;n} & A_{n-1\;n+1} & \ddots \\ \ddots & A_{n\;n-1} & \mathbf{A_{n\,n}} & A_{n\;n+1} & \ddots \\ \ddots & A_{n+1\;n-1} & A_{n+1\;n} & \mathbf{A_{n+1\,n+1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{26}$$

where all the $\{(A)_{ij} := A_{ij} \mid i, j \in \mathbb{Z}\}$ belong to $M_k(R)$ and the (t,s)-entry of A_{ij} is given by $\alpha_{(ik+t-1, jk+s-1)}$. To distinguish between the two notations one will use greek letters for the matrix coefficients of A in R and latin ones for the coefficients in $M_k(R)$. E.g. for the matrices $i_k(F), F \in M_k(R)$ and Λ^k their $k \times k$-block decomposition looks respectively like

$$i_k(F) = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{F} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{F} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{F} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \Lambda^k = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & \mathrm{Id} & 0 & \ddots \\ \ddots & 0 & \mathbf{0} & \mathrm{Id} & \ddots \\ \ddots & 0 & 0 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The basic matrices can be multiplied and satisfy

$$E_{(i,j)}E_{(m,k)} = \delta_{jm}E_{(i,k)}.$$

In general, for arbitrary matrices $A = (\alpha_{ij}) = (A_{ij})$ and $B = (\beta_{ij}) = (B_{ij})$ in $M_{\mathbb{Z}}(R)$, one cannot define the product AB. However, if it exists, its ts-block is given by

$$(AB)_{ts} = \sum_{j\in\mathbb{Z}} A_{tj}B_{js}. \tag{27}$$

In particular, this expression makes sense if both A and B belong to the class of upper or lower triangular matrices that will be introduced in a moment.

Definition 2. An element A in $M_{\mathbb{Z}}(R)$ is called *upper $k\times k$-block triangular of level m*, if it can be written as

$$A = \sum_{j\geq m} a_j\Lambda^{kj}, \text{ with } a_j \in \mathcal{D}_k(R). \tag{28}$$

One calls m the *order* of A in Λ^k, if a_m is nonzero.

The collection of all upper $k\times k$-block triangular elements of level m in Λ^k one denotes by $UT_{\geq m}(\Lambda^k)$ or simply $UT_{\geq m}$. For the set of all upper $k\times k$-block triangular matrices one uses the notations

$$UT(R) := \bigcup_{m\in\mathbb{Z}} UT_{\geq m}(\Lambda^k) =: UT.$$

In the same spirit one has the notations

$$UT_{<m}(\Lambda^k) := \{A \mid A \in UT, A = \sum_{j<m} a_j\Lambda^{kj}, a_j \in \mathcal{D}_k(R)\} =: UT_{<m},$$

$UT_{\leq m}(\Lambda^k)$ and $UT_{>m}(\Lambda^k)$. Clearly $UT(R)$ is independent of k and one speaks also simply of *upper triangular matrices* if the size of the diagonals needs less emphasis.

Similarly one can introduce the collection of lower triangular matrices $LT(R)$ as the collection of matrices in $M_{\mathbb{Z}}(R)$ for which all the j-th diagonals are zero if j is sufficiently large. For completeness sake one introduces analogous terminology and notations for this class of matrices

Definition 3. An element A in $M_{\mathbb{Z}}(R)$ is called *lower $k \times k$-block triangular of level m*, if it can be written as

$$A = \sum_{j \leq m} d_j \Lambda^{kj}, \quad \text{with} \;\; d_j \in \mathcal{D}_k(R).$$

One calls m the *order* of A in Λ^k, if d_m is nonzero.

The collection of all lower $k \times k$-block triangular elements of level m one denotes by $LT_{\leq m}(\Lambda^k)$. The notations $LT_{>m}(\Lambda^k), LT_{<m}(\Lambda^k)$ and $LT_{\geq m}(\Lambda^k)$ speak for themselves. For the set of all lower $k \times k$-block triangular matrices one uses the notations

$$LT(R) := \bigcup_{m \in \mathbb{Z}} LT_{\leq m}(\Lambda^k) =: LT.$$

The matrices in the intersection $LT(R) \cap UT(R)$ of the two algebras are called *finite band* matrices and they possess the property that their product with any matrix from $M_{\mathbb{Z}}(R)$ is well-defined.

A convenient property of the rings LT and UT is that for each $k \geq 1$, there is a ring isomorphism between UT and LT that is compatible with the $k \times k$-block decomposition. Consider namely the element

$$w_k = \sum_{\alpha=1}^{k} \sum_{j \in \mathbb{Z}} E_{(-jk+\alpha-1, jk+\alpha-1)} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 0 & \mathrm{Id} & \ddots \\ \ddots & 0 & \mathbf{Id} & 0 & \ddots \\ \ddots & \mathrm{Id} & 0 & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

in $M_{\mathbb{Z}}(R)$. Then w_k is an element of order two and acts as follows on elements of $\mathcal{D}_k(R)$ and powers of Λ^k

$$w_k \mathrm{diag}(d(j)) w_k = \mathrm{diag}(d(-j)) \;\; \text{and} \;\; w_k \Lambda^{km} w_k = \Lambda^{-km}. \tag{29}$$

This leads to the following result

Lemma 1. *Conjugation with the element w_k gives a ring isomorphism between UT and LT that transforms the $k \times k$-block decompositions in UT to those in LT*

and vice versa. This isomorphism reinverses the order of elements:

$$\text{order}(w_k \ell w_k^{-1}) = -\text{order}(\ell), \ell \in LT.$$

Inside UT and LT one can describe large classes of invertible elements. Since the multiplication in $\mathcal{D}_k(R)$ corresponds to the multiplication of the $k \times k$-matrices with the same number, one sees that the elements of $\mathcal{D}_k(R)$ that possess an inverse in $\mathcal{D}_k(R)$ consists of all

$$\text{diag}(d(j)), \text{ with all } d(j) \in \text{GL}_k(R).$$

Assume now that one has a $V \in LT$ of the form

$$V = \sum_{i \leq m} v_i \Lambda^{ik},$$

with v_m invertible in $\mathcal{D}_k(R)$ and that one tries to find the inverse

$$V^{-1} = \sum_{j \leq t} d_j \Lambda^{jk},$$

of V in LT. Then the relations

$$V \circ V^{-1} = \text{Id} = V^{-1} \circ V$$

amount first of all to $t = -m$ and the relation

$$v_m \Lambda^{km} d_{-m} \Lambda^{-km} = \text{Id},$$

which determines d_{-m} totally. For all $l \geq 1$ there should hold moreover

$$\sum_{i+j=-l} v_i \Lambda^{ik} d_j \Lambda^{-ik} = \sum_{i=m-l}^{m} v_i \Lambda^{ik} d_{-i-l} \Lambda^{-ik} = 0 \tag{30}$$

and these relations determine in a recursive way the remaining coefficients $\{d_j \mid j < -m\}$ uniquely. Using the isomorphism between LT and UT, we proved

Lemma 2. *Each element V in LT resp. UT with an invertible leading coefficient in its $k \times k$-block decomposition (25) is invertible in LT resp. UT.*

The classes of invertible elements from Lemma 2 form the subgroups $I(LT)$ and $I(UT)$ in respectively the groups of invertible elements LT^* and UT^* and they are used at the *dressing-procedure* in LT resp. UT.

Definition 4. An element $A \in LT$ resp. UT is said to be obtained by *dressing* the element $B \in LT$ resp. UT, if there is a $V = \sum_{i \leq m} v_i \Lambda^{ik}$, resp. an $U = \sum_{i \geq \ell} u_i \Lambda^{ik}$ with v_m resp. u_ℓ invertible in $\mathcal{D}_k(R)$ such that $A = VBV^{-1}$ resp. $A = UBU^{-1}$.

Note that dressing an element does not change its order in Λ^k. Next we show that each element in $I(LT)$ resp. $I(UT)$ that is of nonzero order in Λ^k can be obtained by dressing the corresponding power of Λ^k.

Proposition 1. *(1) For any nonzero $r \in \mathbb{Z}$, let $\mathcal{L}$ in $I(LT)$ have the $k \times k$-block form*

$$\mathcal{L} = \sum_{j \leq r} \alpha_j \Lambda^{jk}$$

with α_r invertible in $\mathcal{D}_k(R)$. Then there exists an element U in LT of the $k \times k$-block form $U = \sum_{i \leq 0} \beta_i \Lambda^{ki}$ with $\beta_0 \in \mathcal{D}_k(R)$ invertible in $\mathcal{D}_k(R)$ such that

$$\mathcal{L} = U \Lambda^{rk} U^{-1}.$$

Hereby, one can choose the coefficients of U to be quotients of polynomial expressions in those of $\mathcal{L}$.

(2) For any nonzero $r \in \mathbb{Z}$, let $\mathcal{M}$ in $I(UT)$ have the $k \times k$-block form

$$\mathcal{M} = \sum_{j \geq r} \alpha_j \Lambda^{jk}$$

with α_r in $\mathcal{D}_k(R)$. Then there exists an element V in UT of the $k \times k$-block form $V = \sum_{i \geq 0} \beta_i \Lambda^{ki}$ with $\beta_0 \in \mathcal{D}_k(R)$ invertible in $\mathcal{D}_k(R)$ such that

$$\mathcal{M} = V \Lambda^{rk} V^{-1}.$$

Hereby, one can choose the coefficients of V to be quotients of polynomial expressions in those of $\mathcal{M}$.

Proof. Due to the isomorphism between the rings LT and UT described in (29), it is sufficient to prove part a) of the proposition. Since the leading coefficient α_r is invertible, one knows that $\mathcal{L}$ is invertible and that $\mathcal{L}^{-1}$ belongs to LT_{-r} with $\Lambda^{-rk}\alpha_r^{-1}\Lambda^{rk}$ as its leading coefficient. Therefore one can restrict oneself to the case $r > 0$. The matrix U one is looking for, should satisfy the equation $\mathcal{L}U = U\Lambda^{kr}$. This amounts to the following equations for the coefficients β_j of U:

$$\beta_t = \sum_{i=t}^{0} \alpha_{t+r-i}\Lambda^{k(t+r-i)}\beta_i\Lambda^{-k(t+r-i)} \tag{31}$$

for all $t \leq 0$. If each $\beta_i = \mathrm{diag}(b_i(j))$ and $\alpha_i = \mathrm{diag}(a_i(j))$, then this equation for $t = 0$ says for all $j \in \mathbb{Z}$

$$\mathrm{diag}(b_0(j)) = \mathrm{diag}(a_r(j))\mathrm{diag}(b_0(j+r)). \tag{32}$$

Thus one sees that in order to solve this equation, one can pick arbitrary invertible elements $b_0(s), 0 \leq s < r$, and then the equation (32) determines the element $\beta_0 \in \mathcal{D}_k(R)$ uniquely. Now that it has been shown how to find a solution for β_0, one may assume to have found all the β_l, with $l > s$. The next coefficient β_s of the operator U is then again fully determined by equation (31), once one has chosen the $b_s(v), 0 \leq v < r$. In this way one finds an operator U that satisfies the equations. All other solutions have the form UU_0, with $U_0 = \sum_{i\geq 0} u_i\Lambda^{ki}$, with all u_i in the image of the map i_k and u_0 invertible. To get the last statement in the proposition, one chooses every time the free matrices to be the identity. This completes the proof of the proposition. □

Remark 1. Proposition 1 implies that any element $\mathcal{L}_0$ in LT of order one in Λ^k with an invertible leading coefficient can be written as $\mathcal{L}_0 = U_0\Lambda^k U_0^{-1}$ with $U_0 \in ULT$ of order zero. Since the conjugation with U_0 is an automorphism of LT, one can decompose any matrix A in LT also uniquely as

$$A = \sum_{i\in\mathbb{Z}} a_i\mathcal{L}_0^i, \tag{33}$$

where all the a_i belong to $U_0\mathcal{D}_k(R)U_0^{-1}$. This kind of "coordinate transforms" occur naturally in integrable hierarchies, see e.g. [49], [18] and [38]. In LT the notions of level and order in $\mathcal{L}_0$ of an element A are defined in the same way as for Λ^k and one uses the notation

$$LT_{\leq m}(\mathcal{L}_0) = \{A \in LT \mid \text{ level of } A \text{ in } \mathcal{L}_0 \text{ is } m\}.$$

Thus one has

$$LT(R) := \bigcup_{m\in\mathbb{Z}} LT_{\leq m}(\mathcal{L}_0).$$

For each $\mathcal{L}_0$ as in remark (1) and the corresponding decomposition (33) one is interested in a specific way to split the elements of LT. If $A = \sum_{j\in\mathbb{Z}} a_j \mathcal{L}_0^j$ as in (33) then one writes

$$\begin{aligned} &A = A_{\geq 0}(\mathcal{L}_0) + A_{<0}(\mathcal{L}_0), \text{ with the components} \\ &A_{\geq 0}(\mathcal{L}_0) = \sum_{j\geq 0} a_j \mathcal{L}_0^j \text{ and } A_{<0}(\mathcal{L}_0) = \sum_{j<0} a_j \mathcal{L}_0^j \end{aligned} \tag{34}$$

From the multiplication rules for the $k \times k$-block diagonals it follows that this decomposition induces on the Lie algebra LT a splitting of LT in the direct sum of two Lie subalgebras

$$\begin{aligned} &LT = LT_{\geq 0}(\mathcal{L}_0) \oplus LT_{<0}(\mathcal{L}_0), \\ &\text{with } LT_{\geq 0}(\mathcal{L}_0) := \{A \in LT \mid A = A_{\geq 0}(\mathcal{L}_0)\} \\ &\text{and } LT_{<0}(\mathcal{L}_0) := \{A \in LT \mid A = A_{<0}(\mathcal{L}_0)\}. \end{aligned} \tag{35}$$

This decomposition lies at the basis of the system of the Lax equations in LT that we consider.

Remark 2. From the Proposition 1 it follows similarly that any element $\mathcal{M}_0$ in UT of order one in Λ^k with an invertible leading coefficient can be written as $\mathcal{M}_0 = V_0 \Lambda^k V_0^{-1}$ with $V_0 \in UT$ of order zero. Since conjugation with V_0 is an automorphism of UT, one can decompose any matrix A in UT also uniquely as

$$A = \sum_{i\in\mathbb{Z}} a_i \mathcal{M}_0^i, \tag{36}$$

where all the a_i belong to $V_0 \mathcal{D}_k(R) V_0^{-1}$. The notions of level and order in $\mathcal{M}_0$ of an element A in UT are defined in the same way as for Λ^k and one uses the notation

$$UT_{\geq m}(\mathcal{M}_0) = \{A \in UT \mid \text{ level of } A \text{ in } \mathcal{M}_0 \text{ is } m\}.$$

Thus one has

$$UT(R) := \bigcup_{m \in \mathbb{Z}} UT_{\geq m}(\mathcal{M}_0).$$

For each $\mathcal{M}_0$ as in remark (2) and the corresponding decomposition (36) one is interested in a specific way to split the elements of UT. If $A = \sum_{j \in \mathbb{Z}} a_j \mathcal{M}_0^j$ as in (36) then one writes

$$A = A_{<0}(\mathcal{M}_0) + A_{\geq 0}(\mathcal{M}_0), \text{ with the components} \tag{37}$$
$$A_{<0}(\mathcal{M}_0) = \sum_{j<0} a_j \mathcal{M}_0^j \text{ and } A_{\geq 0}(\mathcal{M}_0) = \sum_{j \geq 0} a_j \mathcal{M}_0^j$$

From the multiplication rules for the $k \times k$-block diagonals it follows that this decomposition induces on the Lie algebra UT a splitting of UT in the direct sum of two Lie subalgebras

$$UT = UT_{<0}(\mathcal{M}_0) \oplus UT_{\geq 0}(\mathcal{M}_0), \tag{38}$$
$$\text{with } UT_{<0}(\mathcal{M}_0) := \{A \in UT \mid A = A_{<0}(\mathcal{M}_0)\}$$
$$\text{and } UT_{\geq 0}(\mathcal{M}_0) := \{A \in UT \mid A = A_{\geq 0}(\mathcal{M}_0)\}.$$

They give rise to the system of Lax equations in UT considered in section 5 of this thesis. As with the direction Λ^k, one introduces also for $\mathcal{M}_0$ the more general notations

$$UT_{<r}(\mathcal{M}_0) := \{A \in UT \mid A = \sum_{i<r} a_i \mathcal{M}_0^i\} \text{ and}$$
$$UT_{\geq r}(\mathcal{M}_0) := \{A \in UT \mid A = \sum_{i \geq r} a_i \mathcal{M}_0^i\}.$$

3 Cauchy problems

The aim of this chapter is to discuss the formal solvability of an infinite dimensional Cauchy problem related to the hierarchies that we will consider. First the finite dimensional analogue will be treated.

3.1 A finite dimensional Cauchy problem

First one has a look at the scalar case. Consider on an open subset U in $\mathbb{C}^m$ with local coordinates $z_1, \cdots, z_r$ around the point $x_0 \in U$ and a collection of holomorphic functions $\{c_1, \cdots, c_r\}$ on U. It is well-known that the system of differential equations

$$\frac{\partial}{\partial z_i}(g) = c_i g, 1 \leq i \leq r, \tag{39}$$

has for a fixed value $g(x_0) \in \mathbb{C}$ locally a unique solution around x_0 if and only if all the compatibility equations

$$\frac{\partial}{\partial z_i}(c_j) = \frac{\partial}{\partial z_j}(c_i), i \text{ and } j \in \{1, \cdots, r\}, \tag{40}$$

hold. Next one considers the vector case of a collection of constant $n \times n$-matrices $\{C_1, \cdots, C_r\}$ and the set of differential equations for a $\mathbb{C}^n$-valued holomorphic function g

$$\frac{\partial}{\partial z_i}(g) = C_i g, 1 \leq i \leq r. \tag{41}$$

If one integrates this equation one variable at a time one sees that a solution, if it exists, must be given by

$$\exp(z_1 C_1) \cdots \exp(z_r C_r)(g(x_0)) = \exp(z_r C_r) \cdots \exp(z_1 C_1)(g(x_0)),$$

where the exponential factors can also be placed in any other order. This last fact implies that all the matrices $\{C_i\}$ have to commute:

$$[C_i, C_j] = 0, \text{ for all } i \text{ and } j \in \{1, \cdots, m\}. \tag{42}$$

The next step is to consider the equation (41) for matrices $\{C_i\}$ that are no longer constant and depend on a neighbourhood of x_0 in a holomorphic way of the local coordinates $z_1, \cdots, z_m$. The conditions under which one has a unique solution of these equations are a mixture of those in (40) and (42). To see how the conditions look like, assume that one has locally for each $\alpha \in \mathbb{C}^n$ a unique solution. Let the $\{e_i\}$ be the standard basis of the $\mathbb{C}^n$. Then there is in a neighbourhood of zero a unique solution g_i of (41) such that $g_i(x_0) = e_i$. Let G be the matrix with the $\{g_i\}$ as columns. It is a holomorphic matrix-valued function with $G(x_0) = \mathrm{Id}$. Hence it is invertible around zero and there holds:

$$\frac{\partial}{\partial z_i}(G) = C_i G, 1 \le i \le r. \tag{43}$$

From the equations

$$\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_i}(G) = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}(G)$$

one deduces

$$C_i C_j G + \frac{\partial}{\partial z_j}(C_i)G = C_j C_i G + \frac{\partial}{\partial z_i}(C_j)G$$

This is equivalent with

$$\left([C_i, C_j] - (\frac{\partial}{\partial z_i}(C_j) - \frac{\partial}{\partial z_j}(C_i))\right) G = 0$$

and since G is invertible, one gets

$$[C_i, C_j] - (\frac{\partial}{\partial z_i}(C_j) - \frac{\partial}{\partial z_j}(C_i)) = 0. \tag{44}$$

These conditions are necessary, but also sufficient. One way to proceed is to use the condition to show that you have a unique formal power series solution and next to prove that this solution converges as soon as the $\{C_i\}$ are convergent power series. This result goes back to Cauchy and Kovalevskaya, see [32], and for completeness we formulate it in a

Theorem 1. *Let x_0 be a point in $\mathbb{C}^r$ with local coordinates $z_1, \cdots, z_r$ and let the $C_1, \cdots, C_r$ be holomorphic matrix-valued functions on a neighbourhood of x_0. Then the equations*

$$\frac{\partial}{\partial z_i}(g) = C_i g, 1 \le i \le r, \tag{45}$$

possess locally a unique solution for fixed $g(x_0)$ if and only if the equations (44) hold.

The equations (44) are also called zero curvature relations because they relate to the vanishing of the curvature in the following setting: let Ω be the holomorphic matrix differential 1-form on $U \subset \mathbb{C}^r$ defined by

$$\Omega = \sum_{i=1}^{r} C_i dz_i$$

Theorem 1 gives a necessary and sufficient condition for the solvability of the linear Pfaffian system

$$dy = \Omega y, \qquad y(z) \in \mathbb{C}^n. \tag{46}$$

Consider over U the trivial vector bundle $E = U \times \mathbb{C}^n$. Let the $\{e_i\}$ be the standard basis of $\mathbb{C}^n$. They determine the trivializing sections $\{s_i \mid 1 \leq i \leq n\}$ of the bundle E by

$$s_i(x) = (x, e_i), x \in U.$$

The matrix Ω of 1-forms over U defines w.r.t. the $\{s_i\}$ a connection $\nabla = d - \Omega$ on the space of sections of E. This connection ∇ is a map from the space $A^0(E)$ of holomorphic sections of E or zero order forms with values in E to the space $A^1(E)$ of 1-forms with values in E satisfying

$$\nabla(f\sigma) = \sigma df + f\nabla(\sigma), \tag{47}$$

where $\sigma \in A^0(E)$ and $f \in A^0$, the space of holomorphic functions on the manifold. The connection has a natural extension $\nabla : A^p(E) \mapsto A^{p+1}(E)$ from the p-forms $A^p(E)$ with values in E to the $p+1$-forms $A^{p+1}(E)$ with values in E by the formula

$$\nabla(f\sigma) = \sigma \wedge df + f\nabla(\sigma),$$

with f as above and $\sigma \in A^p(E)$. The space of horizontal sections, i.e. those $y = \sum_{i=1}^{p} y_i s_i$ with $\nabla(y) = 0$, is p-dimensional, see [30], if and only if the curvature

$$\nabla \circ \nabla = -d\Omega + (-\Omega) \wedge (-\Omega)$$

of ∇ is zero, which is equivalent to the Pfaffian system being integrable in the sense of Frobenius, see [16] and [56]. The connection ∇ is then also called *integrable*. In terms of the matrices $\{C_i\}$ the integrability of ∇ amounts to the so-called *zero curvature* equations (44) for the$\{C_i\}$.

3.2 The Cauchy problem: infinite dimensional case

Here we want to discuss a formal power series version of the Cauchy problem discussed in the foregoing subsection, where both the size of the matrices and the number of variables is infinite. As for the first type of infinity, we treat here only $\mathbb{Z} \times \mathbb{Z}$-matrices, but note that also the semi-infinite case of $\mathbb{N} \times \mathbb{N}$-matrices is important, see [2] and similar results hold in that case. To stress the algebraic nature of our considerations and to give an equal footing to real and complex solutions we replace the complex numbers by a field F of characteristic zero.

We use the notation $F^{\mathbb{Z}}$ for the F-vector space of all $\mathbb{Z} \times 1$-matrices with coefficients from F. The product of any $\mathbb{Z} \times \mathbb{Z}$-matrix with a vector from $F^{\mathbb{Z}}$ with only a finite number of nonzero coordinates, is well-defined. Let e_j be the vector with its j-th coordinate equal to one and all others equal to zero. Then the $\{e_j \mid j \in \mathbb{Z}\}$ are a basis of the finite vectors $F^{\mathbb{Z}}_{fin}$ in $F^{\mathbb{Z}}$. The variables in the present set-up are the

$$\{z_i \mid i \in I\}, \text{ for some finite or countable index set } I = \{i_1.\cdots, i_n, \cdots\}.$$

One will use a multi-index notation for monomials in these variables: take any $\alpha = (\alpha_i) \in \mathbb{Z}^I_{\geq 0}$ with only a finite number of α_i nonzero. Then one writes

$$z^\alpha := \prod_{i \in I} z_i^{\alpha_i}.$$

On these multi-indices one uses the order relation

$$\alpha \leq \beta \Leftrightarrow \alpha_i \leq \beta_i \text{ for all } i \in I.$$

and the inequality $\alpha < \beta$ means that for one index $\alpha_i < \beta_i$. For simplicity the zero index is denoted by 0. The degree $\deg(\alpha)$ of the multi-index α is given by

$$\deg(\alpha) := \sum_{i \in I} \alpha_i.$$

Assume one has for each $i \in I$ an infinite matrix C_i with coefficients from the formal power series in the variables $\{z_i\}$, i.e.

$$C_i = \sum_{\alpha \geq 0} C_i(\alpha) z^\alpha, \tag{48}$$

where each $C_i(\alpha)$ is a matrix in $M_{\mathbb{Z}}(F)$. Now one looks for formal power series solutions of the system of equations

$$\frac{\partial}{\partial z_i}(g) = C_i g, i \in I, \tag{49}$$

where $g \in F[[z_i]]^{\mathbb{Z}}$ i.e.

$$g = \sum_{\alpha \geq 0} g(\alpha) z^{\alpha}, g(\alpha) \in F^{\mathbb{Z}}.$$

In particular, one needs that all the products $C_i g$ are well-defined vectors in $F[[z_i]]^{\mathbb{Z}}$ and that is the case if all the $C_i(\alpha)g(\beta)$ are well-defined vectors, for then the right hand side of equation (49) becomes

$$\sum_{\gamma \geq 0} \left(\sum_{0 \leq \alpha \leq \gamma} C_i(\alpha) g(\gamma - \alpha) \right) z^{\gamma}.$$

Let for each $i \in I$, the multi-index $1(i)$ be defined by

$$1(i)_j = \begin{cases} 0 \text{ for } i \neq j \\ 1 \text{ for } j = i \end{cases}$$

Then the left hand side of equation (49) for the i-th case equals

$$\sum_{\alpha \geq 1(i)} \alpha_i g(\alpha) z^{\alpha - 1(i)}$$

and this leads for all $\gamma \geq 0$ to the identities

$$(\gamma_i + 1) g(\gamma + 1(i)) = \sum_{0 \leq \alpha \leq \gamma} C_i(\alpha) g(\gamma - \alpha). \tag{50}$$

In particular, all the coefficients of g corresponding to degree one multi-indices have to satisfy

$$g(1(i)) = C_i(0) g(0)$$

and with the relations (50) one shows with induction on the degree of the multi-index α that each coefficient $g(\alpha)$ is a polynomial expression in the matrices $\{C_i(\beta) \mid i \in I, \beta \geq 0\}$ acting on the vector $g(0)$. This proofs that, if g exists, it is uniquely determined by $g(0)$. Thus one sees that the condition on the existence of the product of all the $C_i(\alpha)g(\beta)$ is fulfilled if all finite products of the matrices $\{C_i(\beta) \mid i \in I, \beta \geq 0\}$ exist. This last condition is also necessary to discuss zero curvature relations, so it is natural to make it a general

Assumption 1. All finite products of the $\{C_i(\beta) \mid i \in I, \beta \geq 0\}$ are well-defined.

The assumption 1 is clearly satisfied if all the $C_i(\beta)$ belong to $UT(F)$ or $LT(F)$. Assuming that one has for each element e_j in the basis a solution g_j such that $g_j(0) = e_j$, then one can form the matrix G with the g_j as its columns. It decomposes w.r.t. the variables $\{z_i\}$ as

$$G = G(z) = G(z_{i_1}, z_{i_2}, \cdots) = \sum_{\alpha \geq 0} G(\alpha) z^\alpha = \mathrm{Id} + \sum_{\alpha > 0} G(\alpha) z^\alpha$$

The matrix G is called the formal *fundamental matrix* of the Cauchy problem (49) with $G(0) = \mathrm{Id}$. It satisfies for all $i \in I$

$$\frac{\partial}{\partial z_i}(G) = C_i G. \tag{51}$$

In particular, being able to solve the equations (51) is equivalent to having a unique solution of (49) for each $v \in F^{\mathbb{Z}}_{fin}$. Note that the equations (51) imply that the coefficients $G(\alpha)$ are polynomial expressions in the matrices $C_i(\beta)$ and therefore all finite products of the coefficients $G(\alpha)$ exist. This fact allows you to prove another property that G shares with the fundamental matrix from the finite-dimensional context

Lemma 3. *The matrix G has a formal inverse, i.e. there exists an*

$$H = \sum_{\beta \geq 0} H(\beta) z^\beta = \mathrm{Id} + \sum_{\beta > 0} H(\beta) z^\beta$$

such that $GH = \mathrm{Id}$.

Proof. Multiplying the two formal series G en H yields for each $\gamma > 0$:

$$\sum_{0 \leq \alpha \leq \gamma} G(\alpha) H(\gamma - \alpha) = 0 \Leftrightarrow H(\gamma) = - \sum_{0 < \alpha \leq \gamma} G(\alpha) H(\gamma - \alpha).$$

For the multi-indices α of degree -1 this gives you

$$H(\alpha) = -G(\alpha)$$

and by induction on the degree of the relevant multi-indices one shows that each $H(\alpha)$ with $\deg(\alpha) = k$ is a polynomial expression of degree utmost k in the coefficients $\{G(\beta) \mid \deg(\beta) \leq k\}$. $\square$

Due to Assumption 1 it makes sense to consider the equations

$$\frac{\partial}{\partial z_j}\frac{\partial}{\partial z_i}(G) = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j}(G)$$

and they yield

$$C_iC_jG + \frac{\partial}{\partial z_j}(C_i)G = C_jC_iG + \frac{\partial}{\partial z_i}(C_j)G$$

This is the same as

$$\left([C_i, C_j] - (\frac{\partial}{\partial z_i}(C_j) - \frac{\partial}{\partial z_j}(C_i))\right) G = 0$$

and since G is invertible, one gets the formal version of the zero curvature relations

$$[C_i, C_j] - (\frac{\partial}{\partial z_i}(C_j) - \frac{\partial}{\partial z_j}(C_i)) = 0, \text{ for all } i \text{ and } j \in I. \tag{52}$$

We have seen now that under the Assumption 1 the equations (52) are necessary to solve the system (49). They are also sufficient as we will show now. What one needs to show is that the power series coefficients $G(\alpha)$ of G are defined in an unambiguous way by the relations: for all $\gamma \geq 0$

$$(\gamma_i + 1)G(\gamma + 1(i)) = \sum_{0 \leq \alpha \leq \gamma} C_i(\alpha)G(\gamma - \alpha). \tag{53}$$

First we have a look at the multi-indices α in which only one variable occurs. Then one gets the recursion

$$(k_i + 1)G((k_i + 1)1(i)) = \sum_{0 \leq m \leq k_i} C_i(m1(i))G((k_i - m)1(i)) \tag{54}$$

and this determines the $G(k_i 1(i))$ inductively in an unambiguous way starting from $G(0) = \mathrm{Id}$. Moreover the power series $G(\cdots, 0, z_i, 0, \cdots)$ is built such that it satisfies

$$\frac{\partial}{\partial z_i}(G(\cdots, 0, z_i, 0, \cdots)) = C_i(\cdots, 0, z_i, 0, \cdots)G(\cdots, 0, z_i, 0, \cdots)$$

The idea is now to use induction w.r.t. n, the number of variables really present in the multi-index α, i.e. the number of elements in $\{i \mid \alpha_i > 0\}$. Thus we may assume that any coefficient $G(\beta)$ corresponding to n or less variables is well-defined and one considers a coefficient $G(\alpha)$ in which $n + 1$ variables occur. One may assume that

$$\alpha = \sum_{k=1}^{n+1} \alpha_{i_k} 1(i_k),$$

otherwise one can reduce to this case by rearranging. It is sufficient to show that the compatibility conditions allow you to introduce a well-defined power series

$$G(z_{i_1}, \cdots, z_{i_{n+1}}, 0, \cdots)$$

that satisfies for all $k, 1 \leq k \leq n+1$,

$$\frac{\partial}{\partial z_{i_k}}(G(z_{i_1}, \cdots, z_{i_{n+1}}, 0, \cdots)) = C_{i_k}(z_{i_1}, \cdots, z_{i_{n+1}}, 0, \cdots) G(z_{i_1}, \cdots, z_{i_{n+1}}, 0, \cdots).$$

To define the coefficients $G(\alpha), \alpha_{i_k} > 0$ for all $k, 1 \leq k \leq n+1$, one has $n+1$ possibilities each corresponding to the variable z_{i_k} for which one uses the recursion

$$(\alpha_{i_k}) G(\alpha) = \sum_{0 \leq \gamma \leq \alpha - 1(i_k)} C_{i_k}(\gamma) G(\alpha - 1(i_k) - \gamma). \tag{55}$$

Each choice gives you a power series $G_k(z_{i_1}, \cdots, z_{i_{n+1}}, 0, \cdots)$ Note that each G_k is constructed in such a way that it satisfies for the variable z_{i_k} the equation

$$\frac{\partial}{\partial z_{i_k}}(G_k) = C_{i_k}(z_{i_1}, \cdots, z_{i_{n+1}}, 0, \cdots) G_k. \tag{56}$$

Next we show that all the G_k are equal to G_{n+1}. Note that the two series agree on terms with n variables or less so that they agree in particular in the terms with only one variable and hence it is sufficient to prove the equality

$$\frac{\partial}{\partial z_{i_{n+1}}} \frac{\partial}{\partial z_{i_k}}(G_k) - \frac{\partial}{\partial z_{i_k}} \frac{\partial}{\partial z_{i_{n+1}}}(G_{n+1}) = 0$$

To the left hand side one applies the equations (56) and substitutes the zero curvature relations and this gives for the left hand side

$$\frac{\partial}{\partial z_{i_{n+1}}}(C_{i_k})G_k + C_{i_k}\frac{\partial}{\partial z_{i_{n+1}}}(G_k) - \frac{\partial}{\partial z_{i_k}}(C_{i_{n+1}})G_k -$$
$$C_{i_{n+1}}\frac{\partial}{\partial z_{i_k}}(G_{n+1}) = \frac{\partial}{\partial z_{i_{n+1}}}(C_{i_k})G_{n+1} + C_{i_k}\frac{\partial}{\partial z_{i_{n+1}}}(G_{n+1})+$$
$$\frac{\partial}{\partial z_{i_{n+1}}}(C_{i_k})(G_k - G_{n+1}) + C_{i_k}\frac{\partial}{\partial z_{i_{n+1}}}(G_k - G_{n+1}) - -\frac{\partial}{\partial z_{i_k}}(C_{i_{n+1}})G_{n+1} -$$
$$C_{i_{n+1}}\frac{\partial}{\partial z_{i_k}}(G_k) - \frac{\partial}{\partial z_{i_k}}(C_{i_{n+1}})(G_k - G_{n+1}) - C_{i_{n+1}}\frac{\partial}{\partial z_{i_k}}(G_{n+1} - G_k) =$$
$$\frac{\partial}{\partial z_{i_{n+1}}}(C_{i_k})G_{n+1} + C_{i_k}C_{i_{n+1}}G_{n+1} + \frac{\partial}{\partial z_{i_{n+1}}}(C_{i_k})(G_k - G_{n+1})+$$
$$C_{i_k}\frac{\partial}{\partial z_{i_{n+1}}}(G_k - G_{n+1}) - -\frac{\partial}{\partial z_{i_k}}(C_{i_{n+1}})G_{n+1} - C_{i_{n+1}}C_{i_k}G_k -$$
$$\frac{\partial}{\partial z_{i_k}}(C_{i_{n+1}})(G_k - G_{n+1}) - C_{i_{n+1}}\frac{\partial}{\partial z_{i_k}}(G_{n+1} - G_k) = \frac{\partial}{\partial z_{i_{n+1}}}(C_{i_k})(G_k - G_{n+1})+$$
$$C_{i_k}\frac{\partial}{\partial z_{i_{n+1}}}(G_k - G_{n+1}) - \frac{\partial}{\partial z_{i_k}}(C_{i_{n+1}})(G_k - G_{n+1}) - C_{i_{n+1}}\frac{\partial}{\partial z_{i_k}}(G_{n+1} - G_k)$$

Assume that $G_k - G_{n+1}$ is nonzero and has a non trivial coefficient for the multi-index γ, then $\frac{\partial}{\partial z_{i_{n+1}}}\frac{\partial}{\partial z_{i_k}}(G_k) - \frac{\partial}{\partial z_{i_k}}\frac{\partial}{\partial z_{i_{n+1}}}(G_{n+1})$ has a non trivial coefficient for the multi-index $\gamma - 1(i_k) - 1(i_{n+1})$. If one chooses γ to be one of the lowest mult-indices of $G_k - G_{n+1}$ for which there is a nonzero coefficient, then this contradicts the fact that $G_k - G_{n+1}$, $\frac{\partial}{\partial z_{i_k}}(G_{n+1} - G_k)$ and $\frac{\partial}{\partial z_{i_{n+1}}}(G_{n+1} - G_k)$ do not possess such coefficients. Hence $G_k - G_{n+1}$ has to be zero. We resume this result in a

Theorem 2. *Under Assumption 1 the equations (51) possess a unique formal power series solution G with $G(0) = \mathrm{Id}$ if and only if the zero curvature relations (52) hold for the formal power series $\{C_i\}$.*

One can consider a variant of the equations (51) with the starting value $G(0) = D$, where D is an invertible matrix different from the identity and in that case one has the same theorem if one makes another assumption

Assumption 2. All finite products of the $C_i(\alpha)$ multiplied from the right with the matrix D give you well-defined matrices.

This condition is superfluous if all the $C_i(\alpha)$ are finite band matrices.

Remark 3. Now that we have shown that the zero curvature relations are a necessary and sufficient condition for the solvability of the Cauchy problem in the formal power series setting, a natural step would be to develop convergence estimates. We will not pursue this line of approach, but instead we will develop geometric frameworks to obtain convergent solutions, an approach which is also successful in a more general context, see the next remark.

Remark 4. In this subsection we have shown under mild assumptions the existence of a fundamental matrix for a collection of $\mathbb{Z} \times \mathbb{Z}$-matrices $\{C_i \mid i \in I\}$ with coefficients in $F[[z_i]]$ if the zero curvature equations hold for this collection of matrices. This fundamental matrix was moreover unique up to multiplication from the right with an invertible matrix from $M_{\mathbb{Z}}(R)$. A natural generalization would be to allow singularities in the set-up and to consider localizations $R = S^{-1}F[[z_i]]$ of the ring $F[[z_i]]$ with S a multiplicative subset of $F[[z_i]]$. All the partial derivatives $\frac{\partial}{\partial z_i}$ are then derivations of R. Assuming then that all the matrices $\{C_i \mid i \in I\}$ have coefficients in this bigger algebra R and that all finite products of the $\{C_i\}$ exist , then one might wonder if there exists an invertible matrix $G \in M_{\mathbb{Z}}(R)$ that satisfies the equations (51). Such a matrix G will be called a *fundamental matrix* corresponding to the $\{C_i \mid i \in I\}$. As in the case of $R = F[[z_i]]$ one derives that a necessary condition for the existence of a fundamental matrix are the zero curvature relations (52) for the $\{C_i\}$.

A solution to this problem does not have to exist. Consider namely the case of one variable z_1, the multiplicative set $S = \{z_1^m \mid m \in \mathbb{N}\}$ and the function $C_1(z_1) = \frac{a}{z_1}, a \notin \mathbb{Z}$. Then one verifies directly that the equation

$$\frac{\partial}{\partial z_1}(F(z_1)) = \frac{a}{z_1}F(z_1)$$

does not have a solution of the form $F(z_1) = \sum_{i=m}^{\infty} a_i z_1^i, m \in \mathbb{Z}$.

As for uniqueness, let G_1 and G_2 be fundamental matrices for the $\{C_i\}$ and let $D = G_1^{-1}G_2$ be a well-defined matrix, the analogue of the Assumption 2 in the present setting. Then there holds

$$\frac{\partial}{\partial z_i}(G_2) = \frac{\partial}{\partial z_i}(G_1)D + G_1\frac{\partial}{\partial z_i}(D) = C_iG_2 + G_1\frac{\partial}{\partial z_i}(D) = C_iG_2 \qquad (57)$$

and thus the matrix D is constant, i.e. $\frac{\partial}{\partial z_i}(D) = 0$ for all i. Despite the lack of a general existence theorem, one can can often use geometry to construct fundamental matrices. Various examples of this phenomenon can be found in this thesis.

4 Lower triangular hierarchies

4.1 Lax equations in LT

The aim of this section is to describe the algebraic structure that lies at the basis of systems of equations like (17). The coefficients of the $\mathbb{Z} \times \mathbb{Z}$-matrices in these equations will satisfy a mixture of differential-difference equations and are taken from a commutative ring R. Since this structure is similar for real and complex solutions, we assume that R is an algebra over a field F of characteristic zero, whose elements are constant for the derivations involved. Let $\mathcal{L}_0$ in $LT(F)$ be an element of order one in Λ^k as in Remark 1. It is one of the basic matrices in $LT(F)$ on which the system that we will consider is based. The system consists of compatible equations of the form

$$\frac{\partial A}{\partial t} = [P, A], \tag{58}$$

where P belongs to $LT_{\geq 0}(\mathcal{L}_0)$ and A has order ≥ 0 in $\mathcal{L}_0$. From equation (58) one can see directly that something special is going on. Namely, the order of $\frac{\partial A}{\partial t}$ in $\mathcal{L}_0$ will be less or equal to that of A, while the order in $\mathcal{L}_0$ of the right hand side $[P, A]$ is less or equal to the sum of order(A) and order(P).

Thus one arrives at the natural question: given a matrix $A = \sum_{j \leq l} \alpha_j \mathcal{L}_0^j$ of positve order in $\mathcal{L}_0$, how to find matrices P in $LT_{\geq 0}(\mathcal{L}_0)$, such that its commutator with A has order in $\mathcal{L}_0$ smaller or equal to that of A. For solutions to this question it makes sense to consider equations like (58) for A.

There is a natural choice for such matrices. Consider namely the centralizer $Z_{LT}(A)$ of A in LT. Let Z be an element in $Z_{LT}(A)$. Because of the relation

$$[Z_{\geq 0}(\mathcal{L}_0) + Z_{<0}(\mathcal{L}_0), A] = 0,$$

one sees that

$$[Z_{\geq 0}(\mathcal{L}_0), A] = -[Z_{<0}(\mathcal{L}_0), A].$$

Hence $[Z_{\geq 0}(\mathcal{L}_0), A]$ has order $\leq$ order(A)-1. For such matrices Z it makes sense to look for suitable F-linear derivations ∂_Z of R such that the equation

$$\partial_Z(A) = [Z_{\geq 0}(\mathcal{L}_0), A] \tag{59}$$

holds. Recall that the action of ∂_Z on elements of $M_{\mathbb{Z}}(R)$ is defined coefficientwise. Note that the equation (59) becomes trivial, if one takes an element Z in $Z_{LT}(A)$ of negative order in $\mathcal{L}_0$. Therefore, the equation (59) will only be considered for $Z \in Z_{LT}(A)$ that have a positive order in $\mathcal{L}_0$. Before one continues the discussion, note the following property:

Remark 5. If a matrix A satisfies the equation (59), then this equation implies for its leading coefficient α_l that it is a constant for ∂_P, i.e. $\partial_P(\alpha_l) = 0$. The commutator $[P, A]$ has then namely order smaller or equal then order(A)-1 in $\mathcal{L}_0$.

The equation (59) will be considered for various choices of A and P, starting from a basic set of commuting directions in LT that includes the direction $\mathcal{L}_0$. Thereto we consider the centralizer $Z_{LT}(\mathcal{L}_0)$ of $\mathcal{L}_0$ in LT. If $k = 1$, then $Z_{LT}(\mathcal{L}_0)$ is equal to

$$\{\sum_i a_i \mathcal{L}_0^i \in LT \mid a_i \in R \text{ for all } i\}$$

and this ring is commutative. Hence in that case the powers of $\mathcal{L}_0$ basicly span the space of directions commuting with $\mathcal{L}_0$ and to avoid trivial relations (59) one considers only the basic directions $\{\mathcal{L}_0^i \mid i \geq 0\}$.

However, if $k > 1$, then an element of $Z_{LT}(\mathcal{L}_0)$ has the form

$$\sum_i a_i \mathcal{L}_0^i \in LT, \text{ with } a_i \in U_0 i_k(M_k(R)) U_0^{-1},$$

with $U_0 = \sum_{i \leq 0} u_i \Lambda^{ik} \in I(LT(F))$ such that $\mathcal{L}_0 = U_0 \Lambda^k U_0^{-1}$. In particular, the centralizer $Z_{LT}(\mathcal{L}_0)$ is no longer commutative. Therefore, one chooses a number of matrices

$$\mathcal{U}_{\alpha,0} = U_0 i_k(E_\alpha) U_0^{-1}, E_\alpha \in M_k(F),$$

in LT of order zero in $\mathcal{L}_0$. They commute clearly with $\mathcal{L}_0$ and one requires that they also commute between each other

$$[\mathcal{U}_{\alpha,0}, \mathcal{U}_{\beta,0}] = 0 \text{ for all } \alpha \text{ and } \beta.$$

To include as many directions as possible, one assumes that the

$$\{E_\alpha \mid 1 \leq \alpha \leq m_0\}$$

are a basis over F of a maximal commutative F-submodule $\mathfrak{h}$ of $M_k(F)$. The $\{\mathcal{U}_{\alpha,0}\}$ are then a basis of the maximal commutative F-submodule $\mathbf{h}$ of the twisted algebra $U_0 M_k(F) U_0^{-1}$. Here one can think of the diagonal matrices, but also of algebras like

$$\mathfrak{h} = \left\{ h = \sum_{i=0}^{k-1} a_i B^i \text{ with } B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \right\} \tag{60}$$

with a significant nilpotent component. The interesting commuting directions for the equation (59) are then the

$$\{\mathcal{U}_{\alpha,0}\mathcal{L}_0^i \mid i \geq 0, \alpha \in \{1, \cdots, m_0\}\}.$$

To get consistency of notation for the cases $k = 1$ and $k > 1$, one puts $\mathcal{U}_{1,0} = \mathrm{Id}$ if $k = 1$. Now that the basic directions have been fixed, one makes several choices for the matrices A. In fact, one takes for A deformations of the basic directions. In particular, the deformation of $\mathcal{L}_0$ is obtained by dressing this direction with an element from $I(LT)$ of order zero. In order to avoid dressing the leading coefficients with constant matrices, see Remark 5, one gauges the deformations in such a way that the leading coefficients are not effected. One starts with the deformation of the direction $\mathcal{L}_0$

$$\mathcal{L} := \sum_{i \leq 1} l_i \mathcal{L}_0^i = \mathcal{L}_0 + \sum_{i \leq 0} l_i \mathcal{L}_0^i \tag{61}$$

It can be written according to Proposition 27 in the form

$$\mathcal{L} = U\mathcal{L}_0 U^{-1}, \text{ with } U = \mathrm{Id} + \sum_{i \leq -1} u_i \mathcal{L}_0^i. \tag{62}$$

If $k = 1$, then the centralizer of $\mathcal{L}$ in LT is

$$\{\sum_i a_i \mathcal{L}^i \in LT \mid a_i \in R \text{ for all } i\},$$

which is clearly commutative and it suffices to consider the equations (59) only for the generator $\mathcal{L}$. For $k > 1$, we also deform in the assigned way the basic directions of order zero

$$\mathcal{U}_\alpha = \sum_{i \le 0} u_{i,\alpha} \mathcal{L}_0^{ki} = \mathcal{U}_{\alpha,0} + \sum_{i<0} u_{i,\alpha} \mathcal{L}_0^{ki} \tag{63}$$

The element A is chosen to be one of these deformations $(\mathcal{L}, \mathcal{U}_\alpha)$ of the basic generators. Now one requires that the $(\mathcal{L}, \mathcal{U}_\alpha)$ as deformations of the basic generators $(\mathcal{L}_0, \mathcal{U}_{\alpha,0})$ should respect the algebraic relations

$$[\mathcal{L}, \mathcal{U}_\alpha] = 0 \text{ and } [\mathcal{U}_\alpha, \mathcal{U}_\beta] = 0 \text{ for all } \alpha \text{ and } \beta, \tag{64}$$

These relations are trivially satisfied if the deformations $(\mathcal{L}, \mathcal{U}_\alpha)$ are obtained by dressing the basic generators with the same matrix

$$\mathcal{L} = U \mathcal{L}_0 U^{-1} \text{ and } \mathcal{U}_\alpha = U \mathcal{U}_{\alpha,0} U^{-1} \tag{65}$$

For each $i \ge 0$ and all $\alpha \in \{1, \cdots, m_0\}$ one takes as the deformation of $\mathcal{U}_{\alpha,0}\mathcal{L}_0^i$ then the matrix $P_{i\alpha} := \mathcal{L}^i \mathcal{U}_\alpha$. Because of the relations (64) one can choose for the matrices Z in the centralizer of A all the $P_{i\alpha}$. If one has for all $i \ge 0$ and all $\alpha \in \{1, \cdots, m_0\}$ an F-linear derivation $\partial_{P_{i\alpha}}$ of R, then one can consider all these equations (59) simultaneously and the nonlinear differential equations that one wants the $\mathcal{L}$ and the $\mathcal{U}_\beta$ to satisfy, are

$$\partial_{P_{i\alpha}}(\mathcal{L}) = [(P_{i\alpha})_{\ge 0}(\mathcal{L}_0), \mathcal{L}] \text{ and } \partial_{P_{i\alpha}}(\mathcal{U}_\beta) = [(P_{i\alpha})_{\ge 0}(\mathcal{L}_0), \mathcal{U}_\beta]. \tag{66}$$

The basic generators $(\mathcal{L}_0, \mathcal{U}_{\beta,0})$ satisfy these equations trivially. For one has for all $i \ge 0$ and all $\alpha, 1 \le \alpha \le m_0$, that

$$(\mathcal{L}_0^i \mathcal{U}_{\alpha,0})_{\ge 0}(\mathcal{L}_0) = \mathcal{L}_0^i \mathcal{U}_{\alpha,0}$$

and therefore the action of all the derivations $\partial_{P_{i\alpha}}$ on them is trivial. The equations (64) and (66) for matrices $\mathcal{L}$ and the $\mathcal{U}_\beta$ form the equations of the $(\mathcal{L}_0, \mathbf{h}_{\ge 0})$-hierarchy as they correspond to the commuting flows with as generators the

$$\{\mathcal{U}_{\alpha,0}\mathcal{L}_0^i \mid i \ge 0, \alpha \in \{1, \cdots, m_0\}\},$$

which generate a maximal commutative subalgebra of the F-matrices in $LT_{\geq 0}(\mathcal{L}_0)$. The equations (66) are called the *Lax equations* of the hierarchy because of their similarity to the equations (17).

Example 1. A concrete context, where one can look for realizations of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy is the following: for the ring R one takes the algebra $F[t_{i\alpha}]$ of all polynomials in the parameters $\{t_{i\alpha} \mid i \geq 0, 1 \leq \alpha \leq m_0\}$ or, more generally, that of all formal power series $F[[t_{i\alpha}]]$ in these variables. You consider matrices $\mathcal{L}$ and the U_β in $M_{\mathbb{Z}}(R)$ that have first of all the form (61) resp. (63) and secondly satisfy the commutation relations (64). For the derivations $\partial_{P_{i\alpha}}$ one takes then the partial derivative $\partial_{t_{i\alpha}}$ w.r.t. the variable $t_{i\alpha}$. The nonlinear equations one wants the $\mathcal{L}$ and the U_β to satisfy are then

$$\partial_{t_{i\alpha}}(\mathcal{L}) = [(P_{i\alpha})_{\geq 0}(\mathcal{L}_0), \mathcal{L}] \text{ and } \partial_{t_{i\alpha}}(U_\beta) = [(P_{i\alpha})_{\geq 0}(\mathcal{L}_0), U_\beta] \qquad (67)$$

Note that one has silently assumed in this example that the derivations $\partial_{P_{i\alpha}}$ commute among themselves. We come back to this issue in the next subsection.

Example 2. Other examples of possible algebras R are the localizations $S^{-1}F[[t_{i\alpha}]]$ with S a multiplicative subset of $F[[t_{i\alpha}]]$. As in example 1 one takes for $\partial_{P_{i\alpha}}$ the partial derivative $\partial_{t_{i\alpha}}$ and one can look for commuting matrices $\mathcal{L}$ and the U_β in $M_{\mathbb{Z}}(S^{-1}F[[t_{i\alpha}]])$ that have the form (61) and (63) and satisfy the equations (119). By allowing singularities in the solutions one may fail to make the specialization $t_{i\alpha} = 0$ for all i and α.

Remark 6. The dependence of the solutions of the hierarchy of the basic directions is as follows: if $\mathfrak{h}$ is a maximal commutative F-submodule of $i_k(M_k(F))$, then there is a bijection between solutions $\mathcal{L}$ and $\{\mathcal{U}_\alpha\}$ of the $(\Lambda^k, \mathfrak{h}_{\geq 0})$-hierarchy and those of the $(\mathcal{L}_0, (U_0\mathfrak{h}U_0^{-1})_{\geq 0})$-hierarchy , namely

$$\mathcal{L}, \{\mathcal{U}_\alpha\} \mapsto U_0\mathcal{L}U_0^{-1}, \{U_0\mathcal{U}_\alpha U_0^{-1}\}.$$

The reason is that for all $i \geq 0$ and all $\alpha \in \{1, \cdots, m_0\}$

$$U_0(\mathcal{L}^i\mathcal{U}_\alpha)_{\geq 0}(\Lambda)U_0^{-1} = (U_0\mathcal{L}^i\mathcal{U}_\alpha U_0^{-1})_{\geq 0}(\mathcal{L}_0).$$

Since the general case gives better insight in the full geometric picture and it requires no additional effort, one presents the general case here.

4.2 A minimal realization of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy

In this subsection one discusses a minimal realization of the relations (64) and (66) in the style of [55], where minimal refers to the number of relations between the solutions. Start with a collection of $\mathbb{Z} \times \mathbb{Z}$-matrices of the right form

$$\tilde{\mathcal{L}} := \mathcal{L}_0 + \sum_{j \leq 0} \tilde{l}_j \mathcal{L}_0^j, \;\; \tilde{\mathcal{U}}_\alpha := \mathcal{U}_{\alpha,0} + \sum_{i<0} \tilde{u}_{i,\alpha} \mathcal{L}_0^i.$$

Then the matrices $\{\tilde{l}_j\}$ and the $\{\tilde{u}_{i,\alpha}\}$ can be written as

$$\tilde{l}_j = U_0 \mathrm{diag}(\tilde{l}_j(s)) U_0^{-1} \text{ and } \tilde{u}_{i,\alpha} = U_0 \mathrm{diag}(\tilde{u}_{i,\alpha}(t)) U_0^{-1} \tag{68}$$

and this shows that the matrix coefficients of the $\tilde{l}_j$ resp. the $\tilde{u}_{i,\alpha}$ are polynomial expressions with coefficients from the field F in the coefficients $\tilde{l}_j(s)_{\gamma\delta}$ resp. $\tilde{u}_{i,\alpha}(t)_{\rho\sigma}$ of the $\{\tilde{l}_j(s)\}$ resp. $\{\tilde{u}_{i,\alpha}(t)\}$. The idea is now to take these matrix coefficients of all the $\{\tilde{l}_j(s)\}$ and $\{\tilde{u}_{i,\alpha}(t)\}$ as independent as possible. One starts with taking them totally independent. Observe thereto that the coefficients of the matrices $\tilde{\mathcal{L}}$ and the $\{\tilde{\mathcal{U}}_\alpha\}$ belong to $M_{\mathbb{Z}}(R_{\geq 0})$, where the ring $R_{\geq 0}$ is defined as

$$R_{\geq 0} := F[\tilde{l}_j(s)_{\gamma\delta}, \tilde{u}_{i,\alpha}(t)_{\rho\sigma}],$$

with all $\{\gamma,\ \delta,\ \rho,\ \sigma\}$ in $\{1, \ldots, k\}$; both $s,$ and $t \in \mathbb{Z}; j < 1, i < 0$. By conjugating with U_0^{-1} one sees that the matrix coefficients of all the $\{\tilde{l}_j\}$ and the $\{\tilde{u}_{i,\alpha}\}$ are also generators of the ring $R_{\geq 0}$.

Note that every F-linear derivation $\Delta \in \mathrm{Der}_F(R_{\geq 0})$ is completely determined by describing freely all the

$$\{\Delta(\tilde{l}_j(s)_{\gamma\delta}),\ \Delta(\tilde{u}_{i,\alpha}(t)_{\rho\sigma})\}.$$

As the matrix coefficients of the leading terms of $\tilde{\mathcal{L}}$ and the $\tilde{\mathcal{U}}_\alpha$ are constants for all F-linear derivations, every F-linear derivation of $R_{\geq 0}$ acting coefficientwise on these matrices renders a matrix of order in $\mathcal{L}_0$ at least one less than the original. The same property holds for the commutator of one of these matrices with a matrix of order ≤ -1 in $\mathcal{L}_0$. Taking these considerations into account and since the matrix coefficients of $\tilde{l}_j$ and the $\tilde{u}_{i,\alpha}$ are also generators of the ring $R_{\geq 0}$, one can define for each $\tilde{P}$ in $LT(R_{\geq 0})$ of degree ≥ 0 in $\mathcal{L}_0$ a unique derivation $\tilde{\partial}_{\tilde{P}}: R_{\geq 0} \to R_{\geq 0}$ by the matrix equalities

$$\tilde{\partial}_{\tilde{P}}(\tilde{\mathcal{L}}) := -[\tilde{P}_{<0}(\mathcal{L}_0), \tilde{\mathcal{L}}] \text{ and } \tilde{\partial}_{\tilde{P}}(\tilde{\mathcal{U}}_\alpha) := -[\tilde{P}_{<0}(\mathcal{L}_0), \tilde{\mathcal{U}}_\alpha]. \quad (69)$$

The specific $\tilde{P}$ we are interested in of course are the $\tilde{P}_{i\alpha} = \tilde{\mathcal{L}}^i\tilde{\mathcal{U}}_\alpha$, $i \geq 0$ and $\alpha, 1 \leq \alpha \leq m_0$. Now, one wants the $\tilde{\mathcal{L}}$ and the $\{\tilde{\mathcal{U}}_\alpha\}$ to commute and therefore one imposes on the independent variables $\tilde{l}_j(s)_{\gamma\delta}$ and $\tilde{u}_{i,\alpha}(t)_{\rho\sigma}$ first of all the relations

$$[\tilde{\mathcal{L}}, \tilde{\mathcal{U}}_\alpha] = 0 \text{ and } [\tilde{\mathcal{U}}_\alpha, \tilde{\mathcal{U}}_\beta] = 0 \text{ for all } \alpha \text{ and } \beta. \quad (70)$$

The quotient of the ring $R_{\geq 0}$ by the ideal $I_{\geq 0}$ generated by the matrix coefficients of the left hand sides of the relations (70) is denoted by $\underline{R}_{\geq 0}$. We will use the same notation for the natural images in $M_{\mathbb{Z}}(\underline{R}_{\geq 0})$ of the matrices $\tilde{\mathcal{L}}$, the $\{\tilde{\mathcal{U}}_\alpha\}$ and the $\tilde{P}_{i\alpha}$. Note that the derivations $\tilde{\partial}_{\tilde{P}}$ also factorize to derivations of $\underline{R}_{\geq 0}$, for one has

$$\begin{aligned} \tilde{\partial}_{\tilde{P}}([\tilde{\mathcal{L}}, \tilde{\mathcal{U}}_\alpha]) &= [-\tilde{P}_{<0}(\mathcal{L}_0), [\tilde{\mathcal{L}}, \tilde{\mathcal{U}}_\alpha]], \\ \tilde{\partial}_{\tilde{P}}([\tilde{\mathcal{U}}_\alpha, \tilde{\mathcal{U}}_\beta]) &= [-\tilde{P}_{<0}(\mathcal{L}_0), [\tilde{U}_\alpha, \tilde{\mathcal{U}}_\beta]] \end{aligned} \quad (71)$$

and thus $\tilde{\partial}_{\tilde{P}}$ maps the ideal I into itself. One uses the same notation for the corresponding derivation of $\underline{R}_{\geq 0}$. In particular there holds in $M_{\mathbb{Z}}(\underline{R}_{\geq 0})$

$$\begin{aligned} [\tilde{\partial}_{\tilde{P}}(\tilde{\mathcal{L}}), \tilde{\mathcal{U}}_\beta] + [\tilde{\mathcal{L}}, \tilde{\partial}_{\tilde{P}}(\tilde{\mathcal{U}}_\beta)] &= 0 \\ [\tilde{\partial}_{\tilde{P}}(\tilde{\mathcal{U}}_\beta), \tilde{\mathcal{U}}_\gamma] + [\tilde{\mathcal{U}}_\beta, \tilde{\partial}_{\tilde{P}}(\tilde{\mathcal{U}}_\gamma)] &= 0 \end{aligned} \quad (72)$$

Since the $\tilde{P}_{i\alpha}$ commute in $M_{\mathbb{Z}}(\underline{R}_{\geq 0})$ with the $\tilde{\mathcal{L}}$ and the $\{\tilde{\mathcal{U}}_\beta\}$, we have a set of derivations $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ of $\underline{R}_{\geq 0}$ for which the equations (66) hold. It will be clear that the number of relations between the coefficients of the $\tilde{\mathcal{L}}$ and the $\{\tilde{\mathcal{U}}_\alpha\}$ is minimal. Though the Lax equations of the hierarchy in this minimal setting hold practically by definition, they possess interesting consequences. Before discussing them, it is convenient to introduce a notation: for all $i \geq 0$, and all α, $1 \leq \alpha \leq m_0$, one writes

$$\tilde{B}_{i\alpha} := (\tilde{\mathcal{L}}^i\tilde{\mathcal{U}}_\alpha)_{\geq 0}(\mathcal{L}_0).$$

These matrices satisfy the following relations:

Proposition 2. *The matrices* $\{\tilde{B}_{n\alpha}\}$ *in* $M_{\mathbb{Z}}(\underline{R}_{\geq 0})$ *satisfy for all* n *and* $m \geq 0$ *and all* β *and* α *in* $\{1, \cdots, m_0\}$ *the equations*

$$\tilde{\partial}_{\tilde{P}_{n\alpha}}(\tilde{B}_{m\beta}) - \tilde{\partial}_{\tilde{P}_{m\beta}}(\tilde{B}_{n\alpha}) - [\tilde{B}_{n\alpha}, \tilde{B}_{m\beta}] = 0. \tag{73}$$

Proof. The idea is to show that the left hand side of equation (73) belongs both to $LT_{\geq 0}(\mathcal{L}_0)$ and to $LT_{\leq -1}(\mathcal{L}_0)$ and thus has to be zero. The left hand side of equation (73) clearly belongs to $LT_{\geq 0}(\mathcal{L}_0)$. To get the other property, note that for all n and $m \geq 0$ and all α and $\beta \in \{1, \ldots, m_0\}$ there holds

$$\tilde{\partial}_{\tilde{P}_{n\alpha}}(\tilde{P}_{m\beta}) = [\tilde{B}_{n\alpha}, \tilde{P}_{m\beta}] = -[(\tilde{P}_{n\alpha})_{<0}(\mathcal{L}_0), \tilde{P}_{m\beta}]. \tag{74}$$

This is a direct consequence of the fact that both $\tilde{\partial}_{P_{n\alpha}}$ and $[\tilde{B}_{n\alpha}, -]$ are derivations of $LT(R_{\geq 0})$ and that their action on $\tilde{\mathcal{L}}$ and $\tilde{U}_\beta$ is equal. Now one substitutes the equation $\tilde{B}_{n\alpha} = \tilde{P}_{n\alpha} - (\tilde{P}_{n\alpha})_{<0}(\mathcal{L}_0)$ in the left hand side of (73) and one uses equation (74) to obtain

$$\begin{aligned}
&\tilde{\partial}_{\tilde{P}_{n\alpha}}(\tilde{P}_{m\beta}) - \tilde{\partial}_{\tilde{P}_{n\alpha}}((\tilde{P}_{m\beta})_{<0}(\mathcal{L}_0)) - \tilde{\partial}_{\tilde{P}_{m\beta}}(\tilde{P}_{n\alpha}) + \tilde{\partial}_{\tilde{P}_{m\beta}}((\tilde{P}_{n\alpha})_{<0}(\mathcal{L}_0)) - \\
&[\tilde{P}_{n\alpha} - (\tilde{P}_{n\alpha})_{<0}(\mathcal{L}_0), \tilde{P}_{m\beta} - (\tilde{P}_{m\beta})_{<0}(\mathcal{L}_0)] = \\
&\tilde{\partial}_{\tilde{P}_{m\beta}}((\tilde{P}_{n\alpha})_{<0}(\mathcal{L}_0)) - \tilde{\partial}_{\tilde{P}_{n\alpha}}((\tilde{P}_{m\beta})_{<0}(\mathcal{L}_0)) - [(\tilde{P}_{n\alpha})_{<0}(\mathcal{L}_0), (\tilde{P}_{m\beta})_{<0}(\mathcal{L}_0)].
\end{aligned}$$

This last expression belongs clearly to $LT_{\leq -1}(\mathcal{L}_0)$, which proves the statement. □

This proposition allows you to show that it was not so strange to assume in example 1 that the basic derivations commute and it furnishes at the same time a unifying property of the equations that belong to the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy

Corollary 1. *All the derivations* $\{\tilde{\partial}_{\tilde{P}_{n\alpha}} \mid n \geq 0, 1 \leq \alpha \leq m_0\}$ *of the algebra* $\underline{R}_{\geq 0}$ *commute.*

Proof. As mentioned above the matrix coefficients of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{U}}_\beta$ generate the algebra $\underline{R}_{\geq 0}$ so that one merely has to show respectively that

$$(\tilde{\partial}_{\tilde{P}_{k\gamma}} \circ \tilde{\partial}_{\tilde{P}_{n\alpha}} - \tilde{\partial}_{\tilde{P}_{n\alpha}} \circ \tilde{\partial}_{\tilde{P}_{k\gamma}})(\tilde{\mathcal{L}}) = 0 \text{ and } (\tilde{\partial}_{\tilde{P}_{k\gamma}} \circ \tilde{\partial}_{\tilde{P}_{n\alpha}} - \tilde{\partial}_{\tilde{P}_{n\alpha}} \circ \tilde{\partial}_{\tilde{P}_{k\gamma}})(\tilde{\mathcal{U}}_\beta) = 0.$$

One gets the desired identities by applying the following property: let ∂_1 and ∂_2 be derivations of a ring $\mathcal{R}$ and let X be a matrix in $M_{\mathbb{Z}}(\mathcal{R})$ such that for $i = 1, 2$

$$\partial_i(X) = [D_i, X] \text{ for some } D_i \in M_{\mathbb{Z}}(\mathcal{R}).$$

Then a straightforward computation shows that

$$(\partial_1 \circ \partial_2 - \partial_2 \circ \partial_1)(X) = [\partial_1(D_2) - \partial_2(D_1) - [D_1, D_2], X] \tag{75}$$

By inserting the zero curvature relations from Proposition 2 in this identity for the operators and derivations listed above, one obtains the statements in the corollary. Hereby it is tacitly assumed that all the commutators in this reasoning make sense, a fact that holds in our situation, where the matrices X, D_1 and D_2 belong to $LT(\underline{R}_{\geq 0})$. □

Since the basic derivations in the Lax equations of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy commute, it is appropriate to call the equations in Proposition 2 the so-called *zero curvature equations* of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy, see section 3.2.

4.3 The zero curvature form of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy

In the last section one has built a minimal context in which the equations of the form (59) hold. Here one wants to discuss other realizations and the zero curvature form of the hierarchy.

The first step in realizing solutions of the hierarchy is to come up with matrices of the right shape. Since the leading terms are fixed, these matrices are determined by assigning to each matrix coefficient $\tilde{l}_j(s)_{\gamma\delta}, j \leq 0$, of $\tilde{\mathcal{L}}$ an element $l_j(s)_{\gamma\delta}$ in some F-algebra R and to each matrix coefficient $\tilde{u}_{i,\alpha}(t)_{\rho\sigma}, i < 0$, of $\tilde{\mathcal{U}}_\alpha$ an element $u_{i,\alpha}(t)_{\rho\sigma}$ in the same F-algebra R. This assignment is nothing but an F-algebra morphism $\lambda : R_{\geq 0} \to R$. The morphism λ determines an F- linear map from $M_{\mathbb{Z}}(R_{\geq 0})$ to $M_{\mathbb{Z}}(R)$ that is denoted by the same letter. On the level of the lower triangular matrices the map λ is even an F-algebra morphism from $LT(R_{\geq 0})$ to $LT(R)$. It furnishes you a set of matrices

$$\mathcal{L} := \lambda(\tilde{\mathcal{L}}) \text{ and the } \mathcal{U}_\alpha := \lambda(\tilde{\mathcal{U}}_\alpha) \tag{76}$$

of the right shape. To keep notations consistent, one writes for all $i \geq 0$ and all $\alpha, 1 \leq \alpha \leq m_0$,

$$P_{i\alpha} := \mathcal{L}^i \mathcal{U}_\alpha \text{ and } B_{i\alpha} := (P_{i\alpha})_{\geq 0}(\mathcal{L}_0).$$

If the assignment is chosen such that the matrices $\mathcal{L}$ and the $\mathcal{U}_\alpha$ commute among each other, then one has in fact an F-algebra morphism $\lambda : \underline{R}_{\geq 0} \to R$. This we assume from now on.

Next one wants to transfer the Lax structure to R and in that light one needs that R is equipped with a number of F-linear derivations $\partial_{P_{i\alpha}} : R \to R$, with $P_{i\alpha}$ as above. What one wants then algebraically is a compatibility of the derivations $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ and $\partial_{P_{i\alpha}}$ according to

$$\partial_{P_{i\alpha}} \circ \lambda = \lambda \circ \tilde{\partial}_{\tilde{P}_{i\alpha}}, \tag{77}$$

Now the compatibility conditions (77) imply all the Lax equations for the matrices $\mathcal{L}$ and $\mathcal{U}_\alpha$ in $M_{\mathbb{Z}}(R)$:

$$\partial_{P_{i\alpha}}(\mathcal{L}) = [B_{i\alpha}, \mathcal{L}], \tag{78}$$

$$\partial_{P_{i\alpha}}(\mathcal{U}_\alpha) = [B_{i\alpha}, \mathcal{U}_\alpha]. \tag{79}$$

Since the matrix coefficients of $\tilde{\mathcal{L}}$ and the $\tilde{\mathcal{U}}_\alpha$ generate $\underline{R}_{\geq 0}$, the equations (78) and (79) are also sufficient for having the compatibility conditions (77). Thus finding such an R, the morphism $\lambda : \underline{R}_{\geq 0} \to R$ and a set of F-linear derivations $\partial_{P_{i\alpha}}$ such that the relations (77) hold, gives you a solution, namely $\mathcal{L}$ and the $\mathcal{U}_\alpha$, of the equations of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy. Note that the relations (77) also imply that the subring $\lambda(\underline{R}_{\geq 0})$ is stable under all the derivations $\partial_{P_{i\alpha}}$ and that their restrictions to $\lambda(\underline{R}_{\geq 0})$ all commute.

Keeping in mind that λ is an F-algebra morphism, one may conclude from relation (77) and Proposition 2 that the zero curvature relations also hold in R

$$\partial_{P_{n\alpha}}(B_{m\beta}) - \partial_{P_{m\beta}}(B_{n\alpha}) - [B_{n\alpha}, B_{m\beta}] = 0. \tag{80}$$

There is still a second set of zero curvature relations related to the above situation, for there holds

Lemma 4. *Assume one has an F-algebra R, a morphism $\lambda : \underline{R}_{\geq 0} \to R$ and a set of F-linear derivations $\partial_{P_{i\alpha}}$ such that the relations (77) hold. Let $\mathcal{L}$ and the $\mathcal{U}_\alpha$ be given by (76). Then the $\{D_{m\beta} := B_{m\beta} - P_{m\beta}\}$ satisfy*

$$\partial_{P_{n\alpha}}(D_{m\beta}) - \partial_{P_{m\beta}}(D_{n\alpha}) - [D_{n\alpha}, D_{m\beta}] = 0. \tag{81}$$

Proof. Because of the Lax equations for $\mathcal{L}$ and the $\mathcal{U}_\alpha$ each $P_{m\beta}$ satisfies the same Lax equations

$$\partial_{P_{i\alpha}}(P_{m\beta}) = [B_{i\alpha}, P_{m\beta}]. \tag{82}$$

Thus we get

$$\begin{aligned}
&\partial_{P_{n\alpha}}(D_{m\beta}) - \partial_{P_{m\beta}}(D_{n\alpha}) - [D_{n\alpha}, D_{m\beta}] = \\
&\partial_{P_{n\alpha}}(B_{m\beta}) - \partial_{P_{n\alpha}}(P_{m\beta}) - \partial_{P_{m\beta}}(B_{n\alpha}) + \partial_{P_{m\beta}}(P_{n\alpha}) - \\
&[B_{n\alpha}, B_{m\beta}] + [B_{n\alpha}, P_{m\beta}] + [P_{n\alpha}, B_{m\beta}] - [P_{n\alpha}, P_{m\beta}] = 0,
\end{aligned}$$

thanks to the Lax equations (82). □

Next it will be shown that, reversely, the equations (80) also imply the Lax equations (78) and (79). So to describe the stage, let $\mathcal{R}$ be an F-algebra and let $\lambda : \underline{R}_{\geq 0} \to \mathcal{R}$ be an F-algebra morphism. The induced F- linear map from $M_{\mathbb{Z}}(\underline{R}_{\geq 0})$ to $M_{\mathbb{Z}}(\mathcal{R})$ is also denoted by λ. Consider the commuting matrices $\mathcal{L} := \lambda(\tilde{\mathcal{L}})$ and $\mathcal{U}_\alpha := \lambda(\tilde{\mathcal{U}}_\alpha) \in M_{\mathbb{Z}}(\mathcal{R})$, where $\tilde{\mathcal{L}}$ and the $\tilde{\mathcal{U}}_\alpha$ are as in the foregoing subsection. Assume $\mathcal{R}$ is equipped with a number of F-linear derivations $\partial_{P_{i\alpha}} : \mathcal{R} \to \mathcal{R}, i \geq 0, 1 \leq \alpha \leq m_0$, then we can say

Theorem 3. *Under the assumption just made and notations being as above, the Lax equations (78) and (79) for the matrices $\mathcal{L}$ and the $\mathcal{U}_\alpha$ are equivalent to the zero curvature relations (80) for the matrices $\{B_{n\alpha}\}$.*

Proof. One merely has to prove sufficiency still. Recall that for all $m \geq 0$ and all $D_{m\beta} := B_{m\beta} - P_{m\beta}$. To get the Lax equations for $\mathcal{L}$ and $\mathcal{U}_\beta$ one considers for all $m \geq 1$ the matrices $\partial_{P_{i\alpha}}(\mathcal{L}^m \mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{L}^m \mathcal{U}_\beta]$. By substituting in it the identity $\mathcal{L}^m \mathcal{U}_\beta = B_{m\beta} - D_{m\beta}$ and by using the zero curvature equations, one gets the equality

$$\partial_{P_{i\alpha}}(\mathcal{L}^m \mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{L}^m \mathcal{U}_\beta] = -\partial_{P_{i\alpha}}(D_{m\beta}) + \partial_{P_{m\beta}}(B_{i\alpha}) + [B_{i\alpha}, D_{m\beta}]$$

From the left hand side of this equality one sees that the order in $\mathcal{L}_0$ for all the

$$\partial_{P_{i\alpha}}(\mathcal{L}^m \mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{L}^m \mathcal{U}_\beta]$$

is uniformly bounded above by $i-1$. Consider first the operator $\mathcal{L}$. Assume now that

$$\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}] = \alpha \mathcal{L}_0^r + \text{ lower order in } \mathcal{L}_0, \tag{83}$$

with $\alpha \in U_0 \mathcal{D}_k(R) U_0^{-1}$ nonzero. Since both $\partial_{P_{i\alpha}}$ and $[B_{i\alpha}, -]$ are derivations, there follows with induction that

$$\partial_{P_{i\alpha}}(\mathcal{L}^m) - [B_{i\alpha}, \mathcal{L}^m] = \sum_{i=0}^{m-1} \mathcal{L}^i \{\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}]\} \mathcal{L}^{m-i-1}.$$

Now one focusses on the leading term of this equality and one gets

$$\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}] = (\sum_{i=0}^{m-1} \mathcal{L}_0^i \alpha \mathcal{L}_0^{-i}) \mathcal{L}_0^{r+m-1} + \text{ lower order in } \mathcal{L}_0.$$

If one lets m tend to infinity this contradicts the fact that the left hand side belongs to $LT_{\leq i-1}(\mathcal{L}_0)$, unless one has for all sufficiently large m that

$$\sum_{i=0}^{m-1} \mathcal{L}_0^i \alpha \mathcal{L}_0^{-i} = 0.$$

This identity would, however, immediately imply that for sufficiently large m,

$$\mathcal{L}_0^m \alpha \mathcal{L}_0^{-m} = 0$$

and hence $\alpha = 0$. This contradicts the assumption that has been made, so there holds for all $i \geq 0$, and all $\alpha, 1 \leq \alpha \leq m_0$,

$$\partial_{P_{i\alpha}}(\mathcal{L}) - [B_n, \mathcal{L}] = 0.$$

Hence the Lax equations (78) hold for $\mathcal{L}$. Next one considers the operator $\mathcal{L}^m \mathcal{U}_\beta$ and applies the Lax equations for $\mathcal{L}^m$ to get

$$\partial_{P_{i\alpha}}(\mathcal{L}^m \mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{L}^m \mathcal{U}_\beta] = (\partial_{P_{i\alpha}}\{\mathcal{L}^m) - [B_{i\alpha}, \mathcal{L}^m]\}\mathcal{U}_\beta + \tag{84}$$
$$\mathcal{L}^m\{\partial_{P_{i\alpha}}(\mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{U}_\beta]\} = 0 + \mathcal{L}^m(\partial_{P_{i\alpha}}(\mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{U}_\beta])$$

If the operator $\mathcal{U}_\beta$ would not satisfy the Lax equation (79), then there holds

$$\partial_{P_{i\alpha}}(\mathcal{U}_\beta) - [B_{i\alpha}, \mathcal{U}_\beta] = \beta\mathcal{L}_0^s + \text{ lower order in } \mathcal{L}_0, \tag{85}$$

with $\beta \in U_0\mathcal{D}_k(R)U_0^{-1}$ nonzero. Hence the right hand side of (84) would have a nonzero leading coefficient in $\mathcal{L}_0$ of order $m+s$. This contradicts the fact that the order in $\mathcal{L}_0$ of the left hand side is bounded above. Therefore the Lax equations (79) have to hold for $\mathcal{U}_\beta$ as well. □

Remark 7. The zero curvature equations are also often called *Zakharov-Shabat equations* after Zakharov and Shabat who introduced this form for many integrable equations, see e.g. [57].

4.4 Wave matrices for the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy

In this section the *linearization* of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy will be treated. Recall, see [30], that in the finite dimensional situation such as described in subsection 3.1, the vanishing of the curvature of the connection with the connection form

$$\Omega = \sum_{i=1}^{r} C_i dz_i$$

is equivalent to having n independent functions φ_j from U to the $\mathbb{C}^n$ that satisfy

$$\frac{\partial}{\partial z_i}(\varphi_j) = C_i\varphi_j.$$

Hence, if Φ is the $n \times n$-matrix with φ_j as its j-th column, then there holds for this map from U to $\mathrm{GL}_n(\mathbb{C})$

$$\frac{\partial}{\partial z_i}(\Phi) = C_i\Phi \Leftrightarrow C_i = \frac{\partial}{\partial z_i}(\Phi)\Phi^{-1}. \tag{86}$$

Note that, if L_0 is a constant $n \times n$-matrix, then $L := \Phi L_0 \Phi^{-1}$ is a holomorphic matrix on U whose evolution w.r.t. all the variables z_i is given by a set of Lax equations

$$\begin{aligned}\frac{\partial}{\partial z_i}(L) &= \frac{\partial}{\partial z_i}(\Phi)L_0\Phi^{-1} - \Phi L_0\Phi^{-1}\frac{\partial}{\partial z_i}(\Phi)\Phi^{-1}\\ &= [\frac{\partial}{\partial z_i}(\Phi)\Phi^{-1}, L] = [C_i, L].\end{aligned} \tag{87}$$

The linearization of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy will be an algebraic substitute for these horizontal sections from which follow the Lax equations for the hierarchy. One starts out with an F-algebra R equipped with a collection of F-linear commuting derivations $\{\partial_{P_{i\alpha}}, i \geq 0, \alpha \in \{1, \cdots, m_0\}\}$. Further one has the corresponding potential solutions, namely commuting operators $\mathcal{L}$ and $\mathcal{U}_\alpha$ in $LT(R)$ of the form (61) resp. (63). The *linearization of the* $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-*hierarchy* consists of the following equations for $\mathcal{L}$ and $\mathcal{U}_\alpha$

$$\mathcal{L}\phi = \phi\mathcal{L}_0, \mathcal{U}_\alpha\phi = \phi\mathcal{U}_{\alpha,0} \text{ and } \partial_{P_{i\alpha}}(\phi) = B_{i\alpha}\phi, \tag{88}$$

where ϕ is a not yet specified object for which all the operations in (88), like multiplying from the left and right with $\mathbb{Z} \times \mathbb{Z}$-matrices and applying all the derivations $\partial_{P_{i\alpha}}$ make sense. To get the Lax equations for $\mathcal{L}$ one applies the derivation $\partial_{P_{i\alpha}}$ to the first equation in (88) and substitutes the last one. This leads to the following manipulations

$$\begin{aligned} \partial_{P_{i\alpha}}(\mathcal{L}\phi - \phi\mathcal{L}_0) = \partial_{P_{i\alpha}}(\mathcal{L})\phi + \mathcal{L}(\partial_{P_{i\alpha}}(\phi)) - (\partial_{P_{i\alpha}}(\phi))\mathcal{L}_0 = \\ \partial_{P_{i\alpha}}(\mathcal{L})\phi + \mathcal{L}B_{i\alpha}\phi - B_{i\alpha}\phi\mathcal{L}_0 = \{\partial_{P_{i\alpha}}(\mathcal{L}) - [B_{i\alpha}, \mathcal{L}\,]\}\phi = 0. \end{aligned} \tag{89}$$

Hence, if it is allowed to scratch the function ϕ from the foregoing equation, one obtains the Lax equations for $\mathcal{L}$. For the operator $\mathcal{U}_\alpha$ one applies $\partial_{P_{i\alpha}}$ to the second equation in (88) and substitutes the last one. Thus one gets

$$\begin{aligned} \partial_{P_{i\alpha}}(\mathcal{U}_\alpha\phi - \phi\mathcal{U}_{\alpha,0}) = \partial_{P_{i\alpha}}(\mathcal{U}_\alpha)\psi + \mathcal{U}_\alpha(\partial_{P_{i\alpha}}(\phi)) - (\partial_{P_{i\alpha}}(\phi))\mathcal{U}_{\alpha,0} = \\ = \partial_{P_{i\alpha}}(\mathcal{U}_\alpha)\phi + \mathcal{U}_\alpha B_{i\alpha}\phi - B_{i\alpha}\phi\mathcal{U}_{\alpha,0} = \{\partial_{P_{i\alpha}}(\mathcal{U}_\alpha) - [B_{i\alpha}, \mathcal{U}_\alpha\,]\}\phi = 0. \end{aligned} \tag{90}$$

and if one can leave out ϕ again, this yields the Lax equations for $\mathcal{U}_\alpha$.

For the linearization (88) one needs a left action of $\mathcal{L}$, the $\mathcal{U}_\beta$ and all the $B_{i\alpha}$ on the functions ϕ and a right one for matrices like $\mathcal{L}_0$ and the $\mathcal{U}_{\alpha,0}$. To realize the first, one builds a left $UT(R)$-module. The actual form of the elements in the module is guided by the trivial solution $\mathcal{L} = \mathcal{L}_0$ and $\mathcal{U}_\alpha = \mathcal{U}_{\alpha,0}$ of the hierarchy. In that case the equations (88) become

$$\mathcal{L}_0\phi = \phi\mathcal{L}_0, \mathcal{U}_{\alpha,0}\phi = \phi\mathcal{U}_{\alpha,0} \text{ and } \partial_{P_{i\alpha}}(\phi) = (\mathcal{L}_0^i\mathcal{U}_{\alpha,0})\phi. \tag{91}$$

In particular, one sees that the first order approximation of the flow corresponding to each $\partial_{P_{i\alpha}}$ is multiplying from the left with the constant basic direction $\mathcal{L}_0^i \mathcal{U}_{\alpha,0}$. Let $t_{i\alpha}$ be the parameter for the flow corresponding to $\partial_{P_{i\alpha}}$ so that $\partial_{P_{i\alpha}}$ acts on ϕ as taking the partial derivative $\frac{\partial}{\partial t_{i\alpha}} =: \partial_{t_{i\alpha}}$. The equations (91) can then formally be integrated simultaneously. For, consider the formal series

$$\phi_\infty := \exp(\sum_{i=0}^{\infty} \sum_{\alpha=1}^{m_0} t_{i\alpha} \mathcal{U}_{\alpha,0} \mathcal{L}_0^i). \tag{92}$$

With the action of $\partial_{P_{i\alpha}}$ just defined, it satisfies the equations (91). As a formal powerseries in the parameters $\{t_{i\alpha} \mid i \geq 0, \alpha \in \{1, \cdots, m_0\}\}$, one can write

$$\phi_\infty = \sum_\delta \phi_\infty(\delta) t^\delta, \text{ with all } \phi_\infty(\delta) \in LT(F) \tag{93}$$

and where $\delta = \{\delta(i\alpha)\}$, with $\delta(i\alpha) \in \mathbb{N}$ equal to zero, for all but a finite number of them. Note that in general ϕ_∞ is not a well-defined matrix in $M_{\mathbb{Z}}(R)$. That would require minimally a natural embedding of the algebra $F[t_{i\alpha}]$ into R and convergence conditions on the element $\mathcal{L}_0$.

The module for the linearization will consist of perturbations in $LT(R)$ of this trivial solution. Consider namely the collection $M^{(\infty)}(\mathcal{L}_0)$ consisting of formal products

$$\{\sum_{j=-\infty}^{N} d_j \mathcal{L}_0^j\} \exp(\sum_{i=0}^{\infty} \sum_{\alpha=1}^{m_L} t_{i\alpha} \mathcal{U}_{\alpha,0} \mathcal{L}_0^i), \text{ where } d_j \in U_0 \mathcal{D}_k(R) U_0^{-1}. \tag{94}$$

The elements of $M^{(\infty)}(\mathcal{L}_0)$ are called *oscillating matrices at infinity*, since they are formal products of a series in $\mathcal{L}_0$ that has a pole around infinity and the exponential term, which is an essential singularity in $\mathcal{L}_0$. If $F[t_{i\alpha}] \subset R$ and $\mathcal{L}_0$ is sufficiently nice so that ϕ_∞ belongs to $M_{\mathbb{Z}}(R)$, then it requires still convergence considerations to make sense of the product in (94) as a matrix in $M_{\mathbb{Z}}(R)$. Such a setting is described in the remaining subsections of this chapter. So in general these formal products do not give a well-defined element of $M_{\mathbb{Z}}(R)$. However, if $R = F[[t_{i\alpha}]]$ and $\partial_{P_{i\alpha}} = \partial_{t_{i\alpha}}$ as in example (1), then we know from (148) that both ϕ_∞ and any element $\sum_{j=-\infty}^{N} d_j \mathcal{L}_0^j$ are formal power series in the variables $t_{i\alpha}$ with coefficients from $LT(F)$ and their product is also a well-defined formal power series in the variables $t_{i\alpha}$ with coefficients from $LT(F)$.

Nevertheless there is in the general case a well-defined left action of $LT(R)$ on $M^{(\infty)}(\mathcal{L}_0)$. For all u_1 and $u_2 \in LT(R)$ one puts namely

$$u_1\{u_2\}\phi_\infty = \{u_1u_2\}\phi_\infty. \tag{95}$$

Multiplying elements of $M^{(\infty)}(\mathcal{L}_0)$ from the right with the matrices $\mathcal{L}_0$ and $\mathcal{U}_{\beta,0}$ is defined as if the formal product is a real product of matrices and as if $\mathcal{L}_0$ and $\mathcal{U}_{\beta,0}$ commute with ϕ_∞

$$\{u\}\phi_\infty\,\mathcal{L}_0 := \{u\,\mathcal{L}_0\}\phi_\infty \text{ and } \{u\}\phi_\infty\,\mathcal{U}_{\beta,0} := \{u\,\mathcal{U}_{\beta,0}\}\phi_\infty$$

The action of the derivations $\partial_{P_{i\alpha}}$ on $M^{(0)}(\mathcal{L}_0)$ is defined as if the product in the module $M^{(\infty)}(\mathcal{L}_0)$ is a real one

$$\partial_{P_{i\alpha}}\{\sum_{t=-\infty}^{N} d_t\mathcal{L}_0^t\}\phi_\infty = \{\sum_{t=-\infty}^{N}\partial_{P_{i\alpha}}(d_t)\mathcal{L}_0^t + \sum_{t=-\infty}^{N} d_t\mathcal{L}_0^t\,\mathcal{L}_0^i\,\mathcal{U}_{\alpha,0}\}\phi_\infty. \tag{96}$$

All the actions occurring in the linearization have been introduced now. Note that $M^{(\infty)}(\mathcal{L}_0)$ is a free $LT(R)$-module with generator ϕ_∞. Hence scratching ϕ from the equations (89) and (90) is permitted as soon as one knows that $\phi = \hat{\phi}\phi_\infty$ with $\hat{\phi} \in LT(R)$ invertible. In this last case the equation $\mathcal{L}\phi = \phi\mathcal{L}_0$ implies then that

$$\mathcal{L} = \mathcal{L}(\hat{\phi}) := \hat{\phi}\mathcal{L}_0\hat{\phi}^{-1}$$

and the equation $\mathcal{U}_\alpha\phi = \phi\mathcal{U}_{\alpha,0}$ implies that

$$\mathcal{U}_\alpha = U_\alpha(\hat{\phi}) := \hat{\phi}\mathcal{U}_{\alpha,0}\hat{\phi}^{-1}.$$

An oscillating matrix at infinity $\phi = \hat{\phi}\phi_\infty$, with $\hat{\phi} = \mathrm{Id} + \sum_{i\leq -1} d_i\mathcal{L}_0^i$, is called *a wave matrix at infinity* for the matrices $\mathcal{L}(\hat{\phi})$ and $U_\alpha(\hat{\phi})$, if it satisfies the equations (88). Since the manipulations to get the Lax equations are well-defined on such a ϕ, the matrices $\mathcal{L}(\hat{\phi})$ and the $\mathcal{U}_\alpha(\hat{\phi})$ form a solution of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy. If one wants to prove the equations (88) for an oscillating matrix at infinity ϕ of the right form, it suffices to prove a weaker result, for there holds

Proposition 3. *Let $\phi = \hat{\phi}\phi_\infty$, with $\hat{\phi} - \mathrm{Id} \in LT_{\leq -1}(\mathcal{L}_0)$, be an oscillating matrix at infinity. If it satisfies for all $i \geq 0$ and all $\alpha \in \{1, \cdots, m_0\}$*

$$\partial_{P_{i\alpha}}(\phi) = G_{i\alpha}\psi, \text{ with } G_{i\alpha} \in LT(R)_{\geq 0},$$

then $G_{i\alpha} = (\mathcal{L}(\hat{\phi})^i \mathcal{U}_\alpha(\hat{\phi}))_{\geq 0}(\mathcal{L}_0)$. *In particular* $\mathcal{L}(\hat{\phi})$ *and the* $\mathcal{U}_\beta(\hat{\phi})$ *form a solution to the* $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$*-hierarchy*

Proof. From the definition of the action of $\partial_{P_{i\alpha}}$ on $M^{(\infty)}(\mathcal{L}_0)$ and the fact that $M^{(\infty)}(\mathcal{L}_0)$ is a free $LT(R)$-module with generator ϕ_∞, one gets the operator equation

$$\partial_{P_{i\alpha}}(\hat{\phi}) + \hat{\phi}\mathcal{L}_0^i \mathcal{U}_{\alpha,0} = G_{i\alpha}\hat{\phi}. \tag{97}$$

Multiplying this equation from the right with $\hat{\phi}^{-1}$ and taking the upper triangular part "≥ 0" in $\mathcal{L}_0$ gives the desired result. □

Remark 8. The wave matrices at infinity are in the present context the analogue of the Baker-Akhiezer functions for the KP-hierarchy, see [49].

Remark 9. Note that if one has an oscillating matrix at infinity of the right form, i.e. $\phi = \hat{\phi}\phi_\infty$ with $\hat{\phi} - \mathrm{Id} \in LT_{\leq -1}(\mathcal{L}_0)$, then the condition to ensure that ϕ is a wave matrix at infinity can totally be expressed in the perturbation $\hat{\phi}$. There always holds

$$\partial_{P_{i\alpha}}(\phi) = (\partial_{P_{i\alpha}}(\hat{\phi}) + \hat{\phi}\mathcal{L}_0^i \mathcal{U}_{\alpha,0})\phi_\infty$$

and if this has to equal $B_{i\alpha}\hat{\phi}\phi_\infty$, then there holds, since $\partial_{P_{i\alpha}}(\hat{\phi})\hat{\phi}^{-1} \in LT_{\leq -1}(R)$ that

$$\partial_{P_{i\alpha}}(\hat{\phi}) = D_{i\alpha}\hat{\phi}, \tag{98}$$

where $D_{i\alpha} = -(\mathcal{L}^i \mathcal{U}_\alpha)_{<0}(\mathcal{L}_0)$ as in subsection 4.3. Reversely, if the equations (98) hold, then ϕ is a wave matrix at infinity for the operators $\mathcal{L}(\hat{\phi})$ and the $\mathcal{U}_\alpha(\hat{\phi})$. The equations (98) form the analogue of the Sato equations in the context of the KP-hierarchy.

Different wave matrices at infinity may lead to the same solution of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy. Assume namely that

$$\mathcal{L} = \hat{\phi}_1 \mathcal{L}_0 \hat{\phi}_1^{-1} = \hat{\phi}_2 \mathcal{L}_0 \hat{\phi}_2^{-1}$$

and

$$\mathcal{U}_\beta = \hat{\phi}_1 \mathcal{U}_{\beta,0} \hat{\phi}_1^{-1} = \hat{\phi}_2 \mathcal{U}_{\beta,0} \hat{\phi}_2^{-1},$$

where both ϕ_1 and ϕ_2 are wave matrices at infinity. Then one has first of all that $\hat{\phi_1}^{-1}\hat{\phi_2}$ commutes with $\mathcal{L}_0$ and the $\mathcal{U}_{\beta,0}$ and thus has to have the form

$$\hat{\phi_1}^{-1}\hat{\phi_2} = \sum_{r\leq 0} v_r \mathcal{L}_0^r,$$

where $v_i \in U_0 i_k(M_k(R))U_0^{-1}$ commutes with the $\mathcal{U}_{\beta,0}$. One has seen in the proof of Proposition 10 that for all $i \geq 0$, all $\alpha \in \{1, \cdots, m_L\}$ and $s = 1, 2$,

$$\partial_{P_{i\alpha}}(\hat{\phi_s}) = (\mathcal{L}^i U_\alpha)_{\geq 0}(\mathcal{L}_0)\hat{\phi_s} - \hat{\phi_s}\mathcal{L}_0^i \mathcal{V}_{\alpha,0}.$$

Hence, if one applies the operator $\partial_{P_{i\alpha}}$ to the equality $\hat{\phi_2} = \hat{\phi_1}\sum_r v_r \mathcal{L}_0^r$, then one obtains

$$\partial_{P_{i\alpha}}(\hat{\phi_2}) = \partial_{P_{i\alpha}}(\hat{\phi_1})\sum_{r\leq 0} v_r \mathcal{L}_0^r + \hat{\phi_1}\sum_{r\leq 0}\partial_{P_{i\alpha}}(v_r)\mathcal{L}_0^r =$$
$$((\mathcal{L}^i U_\alpha)_{\geq 0}(\mathcal{L}_0)\hat{\phi_1} - \hat{\phi_1}(\mathcal{L}_0^i \mathcal{U}_{\alpha,0})\sum_{r\leq 0} v_r \mathcal{L}_0^r + \hat{\phi_1}\sum_{r\leq 0}\partial_{P_{i\alpha}}(v_r)\mathcal{L}_0^r =$$
$$(\mathcal{L}^i U_\alpha)_{\geq 0}(\mathcal{L}_0)\hat{\phi_2} - \hat{\phi_2}\mathcal{L}_0^i \mathcal{U}_{\alpha,0} + \hat{\phi_1}\sum_{r\leq 0}\partial_{P_{i\alpha}}(v_r)\mathcal{L}_0^r.$$

Hence one must have for all $r \leq 0$, all $i \geq 0$ and all $\alpha \in \{1, \cdots, m_0\}$ that

$$\partial_{P_{i\alpha}}(v_r) = 0.$$

For completeness sake, this result is resumed in a corollary

Corollary 2. *If ϕ_1 and ϕ_2 are wave matrices at infinity w.r.t. the same set of operators $\mathcal{L}$ and the $\mathcal{U}_\alpha$, then there holds*

$$\phi_2 = \phi_1 \sum_{r\leq 0} v_r \mathcal{L}_0^r,$$

where all the v_r are constant for the derivations $\partial_{P_{i\alpha}}$, i.e. $\partial_{P_{i\alpha}}(v_r) = 0$.

Remark 10. In the case of the Example 1 of the formal power series in the variables $\{t_{i\alpha}\}$, a wave matrix at infinity $\phi = \hat{\phi}\phi_\infty$ for the solutions $\mathcal{L}$ and the $\mathcal{U}_\alpha$ is a formal power series in these variables with coefficients from $LT(F)$ and a constant term equal to the identity. Note that all the matrices $B_{i\alpha}$ are formal power series in the variables $\{t_{i\alpha}\}$ with as coefficients matrices in $LT_{\leq i}(F)$. Therefore these coefficients satisfy the Assumption 1 and the zero curvature relations. Hence, since ϕ satisfies

$$\partial_{t_{i\alpha}}(\phi) = B_{i\alpha}\phi,$$

the wave matrix at infinity ϕ is nothing but the fundamental matrix for the corresponding Cauchy problem. Note that thanks to Remark 9 the matrix $\hat{\phi}$ is also a fundamental matrix for a Cauchy problem, namely the one corresponding to the matrices $\{D_{i\alpha}\}$.

Remark 11. To actually construct solutions of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy, one will describe in the next subsection a geometric setting from which one can construct oscillating functions at infinity for which the exponential factor determines a well-defined element in $U_0 UT(R) U_0^{-1}$, the formal product of this exponential factor and the perturbation factor in $LT(R)$ is real, i.e. it converges, and the oscillating functions at infinity satisfy Proposition 3.

4.5 A geometric setting for the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy

All the relevant $\mathbb{Z} \times \mathbb{Z}$-matrices that will be produced in the sequel correspond to bounded operators on a separable complex Hilbert space and therefore one takes the basic field F here equal to $\mathbb{C}$, so that the ring R will be a $\mathbb{C}$-algebra.

Let S^1 be the unit circle in the complex plane. Throughout this thesis one will work with the Hilbert space $H = L^2(S^1, \mathbb{C}^k)$ of square integrable $\mathbb{C}^k$-valued functions on S^1. Its elements are represented by their Fourier series

$$h = \sum_{n \in \mathbb{Z}} a(n) z^n, \text{ where } a(n) \in \mathbb{C}^k \text{ for all } n \in \mathbb{Z}.$$

Let $(\cdot \mid \cdot)$ be the standard inner product on $\mathbb{C}^k$. The inner product on H is expressed in the Fourier coefficients of its elements by

$$< \sum_{n \in \mathbb{Z}} a(n) z^n \mid \sum_{n \in \mathbb{Z}} b(n) z^n > := \sum_{n \in \mathbb{Z}} (a(n) \mid b(n)).$$

Let $\{f_s \mid 0 \leq s \leq k-1\}$ denote the standard basis of $\mathbb{C}^k$, where f_s has a one on the $s+1$-th entry and zeros elsewhere. Then one gets a Hilbert basis $\{e_i \mid i \in \mathbb{Z}\}$ of H, if one puts for all $j \in \mathbb{Z}$ and all $s, 0 \leq s \leq k-1$,

$$e_{s+kj} := f_s z^j.$$

To each bounded operator $B \in B(H)$ one associates the $\mathbb{Z} \times \mathbb{Z}$-matrix $[B]$ w.r.t. this basis. In this paper one will mainly work with a decomposition of $[B]$ in $k \times k$-blocks. Thereto one considers inside H a number of subspaces. For each $i \in \mathbb{Z}$, let $H^{(i)}$ be the complex subspace of H spanned by the

$$\{f_s z^i \mid 0 \leq s \leq k-1\}.$$

The projection $H \mapsto H^{(i)}$ given by $\sum_{j \in \mathbb{Z}} h(j) z^j \mapsto h(i) z^i$ is denoted by $p^{(i)}$. The space H decomposes as the direct sum

$$H = \oplus_{i \in \mathbb{Z}} H^{(i)}$$

and this determines for each bounded linear operator $B \in B(H)$ the associated block decomposition $B = (B_{ij})$, where $B_{ij} := p^{(i)} \circ B \mid H^{(j)}$ and correspondingly the matrix decomposition $[B] = ([B_{ij}])$ in $k \times k$-blocks. E.g. if $A \in \mathrm{gl}_k(\mathbb{C})$, then multiplying from the left with A defines a bounded map $M_A : H \mapsto H$ such that the $k \times k$-block decomposition of its $\mathbb{Z} \times \mathbb{Z}$-matrix looks like

$$[M_A] = i_k(A) = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{A} & 0 & 0 & \ddots \\ \ddots & 0 & \mathbf{A} & 0 & \ddots \\ \ddots & 0 & 0 & \mathbf{A} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

Similarly, the $\mathbb{Z} \times \mathbb{Z}$-matrix of the bounded map $M_z : H \mapsto H$, defined by

$$M_z(\sum_{i \in \mathbb{Z}} h_i z^i) = \sum_{i \in \mathbb{Z}} h_i z^{i+1},$$

has a $k \times k$-block decomposition of the form

$$[M_z] := \Lambda^{-k} = \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathbf{0} & 0 & 0 & \ddots \\ \ddots & \mathrm{Id} & \mathbf{0} & 0 & \ddots \\ \ddots & 0 & \mathrm{Id} & \mathbf{0} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

with Λ the $\mathbb{Z}\times\mathbb{Z}$-matrix corresponding to the shift operator $\sum_i \alpha_i e_i \mapsto \sum_i \alpha_i e_{i-1}$. A direct computation shows that conjugating with M_z corresponds in the matrix picture with shifting each $k \times k$-block one step along the diagonal. Hence all operators in $B \in B(H)$ that commute with M_z can be written in the form

$$B = \sum_{i\in\mathbb{Z}} M_{B_i} M_z^i, \text{ with } B_i \in \mathrm{gl}_k(\mathbb{C}). \tag{99}$$

Another subspace that plays a role in the sequel is the subspace H_j, $j \in \mathbb{Z}$, defined by

$$H_j = \oplus_{i\leq j} H^{(i)},$$

with its orthogonal projection $p_j := \oplus_{i\leq j} p^{(i)}$. One will use a special notation for the decomposition of any element $b \in B(H)$ w.r.t. the splitting $H = H_j \oplus H_j^{\perp}$, namely

$$b = \begin{pmatrix} b_{++}(j)) & b_{+-}(j) \\ b_{-+}(j) & b_{--}(j) \end{pmatrix}. \tag{100}$$

Now one is ready to introduce a Lie group in which the flows corresponding to the basic directions

$$\{\mathcal{L}_0^i \mathcal{U}_{\beta,0} \mid i \geq 0, 1 \leq \beta \leq m_0\}$$

can be realized and from which one can construct solutions of the hierarchy. Here one has some freedom and to illustrate that the group one is about to introduce, depends of a class of compact operators of H and for each class the construction works. For each integer $r \geq 1$, let S_r be the Schatten ideal, see [47], of bounded operators $A : H \mapsto H$ such that

$$||A||_r^r := \mathrm{trace}((A^*A)^{\frac{r}{2}}) = \mathrm{trace}(|A|^r) < \infty,$$

where A^* denotes the adjoint of A. For $r = 1$ this gives you the trace-class operators and for $r = 2$ the Hilbert-Schmidt operators. For each such a r one introduces the group $G_-(r)$ by

$$G_-(r) = \left\{ g = (g_{ij}) \in \mathrm{GL}(H) \;\middle|\; \begin{matrix} \oplus_{i>j} g_{ij} \in S_r \\ \oplus_{i>j} (g^{-1})_{ij} \in S_r \end{matrix} \right\}.$$

It consists of the invertible elements in the Banach algebra

$$\mathcal{G}_-(r) = \left\{ b = (b_{ij}) \in B(H) \;\middle|\; \oplus_{i>j} b_{ij} \in S_r \right\}$$

equiped with the norm $||\cdot||_{res}$ defined by

$$||b||_{res} = ||(b_{ij})||_{res} := ||b|| + ||\oplus_{i>j} b_{ij}||_r.$$

Here $||\cdot||$ is the operator norm and $||\cdot||_r$ the Schatten norm. This turnes $G_-(r)$ in a natural way into a Banach Lie group with $\mathcal{G}_-(r)$ as its Lie algebra. For each $N \geq 1$, one has the subgroup

$$G_N := \left\{ g = (g_{ij}) \in G_-(r) \,\middle|\, \begin{array}{c} g_{ij} = 0 \text{ for } i > j \text{ and } |j| > N \\ g_{ij} = 0 \text{ for } i < -N \text{ and } -N \leq j \leq N \\ g_{jj} \text{ is invertible for } |j| > N \end{array} \right\} \quad (101)$$

By using the fact that in each $\mathrm{GL}(n, \mathbb{C})$ any element can be linked to the identity by a continuous path, one proves the same for the group G_N, in other words G_N is connected. Since each element in $G_-(r)$ differs from an element in G_N for some N by a small operator in the Schatten class S_r, one can connect any element in $G_-(r)$ by a continuous path with some element from G_N and thus one may conclude

Lemma 5. *The group $G_-(r)$ is connected.*

Next a number of ways to split the Lie algebra $\mathcal{G}_-(r)$ will be discussed. First of all the Lie algebra $\mathcal{G}_-(r)$ can be split into the sum of the Lie subalgebras

$$\mathcal{P} := \left\{ p = (p_{ij}) \in \mathcal{G}_-(r) \,\middle|\, p_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$\mathcal{U}_- := \left\{ u = (u_{ij}) \in \mathcal{G}_-(r) \,\middle|\, u_{ij} = 0 \text{ for all } i \leq j \right\}.$$

Their corresponding Lie groups are

$$P := \left\{ p = (p_{ij}) \in G_-(r) \,\middle|\, p_{ij} = 0 \text{ and } (p^{-1})_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$U_- := \left\{ u = (u_{ij}) \in G_-(r) \,\middle|\, \begin{array}{c} u_{ij} = 0 \text{ for all } i < j \\ u_{ii} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \end{array} \right\}.$$

The group $P \cap P_-$ of diagonal $k \times k$-blocks in $G_-(r)$ is denoted by D_k. Let p_- be an element of the group P_-. Then its matrix $[p_-] = U_0$ is a lower triangular matrix of order zero in Λ^k. It has an invertible leading coefficient and determines the direction $\mathcal{L}_0$. Now conjugation with p_- determines another way to split $\mathcal{G}_-(r)$ namely

$$\mathcal{G}_-(r) = \mathcal{U}_- \oplus p_- \, \mathcal{P} p_-^{-1}.$$

All the commuting directions

$$\{\mathcal{U}_{\alpha,0}\mathcal{L}_0^i \mid \alpha \in \{1, \cdots, m_L\}, i \geq 0\}$$

that are relevant for $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy belong to the Lie algebra $p_- \, \mathcal{P} p_-^{-1}$. The Lie group corresponding to the Lie algebra $p_- \, \mathcal{P} p_-^{-1}$ is clearly the group $p_- \, P p_-^{-1}$. As the map from $\mathcal{U}_- \times p_- \, \mathcal{P} p_-^{-1}$ to $G_-(r)$ defined by

$$(u, p) \mapsto \exp(u) \exp(p)$$

is a local diffeomorphism at $(0, 0)$, the set $U_- p_- \, P p_-^{-1}$ is an open subset of $G_-(r)$. It is called the *big cell* in $G_-(r)$ w.r.t. U_- and $p_- \, P p_-^{-1}$. Following the terminology for loop groups, see [42], we speak of the Bruhat decomposition of the big cell. The component in $p_- \, P p_-^{-1}$ is called the parabolic component of the decomposition and U_- the unipotent component of the decomposition. Each big cell is a dense open subset of $G_-(r)$ since this holds for the case $p_- = \mathrm{Id}$ and in that case it is a consequence of the following characterization, which is similar to the finite dimensional case

Proposition 4. *Let $\Omega \subset G_-(r)$ be the collection of all $g \in G_-(r)$ such that $g_{++}(i)$ is invertible for all $i \in \mathbb{Z}$. Then Ω is equal to $U_- P$ and is a dense open subset of $G_-(r)$.*

Proof. One starts with the alternative description of the big cell. Since for all $i \in \mathbb{Z}$ and all $p \in P$ and $u \in U_-$ one has the decompositions

$$p = \begin{pmatrix} p_{++}(i) & p_{+-}(i) \\ 0 & p_{--}(i) \end{pmatrix} \text{ and } u = \begin{pmatrix} u_{++}(i) & 0 \\ u_{-+}(i) & u_{--}(i) \end{pmatrix},$$

the inclusion $U_- P \subset \Omega$ is clear.

Reversely, a direct computation shows that Ω is invariant under right multiplication with elements of P and under left multiplication with elements from U_-.

For each $j > i$, consider the decomposition of an operator $g \in \Omega$ w.r.t. the splitting $H = H_i \oplus \{H_j \cap H_i^{\perp}\} \oplus H_j^{\perp}$

$$g = \begin{pmatrix} g_{++}(i) & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{--}(j) \end{pmatrix}.$$

By definition of the subset Ω the operator $g_{++}(i)$ is invertible and one can form the element

$$u = \begin{pmatrix} \mathrm{Id} & 0 & 0 \\ -g_{2,1}g_{++}(i)^{-1} & \mathrm{Id} & 0 \\ 0 & 0 & \mathrm{Id} \end{pmatrix}$$

in U_-. By left multiplication with u one reduces g to the form

$$g = \begin{pmatrix} g_{++}(i) & \tilde{g}_{1,2} & g_{1,3} \\ 0 & \tilde{g}_{2,2} & \tilde{g}_{2,3} \\ g_{3,1} & g_{3,2} & g_{--}(j) \end{pmatrix}.$$

Similarly one can get rid of $g_{3,1}$. Note now that the defining property of Ω ensures that $\tilde{g}_{2,2}$ is invertible and the same procedure gives that one may also assume $g_{3,2}$ to be zero. Thus the operator g is brought back to the form

$$g = \begin{pmatrix} g_{++}(i) & * & * \\ 0 & \tilde{g}_{2,2} & * \\ 0 & 0 & g_{--}(j) \end{pmatrix}.$$

The operator $\tilde{g}_{2,2}$ is finite-dimensional and it satisfies on the $k \times k-$ block level the finite-dimensional analogue of the defining property of Ω, which results in the required splitting. As for the other operators on the diagonal, $g_{++}(i)$ decomposes as

$$g_{++}(i) = u_i(g) + p_i(g),$$

where the $k \times k$-block decompositions of these components are given by

$$u_i(g) := \begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & g_{i-2\, i-2} & 0 & \ddots \\ \ddots & g_{i-1\, i-2} & g_{i-1\, i-1} & 0 \\ \cdots & g_{i\, i-2} & g_{i\, i-1} & g_{i\, i} \end{pmatrix}, p_i(g) := \begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & g_{i-2\, i-1} & g_{i-2\, i} \\ \ddots & 0 & 0 & g_{i-1\, i} \\ \cdots & 0 & 0 & 0 \end{pmatrix}.$$

By choosing i sufficiently small, one can get the Schatten norm of $p_i(g)$ arbitrary small such that for all $t < i$

$$\max(||u_i(g)g_{++}(t))^{-1}p_i(g)||_{res}, ||g_{++}(t))^{-1}p_i(g)||_{res}) < \alpha \text{ for some } 0 < \alpha < 1,$$

independent of t. Now $g_{++}(i)$ can be split as follows:

$$\begin{pmatrix} g_{++}(i+1) & v \\ u & g_{ii} \end{pmatrix} = \begin{pmatrix} g_{++}(i+1) & 0 \\ u & g_{ii} - ug_{++}(i+1)^{-1}v \end{pmatrix} \begin{pmatrix} \mathrm{Id} & g_{++}(i+1)^{-1}v \\ 0 & \mathrm{Id} \end{pmatrix}.$$

By the estimate above, repeating this process determines a well-defined operator of the form

$$\begin{pmatrix} \ddots & \ddots & \ddots & \ddots \\ \ddots & \mathrm{Id} & * & \ddots \\ \ddots & 0 & \mathrm{Id} & * \\ \dots & 0 & 0 & \mathrm{Id} \end{pmatrix}.$$

Then one can reduce the operator $g_{++}(i)$ to an operator of the form of $u_i(g)$ by right multiplication with this operator. By extending this operator with the identity on $H_i^{\perp}$, this yields elements of U_-. Likewise the operator $g_{--}(j)$ decomposes as

$$g_{--}(j) = u^{(j)}(g) + p^{(j)}(g),$$

where the $k \times k$-block decompositions of these components are respectively given by

$$u^{(j)}(g) := \begin{pmatrix} g_{j+1\,j+1} & 0 & 0 & \dots \\ g_{j+2\,j+1} & g_{j+2\,j+2} & 0 & \ddots \\ g_{j+3\,j+1} & g_{j+3\,j+2} & g_{j+3\,j+3} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix},$$

$$p^{(j)}(g) := \begin{pmatrix} 0 & g_{j+1\,j+2} & g_{j+1\,j+3} & \dots \\ 0 & 0 & g_{j+2\,j+3} & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

By taking j sufficiently big, the Schatten norm of $p^{(j)}(g)$ can be made arbitrary small such that all the $\{g_{kk} \mid k \geq j+1\}$ are invertible. Now $g_{--}(j)$ can be split similarly as:

$$\begin{pmatrix} g_{j+1j+1} & x \\ y & g_{--}(j+1) \end{pmatrix} = \begin{pmatrix} g_{j+1j+1} & 0 \\ y & g_{--}(j+1) - yg_{j+1j+1}^{-1}x \end{pmatrix} \begin{pmatrix} \mathrm{Id} & g_{j+1j+1}^{-1}x \\ 0 & \mathrm{Id} \end{pmatrix}.$$

The infinite product

$$\cdots \begin{pmatrix} \mathrm{Id}_k & 0 & 0 & \cdots \\ 0 & \mathrm{Id}_k & u_2 & \cdots \\ 0 & 0 & \mathrm{Id} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \mathrm{Id}_k & u_1 & * & \cdots \\ 0 & \mathrm{Id}_k & 0 & \ddots \\ 0 & 0 & \mathrm{Id} & 0 \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_k & u_1 & * & \cdots \\ 0 & \mathrm{Id}_k & u_2 & \cdots \\ 0 & 0 & \mathrm{Id} & * \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

thus one can reduce $g_{--}(j))$ to an operator of the form of $u^j(g)$ by right multiplication with operators of the form

$$\begin{pmatrix} \mathrm{Id} & * & * & \cdots \\ 0 & \mathrm{Id} & * & \ddots \\ 0 & 0 & \mathrm{Id} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

By extending these with the identity on H_j one obtains again operators in U_- and thus one has reduced g to an element of $P_{-}- = U_- D_k$. This proves the equality $\Omega = U_- P$. It implies directly that the big cell is open.

To show that it is dense, note that we have seen from the foregoing discussion that for sufficiently small i all the $g_{++}(i)$ are invertible. The idea is now that by adding a suitable small $k \times k$-block at the $(i+1, i+1)$-entry one can make $g_{++}(i+1)$ also invertible. This can be seen as follows: the operator $g_{++}(i+1)$ is Fredholm of index zero, since $g_{++}(i)$ is invertible. Hence it suffices to show that such a small perturbation can make its kernel equal to zero. A vector v in the kernel of $g_{++}(i+1)$ satisfies:

$$g_{++}(i+1)(v) = \begin{pmatrix} g_{++}(i) & b \\ c & g_{i+1,i+1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

and since $g_{++}(i)$ is invertible, this amounts to the equation

$$(-c g_{++}(i)^{-1} b + g_{i+1,i+1}) y = 0.$$

Hence, adding a suitable $k \times k$-block to $g_{i+1,i+1}$, can arrange that this problem has only the trivial solution $y = 0$. Continuing this process a finite number of steps, it can be arranged for a sufficiently large N that for all $i \leq N$ the perturbed matrices $\tilde{g}_{++}(i)$ are invertible. If N is sufficiently large then the operator $\tilde{g}_{+-}(N)$

is small, just like the part $p^N(\tilde{g})$ of the matrix and all the $\tilde{g}_{jj}$ for $j \geq N+1$ are invertible. Thus one sees that for all $j \geq N+1$ that $\tilde{g}_{++}(j)$ is the matrix of a Fredholm operator that differs by a small operator from the operator with matrix

$$\begin{pmatrix} \tilde{g}_{++}(j) & 0 \\ \tilde{g}_{-+}(j) & \tilde{g}_{j+1,j+1} \end{pmatrix}. \tag{102}$$

In particular it is invertible if $\tilde{g}_{++}(j))$ and $\tilde{g}_{j+1,j+1}$ are invertible. This implies that all the $\tilde{g}_{++}(i), i \in \mathbb{Z}$, are invertible for the slightly perturbed matrix $\tilde{g}$, in other words $\tilde{g} \in \Omega$. □

To demonstrate how wave matrices at infinity can be constructed in practice, one makes first a choice for the basic commuting directions. Let $\mathfrak{h}$ as in subsection 4.1 be a maximal commutative subalgebra of $M_k(\mathbb{C})$ with the basis $\{E_\rho \mid 1 \leq \rho \leq m_0\}$ and let $\mathbf{h}$ be equal to the span of the twisted directions $U_0(i_k(\mathfrak{h}))U_0^{-1}$, where $U_0 = [p_-]$ is the lower triangular matrix of an element p_- in P_- that determines the direction $\mathcal{L}_0$.

There is a natural group of commuting flows associated with $\mathfrak{h}$ that embeds into $\mathrm{GL}(H)$. Let namely U be any open connected neighbourhood in the complex plane of the unit circle S^1. Then one writes $\Gamma(U, \mathfrak{h})$ for the set of holomorpic maps $\gamma : U \mapsto \mathfrak{h}$ such that

$$\det(\gamma(u)) \neq 0 \text{ for all } u \in U.$$

It is a group w.r.t. the pointwise multiplication in $\mathrm{GL}_k(\mathbb{C})$. If two such neighbourhoods U_1 and U_2 satisfy $U_2 \subset U_1$ then one has a natural embedding of $\Gamma(U_1, \mathfrak{h})$ into $\Gamma(U_2, \mathfrak{h})$ and the inductive limit is denoted by $\Gamma(\mathfrak{h})$. For each $r \geq 1$, one writes $\Gamma(\mathfrak{h}, r)$ for the subgroup of $\Gamma(\mathfrak{h})$ consisting of the inductive limit of all the $\Gamma(U, \mathfrak{h})$, with U containing the disk with radius r. Each $\gamma \in \Gamma(\mathfrak{h})$ has a Fourier series

$$\sum_{i \in \mathbb{Z}} \gamma_i z^i, \text{ with } \gamma_i \in \mathfrak{h}.$$

Let $\mathfrak{h}_{ss}$ denote the subset of semi simple elements in $\mathfrak{h}$ and let $\mathfrak{h}_n$ be the collection of nilpotent elements in $\mathfrak{h}$. From the fact that $\mathfrak{h}$ is the direct sum of these subspaces one deduces that the group $\Gamma(\mathfrak{h})$ is the direct product of the groups

$$\Gamma(\mathfrak{h})_{ss} := \{\gamma \mid \gamma \in \Gamma(\mathfrak{h}), \gamma(u) \in \mathbf{h}_{ss} \text{ for all } u\}$$

and

$$\Gamma(\mathfrak{h})_u := \{\gamma \mid \gamma \in \Gamma(\mathfrak{h}), \gamma(u) \text{ is unipotent for all } u\}.$$

Now it is easy to see that any $\gamma \in \Gamma(\mathfrak{h})_u$ can be written as

$$\gamma = \exp(\sum_{s\in\mathbb{Z}} k_s z^s) = \exp(\sum_{s>0} k_s z^s) \exp(\sum_{s\leq 0} k_s z^s),$$

where $k_s \in \mathbf{h}_n$ for all $s \in \mathbb{Z}$. This shows that the elements of $\Gamma(\mathfrak{h})_u$ split up perfectly in those that have an analytic continuation to the interior of S^1 and those that extend holomorphically around "infinity".

As for the group $\Gamma(\mathfrak{h})_{ss}$, recall that, if U is an open connected neighbourhood of S^1, any holomorphic $f : U \mapsto \mathbb{C}^*$ decomposes as

$$f(z) = \{1 + \sum_{i>0} b_i z^i\} z^m \{\sum_{j\leq 0} c_j z^j\}, \text{ with } c_0 \neq 0 \text{ and } m \in \mathbb{Z}.$$

By applying this to the group $\Gamma(\mathfrak{h})_{ss}$, one arrives at the following decomposition of $\Gamma(\mathfrak{h})$

Proposition 5. *There is a subgroup $\Delta(\mathfrak{h})$ of $\Gamma(\mathfrak{h})_{ss}$ isomorphic to $\mathbb{Z}^r$, where r is the dimension of $\mathfrak{h}_{ss}$, such that $\Gamma(\mathfrak{h}) = \Gamma(\mathfrak{h})_{\geq 0}\ \Delta(\mathfrak{h})\ \Gamma(\mathfrak{h})_{<0}$, where*

$$\Gamma(\mathfrak{h})_{\geq 0} = \{\gamma \mid \gamma = \exp(\sum_{s\leq 0} \gamma_s z^s),\ \textit{with } \gamma_s \in \mathfrak{h} \textit{ for all } s \leq 0\}$$

and

$$\Gamma(\mathfrak{h})_{<0} = \{\gamma \mid \gamma = \exp(\sum_{s>0} \gamma_s z^s),\ \textit{with } \gamma_s \in \mathfrak{h} \textit{ for all } s > 0\}.$$

In the case that $\mathfrak{h}$ equals the diagonal matrices, one can take

$$\Delta(\mathfrak{h}) = \left\{ \operatorname{diag}(z^{m_1}, \ldots, z^{m_k}) := \begin{pmatrix} z^{m_1} & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & \ldots & 0 & z^{m_k} \end{pmatrix} \middle| \ m_i \in \mathbb{Z} \right\} \tag{103}$$

However if one takes the unipotent algebra (60), then the group $\Delta(\mathfrak{h})$ shrinks to the subgroup of (103) consisting of the elements with all m_i equal to some $m \in \mathbb{Z}$.

A direct embedding of $\Gamma(\mathfrak{h})$ into $\mathrm{GL}(H)$ is obtained by letting it act on H through multiplication from the left with this series. It defines then the bounded operator $M_\gamma : H \mapsto H$. In general the operators $M_\gamma : H \mapsto H$ do not belong to $G_-(r)$. E.g. let γ_0 denote the element $u \mapsto \mathrm{Id}\, u$, then $[M_{\gamma_0}] = \Lambda^{-k}$ and M_{γ_0} does not belong to $G_-(r)$. Since the elements of the commutator subgroup of M_{γ_0} have the form

$$\sum_{i\in\mathbb{Z}} M_{A_i} M_{\gamma_0}^i, \text{ with } A_i \in \mathrm{gl}_k(\mathbb{C}),$$

one sees that the image of $\Gamma(\mathfrak{h})$ contains pretty much all the directions that commute with $\mathfrak{h}$ and M_{γ_0}. Let B be an operator in $\mathrm{GL}(H)$ with inverse B^{-1}, then one writes $\alpha(B)$ for the maximum of one and $||B||||B^{-1}||$. Then one has the twisted embedding i_B of the subgroup $\Gamma(\mathfrak{h}, \alpha(B))$ into $\mathrm{GL}(H)$ defined by

$$\gamma \mapsto B M_\gamma B^{-1}. \tag{104}$$

Likewise the image of $\Gamma(\mathfrak{h}, \alpha(B))$ under the map i_B takes care of all the directions that commute with $B\mathfrak{h}B^{-1}$ and $BM_{\gamma(0)}B^{-1}$.

The flows from $\Gamma_{<0}(\mathfrak{h})$ do not embed into $G_-(r)$ in the way just described, since they fail the Schatten condition, but those of $\Gamma_{\geq 0}(\mathfrak{h})$ do. Take an operator $p_- \in P_-$ that determines the direction $\mathcal{L}_0$. The group $\Gamma_{\geq 0}(\mathfrak{h}, \alpha(p_-))$ embeds through i_{p_-} into the subgroup $p_- P_- p_-^{-1}$ of $G_-(r)$. Its image is the group of commuting flows $\Gamma(\mathbf{h}_{\geq 0})$ connected to the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy. After this embedding there is for each element $\gamma_{\geq 0}$ of $\Gamma_{\leq 0}(\mathfrak{h}, \alpha(p_-))$ an $N_1 > \alpha(p_-)$ such that

$$[i_{p_-}(\gamma_{\geq 0})] = \exp(\sum_{i=0}^{\infty}\sum_{\alpha=1}^{m_0} t_{i\alpha}\mathcal{U}_{\alpha,0}\mathcal{L}_0^i), t_{i\alpha} \in \mathbb{C}, \sum_{i,\alpha} |t_{i\alpha}| N_1^i < \infty$$

This illustrates once more that the $\{t_{i\alpha}\}$ are the coordinates w.r.t. the basic directions of the relevant group of commuting flows. Thus one has for $\gamma_{\geq 0}$ from $\Gamma_{\geq 0}(\mathfrak{h}, \alpha(p_-))$ a well-defined matrix $[i_{p_-}(\gamma_{\geq 0})]$ that can be used as the exponential factor in the wave matrices at infinity.

4.6 The construction of solutions of the hierarchy

One starts with an element $g \in G_-(r)$. Inside the group of commuting flows $\Gamma_{\geq 0}(\mathfrak{h}, \alpha(p_-))$ one considers the subset

$$\Gamma_{\geq 0}(g, \mathfrak{h}, \alpha(p_-)) = \{\gamma_{\geq 0} \in \Gamma_{\geq 0}(\mathfrak{h}, \alpha(p_-)) \mid i_{p_-}(\gamma_{\geq 0})g \in p_- \Omega p_-^{-1}\}.$$

Now one takes for R the ring of holomorphic functions on $\Gamma_{\geq 0}(g, \mathfrak{h}, \alpha(p_-))$. On this ring one clearly has the set of derivations

$$\{\partial_{t_{i\alpha}} := \frac{\partial}{\partial t_{i\alpha}} \mid i \geq 0, 1 \leq \alpha \leq m_0\}$$

The decomposition $p_- \Omega p_-^{-1} = U_- p_- P p_-^{-1}$ is basic for the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy: if an element $\gamma_{\geq 0}$ lies in $\Gamma_{\geq 0}(g, \mathfrak{h}, \alpha(p_-))$, then there holds

$$i_{p_-}(\gamma_{\geq 0})g = u_-(g, \gamma_{\geq 0}, p_-)^{-1} p(g, \gamma_{\geq 0}, p_-),$$

where the component $u_-(g, \gamma_{\geq 0}, p_-)$ belongs to U_- and the component $p(g, \gamma_{\geq 0}, p_-)$ to $p_- P p_-^{-1}$. Thus one has obtained an element of $M^{(\infty)}(\mathcal{L}_0)$, namely

$$[u_-(g, \gamma_{\geq 0}, p_-)][i_{p_-}(\gamma_{\geq 0})] =: \hat{\Phi}\phi_\infty = \Phi$$

for which the formal product in the definition of these spaces is a well-defined one. Note that on the matrix level one has the relations

$$[u_-(g, \gamma_{\geq 0}, p_-)][i_{p_-}(\gamma_{\geq 0})] = [p(g, \gamma_{\geq 0}, p_-)][g]^{-1} \tag{105}$$

Now one takes the derivative of $[u_-(g, \gamma_{\geq 0}, p_-)][i_{p_-}(\gamma_{\geq 0})]$ w.r.t. the parameter $t_{i\alpha}$. On one hand there holds

$$\begin{aligned} \partial_{t_{i\alpha}}(\hat{\Phi}[i_{p_-}(\gamma_{\geq 0})]) &= \partial_{t_{i\alpha}}(\hat{\Phi})[i_{p_-}(\gamma_{\geq 0})] + \hat{\Phi}\mathcal{U}_{\alpha,0}\mathcal{L}_0^i[i_{p_-}(\gamma_{\geq 0})] \\ &= \{\partial_{t_{i\alpha}}(\hat{\Phi})\hat{\Phi}^{-1} + \mathcal{L}(\hat{\Phi})^i \mathcal{U}_\alpha(\hat{\Phi})\}\Phi, \end{aligned} \tag{106}$$

where the matrices $\mathcal{L}(\hat{\Phi})$ and $\mathcal{U}_\alpha(\hat{\Phi})$ are obtained by dressing the basic directions with $\hat{\Phi}$:

$$\mathcal{L}(\hat{\Phi}) = \hat{\Phi}\mathcal{L}_0\hat{\Phi}^{-1} \text{ and } \mathcal{U}_\alpha(\hat{\Phi}) = \hat{\Phi}\mathcal{U}_{\beta,0}\hat{\Phi}^{-1}.$$

On the other hand, if one writes $\Psi := [p(g, \gamma_{\leq 0}, p_-)]^{-1}$, then differentiating the right hand side of equation (105) w.r.t. $t_{i\alpha}$ results in

$$\partial_{t_{i\alpha}}(\Phi) = \partial_{t_{i\alpha}}(\Psi)[g]^{-1} = \{\partial_{t_{i\alpha}}(\Psi)\Psi^{-1}\}\Phi. \tag{107}$$

As $M^{(\infty)}(\mathcal{L}_0)$ is a free $LT(R)$-module, the equations (106) and (107), yield

$$\partial_{t_{i\alpha}}(\hat{\Phi})\hat{\Phi}^{-1} + \mathcal{L}(\hat{\Phi})^i\mathcal{U}_\alpha(\hat{\Phi}) = \partial_{t_{i\alpha}}(\Psi)\Psi^{-1}.$$

Since the right hand side of this equation belongs to $UT_{\geq 0}(R)$, the left hand side to $LT_{\leq i}(R)$ and the matrix $\partial_{t_{i\alpha}}(\hat{\Phi})\hat{\Phi}^{-1}$ to $LT_{<0}(R)$ one may conclude that the following two identities hold:

$$\partial_{t_{i\alpha}}(\Psi)\Psi^{-1} = (\mathcal{L}(\hat{\Phi})^j\mathcal{U}_\beta(\hat{\Phi}))_{\geq 0}(\mathcal{L}_0) \tag{108}$$

$$\partial_{t_{i\alpha}}(\hat{\Phi})\hat{\Phi}^{-1} = -(\mathcal{L}(\hat{\Phi})^i\mathcal{U}_\beta(\hat{\Phi}))_{<0}(\mathcal{L}_0). \tag{109}$$

In particular the matrix Φ satisfies the linearization equations for the set of matrices $(\mathcal{L}(\hat{\Phi}), \mathcal{U}_\alpha(\hat{\Phi}))$. If one replaces $g \in G_-(r)$ by $gp^{(0)}$ with an arbitrary $p^{(0)} \in p_-Pp_-^{-1}$, then the lower triangular component does not change:

$$u_-(gp^{(0)}, \gamma_{\geq 0}, p_-) = u_-(g, \gamma_{\geq 0}, p_-)$$

and consequently the corresponding solutions of the hierarchy are the same. One resumes the results obtained in a

Theorem 4. *Let $p_- \in P_-$ determine the direction $\mathcal{L}_0$. Consider operators $g \in G_-(r)$ and $\gamma_{\geq 0} \in \Gamma_{\geq 0}(g, \mathfrak{h}, \alpha(p_-))$. Denote the inverse of the unipotent component of $i_{p_-}(\gamma_{\geq 0})g$ in the Bruhat decomposition of the big cell by $u_-(g, \gamma_{\geq 0}, p_-)$. Then the matrix $[u_-(g, \gamma_{\geq 0}, p_-)][i_{p_-}(\gamma_{\leq 0})]$ is a wave matrix at infinity for the set of matrices $(\mathcal{L}([u_-(g, \gamma_{\geq 0}, p_-)]), \mathcal{U}_\alpha([u_-(g, \gamma_{\geq 0}, p_-)]))$ and these matrices form a solution of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy. Moreover, there holds for each $p^{(0)}$ in the parabolic subgroup $p_-Pp_-^{-1}$ of the decomposition that*

$$\mathcal{L}([u_-(g, \gamma_{\geq 0}, p_-)]) = \mathcal{L}([u_-(gp^{(0)}, \gamma_{\geq 0}, p_-)]) \text{ and}$$
$$\mathcal{U}_\beta([u_-(g, \gamma_{\geq 0}, p_-)]) = \mathcal{U}_\beta([u_-(gp^{(0)}, \gamma_{\geq 0}, p_-)]).$$

Remark 12. The theorem shows that the variety $G_-(r)/p_- P p_-^{-1}$ determines the solutions of the $(\mathcal{L}_0, \mathbf{h}_{\geq 0})$-hierarchy. The isomorphic variety $G_-(r)/P$ is a flag variety consisting of all infinite flags

$$\cdots W_{i-1} = gH_{i-1} \subset W_i = gH_i \subset W_{i+1} = gH_{i+1} \cdots$$

which are perturbations of the basic flag

$$\cdots H_{i-1} \subset H_i \subset H_{i+1} \cdots$$

5 Upper triangular hierarchies

In this chapter we consider deformations of a number of basic commuting directions in the upper triangular matrices but this time we also allow a perturbation of the leading coefficient.

5.1 Lax equations in UT

As in chapter 4 we will denote by R a commutative F-algebra over a field F of characterictic zero. Let $\mathcal{M}_0 = V_0 \Lambda^k V_0^{-1}$ in $UT(F)$ be an element of order one in Λ^k as in Remark 2. It is an analogue of the direction $\mathcal{L}_0$ in the lower triangular matrices and the matrix $\mathcal{M}_0^{-1}$ will be one of the basic directions in $UT(F)$ on which the system that we will consider is based. The evolution of the deformations of the basic directions will consist of compatible systems of equations that have the form

$$\frac{\partial B}{\partial t} = [Q, B] = QB - BQ. \tag{110}$$

In particular one is interested in equations of the form (110) with $B \in UT$ being an element of negative order in $\mathcal{M}_0$ and Q an element from $UT_{<0}(\mathcal{M}_0)$. Note the special character of the equation: the order of $\frac{\partial B}{\partial t}$ in $\mathcal{M}_0$ in equation (110) will be higher or equal to that of B, while the order in $\mathcal{M}_0$ of the right hand side $[Q, B]$ is higher or equal to the sum of order(Q) and order(B).

Thus one arrives like in the case of the lower triangular matrices at a natural question: given a matrix B of negative order in $\mathcal{M}_0$, is there a systematic way to find a matrix Q in $UT_{<0}(\mathcal{M}_0)$ such that $[Q, B]$ has an order in $\mathcal{M}_0$ bigger or equal to that of B? For solutions Q to this question, it makes sense to consider the equations (110) for B.

Also here the answer to the question is affirmative: consider namely the centralizer $Z_{UT}(B)$ of B in UT and choose an element Z of a strictly negative order in $\mathcal{M}_0$ in the centralizer $Z_{UT}(B)$ of B in UT , then one has that

$$[Z_{<0}(\mathcal{M}_0), B] = -[Z_{\geq 0}(\mathcal{M}_0), B]$$

and the order in $\mathcal{M}_0$ of the right hand side is clearly $\geq$ order(B). For the matrix $Q = Z_{<0}(\mathcal{M}_0)$, it makes sense to look for a suitable derivation ∂_Z of R such that the equations

$$\partial_Z(B) = [Z_{<0}(\mathcal{M}_0), B] \tag{111}$$

hold. Here the action of ∂_Z on elements of $M_{\mathbb{Z}}(R)$ is defined coefficientwise. Note that the equation (111) becomes trivial, if one takes an element Z in $Z_{UT}(B)$ of order bigger or equal to zero in $\mathcal{M}_0$. Therefore, we will consider the equation (111) only for $Z \in Z_{UT}(B) \cap UT_{\leq -1}(\mathcal{M}_0)$.

One makes various choices for the matrices B and also for the matrices Z in the centralizer $Z_{UT}(B)$ of B in UT to produce sensible equations (111). First we fix a set of generators of a commuting set of matrices in $UT(F) \cap UT_{<0}(\mathcal{M}_0)$, including $\mathcal{M}_0^{-1}$. Consider thereto the centralizer $Z_{UT}(\mathcal{M}_0)$ of $\mathcal{M}_0$ in UT. If $k = 1$, $Z_{UT}(\mathcal{M}_0)$ is equal to

$$\{\sum_i b_i \mathcal{M}_0^i \in UT \mid b_i \in R, \text{ for all } i\}$$

and this ring is commutative. Hence in that case the powers of $\mathcal{M}_0$ basically span the space of directions commuting with $\mathcal{M}_0$ and hence one considers only the basic directions $\{\mathcal{M}_0^{-j} \mid j \geq 1\}$.

However, if $k > 1$, then an element of $Z_{UT}(\mathcal{M}_0)$ has the form

$$\sum_i b_i \mathcal{M}_0^i \in UT, \text{ with } b_i \in V_0 i_k(M_k(R)) V_0^{-1}$$

and $Z_{UT}(\mathcal{M}_0)$ is no longer commutative. Therefore, one chooses a number of matrices

$$\mathcal{V}_{\alpha,0} = V_0 i_k(F_\alpha) V_0^{-1}, F_\alpha \in M_k(F),$$

in UT of order zero in $\mathcal{M}_0$. They commute clearly with $\mathcal{M}_0$ and one requires that they also commute among each other

$$[\mathcal{V}_{\alpha,0}, \mathcal{V}_{\beta,0}] = 0 \text{ for all } \alpha \text{ and } \beta.$$

To include as many directions as possible, one assumes that the $\{\mathcal{V}_{\alpha,0}\}$ form a basis over F of a maximal commutative F-algebra $\mathbf{k}$ of $V_0 i_k(M_k(F)) V_0^{-1}$. The interesting commuting directions for the equation (111) are then the

$$\{\mathcal{V}_{\alpha,0} \mathcal{M}_0^{-j} \mid j \geq 1, \alpha \in \{1, \cdots, m_0\}\}.$$

To get compatibility in notation of the cases $k = 1$ and $k > 1$, one puts $\mathcal{V}_{1,0} = \mathrm{Id}$ if $k = 1$. Having fixed the basic commuting directions, one takes for B a number of generators of a commutative ring in UT, starting with a generator $\mathcal{M}$ of order minus one in $\mathcal{M}_0$

$$\mathcal{M} := \sum_{i \geq -1} m_i \mathcal{M}_0^i, \text{ with } m_{-1} \text{ invertible.} \tag{112}$$

The matrix $\mathcal{M}$ is regarded as a deformation of the direction $\mathcal{M}_0^{-1}$. The coefficients in this decomposition of $\mathcal{M}$ all have the form

$$m_i = V_0 \mathrm{diag}(m_i(s)) V_0^{-1}.$$

According to Proposition 27 the matrix $\mathcal{M}$ can be written in the form

$$\mathcal{M} = V \mathcal{M}_0^{-1} V^{-1}, \text{ with } V = \sum_{i \geq 0} v_i \mathcal{M}_0^i, \tag{113}$$

where the leading coefficient v_0 of V is invertible. Concretely it can be taken of the form $v_0 = V_0 w_0 V_0^{-1}$, with $w_0 = \mathrm{diag}(w_0(s))$ the $k \times k$-block diagonal matrix given by $w_0(0) = \mathrm{Id}$ and

$$\begin{aligned} w_0(t) &= m_{-1}(t) \ldots m_{-1}(1) \text{ for } t \geq 1 \text{ and} \\ w_0(t) &= m_{-1}(t+1)^{-1} \ldots m_{-1}(0)^{-1} \text{ for } t < 0. \end{aligned} \tag{114}$$

For $k = 1$ the centralizer of $\mathcal{M}$ in UT equals

$$\{\sum_i b_i \mathcal{M}^i \in UT \mid b_i \in R \text{ for all } i\},$$

which is commutative and one merely chooses $Z = \mathcal{M}^{-j}, j \geq 1$, to get the equations

$$\partial_{\mathcal{M}^{-j}}(\mathcal{M}) = [\mathcal{M}^{-j}_{<0}(\mathcal{M}_0), \mathcal{M}] \tag{115}$$

Since ∂_Z and $[Z_{<0}(\mathcal{M}_0), -]$ are derivations, one sees that as soon as the equations (111) holds for $\mathcal{M}$, then they are also valid for all the

$$\{\sum_i b_i \mathcal{M}^i \in UT \mid b_i \in R \text{ and } \partial_Z(b_i) = 0 \text{ for all } i\}.$$

Therefore it suffices to consider the equations (115) only for the generator $\mathcal{M}$. However, for $k > 1$, one also considers deformations of the other basic directions $\{\mathcal{V}_{\alpha,0}\mathcal{M}_0^{-1}\}$. One assumes again as in the lower triangular case that the deformations of each $\mathcal{V}_{\alpha,0}\mathcal{M}_0^{-1}$ is the product $\mathcal{V}_\alpha\mathcal{M}$ of a deformation $\mathcal{V}_\alpha$ of the zero-th order matrix $\mathcal{V}_{\alpha,0}$ and the deformation $\mathcal{M}$ of $\mathcal{M}_0^{-1}$. Moreover, the type of deformations of the $\mathcal{V}_{\alpha,0}$ should in first order be the same as that of $\mathcal{M}_0$. More precisely, one wants the $\mathcal{V}_\alpha$ to have the form

$$\mathcal{V}_\alpha = \sum_{i\geq 0} v_{i,\alpha}\mathcal{M}_0^i, \text{ with } v_{0,\alpha} = v_0\mathcal{V}_{\alpha,0}v_0^{-1}, \tag{116}$$

where v_0 is the leading coefficient of the dressing matrix of $\mathcal{M}$. Again the deformation should preserve the commutativity relations:

$$[\mathcal{M}, \mathcal{V}_\alpha] = 0 \text{ and } [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0 \text{ for all } \alpha \text{ and } \beta \in \{1, \cdots, m_0\}. \tag{117}$$

This condition is trivially fulfilled if the matrices $(\mathcal{M}, \mathcal{V}_\alpha)$ are obtained by dressing the basic directions with the same matrix, i.e.

$$\mathcal{M} = V\mathcal{M}_0^{-1}V^{-1} \text{ and } \mathcal{V}_\alpha = V\mathcal{V}_{\alpha,0}V^{-1}.$$

For the matrices Z in the centralizers of both $\mathcal{M}$ and the $\{\mathcal{V}_\alpha\}$ one can choose now the $Z = Q_{j\beta} := \mathcal{M}^j\mathcal{V}_\beta$ for all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$. The nonlinear equations that one wants the $\mathcal{M}$ and the $\mathcal{V}_\alpha$ to satisfy, are

$$\partial_{Q_{j\beta}}(\mathcal{M}) = [(Q_{j\beta})_{<0}(\mathcal{M}_0), \mathcal{M}] \text{ and } \partial_{Q_{j\beta}}(\mathcal{V}_\alpha) = [(Q_{j\beta})_{<0}(\mathcal{M}_0), \mathcal{V}_\alpha]. \tag{118}$$

Note that, if the equations (118) hold for the generators $\mathcal{M}$ and the $\{\mathcal{V}_\alpha\}$, then they hold for all the $\{\mathcal{M}^m\mathcal{V}_\alpha \mid m \in \mathbb{Z}\}$, since $\partial_{Q_{j\beta}}$ and $[(Q_{j\beta})_{<0}(\mathcal{M}_0), -]$ are derivations of UT. Therefore it suffices to consider the equations (118) just for the generators.

The equations (117) and (118) for matrices $\mathcal{M}$ and the $\mathcal{V}_\alpha$ form the equations of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy since they correspond to the commuting flows with infinitesimal generators the basic directions

$$\{\mathcal{V}_{\alpha,0}\mathcal{M}_0^{-j} \mid j \geq 1, \alpha \in \{1, \cdots, m_0\}\}.$$

that determine a maximal commutative subring of F-matrices in $UT_{<0}(\mathcal{M}_0)$. The equations (118) are called the *Lax equations* of the hierarchy. The $(\mathcal{M}_0^{-1}, \mathcal{V}_{\alpha,0})$ are a trivial solution of the Lax equations of the $(\mathcal{M}_0, \mathbf{k}_{<0})$-hierarchy, meaning that

$$\partial_{Q_{j\beta}}(\mathcal{M}_0^{-1}) = 0 \text{ and } \partial_{Q_{j\beta}}(\mathcal{V}_{\alpha,0}) = 0 \text{ for all } j \text{ and } \beta.$$

The reason is that one has for all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$ that

$$(\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0})_{<0}(\mathcal{M}_0) = \mathcal{M}_0^{-j}\mathcal{V}_{\beta,0}.$$

Remark 13. Here one wants to give an idea of the differential difference character of the equations in the hierarchy. For simplicity one takes $k = 1$ and $\mathcal{M}_0 = \Lambda$ and one concentrates on the derivation ∂_1 corresponding to the direction Λ^{-1}. Then one has

$$\mathcal{M} = \sum_{i \geq -1} m_i \Lambda^i, \text{ with } m_i = \text{diag}(m_i(s))$$

and the Lax equation w.r.t. ∂_1 reads

$$\sum_{i \geq -1} \partial_1(m_i)\Lambda^i = \sum_{j \geq 0}(m_{-1}\Lambda^{-1}m_j\Lambda - m_j\Lambda^j m_{-1}\Lambda^{-j})\Lambda^{j-1}.$$

Componentwise this amounts to: for all $i \geq -1$ and all $s \in \mathbb{Z}$

$$\partial_1(m_i(s)) = m_{-1}(s)m_{i+1}(s-1) - m_{i+1}(s)m_{-1}(s+i+1).$$

In particular, this gives for the $m_{-1}(s)$

$$\partial_1(m_{-1}(s)) = m_{-1}(s)(m_0(s-1) - m_0(s)) \Leftrightarrow \partial_1(\ln(m_{-1}(s)) = m_0(s-1) - m_0(s),$$

a mixture of a differential and a difference equation well-known from Toda equations like (15).

Remark 14. The dependence of the solutions of the hierarchy of the basic directions is as follows: if $\mathfrak{h}$ is a maximal commutative F-subalgebra of $i_k(M_k(F))$, then there is a bijection between solutions $\mathcal{M}$ and $\{\mathcal{V}_\alpha\}$ of the $(\Lambda^{-k}, \mathfrak{h}_{<0})$-hierarchy and those of the $(\mathcal{M}_0, (V_0\mathfrak{h}V_0^{-1})_{<0})$-hierarchy , namely

$$\mathcal{M}, \{\mathcal{V}_\alpha\} \mapsto V_0\mathcal{M}V_0^{-1}, \{V_0\mathcal{V}_\alpha V_0^{-1}\}.$$

The reason is that for all $j > 0$ and all $\alpha \in \{1, \cdots, m_0\}$

$$V_0(\mathcal{M}^j \mathcal{V}_\alpha)_{<0}(\Lambda^{-k})V_0^{-1} = (V_0 \mathcal{M}^j \mathcal{V}_\alpha V_0^{-1})_{<0}(\mathcal{M}_0).$$

Since the general case gives better insight in the full geometric picture and it requires no additional effort, one presents the general case here.

Example 3. Also in the upper triangular case there is a concrete setting, where one can look for a realization of the equations of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy: take for the ring R the algebra $F[s_{j\beta}]$ of all polynomials in the parameters

$$\{s_{j\beta} \mid j \geq 1, 1 \leq \beta \leq m_0\}$$

or, more generally, that of all formal power series $F[[s_{j\beta}]]$ in these variables. You consider matrices $\mathcal{M}$ and the V_α in $M_{\mathbb{Z}}(R)$ that have first of all the form (112) resp. (116) and secondly satisfy the commutation relations (117). For the derivations $\partial_{Q_{j\beta}}$ one takes then the partial derivative $\partial_{s_{j\beta}}$ w.r.t. the variable $s_{j\beta}$. The nonlinear equations one wants the $\mathcal{M}$ and the V_α to satisfy are then

$$\partial_{s_{j\beta}}(\mathcal{M}) = [(Q_{j\beta})_{\geq 0}(\mathcal{M}_0), \mathcal{M}] \text{ and } \partial_{s_{j\beta}}(V_\alpha) = [(Q_{j\beta})_{\geq 0}(\mathcal{M}_0), V_\alpha] \qquad (119)$$

Again one has silently assumed in this example that the derivations $\partial_{Q_{j\beta}}$ commute among themselves. We come back to this issue in the next subsection.

Example 4. A wider range of solutions of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy is obtained if one localizes the F-algebra from example 3. Take a multiplicative subset S in the algebra $F[[s_{j\beta}]]$ and consider $R = S^{-1}F[[s_{j\beta}]]$. On this F-algebra one has the same set of derivations as in example 3 and one can search for solutions inside $M_{\mathbb{Z}}(S^{-1}F[[s_{j\beta}]])$ of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy and this may lead to solutions that have singularities in their $s_{j\beta}$-dependence.

5.2 A minimal realization of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy

In this section one discusses a minimal realization of the relations (118), where minimality refers again to the number of relations between the solutions. For the minimal model of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy, one starts with matrices of the right shape. On the first place there is the deformation of $\mathcal{M}_0^{-1}$

$$\tilde{\mathcal{M}} := \sum_{r\geq -1} \tilde{m}_r \mathcal{M}_0^r, \text{ with } \tilde{m}_r = V_0 \mathrm{diag}(\tilde{m}_r(s)) V_0^{-1} \text{ and } \tilde{m}_{-1} \text{ invertible} . \tag{120}$$

Then, one has thanks to Proposition 1,

$$\tilde{\mathcal{M}} = V \mathcal{M}_0^{-1} V^{-1}, \text{ with } V = \sum_{i=0}^{\infty} v_i \mathcal{M}_0^i \text{ and } v_0 = V_0 \tilde{w}_0 V_0^{-1},$$

where $\tilde{w}_0 = \mathrm{diag}(\tilde{w}_0(s))$ is the $k \times k$-block diagonal matrix given by $\tilde{w}_0(0) = \mathrm{Id}$ and

$$\tilde{w}_0(t) = \tilde{m}_{-1}(t) \ldots \tilde{m}_{-1}(1) \text{ for } t \geq 1 \text{ and} \tag{121}$$

$$\tilde{w}_0(t) = \tilde{m}_{-1}(t+1)^{-1} \ldots \tilde{m}_{-1}(0)^{-1} \text{ for } t < 0.$$

Besides $\tilde{\mathcal{M}}$ one has all the deformations of the $\mathcal{V}_{\alpha,0}$

$$\tilde{\mathcal{V}}_\alpha := V_0 \tilde{w}_0 V_0^{-1} \mathcal{V}_{\alpha,0} V_0 \tilde{w}_0^{-1} V_0^{-1} + \sum_{i>0} \tilde{v}_{i,\alpha} \mathcal{M}_0^{-i}, \tag{122}$$

with $\alpha \in \{1, \cdots, m_0\}$ and $\tilde{w}_0$ as in (121). One writes $\tilde{Q}_{j\beta}$ for each $\tilde{\mathcal{M}}^j \tilde{\mathcal{V}}_\beta$. Also the matrices $\{\tilde{v}_{i,\alpha}\}$ in $V_0 \mathcal{D}_k(R) V_0^{-1}$ can be written as

$$\tilde{v}_{i,\alpha} = V_0 \mathrm{diag}(\tilde{v}_{i,\alpha}(t)) V_0^{-1}. \tag{123}$$

Combining the formulae (120), (121) and (122) yields that all the matrix coefficients of the $\{\tilde{m}_r\}$ and the $\{\tilde{v}_{i,\alpha}\}$ are polynomials over F of the matrix coefficients $\tilde{m}_n(p)_{\epsilon\eta}$ of the $\tilde{m}_n(p)$, the matrix coefficients $\tilde{v}_{m,\beta}(t)_{\mu\nu}$ of the $\tilde{v}_{m,\beta}(t)$ and the inverses of the determinants of all the $\tilde{m}_{-1}(p)$. Recall that the aim was to construct a minimal realization of the equations of the hierarchy and in that light one wants to take the coefficients of $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ as independent as possible. Consider the free polynomial ring $F[\tilde{m}_r(p)_{\epsilon\eta}, \tilde{v}_{m,\beta}(t)_{\mu\nu}]$, where all the indices $\{\epsilon,\ \eta,\ \mu,\ \nu\}$ belong to $\{1, \ldots, k\}$; both p and $t \in \mathbb{Z}$; $m \geq 1$ and $r \geq -1$. In this ring one has the multiplicative subset S generated by the determinants of the matrices $\tilde{m}_{-1}(s), s \in \mathbb{Z}$. Then the coefficients of $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ belong to the localization

$$R_{<0} := S^{-1} F[\tilde{m}_r(p)_{\epsilon\eta}, \tilde{v}_{m,\beta}(t)_{\mu\nu}],$$

of the ring $F[\tilde{m}_r(p)_{\epsilon\eta}, \tilde{v}_{m,\beta}(t)_{\mu\nu}]$ w.r.t. S. By conjugating with V_0^{-1} one sees that the matrix coefficients of $\tilde{m}_r$ and the $\tilde{v}_{i,\alpha}$ are also generators of the ring $R_{<0}$. Every F-linear derivation $\Delta \in \mathrm{Der}_F(R_{<0})$ is completely determined by describing freely all the

$$\{\Delta(\tilde{m}_r(p)_{\epsilon\eta}),\ \Delta(\tilde{v}_{m,\beta}(t)_{\mu\nu})\}.$$

Since the commutator of an element in $UT_{\geq 0}(\mathcal{M}_0)$ with an element $U \in UT$ has order in $\mathcal{M}_0$ bigger or equal to that of U, one can therefore define for each $\tilde{Q}$ in UT of degree < 0 in $\mathcal{M}_0$ a unique derivation $\tilde{\partial}_{\tilde{Q}}:\ R_{<0} \to R_{<0}$ by the matrix equalities

$$\tilde{\partial}_{\tilde{Q}}(\tilde{\mathcal{M}}) := -[\tilde{Q}_{\geq 0}(\mathcal{M}_0), \tilde{\mathcal{M}}] \text{ and } \tilde{\partial}_{\tilde{Q}}(\tilde{V}_\alpha) := -[\tilde{Q}_{\geq 0}(\mathcal{M}_0), \tilde{V}_\alpha]. \tag{124}$$

In particular, one has for all $s \in \mathbb{Z}$

$$\tilde{\partial}_{\tilde{Q}}(\det(\tilde{m}_{-1}(s))^{-1}) = -\frac{1}{\det(\tilde{m}_{-1}(s))^2}\tilde{\partial}_{\tilde{Q}}(\det(\tilde{m}_{-1}(s))).$$

A priori the matrices $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ do not commute between each other, so one sanctions upon all the independent variables $\tilde{m}_r(p)_{\epsilon\eta}$ and $\tilde{v}_{m,\beta}(t)_{\mu\nu}$ the set of relations

$$[\tilde{\mathcal{M}}, \tilde{\mathcal{V}}_\alpha] = 0 \text{ and } [\tilde{\mathcal{V}}_\alpha, \tilde{\mathcal{V}}_\beta] = 0, \text{ for all } \alpha \text{ and } \beta \tag{125}$$

meaning that we pass over to the quotient ring $\underline{R}_{<0}$ of $R_{<0}$ by the ideal J generated by the matrix coefficients of the left hand sides in the relations (125). For simplicity, the natural image in $M_{\mathbb{Z}}(\underline{R}_{<0})$ of the matrices $\tilde{\mathcal{M}}$, the $\tilde{\mathcal{V}}_\alpha$ and the $\tilde{Q}_{j\beta}$ is denoted by the same notation. Observe that the derivations $\tilde{\partial}_{\tilde{Q}}$ also factorize to derivations of $\underline{R}_{<0}$ for there holds

$$\begin{aligned} \tilde{\partial}_{\tilde{Q}}([\tilde{\mathcal{M}}, \tilde{\mathcal{V}}_\alpha]) &= [-\tilde{Q}_{\geq 0}(\mathcal{M}_0), [\tilde{\mathcal{M}}, \tilde{\mathcal{V}}_\alpha]], \\ \tilde{\partial}_{\tilde{Q}}([\tilde{\mathcal{V}}_\alpha, \tilde{\mathcal{V}}_\beta]) &= [-\tilde{Q}_{\geq 0}(\mathcal{M}_0), [\tilde{\mathcal{V}}_\alpha, \tilde{\mathcal{V}}_\beta]] \end{aligned} \tag{126}$$

and thus $\tilde{\partial}_{\tilde{Q}}$ maps the ideal J into itself. One uses the same notation for the corresponding derivation of $\underline{R}_{<0}$. In particular there holds in $M_{\mathbb{Z}}(\underline{R}_{<0})$ the first order consequences of the commutativity relations

$$\begin{aligned} [\tilde{\partial}_{\tilde{Q}}(\tilde{\mathcal{M}}), \tilde{\mathcal{V}}_\beta] + [\tilde{\mathcal{M}}, \tilde{\partial}_{\tilde{Q}}(\tilde{\mathcal{V}}_\beta)] &= 0 \\ [\tilde{\partial}_{\tilde{Q}}(\tilde{\mathcal{V}}_\beta), \tilde{\mathcal{V}}_\gamma] + [\tilde{\mathcal{V}}_\beta, \tilde{\partial}_{\tilde{Q}}(\tilde{\mathcal{V}}_\gamma)] &= 0 \end{aligned} \tag{127}$$

and likewise all the higher order ones. The $\tilde{Q}_{j\beta} = \tilde{\mathcal{M}}^j \tilde{\mathcal{V}}_\beta$, $j \geq 1$ and $\beta \in \{1, \cdots, m_0\}$, all commute in $M_{\mathbb{Z}}(\underline{R}_{<0})$ with the $\tilde{\mathcal{M}}$ and the $\{\tilde{\mathcal{V}}_\beta\}$ and if one takes for $\tilde{Q}$ all the elements $\tilde{Q}_{j\beta}$, then we have a set of derivations $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ of $\underline{R}_{<0}$ for which the Lax equations of the hierarchy

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{\mathcal{M}}) = [(\tilde{Q}_{j\beta})_{<0}(\mathcal{M}_0), \tilde{\mathcal{M}}] \text{ and } \tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{\mathcal{V}}_\alpha) = [(\tilde{Q}_{j\beta})_{<0}(\mathcal{M}_0), \tilde{\mathcal{V}}_\alpha]. \tag{128}$$

hold nearly by definition.

As with the lower triangular hierarchies one concludes also here with a discussion of some consequences from this minimal realization of the Lax equations of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy. Before doing so, it is convenient to introduce a notation: for all $n \geq 1$, and all $\gamma \in \{1, \cdots, m_0\}$, one writes

$$\tilde{C}_{n\gamma} := (\tilde{\mathcal{M}}^n \tilde{\mathcal{V}}_\gamma)_{<0}(\mathcal{M}_0).$$

Also these matrices satisfy the following so-called *zero curvature equations*

Proposition 6. *The operators $\{\tilde{C}_{n\gamma}\}$ in $M_{\mathbb{Z}}(\underline{R}_{<0})$ satisfy for all $n, m \in \mathbb{N}$ and all α and γ in $\{1, \cdots, m_0\}$*

$$\tilde{\partial}_{\tilde{Q}_{n\alpha}}(\tilde{C}_{m\gamma}) - \tilde{\partial}_{\tilde{Q}_{m\gamma}}(\tilde{C}_{n\alpha}) - [\tilde{C}_{n\alpha}, \tilde{C}_{m\gamma}] = 0. \tag{129}$$

Proof. One will show that the left hand side belongs both to $UT_{<0}(\mathcal{M}_0)$ and to $UT_{\geq 0}(\mathcal{M}_0)$ and thus has to be zero.

Since the $\tilde{C}_{m\gamma}$ belongs to $UT_{<0}(\mathcal{M}_0)$ for all $m \geq 1$, it is clear that the left hand side of the zero curvature equations for the $\{\tilde{C}_{m\gamma}\}$ belongs to $UT_{<0}(\mathcal{M}_0)$. Next one uses again the Lax equations for $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ to get the relations: for all n and $m \in \mathbb{N}$ and all α and $\gamma \in \{1, \ldots, m_0\}$

$$\tilde{\partial}_{\tilde{Q}_{n\gamma}}(\tilde{Q}_{m\alpha}) = [\tilde{C}_{n\gamma}, \tilde{Q}_{m\alpha}] = -[(\tilde{Q}_{n\gamma})_{\geq 0}(\mathcal{M}_0), \tilde{Q}_{m\alpha}]. \tag{130}$$

If one substitutes in the zero curvature equation $\tilde{C}_{m\gamma} = \tilde{Q}_{m\gamma} - (\tilde{Q}_{m\gamma})_{\geq 0}(\mathcal{M}_0)$ and uses property (130) , then this leads to

$$\begin{aligned} &\tilde{\partial}_{\tilde{Q}_{n\alpha}}(\tilde{Q}_{m\gamma}) - \tilde{\partial}_{\tilde{Q}_{n\alpha}}((\tilde{Q}_{m\gamma})_{\geq 0}(\mathcal{M}_0)) - \tilde{\partial}_{\tilde{Q}_{m\gamma}}(\tilde{Q}_{n\alpha}) + \tilde{\partial}_{\tilde{Q}_{m\gamma}}((\tilde{Q}_{n\alpha})_{\geq 0}(\mathcal{M}_0)) \\ &- [\tilde{Q}_{n\alpha} - (\tilde{Q}_{n\alpha})_{\geq 0}(\mathcal{M}_0), \tilde{Q}_{m\gamma} - (\tilde{Q}_{m\gamma})_{\geq 0}(\mathcal{M}_0)] = -\tilde{\partial}_{\tilde{Q}_{n\alpha}}((\tilde{Q}_{m\gamma})_{\geq 0}(\mathcal{M}_0)) \\ &+ \tilde{\partial}_{\tilde{Q}_{m\gamma}}((\tilde{Q}_{n\alpha})_{\geq 0}(\mathcal{M}_0)) - [(\tilde{Q}_{n\alpha})_{\geq 0}(\mathcal{M}_0), (\tilde{Q}_{m\gamma})_{\geq 0}(\mathcal{M}_0)]. \end{aligned}$$

The right hand side of this expression belongs to $UT_{\geq 0}(\mathcal{M}_0)$ and one has shown the desired property. □

This proposition allows you to show the following property that unites the equations that belong to this hierarchy

Corollary 3. *All derivations* $\{\tilde{\partial}_{\tilde{Q}_{m\beta}} \mid m \geq 1, \beta \in \{1, \cdots, m_0\}\}$ *of the algebra* $\underline{R}_{<0}$ *commute.*

Proof. The matrix coefficients of $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ generate the algebra $\underline{R}_{<0}$. Therefore it suffices to show

$$(\tilde{\partial}_{\tilde{Q}_{m\beta}} \circ \tilde{\partial}_{\tilde{Q}_{r\delta}} - \tilde{\partial}_{\tilde{Q}_{r\delta}} \circ \tilde{\partial}_{\tilde{Q}_{m\beta}})(\tilde{\mathcal{M}}) = 0 \text{ and } (\tilde{\partial}_{\tilde{Q}_{m\beta}} \circ \tilde{\partial}_{\tilde{Q}_{r\delta}} - \tilde{\partial}_{\tilde{Q}_{r\delta}} \circ \tilde{\partial}_{\tilde{Q}_{m\beta}})(\tilde{\mathcal{V}}_\alpha) = 0.$$

One gets the desired identities by applying the following property already mentioned in Corollary 1 in chapter 4: let ∂_1 and ∂_2 be derivations of some ring $\mathcal{R}$ and let X be a matrix in $M_{\mathbb{Z}}(\mathcal{R})$ such that for $i = 1, 2$

$$\partial_i(X) = [D_i, X] \text{ for some } D_i \in M_{\mathbb{Z}}(\mathcal{R}).$$

Then a straightforward computation shows that

$$(\partial_1 \circ \partial_2 - \partial_2 \circ \partial_1)(X) = [\partial_1(D_2) - \partial_2(D_1) - [D_1, D_2], X] \tag{131}$$

By inserting the zero curvature relations from Proposition 9 in this identity for the operators and derivations listed above, one obtains the statements in the corollary. Note that all the commutators in this reasoning make sense, because the matrices X, D_1 and D_2 belong to $UT(\underline{R}_{<0})$. □

Since the basic derivations in the Lax equations of the $(\mathcal{M}_0, \mathbf{k}_{<0})$-hierarchy commute, it is appropriate to call the equations in Proposition 9 the so-called *zero curvature equations* of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy, see section 3.

5.3 The zero curvature form of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy

In the last section one has built a minimal context in which the equations of the form (111) hold. Here one discusses other realizations of the equations of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy and the zero curvature form of the hierarchy.

Let $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ be as in subsection 5.2. To get a matrix $\mathcal{M}$ of the right shape we assign to each matrix coefficient $\tilde{m}_n(p)_{\epsilon\eta}$ of $\tilde{\mathcal{M}}$ an element $m_n(p)_{\epsilon\eta}$ in some F-algebra R such that all the $m_{-1}(p), p \in \mathbb{Z}$, are invertible in $M_k(R)$. Those matrices $m_{-1}(p), p \in \mathbb{Z}$, determine also the invertible element $w_0 \in \mathcal{D}_k(R)$ by making the same substitution in the element $\tilde{w}_0 \in \mathcal{D}_k(R_{<0})$ defined in subsection 5.2. For the leading coefficient m_{-1} this amounts to $m_{-1} = V_0 w_0 V_0^{-1} \mathcal{M}_0^{-1} V_0 w_0^{-1} V_0^{-1} \mathcal{M}_0$. Likewise the matrix $\mathcal{V}_\beta$ is determined by assigning to each matrix coefficient $\tilde{v}_{m,\beta}(t)_{\mu\nu}, m \geq 1$, of the $\tilde{\mathcal{V}}_\beta$ an element $v_{m,\beta}(t)_{\mu\nu}$ in the same F-algebra R and by assigning to $\operatorname{diag}(\tilde{v}_{0,\beta}(t))$ the $k \times k$-block matrix $w_0 i_k(F_\beta) w_0^{-1}$. The leading coefficient of each $\mathcal{V}_\beta$ is then in particular $V_0 w_0 V_0^{-1} \mathcal{V}_{\beta,0} V_0 w_0^{-1} V_0^{-1}$. All these assignments are nothing but an F-algebra morphism $\mu : R_{<0} \to R$. The morphism μ determines an F-linear map from $M_{\mathbb{Z}}(R_{<0})$ to $M_{\mathbb{Z}}(R)$ for which one uses the same notation. On the collection of upper triangular matrices the map μ induces an F-algebra morphism from $UT(R_{<0})$ to $UT(R)$. It furnishes you a set of matrices

$$\mathcal{M} := \mu(\tilde{\mathcal{M}}) \text{ and the } \mathcal{V}_\alpha := \mu(\tilde{\mathcal{V}}_\alpha) \tag{132}$$

of the right shape. To keep notations consistent, one writes for all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$,

$$Q_{j\beta} := \mathcal{M}^j \mathcal{V}_\beta \text{ and } C_{j\beta} := (Q_{j\beta})_{<0}(\mathcal{M}_0).$$

If the matrices $\mathcal{M}$ and the $\mathcal{V}_\alpha$ commute among each other, then the assignment corresponds even to a morphism $\mu : \underline{R}_{<0} \to R$. This we assume from now on.

Next we want to transfer the Lax structure to R and in that light we require that R is equipped with a number of F-linear derivations $\partial_{Q_{j\beta}} : R \to R$, with $Q_{j\beta}$ as above, such that

$$\partial_{Q_{j\beta}} \circ \mu = \mu \circ \tilde{\partial}_{\tilde{Q}_{j\beta}}. \tag{133}$$

Clearly, the relation (133) implies relations for the matrices $\mathcal{M}$ and $\mathcal{V}_\alpha$ in $M_{\mathbb{Z}}(R)$. Namely they have to satisfy respectively

$$\partial_{Q_{j\beta}}(\mathcal{M}) = [C_{j\beta}, \mathcal{M}], \tag{134}$$

$$\partial_{Q_{j\beta}}(\mathcal{V}_\alpha) = [C_{j\beta}, \mathcal{V}_\alpha]. \tag{135}$$

Since the matrix coefficients of $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ generate $\underline{R}_{<0}$, the equations (134) and (135) are also sufficient in order that relation (133) holds. Thus finding such an R and a set of derivations $\partial_{Q_{j\beta}}$ such that relation (133) holds, amounts to finding a solution, namely $\mathcal{M}$ and the $\mathcal{V}_\alpha$, of the equations of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy. Keeping in mind that μ is an F-algebra morphism, one may conclude from relation (133) and Proposition 6 that, if $\tilde{\mathcal{M}}$ and the $\tilde{\mathcal{V}}_\alpha$ satisfy (134) and (135), then also the zero curvature relations hold

$$\partial_{Q_{n\alpha}}(C_{m\gamma}) - \partial_{Q_{m\gamma}}(C_{n\alpha}) - [C_{n\alpha}, C_{m\gamma}] = 0. \tag{136}$$

Also in the upper triangular case there is still a second set of zero curvature relations related to the above situation, for there holds

Lemma 6. *Assume one has an F-algebra R, a morphism $\mu : \underline{R}_{<0} \to R$ and a set of F-linear derivations $\partial_{Q_{j\beta}}$ such that the relations (133) hold. Let $\mathcal{M}$ and the V_α be given by (132). Then the $\{H_{m\delta} := C_{m\delta} - Q_{m\delta}\}$ satisfy*

$$\partial_{Q_{n\alpha}}(H_{m\delta}) - \partial_{Q_{m\delta}}(H_{n\alpha}) - [H_{n\alpha}, H_{m\delta}] = 0. \tag{137}$$

Proof. Because of the Lax equations for $\mathcal{M}$ and the V_α each $Q_{m\delta}$ satisfies the same Lax equations

$$\partial_{Q_{j\beta}}(Q_{m\delta}) = [C_{j\beta}, Q_{m\delta}]. \tag{138}$$

Thus we get

$$\begin{aligned}
&\partial_{Q_{n\alpha}}(H_{m\delta}) - \partial_{Q_{m\delta}}(H_{n\alpha}) - [H_{n\alpha}, H_{m\delta}] = \\
&\partial_{Q_{n\alpha}}(C_{m\delta}) - \partial_{Q_{n\alpha}}(Q_{m\delta}) - \partial_{Q_{m\delta}}(C_{n\alpha}) + \partial_{Q_{m\delta}}(Q_{n\alpha}) - \\
&[C_{n\alpha}, C_{m\delta}] + [C_{n\alpha}, Q_{m\delta}] + [Q_{n\alpha}, C_{m\delta}] - [Q_{n\alpha}, Q_{m\delta}] = 0,
\end{aligned}$$

thanks to the Lax equations (138). □

Next it will be shown that, reversely, the equations (136) also imply the Lax equations (134) and (135). So we start with the following setting: let $\mathcal{R}$ be an F-algebra equipped with a collection of F-linear derivations $\partial_{Q_{j\beta}} : \mathcal{R} \to \mathcal{R}, j \geq 1, \beta \in \{1, \cdots, m_0\}$, and let $\mu : \underline{R}_{<0} \to \mathcal{R}$ be an F-algebra morphism. It induces an F- linear map from $M_{\mathbb{Z}}(R_{<0})$ to $M_{\mathbb{Z}}(\mathcal{R})$. The matrices $\mathcal{M} := \mu(\tilde{\mathcal{M}})$ and $\mathcal{V}_\alpha := \mu(\tilde{\mathcal{V}}_\alpha) \in M_{\mathbb{Z}}(\mathcal{R})$ satisfy the relations (117). Then we have

Theorem 5. *With the notations just introduced, there holds that the Lax equations (134) and (135) for $\mathcal{M}$ and the $\mathcal{V}_\alpha$ are equivalent to the zero curvature relations (136) for the matrices $\{C_{r\gamma}\}$.*

Proof. One merely has to prove still that the zero curvature equations imply the Lax equations. To get the Lax equations for $\mathcal{M}$ and the $\{\mathcal{V}_\alpha\}$, consider for all $m \geq 1$ the matrices

$$\partial_{Q_{n\beta}}(\mathcal{M}^m \mathcal{V}_\alpha) - [C_{n\beta}, \mathcal{M}^m \mathcal{V}_\alpha].$$

By substituting in it the formula $\mathcal{M}^m \mathcal{V}_\alpha = C_{m\alpha} + (\mathcal{M}^m \mathcal{V}_\alpha)_{\geq 0}(\mathcal{M}_0)$ and by using the zero curvature equations, one gets the equality

$$\partial_{Q_{n\beta}}(\mathcal{M}^m \mathcal{V}_\alpha) - [C_{n\beta}, \mathcal{M}^m \mathcal{V}_\alpha] = \partial_{Q_{n\beta}}((\mathcal{M}^m \mathcal{V}_\alpha)_{\geq 0}(\mathcal{M}_0)) + \partial_{Q_{m\alpha}}(C_{n\beta}) -$$
$$[C_{n\beta}, (\mathcal{M}^m \mathcal{V}_\alpha)_{\geq 0}(\mathcal{M}_0)].$$

From this equality one sees that the order in $\mathcal{M}_0$ in all the expressions

$$\partial_{Q_{n\beta}}(\mathcal{M}^m \mathcal{V}_\alpha)) - [C_{n\beta}, \mathcal{M}^m \mathcal{V}_\alpha)]$$

is bounded below by $-n$. Consider first the operator $\mathcal{M}$. Assume that one has

$$\partial_{Q_{n\beta}}(\mathcal{M}) - [C_{n\beta}, \mathcal{M}] = \beta \mathcal{M}_0^l + \text{ higher order in } \mathcal{M}_0, \tag{139}$$

with $\beta \in V_0 \mathcal{D}_k(R) V_0^{-1}$ nonzero. With induction with respect to m one shows

$$\partial_{Q_{n\beta}}(\mathcal{M}^m) - [C_{n\beta}, \mathcal{M}^m] = \sum_{i=0}^{m-1} \mathcal{M}^i \{\partial_{Q_{n\beta}}(\mathcal{M}) - [C_{n\beta}, \mathcal{M}]\} \mathcal{M}^{m-i-1}. \tag{140}$$

Hence the leading term in $\mathcal{M}_0$ of the right hand side is

$$\sum_{i=0}^{m-1} (m_{-1}\mathcal{M}_0^{-1})^i \beta \mathcal{M}_0^l (m_{-1}\mathcal{M}_0^{-1})^{-i} (m_{-1}\mathcal{M}_0^{-1})^{m-1}, \tag{141}$$

which is of order $l + 1 - m$ in $\mathcal{M}_0$. If m tends to infinity this contradicts again the fact that the left hand side belongs to $UT_{\geq -n}(\mathcal{M}_0)$, unless for all sufficiently large m

$$\sum_{i=0}^{m-1} (m_{-1}\mathcal{M}_0^{-1})^i \beta \mathcal{M}_0^l (m_{-1}\mathcal{M}_0^{-1})^{-i} = 0.$$

This in its turn implies that the matrix $\beta\mathcal{M}_0^l$ is equal to zero and hence that $\beta = 0$. So, the Lax equations (134) have to hold for $\mathcal{M}$ for all $n \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$. To obtain the Lax equations (135) for $\mathcal{V}_\alpha$ one considers the operator $\mathcal{M}^m\mathcal{V}_\alpha$ and applies the Lax equations for $\mathcal{M}^m$ to get

$$\partial_{Q_{n\beta}}(\mathcal{M}^m\mathcal{V}_\alpha) - [C_{n\beta}, \mathcal{M}^m\mathcal{V}_\alpha)] = 0 + \mathcal{M}^m(\partial_{Q_{n\beta}}(\mathcal{V}_\alpha) - [C_{n\beta}, \mathcal{V}_\alpha]). \tag{142}$$

If the operator $\mathcal{V}_\alpha$ would not satisfy the Lax equation, then the matrix

$$\partial_{Q_{n\beta}}(\mathcal{V}_\alpha) - [C_{n\beta}, \mathcal{V}_\alpha]$$

would have an order s in $\mathcal{M}_0$ and the right hand side of (142) would be of order $-m + s$ in $\mathcal{M}_0$ and this contradicts the fact that it was bounded below by $-n$. Therefore the Lax equations have to hold for the $\mathcal{V}_\alpha$ as well. □

5.4 Wave matrices for the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy

The *linearization* of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy will also be an algebraic substitute for the basis of horizontal sections of the lacking flat connection that would yield the zero curvature relations of the hierarchy.

To set the stage: one starts with an F-algebra R and an F-linear morphism $\mu : \underline{R}_{<0} \to R$. Then one has the corresponding potential solutions, namely operators $\mathcal{M} = \mu(\tilde{\mathcal{M}})$ and $\mathcal{V}_\alpha = \mu(\tilde{\mathcal{V}_\alpha})$ in $UT(R)$ of the form (120) resp. (122). Further one assumes that R is equipped with a collection of F-linear derivations $\{\partial_{Q_{j\beta}}, j \geq 1, \beta \in \{1, \cdots, m_0\}\}$.

The *linearization of the* $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$*-hierarchy* consists of the following equations for $\mathcal{M}$ and $\mathcal{V}_\alpha$

$$\mathcal{M}\phi = \phi\mathcal{M}_0^{-1}, V_\alpha\phi = \phi\mathcal{V}_{\alpha,0} \text{ and } \partial_{Q_{j\beta}}(\phi) = C_{j\beta}\phi, \tag{143}$$

where ϕ is a not yet specified object for which all the operations in (143), like multiplying from the left and right with $\mathbb{Z} \times \mathbb{Z}$-matrices and applying all the derivations $\partial_{Q_{j\beta}}$ make sense. Before specifying ϕ, we show first by which manipulations one can derive the Lax equations of the hierarchy and after that we build a context in which they make sense.

To get the Lax equations for $\mathcal{M}$ one applies the derivation $\partial_{Q_{j\beta}}$ to the first equation in (209) and subtitutes the last one. This leads to the following manipulations

$$\partial_{Q_{j\beta}}(\mathcal{M}\phi - \phi\mathcal{M}_0^{-1}) = \partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))\mathcal{M}_0^{-1} = \quad (144)$$
$$\partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}C_{j\beta}\phi - C_{j\beta}\phi\mathcal{M}_0^{-1} = \{\partial_{Q_{j\beta}}(\mathcal{M}) - [C_{j\beta}, \mathcal{M}\ ]\}\phi = 0.$$

Hence, if it is allowed to eliminate the function ϕ from the foregoing equation, one obtains the Lax equations for $\mathcal{M}$. For the operator $\mathcal{V}_\alpha$ one applies $\partial_{Q_{j\beta}}$ to the second equation in (143) and substitutes the last one. Thus one gets

$$\partial_{Q_{j\beta}}(\mathcal{V}_\alpha\phi - \phi\mathcal{V}_{\alpha,0}) = \partial_{Q_{j\beta}}(\mathcal{V}_\alpha)\psi + \mathcal{V}_\alpha(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))\mathcal{V}_{\alpha,0} = \quad (145)$$
$$= \partial_{Q_{j\beta}}(\mathcal{V}_\alpha)\phi + \mathcal{V}_\alpha C_{j\beta}\phi - C_{j\beta}\phi\mathcal{V}_{\alpha,0} = \{\partial_{Q_{j\beta}}(\mathcal{V}_\alpha) - [C_{j\beta}, \mathcal{V}_\alpha\ ]\}\phi = 0.$$

and if one can leave out ϕ again, this yields the Lax equations for $\mathcal{V}_\alpha$.

For the linearization (143) one needs a left action of $\mathcal{M}$ and all the $C_{j\beta}$ on the functions ϕ and a right one for matrices like $\mathcal{M}_0^{-1}$ and the $V_0 i_k(F_\alpha)V_0^{-1}$. To realize the first, one builds a left $UT(R)$-module. The actual form of the elements in the module is guided by the trivial solution $\mathcal{M} = \mathcal{M}_0^{-1}$ and $\mathcal{V}_\alpha = \mathcal{V}_{\alpha,0}$ of the hierarchy. In that case the equations (143) become

$$\mathcal{M}_0^{-1}\phi = \phi\mathcal{M}_0^{-1}, \mathcal{V}_{\alpha,0}\phi = \phi\mathcal{V}_{\alpha,0} \text{ and } \partial_{Q_{j\beta}}(\phi) = (\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0})\phi. \quad (146)$$

The last equation of (146) tells you again that the action on ϕ of the first order approximation of the flow corresponding to the basic direction $\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0}$ is simply multiplying ϕ from the left with the constant matrix $\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0}$. Let $s_{j\beta}$ be the parameter corresponding to the flow of $\partial_{Q_{j\beta}}$, so that $\partial_{Q_{j\beta}}$ acts as taking the partial derivative $\frac{\partial}{\partial s_{j\beta}} =: \partial_{s_{j\beta}}$ w.r.t. the parameter $s_{j\beta}$. The equations (146) can then formally be integrated simultaneously. For, consider the formal series

$$\phi_0 := \exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_0} s_{j\beta}\mathcal{V}_{\beta,0}\mathcal{M}_0^{-j}). \quad (147)$$

With the action of $\partial_{Q_{j\beta}}$ just mentioned, ϕ_0 satisfies the equations (146). A priori, ϕ_0 does not define a proper matrix in $M_{\mathbb{Z}}(R)$. That would require at least a natural embedding of $F[s_{j\beta}]$ into R and convergence conditions on the element $\mathcal{M}_0$. However, as a formal power series in the parameters $\{s_{j\beta} \mid j \geq 1, \beta \in \{1, \cdots, m_0\}\}$, there holds

$$\phi_0 = \sum_{\gamma} \phi_0(\gamma) t^{\gamma}, \text{ with all } \phi_0(\gamma)) \in UT(F). \tag{148}$$

Here the multi-index $\gamma = \{\gamma(j\beta)\}$ is such that $\gamma(j\beta) \in \mathbb{N}$ is equal to zero, for all but a finite number of indices.

The module for the linearization will consist of perturbations in $UT(R)$ of this trivial solution. Consider namely the collection $M^{(0)}(\mathcal{M}_0)$ consisting of formal products

$$\{\sum_{j=N}^{\infty} d_j \mathcal{M}_0^j\} \exp(\sum_{j=1}^{\infty} \sum_{\beta=1}^{m_U} s_{j\beta} \mathcal{V}_{\beta,0} \mathcal{M}_0^{-j}), \text{ where } d_j \in V_0 \mathcal{D}_k(R) V_0^{-1}. \tag{149}$$

The elements of $M^{(0)}(\mathcal{M}_0)$ are called *oscillating matrices at zero*, since they are formal products of a series in $\mathcal{M}_0$ that has a pole around zero and the exponential term, which is an essential singularity in $\mathcal{M}_0$. If $F[s_{j\beta}] \subset R$ and $\mathcal{M}_0$ is sufficiently nice so that ϕ_0 belongs to $M_{\mathbb{Z}}(R)$, then it requires still convergence considerations to make sense of the product in (94) as a matrix in $M_{\mathbb{Z}}(R)$. Such a setting is described in the remaining subsections of this chapter. So in general these formal products do not give a well-defined element of $M_{\mathbb{Z}}(R)$. However, if $R = F[[s_{j\beta}]]$ and $\partial_{Q_{j\beta}} = \partial_{s_{j\beta}}$ as in Example (1), then we know from (148) that both ϕ_0 and any element $\sum_{j=-\infty}^{N} d_j \mathcal{M}_0^j$ are formal power series in the variables $s_{j\beta}$ with coefficients from $UT(F)$ and their product is also a well-defined formal power series in the variables $s_{j\beta}$ with coefficients from $UT(F)$.

Nevertheless there is in the general case a well-defined left action of $UT(R)$ on it. For all u_1 and $u_2 \in UT(R)$ one puts namely

$$u_1\{u_2\}\phi_0 = \{u_1 u_2\}\phi_0. \tag{150}$$

Also the right multiplication with $\mathcal{M}_0^{-1}$ and $\mathcal{V}_{\beta,0}$ is well-defined on an element $u\phi_0$ of $M^{(0)}(\mathcal{M}_0)$ and well by

$$\{u\phi_0\}\mathcal{M}_0^{-1} = \{u\mathcal{M}_0^{-1}\}\phi_0 \text{ and } \{v\phi_0\}\mathcal{V}_{\beta,0} = \{v\mathcal{V}_{\beta,0}\}\phi_0, \tag{151}$$

as if ϕ_0 is a matrix commuting with these basic matrices. An action of the derivations $\partial_{s_{j\beta}}$ on $M^{(0)}(\mathcal{M}_0)$ is defined as if the product in the module $M^{(0)}(\mathcal{M}_0)$ is a real one

$$\partial_{s_{j\beta}}\{\sum_{t=N}^{\infty} d_t\mathcal{M}_0^t\}\phi_0 = \{\sum_{t=N}^{\infty} \partial_{s_{j\beta}}(d_t)\mathcal{M}_0^t + \sum_{t=N}^{\infty} d_t\mathcal{M}_0^t\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0}\}\phi_0. \tag{152}$$

All the actions occurring in the linearization have been introduced now. Note that $M^{(0)}(\mathcal{M}_0)$ is a free $UT(R)$-module with generator ϕ_0. Hence scratching ϕ from the equations (144) and (145) is permitted as soon as one knows that $\phi = \hat{\phi}\phi_0$ with $\hat{\phi} \in UT(R)$ invertible. In this last case the equation $\mathcal{M}\phi = \phi\mathcal{M}_0^{-1}$ implies then that

$$\mathcal{M} = \mathcal{M}(\hat{\phi}) := \hat{\phi}\mathcal{M}_0^{-1}\hat{\phi}^{-1}$$

and the equation $\mathcal{V}_\alpha\phi = \phi\mathcal{V}_{\alpha,0}$ implies that

$$\mathcal{V}_\alpha = \mathcal{V}_\alpha(\hat{\phi}) := \hat{\phi}\mathcal{V}_{\alpha,0}\hat{\phi}^{-1}.$$

An oscillating matrix at zero $\phi = \hat{\phi}\phi_0$, with $\hat{\phi} = \sum_{i=0}^{\infty} d_i\mathcal{M}_0^i$, with d_0 invertible, is called *a wave matrix at zero* for the matrices $\mathcal{M}(\hat{\phi})$ and $\mathcal{V}_\alpha(\hat{\phi})$, if it satisfies the equations (143). Since the manipulations to get the Lax equations are well-defined on such a ϕ, the matrices $\mathcal{M}(\hat{\phi})$ and the $\mathcal{V}_\alpha(\hat{\phi})$ form a solution of the hierarchy. If one wants to prove the equations (143) for an oscillating matrix at zero ϕ of the right form, it suffices to prove a weaker result, for there holds

Proposition 7. *Let $\phi = \hat{\phi}\phi_0$, with $\hat{\phi} - d_0 \in UT_{\geq 1}(\mathcal{M}_0)$ and $d_0 \in V_0\mathcal{D}_k(R)V_0^{-1}$ invertible, be an oscillating matrix at zero. If it satisfies for all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$*

$$\partial_{s_{j\beta}}(\phi) = G_{j\beta}\psi, \text{ with } G_{j\beta} \in UT(R)_{<0}.$$

then $G_{j\beta} = (\mathcal{M}(\hat{\phi})^j\mathcal{V}_\beta(\hat{\phi}))_{<0}(\mathcal{M}_0)$. In particular $\mathcal{M}(\hat{\phi})$ and the $\mathcal{V}_\beta(\hat{\phi})$ form a solution to the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy

Proof. From the definition of the action of $\partial_{s_{j\beta}}$ on $M^{(0)}(\mathcal{M}_0)$ and the fact that $M^{(0)}(\mathcal{M}_0)$ is a free $UT(R)$-module with generator ϕ_0, one gets the operator equation

$$\partial_{s_{j\beta}}(\hat{\phi}) + \hat{\phi}\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0} = G_{j\beta}\hat{\phi}. \tag{153}$$

Multiplying this equation from the right with $\hat{\phi}^{-1}$ and taking the strict lower triangular part "< 0" in $\mathcal{M}_0$ gives the desired result. □

Remark 15. Note that if one has an oscillating matrix at zero of the right form, i.e. $\phi = \hat{\phi}\phi_0$ where $\hat{\phi} = \sum_{i \geq 0} d_i \mathcal{M}_0^i$ with d_0 invertible, then the condition to ensure that ϕ is a wave matrix at zero can totally be expressed in the perturbation $\hat{\phi}$. There always holds

$$\partial_{Q_{j\beta}}(\phi) = (\partial_{Q_{j\beta}}(\hat{\phi}) + \hat{\phi}\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0})\phi_0$$

and if this has to equal $C_{j\beta}\hat{\phi}\phi_0$, then there holds, since $\partial_{Q_{j\beta}}(\hat{\phi})\hat{\phi}^{-1} \in UT_{\geq 0}(R)$ that

$$\partial_{Q_{j\beta}}(\hat{\phi}) = H_{j\beta}\hat{\phi}, \tag{154}$$

where $H_{j\beta} = -(\mathcal{M}(\hat{\phi})^j \mathcal{V}_\beta(\hat{\phi}))_{\geq 0}(\mathcal{M}_0)$ as in subsection 5.3. Reversely, if the equations (154) hold, then ϕ is a wave matrix at zero for the operators $\mathcal{M}(\hat{\phi})$ and the $\mathcal{V}_\alpha(\hat{\phi})$. The equations (154) form the analogue of the Sato equations in the context of the KP-hierarchy.

Similar to the considerations in Remark 4 in chapter 3 one shows that wave matrices at zero that lead to the same solution of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy differ by a constant perturbation. Moreover this perturbation has a specific form. Assume namely that

$$\mathcal{M} = \hat{\phi}_1\mathcal{M}_0^{-1}\hat{\phi}_1^{-1} = \hat{\phi}_2\mathcal{M}_0^{-1}\hat{\phi}_2^{-1}$$

and

$$\mathcal{V}_\beta = \hat{\phi}_1\mathcal{V}_{\beta,0}\hat{\phi}_1^{-1} = \hat{\phi}_2\mathcal{V}_{\beta,0}\hat{\phi}_2^{-1},$$

where both ϕ_1 and ϕ_2 are wave matrices at zero. Then one has first of all that $\hat{\phi}_1^{-1}\hat{\phi}_2$ commutes with $\mathcal{M}_0$ and the $\mathcal{V}_{\beta,0}$ and thus has to have the form

$$\hat{\phi}_1^{-1}\hat{\phi}_2 = \sum_{i \geq 0} v_i \mathcal{M}_0^i,$$

where $v_i \in V_0 i_k(M_k(R))V_0^{-1}$ commutes with the $\mathcal{V}_{\beta,0}$. One has seen in the proof of Proposition 10 that for all $j \geq 1$, all $\beta \in \{1, \cdots, m_0\}$ and $i = 1, 2$,

$$\partial_{s_{j\beta}}(\hat{\phi}_i) = (\mathcal{M}^j V_\beta)_{<0}(\mathcal{M}_0)\hat{\phi}_i - \hat{\phi}_i\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0}.$$

Hence, if one applies the operator $\partial_{s_{j\beta}}$ to the equality $\hat{\phi}_2 = \hat{\phi}_1 \sum_i v_i \mathcal{M}_0^i$, then one obtains

$$\partial_{s_{j\beta}}(\hat{\phi}_2) = \partial_{s_{j\beta}}(\hat{\phi}_1) \sum_{i\geq 0} v_i \mathcal{M}_0^i + \hat{\phi}_1 \sum_{i\geq 0} \partial_{s_{j\beta}}(v_i)\mathcal{M}_0^i = \quad (155)$$

$$((\mathcal{M}^j V_\beta)_{<0}(\mathcal{M}_0)\hat{\phi}_1 - \hat{\phi}_1(\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0}) \sum_{i\geq 0} v_i \mathcal{M}_0^i + \hat{\phi}_1 \sum_{i\geq 0} \partial_{s_{j\beta}}(v_i)\mathcal{M}_0^i = \quad (156)$$

$$(\mathcal{M}^j V_\beta)_{<0}(\mathcal{M}_0)\hat{\phi}_2 - \hat{\phi}_2\mathcal{M}_0^{-j}\mathcal{V}_{\beta,0} + \hat{\phi}_1 \sum_{i\geq 0} \partial_{s_{j\beta}}(v_i)\mathcal{M}_0^i. \quad (157)$$

Hence one must have for all $i \geq 0$, all $j \geq 1$ and all $\beta \in \{1, \cdots, m_U\}$ that $\partial_{s_{j\beta}}(v_i) = 0$. For completeness sake, this result is resumed in

Corollary 4. *If ϕ_1 and ϕ_2 are wave matrices at zero w.r.t. the same operators $\mathcal{M}$ and $\mathcal{V}_\beta$, then there holds*

$$\phi_2 = \phi_1 \sum_{i\geq 0} v_i \mathcal{M}_0^i,$$

where all the v_i are constant for the derivations $\partial_{s_{j\beta}}$, i.e. $\partial_{s_{j\beta}}(v_i) = 0$.

Remark 16. In the case of the Example 3 of the formal power series in the variables $\{s_{j\beta}\}$, a wave matrix at zero $\phi = \hat{\phi}\phi_0$ for the solutions $\mathcal{M}$ and the $\mathcal{V}_\alpha$ is a formal power series in these variables with coefficients from $UT(F)$ and a constant term equal to the identity. Note that all the matrices $C_{j\beta}$ are formal power series in the variables $\{s_{j\beta}\}$ with as coefficients matrices in $UT_{\geq i}(F)$. Therefore these coefficients satisfy the Assumption 1 from chapter 3 and the zero curvature relations. Hence, since ϕ satisfies

$$\partial_{s_{j\beta}}(\phi) = C_{j\beta}\phi,$$

the wave matrix at zero ϕ is nothing but the fundamental matrix for the corresponding Cauchy problem. Note that thanks to Remark 15 the matrix $\hat{\phi}$ is also a fundamental matrix for a Cauchy problem, namely the one corresponding to the matrices $\{H_{j\beta}\}$.

Remark 17. To actually construct solutions of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy, one will describe in the next subsection a geometric setting from which one can construct oscillating functions at zero for which the exponential factor determines a well-defined element in $V_0 LT(R) V_0^{-1}$, the formal product of this exponential factor and the perturbation factor in $UT(R)$ is real, i.e. it converges, and the oscillating functions at zero satisfy Proposition 10

5.5 The construction of solutions of the hierarchy

As in subsection 4.5 one takes the basic field F here equal to $\mathbb{C}$, so that the ring R will be a $\mathbb{C}$-algebra. Also the solutions of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy to be constructed in subsection 5.5 correspond to $\mathbb{Z} \times \mathbb{Z}$-matrices coming from bounded operators on the Hilbert space $H = L^2(S^1, \mathbb{C}^k)$ and we will freely use the notations and results related to this space from subsection 4.5.

One starts with describing the group in which the flows of the hierarchy can be realized and from which one can construct solutions for the hierarchy. One has some freedom of choice in this matter and we illustrate that again with a group depending of the Schatten class $S_r, r \geq 1$. For each such a r one introduces the group $G_+(r)$ by

$$G_+(r) = \left\{ g = (g_{ij}) \in \mathrm{GL}(H) \;\middle|\; \begin{array}{c} \oplus_{i<j} g_{ij} \in S_r \\ \oplus_{i<j} (g^{-1})_{ij} \in S_r \end{array} \right\}.$$

It consists of the invertible elements in the Banach algebra

$$\mathcal{G}_+(r) = \left\{ b = (b_{ij}) \in B(H) \;\middle|\; \oplus_{i<j} b_{ij} \in S_r \right\}$$

equiped with the norm $|| \cdot ||_{res}$ defined by

$$||b||_{res} = ||(b_{ij})||_{res} := ||b|| + || \oplus_{i<j} b_{ij} ||_r.$$

Here $|| \cdot ||$ is the operator norm and $|| \cdot ||_r$ the Schatten norm. This turnes $G_+(r)$ in a natural way into a Banach Lie group with $\mathcal{G}_+(r)$ as its Lie algebra. Let $\mathbf{w_k}$ be the automorphism of H whose matrix $[\mathbf{w_k}]$ equals the matrix w_k from page 23 in chapter 2 that determined the bijection $A \leftrightarrow w_k A w_k^{-1}$ from $UT(R)$ to $LT(R)$. Then conjugation with $\mathbf{w_k}$ determines an isomorphism between the groups $G_+(r)$ and $G_-(r)$. Hence we may conclude from Lemma 5

Lemma 7. *The group $G_+(r)$ is connected.*

As in the case of the lower triangular matrices there is a number of ways to split the Lie algebra $\mathcal{G}_+(r)$. First of all the Lie algebra $\mathcal{G}_+(r)$ can be split into the sum of the Lie subalgebras

$$\mathcal{P} := \left\{ p = (p_{ij}) \in \mathcal{G}_+(r) \;\middle|\; p_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$\mathcal{U}_- := \left\{ u = (u_{ij}) \in \mathcal{G}_+(r) \;\middle|\; u_{ij} = 0 \text{ for all } i \leq j \right\}.$$

Their corresponding Lie groups are

$$P := \left\{ p = (p_{ij}) \in G_+(r) \;\middle|\; p_{ij} = 0 \text{ and } (p^{-1})_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$U_- := \left\{ u = (u_{ij}) \in G_+(r) \;\middle|\; \begin{array}{c} u_{ij} = 0 \text{ for all } i < j \\ u_{ii} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \end{array} \right\}.$$

The group U_- is the unipotent radical of the opposite parabolic group

$$P_- = \left\{ p = (p_{ij}) \in \mathcal{G}_+(r) \;\middle|\; p_{ij} = 0 \text{ for all } i < j \right\},$$

which has the Lie algebra

$$\mathcal{P}_- := \left\{ p = (p_{ij}) \in \mathcal{G}_+(r) \;\middle|\; p_{ij} = 0 \text{ for all } i < j \right\}.$$

The group $P \cap P_-$ of diagonal $k \times k$-blocks in $G_+(r)$ is denoted by D_k. Let p_+ be an element of the group P. Then its matrix $[p_+] = V_0$ is an upper triangular matrix of order zero in Λ^k. It has an invertible leading coefficient and determines the direction $\mathcal{M}_0^{-1} = V_0 \Lambda^{-k} V_0^{-1}$. Now conjugation with p_+ determines another way to split $\mathcal{G}_+(r)$ namely

$$\mathcal{G}_+(r) = p_+ \, \mathcal{U}_- p_+^{-1} \oplus \mathcal{P}.$$

All the commuting directions

$$\{\mathcal{V}_{\alpha,0} \mathcal{M}_0^{-j} \mid \alpha \in \{1, \cdots, m_0\}, j \geq 1\}$$

that are relevant for $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy belong to the Lie algebra $p_+ \, \mathcal{U}_- p_+^{-1}$. The Lie group corresponding to the Lie algebra $p_+ \, \mathcal{U}_- p_+^{-1}$ is clearly the group $p_+ \, U_- p_+^{-1}$. As the map from $p_+ \, \mathcal{U}_- p_+^{-1} \times \mathcal{P}$ to $G_+(r)$ defined by

$$(u, p) \mapsto \exp(u) \exp(p)$$

is a local diffeomorphism at $(0,0)$, the set $p_+ \, U_- p_+^{-1} P = p_+ \, P_- p_+^{-1} U$ is an open subset of $G_+(r)$. It is called the *big cell* in $G_+(r)$ w.r.t. $p_+ \, U_- p_+^{-1}$ and P . The component in P is called the parabolic components of the decomposition and that in $p_+ \, U_- p_+^{-1}$ the unipotent component of the decomposition. Each big cell is a dense open subset of $G_+(r)$ since this holds for the case $p_+ = \mathrm{Id}$ and that case is proved in a similar way as for the group $G_-(r)$.

Proposition 8. *Let $\Omega \subset G_+(r)$ be the collection of all $g \in G_+(r)$ such that $g_{++}(i)$ is invertible for all $i \in \mathbb{Z}$. Then Ω is equal to $U_- P = P_- U$ and is a dense open subset of $G_+(r)$.*

Recall from subsection 4.5 that we have a natural group $\Gamma(\mathfrak{h})$ of commuting flows associated with $\mathfrak{h}$ consisting of all holomorphic maps from a neighbourhood of the unit circle to $\mathfrak{h}$ that have a nonzero determinant. The flows of

$$\Gamma_{\geq 0}(\mathfrak{h}) = \{\gamma_{\leq 0} \in \Gamma(\mathfrak{h}) \mid \gamma_{\leq 0} = \exp(\sum_{s \geq 0} \gamma_s z^s), \text{ with } \gamma_s \in \mathfrak{h} \text{ for all } s \geq 0\}$$

do not embed into $G_+(r)$, but those of

$$\Gamma_{<0}(\mathfrak{h}) = \{\gamma_{<0} \in \Gamma(\mathfrak{h}) \mid \gamma_{<0} = \exp(\sum_{s \geq 1} \gamma_s z^s), \text{ with } \gamma_s \in \mathfrak{h} \text{ for all } s \geq 1\}$$

do. Let the operator $p_+ \in P$ determine the direction $\mathcal{M}_0^{-1}$ and let $\alpha(p_+)$ as in subsection 4.5 be the maximum of one and $||p_+||||p_+^{-1}||$. To get the relevant flows for the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy one looks at the subgroup

$$\Gamma_{<0}(\mathfrak{h}, \alpha(p_0)) = \{\gamma_{<0} \in \Gamma(\mathfrak{h}, \alpha(p_+)) \mid \gamma_{<0} = \exp(\sum_{s \geq 1} \gamma_s z^s), \gamma_s \in \mathfrak{h} \text{ for } s \geq 1\}.$$

The group $\Gamma_{<0}(\mathfrak{h}, \alpha(p_+))$ embeds through i_{p_+} into the subgroup $p_+ U_- p_-^{-1}$ of $G_+(r)$. Its image is the group of commuting flows $\Gamma(\mathbf{k}_{<0})$. After this embedding there is for each element $\gamma_{<0}$ of $\Gamma_{<0}(\mathfrak{h}, \alpha(p_+))$ an $N > \alpha(p_+)$ such that

$$[i_{p_+}(\gamma_{<0})] = \exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_0} s_{j\beta}\mathcal{V}_{\beta,0}\mathcal{M}_0^{-j}), s_{j\beta} \in \mathbb{C}, \sum_{j,\beta} |s_{j\beta}|N^j < \infty.$$

In other words the $\{s_{j\beta}\}$ are the coordinates w.r.t. the basic directions of the group of commuting flows $\Gamma(\mathbf{k}_{<0})$. For each $\gamma_{<0}$ in $\Gamma_{<0}(\mathfrak{h}, \alpha(p_0))$ one has thus a well-defined matrix $[i_{p_+}(\gamma_{<0})]$ that can be used as the exponential factor in the wave matrices at zero.

To construct solutions one starts with an element $g \in G_+(r)$. Inside the group of commuting flows $\Gamma_{<0}(\mathfrak{h}, \alpha(p_+))$ one considers the open subset

$$\Gamma_{<0}(g, \mathfrak{h}, \alpha(p_+)) = \{\gamma_{<0} \in \Gamma_{<0}(\mathfrak{h}, \alpha(p_+)) \mid g i_{p_+}(\gamma_{<0}^{-1}) \in p_+\Omega p_+^{-1}\}.$$

Now one takes R to be equal to the collection of holomorphic functions on $\Gamma_{<0}(g, \mathfrak{h}, \alpha(p_+))$. On this ring one clearly has the set of derivations

$$\{\partial_{s_{j\beta}} := \frac{\partial}{\partial s_{j\beta}} \mid j \geq 1, 1 \leq \beta \leq m_0\}.$$

Again one uses the decomposition $p_+\Omega p_+^{-1} = p_+U_-p_+^{-1}P$ of the big cell: if an element $\gamma_{<0} \in \Gamma_{<0}(g, \mathfrak{h}, \alpha(p_+))$, then one has

$$g i_{p_+}(\gamma_{<0})^{-1} = u_-(g, \gamma_{<0}, p_+)p(g, \gamma_{<0}, p_+),$$

with one component $p(g, \gamma_{<0}, p_+) \in P$ and the other $u_-(g, \gamma_{<0}, p_+) \in p_+U_-p_+^{-1}$. Thus one has obtained an element of $M^{(0)}(\mathcal{M}_0)$, namely

$$[p(g, \gamma_{<0}, p_+)][i_{p_+}(\gamma_{<0})] =: \hat{\Phi}\phi_0 = \Phi,$$

for which the formal product in the definition of this space is a well-defined one. Note that on the matrix level one has the relations

$$[u_-(g, \gamma_{<0}, p_+)]^{-1}[g] = [p(g, \gamma_-, p_+)][i_{p_+}(\gamma_{<0})] \tag{158}$$

Take the derivative of $[p(g, \gamma, p_+)][i_{p_+}(\gamma_{<0})]$ w.r.t. the parameter $s_{j\beta}$. On one hand there holds

$$\begin{aligned}\partial_{s_{j\beta}}(\hat{\Phi}[i_{p_+}(\gamma_{<0})]) &= \partial_{s_{j\beta}}(\hat{\Phi})[i_{p_+}(\gamma_{<0})] + \hat{\Phi}\mathcal{V}_{\beta,0}\mathcal{M}_0^{-j}[i_{p_+}(\gamma_{<0})] \\ &= \{\partial_{s_{j\beta}}(\hat{\Phi})\hat{\Phi}^{-1} + \mathcal{M}(\hat{\Phi})^j\mathcal{V}_\beta(\hat{\Phi})\}\Phi,\end{aligned} \tag{159}$$

where the matrices $\mathcal{M}(\hat{\Phi})$ and $\mathcal{V}_\beta(\hat{\Phi})$ are obtained by dressing the basic directions with $\hat{\Phi}$:

$$\mathcal{M}(\hat{\Phi}) = \hat{\Phi}\mathcal{M}_0^{-1}\hat{\Phi}^{-1} \text{ and } \mathcal{V}_\beta(\hat{\Phi}) = \hat{\Phi}\mathcal{V}_{\beta,0}\hat{\Phi}^{-1}.$$

On the other hand, if one writes $\Psi := [u_-(g,\gamma_-,p_+)]^{-1}$, then equation (158) results in

$$\partial_{s_{j\beta}}(\Phi) = \partial_{s_{j\beta}}(\Psi)[g] = \{\partial_{s_{j\beta}}(\Psi)\Psi^{-1}\}\Phi. \tag{160}$$

As $M^{(0)}(\mathcal{M}_0)$ is a free $UT(R)$-module, the equations (159) and (160), yield

$$\partial_{s_{j\beta}}(\hat{\Phi})\hat{\Phi}^{-1} + \mathcal{M}(\hat{\Phi})^j\mathcal{V}_\beta(\hat{\Phi}) = \partial_{s_{j\beta}}(\Psi)\Psi^{-1}.$$

Since the right hand side of this equation belongs to $LT_{\leq -1}(R)$, the left hand side to $UT_{\geq -j}(R)$ and the matrix $\partial_{s_{j\beta}}(\hat{\Phi})\hat{\Phi}^{-1}$ to $UT_{\geq 0}(R)$ one may conclude that

$$\partial_{s_{j\beta}}(\Psi)\Psi^{-1} = (\mathcal{M}(\hat{\Phi})^j\mathcal{V}_\beta(\hat{\Phi}))_{<0}(\mathcal{M}_0) \text{ and}$$
$$\partial_{j\beta}(\hat{\Phi})\hat{\Phi}^{-1} = -(\mathcal{M}(\hat{\Phi})^j\mathcal{V}_\beta(\hat{\Phi}))_{\geq 0}(\mathcal{M}_0)$$

In particular Φ satisfies the linearization equations for the collection of matrices $(\mathcal{M}(\hat{\Phi}), \mathcal{V}_\alpha(\hat{\Phi}))$. If one replaces $g \in G_+(r)$ by $u_0 g$ with an arbitrary $u_0 \in p_+U_-p_+^{-1}$, then the upper triangular component in the decomposition (158) does not change: $p(u_0g,\gamma_-,p_+) = p(g,\gamma_-,p_+)$ and consequently the corresponding solutions of the hierarchy are the same. One resumes the results obtained in

Theorem 6. *Let $p_+ \in G_+(r)$ determine the direction $\mathcal{M}_0$. Consider elements $g \in G_+(r)$ and $\gamma_{<0} \in \Gamma_{<0}(g,\mathfrak{h},\alpha(p_+))$. Let $p(g,\gamma_{<0},p_+)$ be the parabolic component of the Bruhat decomposition of the element $gi_{p_+}(\gamma_{<0}^{-1})$ in the corresponding big cell. Then the matrix $[p(g,\gamma_{<0},p_+)][[i_{p_+}(\gamma_{<0})]$ is a wave matrix at zero for the set of matrices $(\mathcal{M}([p(g,\gamma_{<0},p_+)]), \mathcal{V}_\beta([p(g,\gamma_{<0},p_+)]))$. In particular these matrices form a solution of the $(\mathcal{M}_0^{-1},\mathbf{k}_{<0})$-hierarchy. For each u_0 in the unipotent subgroup $p_+U_-p_+^{-1}$ of this decomposition one has*

$$\mathcal{M}([p(g,\gamma_{<0},p_+)]) = \mathcal{M}([p(u_0g,\gamma_{<0},p_+)]) \text{ and}$$
$$\mathcal{V}_\beta([p(g,\gamma_{<0},p_+)]) = \mathcal{V}_\beta([p(u_0g,\gamma_{<0},p_+)]).$$

Remark 18. Theorem 6 shows first of all that the variety $p_+U_-p_+^{-1}\backslash G_+(r)$ determines the solutions of the $(\mathcal{M}_0^{-1}, \mathbf{k}_{<0})$-hierarchy. This homogeneous space is clearly isomorphic to $U_-\backslash G_+(r)$ and $G_+(r)/U_-$. Now the variety $G_+(r)/P_-$ is a flag variety consisting of all infinite flags

$$\cdots W_{i+1} = gH_{i+1}^{\perp} \subset W_i = gH_i^{\perp} \subset W_{i-1} = gH_{i-1}^{\perp} \cdots$$

The natural projection from the manifold $G_+(r)/P_-$ on the manifold $G_+(r)/U_-$ is a fiber bundle over the flag variety $G_+(r)/P_-$ with the diagonal group

$$\left\{ g = (g_{ij}) \in G_+(r) \,\middle|\, g_{ij} = 0 \text{ and } (g^{-1})_{ij} = 0 \text{ for all } i \neq j \right\}$$

as a fiber and this is nothing but the collection of frames of the space $\oplus_{i\in\mathbb{Z}} W_i/W_{i+1}$ that are bounded above and away from zero. Therefore one calls it the *relative frame bundle of the flag variety* $G_+(r)/P_-$.

6 Multi component Toda hierarchies

In this chapter we combine the two types of deformations introduced in the foregoing chapters.

6.1 The equations in Lax form

As in the chapters 4 and 5 we work over a field F of characteristic zero and by R we denote a commutative F-algebra. At the basis of the hierarchies lie a number of commuting directions inside $M_{\mathbb{Z}}(F)$ that include the matrices $\{\Lambda^{km} \mid m \in \mathbb{Z}\}$. Since the matrices that commute with all the $\{\Lambda^{km}\}$ have the form

$$\sum_{s\in\mathbb{Z}} i_k(A_s)\Lambda^{ks}, A_s \in M_k(F),$$

we complement the initial directions with $i_k(\mathfrak{h})$, where $\mathfrak{h}$ is as before a maximal commutative F-subalgebra of $M_k(F)$ that possesses a basis $\{E_\alpha \mid \alpha \in \{1, \cdots, m_0\}\}$. This gives you the basic set of directions

$$BD := \{i_k(E_\alpha)\Lambda^{km} \mid m \in \mathbb{Z}, \alpha \in \{1, \cdots, m_0\}\}$$

in $M_{\mathbb{Z}}(F)$. This set one divides into two parts

$$BD_{\geq 0} = \{i_k(E_\alpha)\Lambda^{ik} \mid i \geq 0, \alpha \in \{1, \cdots, m_0\}\}$$
$$BD_{<0} = \{i_k(E_\beta)\Lambda^{-kj} \mid j > 0, \beta \in \{1, \cdots, m_0\}\}.$$

The directions from $BD_{\geq 0}$ will be deformed in the lower triangular matrices in the way described in chapter 4 and those from $BD_{<0}$ are deformed into the upper triangular matrices in the style of chapter 5.

We start with the deformation of the basic directions from $BD_{\geq 0}$. Each basic element is a product of some $i_k(E_\alpha)$ and a suitable power of Λ^k and the deformation is built up in the same way. Therefore we concentrate first on the deformations of those generating matrices. The deformation $\mathcal{L}$ in LT of the basic direction Λ^k is such that the leading term is unchanged, i.e.

$$\mathcal{L} := \Lambda^k + \sum_{m\leq 0} l_m\Lambda^{km}. \tag{161}$$

Besides $\mathcal{L}$ one considers in LT also deformations of the basic directions $\{i_k(E_\gamma)\}$ that preserve the leading term. This gives you elements

$$\mathcal{U}_\gamma = \sum_{n\leq 0} u_{n,\gamma}\Lambda^{kn} = i_k(E_\gamma) + \sum_{n<0} u_{n,\gamma}\Lambda^{kn}, 1 \leq \gamma \leq m_0. \tag{162}$$

As deformations of the basic generators $(\Lambda^k, i_k(E_\gamma))$ one requires that the $(\mathcal{L},\mathcal{U}_\gamma)$ preserve the commutativity relations

$$[\mathcal{L},\mathcal{U}_\gamma] = 0 \text{ and } [\mathcal{U}_\gamma,\mathcal{U}_\beta] = 0 \text{ for all } \gamma \text{ and } \beta. \tag{163}$$

As deformation of $i_k(E_\alpha)\Lambda^{ik} \in BD_{\geq 0}$ one takes the matrix $P_{i\alpha} := \mathcal{U}_\alpha\mathcal{L}^i$. Its leading coefficient is $i_k(E_\alpha)\Lambda^{ik}$, and $P_{i\alpha}$ belongs to the centralizer of all the elements $(\mathcal{L},\mathcal{U}_\gamma)$ for all α and all $i \geq 0$. This implies the identities

$$\begin{aligned}[(P_{i\alpha})_{\geq 0}(\Lambda^k), \mathcal{L}] &= -[(P_{i\alpha})_{<0}(\Lambda^k), \mathcal{L}], \\ [(P_{i\alpha})_{\geq 0}(\Lambda^k),\mathcal{U}_\gamma] &= -[(P_{i\alpha})_{<0}(\Lambda^k),\mathcal{U}_\alpha].\end{aligned} \tag{164}$$

As we have seen in subsection 4.1 the relations (164) show that it makes sense to look for a F-algebra R that is equipped first of all with a set of F-linear derivations $\{\partial_{P_{i\alpha}} \mid i \geq 0, \alpha \in \{1,\cdots,m_0\}\}$ such that the following nonlinear differential equations hold for $\mathcal{L}$ and the $\{\mathcal{U}_\gamma\}$

$$\partial_{P_{i\alpha}}(\mathcal{L}) = [(\mathcal{L}^i U_\alpha)_{\geq 0}(\Lambda^k), \mathcal{L}] \text{ and } \partial_{P_{i\alpha}}(U_\gamma) = [(\mathcal{L}^i U_\alpha)_{\geq 0}(\Lambda^k), U_\gamma] \tag{165}$$

In other words the matrices $\mathcal{L}$ and the U_γ form a solution of the equations of the $(\Lambda^k, \mathfrak{h}_{\geq 0})$-hierarchy.

Next, one deforms the directions from $BD_{<0}$ into the upper triangular matrices, only this time one allows also certain deformations of the leading term. The deformation is built up in a similar way as in the lower triangular case starting from a number of generators. Since $\mathfrak{h}$ is maximal commutative, this F-algebra contains the identity matrix and there holds:

$$\mathrm{Id} = \sum_{\alpha=1}^{m_0} c_\alpha E_\alpha.$$

Hence $\Lambda^{-k} = \sum_{\alpha=1}^{m_0} c_\alpha i_k(E_\alpha)\Lambda^{-k}$. First, one considers the deformation $\mathcal{M}$ of the direction Λ^{-k} and one assumes that $\mathcal{M}$ like Λ^{-k} is invertible, more precisely one takes $\mathcal{M}$ to be of the form

$$\mathcal{M} := \sum_{i\geq -1} m_i \Lambda^{ki} \text{ with } m_{-1} \text{ invertible.} \tag{166}$$

The coefficients in this decomposition of $\mathcal{M}$ all have the form

$$m_i = \operatorname{diag}(m_i(s)).$$

Note that according to Proposition 27 from chapter 2 the matrix $\mathcal{M}$ can be written in the form

$$\mathcal{M} = V\Lambda^{-k}V^{-1}, \text{ with } V = \sum_{i\geq 0} v_i \Lambda^{ki}, \tag{167}$$

where the leading coefficient $v_0 = \operatorname{diag}(v_0(s))$ of V is chosen in such a way that $v_0(0) = \mathrm{Id}$ and

$$\begin{aligned} v_0(t) &= m_{-1}(t)\dots m_{-1}(1) \text{ for } t \geq 1 \text{ and} \\ v_0(t) &= m_{-1}(t+1)^{-1}\dots m_{-1}(0)^{-1} \text{ for } t < 0. \end{aligned} \tag{168}$$

Next, one considers the deformations of the elements $i_k(E_\beta)\Lambda^{-k}$ and one assumes that they have the form $\mathcal{V}_\beta\mathcal{M}$, where $\mathcal{V}_\beta$ is a deformation of $i_k(E_\beta)$ in the upper triangular matrices of order zero in Λ^k that has the form

$$\mathcal{V}_\beta = \sum_{i\geq 0} v_{i,\beta}\Lambda^{ki}, \text{ with } v_{0,\beta} = v_0 i_k(E_\beta) v_0^{-1}. \tag{169}$$

Here v_0 is the leading coefficient described above of the dressing matrix of $\mathcal{M}$. As deformations of the $i_k(E_\beta)\Lambda^{-kj}$ one takes then the $Q_{j\beta} := \mathcal{V}_\beta\mathcal{M}^j$. Again one wants that this deformation preserves the commutativity relations:

$$[\mathcal{M}, \mathcal{V}_\alpha] = 0 \text{ and } [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0 \text{ for all } \alpha \text{ and } \beta \in \{1, \cdots, m_0\}. \tag{170}$$

These relations imply that the elements $Q_{j\beta}$ belong to the centralizers of both $\mathcal{M}$ and the $\{\mathcal{V}_\alpha\}$ for all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$. In particular, there hold then the following identities

$$[(Q_{j\beta})_{<0}(\Lambda^k), \mathcal{M}] = -[(Q_{j\beta})_{\geq 0}(\Lambda^k), \mathcal{M}], \tag{171}$$

$$[(Q_{j\beta})_{<0}(\Lambda^k), \mathcal{V}_\delta] = -[(Q_{j\beta})_{\geq 0}(\Lambda^k), \mathcal{V}_\delta]. \tag{172}$$

Because of these relations, it makes sense to look for an F- algebra R that is also equipped with a set of F -linear derivations $\{\partial_{Q_{j\beta}} \mid j \geq 1, \beta \in \{1, \cdots, m_0\}\}$ such that the following nonlinear differential equations hold for $\mathcal{M}$ and the $\{\mathcal{V}_\alpha\}$

$$\partial_{Q_{j\beta}}(\mathcal{M}) = [(Q_{j\beta})_{<0}(\mathcal{L}^k), \mathcal{M}] \text{ and } \partial_{Q_{j\beta}}(\mathcal{V}_\delta) = [(Q_{j\beta})_{<0}(\mathcal{L}^k), \mathcal{V}_\delta]. \tag{173}$$

In other words the matrices $\mathcal{M}$ and the $\mathcal{V}_\beta$ form a solution of the $(\Lambda^{-k}, \mathfrak{h}_{<0})$-hierarchy

The next step is to combine both hierarchies. This combination consists of a number of data. First of all one looks for an F-algebra R that is equipped with two sets of F-linear derivations the $\{\partial_{P_{i\alpha}} \mid i \geq 0, \alpha \in \{1, \cdots, m_0\}\}$ and the $\{\partial_{Q_{j\beta}} \mid j \geq 1, \beta \in \{1, \cdots, m_0\}\}$. Further, one needs inside $M_{\mathbb{Z}}(R)$ a solution $(\mathcal{L}, \{U_\gamma\})$ of the $(\Lambda^k, \mathfrak{h}_{\geq 0})$-hierarchy and a solution $(\mathcal{M}, \mathcal{V}_\alpha)$ of the $(\Lambda^{-k}, \mathfrak{h}_{<0})$-hierarchy. These four types of matrices should satisfy besides the Lax equations coupled to these two hierarchies, still some other nonlinear equations. The first set of equations should describe the evolution of the $\mathcal{L}$ and the $\{U_\gamma\}$ w.r.t. the flows corresponding to the derivations $\{\partial_{Q_{j\beta}}\}$ and the second one should give the evolution of the set $(\mathcal{M}, \mathcal{V}_\alpha)$ w.r.t. the flows that correspond to the other set of derivations $\{\partial_{P_{i\alpha}}\}$. This yields the remaining Lax equations. If one uses as in chapter 4 and chapter 5 the short hand notation $B_{i\alpha}$ for $(\mathcal{L}^i U_\alpha)_{\geq 0}(\Lambda^k)$ and $C_{j\beta}$ for $(Q_{j\beta})_{<0}(\Lambda^k)$, then these cross equations are

$$\partial_{P_{i\alpha}}(\mathcal{M}) = [B_{i\alpha}, \mathcal{M}] \text{ and } \partial_{P_{i\alpha}}(\mathcal{V}_\delta) = [B_{i\alpha}, \mathcal{V}_\delta], \tag{174}$$

$$\partial_{Q_{j\beta}}(\mathcal{L}) = [C_{j\beta}, \mathcal{L}] \text{ and } \partial_{Q_{j\beta}}(\mathcal{U}_\gamma) = [C_{j\beta}, \mathcal{U}_\gamma]. \tag{175}$$

Both sets of equations from the two hierarchies together with the cross-differentiation rules (174) and (175) form the equations of the $(\Lambda^k, \mathfrak{h})$-hierarchy. We will also call it a *multi component Toda hierarchy* because of a similar terminology at the KP-hierarchy. The equations (165), (173), (174) and (175) are called the *Lax equations* of the hierarchy.

Example 5. Also in the combined case one gets a concrete setting where one can look for solutions of the hierarchy by combining the Examples 1 and 3 from respectively subsection 4.1 and 5.1. Take for R the F-algebra $F[[t_{i\alpha}, s_{j\beta}]]$ of all formal power series in the variables $t_{i\alpha}$ and $s_{j\beta}$. For the derivation $\partial_{P_{i\alpha}}$ one takes the partial derivative $\partial_{t_{i\alpha}}$ w.r.t. $t_{i\alpha}$ and similarly for $\partial_{Q_{j\beta}}$ one chooses the partial derivative $\partial_{s_{j\beta}}$ w.r.t. $s_{j\beta}$. This context has the advantage that it allows the specialization $t_{i\alpha} = 0$ and $s_{j\beta} = 0$ for all indices. Note that the present example takes for granted the commutativity of all derivations involved in the hierarchy. This matter will be discussed in the following subsection.

Example 6. Again one also has the variant of allowing singularities in the solutions by localizing the Example 5. Let S be a multiplicative subset of $F[[t_{i\alpha}, s_{j\beta}]]$. Then one can take for R also the localization $S^{-1}F[[t_{i\alpha}, s_{j\beta}]]$, choose the derivations as in Example 5 and look for solutions in $M_{\mathbb{Z}}(S^{-1}F[[t_{i\alpha}, s_{j\beta}]])$. The specialization $t_{i\alpha} = 0$ and $s_{j\beta} = 0$ survives in this case only if all the elements of S have a nonzero constant term.

Remark 19. The present hierarchy is built around the basic directions from BD. If one has an invertible matrix $W_0 \in M_{\mathbb{Z}}(F)$ then one could distort BD to the collection of commuting directions

$$\{W_0 i_k(E_\alpha)\Lambda^{km}W_0^{-1} \mid m \in \mathbb{Z}, \alpha \in \{1, \cdots, m_0\}\},$$

split this set into $W_0 BD_{\geq 0} W_0^{-1} \cup W_0 BD_{\geq 0} W_0^{-1}$ and try to introduce a corresponding hierarchy with similar properties. If W_0 and the F-algebra R are such that $W_0 LT(R) W_0^{-1}$ and $W_0 UT(R) W_0^{-1}$ are well-defined algebras in $M_{\mathbb{Z}}(R)$, then this can be carried out, but we refrain from it here due to the lack of a simple visualization of the algebras $W_0 LT(R) W_0^{-1}$ and $W_0 UT(R) W_0^{-1}$.

6.2 A minimal realization of the $(\Lambda^k, \mathfrak{h})$-hierarchy

In this subsection one discusses a minimal realization of the relations in (165), (173), (174) and (175), where minimal refers to the number of relations between the solutions.

For this minimal model one starts with matrices of the right shape. First of all one considers matrices $\tilde{\mathcal{L}}$ and $\tilde{U}_\gamma$ of the right form

$$\tilde{\mathcal{L}} := \Lambda^k + \sum_{m \leq 0} \tilde{l}_m \Lambda^{km}, \quad \tilde{U}_\gamma := \sum_{n \leq 0} \tilde{u}_{n,\gamma} \Lambda^{kn} = i_k(E_\gamma) + \sum_{n < 0} \tilde{u}_{n,\gamma} \Lambda^{kn}.$$

The matrices $\{\tilde{l}_m\}, m \leq 0$, and the $\{\tilde{u}_{n,\gamma}\}$, $n < 0$, are written as

$$\tilde{l}_m = \operatorname{diag}(\tilde{l}_m(s)) \text{ and } \tilde{u}_{n,\gamma} = \operatorname{diag}(\tilde{u}_{n,\gamma}(t)). \tag{176}$$

and their respective matrix coefficients are the $\tilde{l}_m(s)_{\kappa\lambda}$ resp. $\tilde{u}_{n,\gamma}(t)_{\rho\sigma}$.

Next there is the matrix

$$\tilde{\mathcal{M}} := \sum_{r \geq -1} \tilde{m}_r \Lambda^{ki}, \text{ with } \tilde{m}_r = \operatorname{diag}(\tilde{m}_r(s)) \text{ and } \tilde{m}_{-1} \text{ invertible} . \tag{177}$$

Then one has thanks to Proposition 1

$$\tilde{\mathcal{M}} = V\Lambda^{-k}V^{-1}, \text{ with } V = \sum_{i=0}^{\infty} v_i \Lambda^{ki} \text{ and } v_0 = \tilde{w}_0,$$

where $\tilde{w}_0 = \operatorname{diag}(\tilde{w}_0(s))$ is the gauge introduced in subsection 5.2 of a $k \times k$-block diagonal matrix given by $\tilde{w}_0(0) = \text{Id}$ and

$$\begin{aligned} \tilde{w}_0(t) &= \tilde{m}_{-1}(t) \ldots \tilde{m}_{-1}(1) \text{ for } t \geq 1 \text{ and} \\ \tilde{w}_0(t) &= \tilde{m}_{-1}(t+1)^{-1} \ldots \tilde{m}_{-1}(0)^{-1} \text{ for } t < 0. \end{aligned} \tag{178}$$

Besides $\tilde{\mathcal{M}}$ one has all the deformations

$$\tilde{V}_\delta := \tilde{w}_0 i_k(E_\delta)\tilde{w}_0^{-1} + \sum_{l>0} \tilde{v}_{l,\delta}\Lambda^{kl}, \tag{179}$$

with $\delta \in \{1, \cdots, m_0\}$ and $\tilde{w}_0$ as in (178). Also the matrices $\{\tilde{v}_{l,\delta}\}$ in $\mathcal{D}_k(R)$ are written as

$$\tilde{v}_{l,\delta} = \operatorname{diag}(\tilde{v}_{l,\delta}(q)). \tag{180}$$

The $\tilde{m}_r(p)$ resp. $\tilde{v}_{l,\delta}(q)$ have matrix coefficients $\tilde{m}_r(p)_{\epsilon\eta}$ resp. $\tilde{v}_{l,\delta}(q)_{\mu\nu}$.

Recall that the aim was to construct a minimal realization of the Lax equations of the hierarchy and in that light one wants to take the coefficients of $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}$, the $\tilde{U}_\gamma$ and the $\tilde{V}_\alpha$ as independent as possible. Consider thereto the free polynomial ring

$$F[\tilde{m}_r(p)_{\epsilon\eta}, \tilde{v}_{l,\delta}(q)_{\mu\nu}, \tilde{l}_m(s)_{\kappa\lambda}, \tilde{u}_{n,\gamma}(t)_{\rho\sigma}],$$

where all the indices $\{\epsilon,\ \eta,\ \kappa,\ \lambda,\ \mu,\ \nu, \rho,\ \sigma\}$ belong to $\{1, \ldots, k\}$; p, q, s and t are in $\mathbb{Z}$; the remaining indices range as follows: $r \geq -1, l > 0, m \leq 0$ and $n < 0$.

In this ring one has the multiplicative subset S generated by the determinants of the matrices $\tilde{m}_{-1}(s), s \in \mathbb{Z}$. Then the matrix coefficients of $\tilde{\mathcal{L}}$, $\tilde{\mathcal{M}}$, the $\tilde{U}_\gamma$ and the $\tilde{V}_\alpha$ belong to the localization

$$R_{\mathbb{Z}} := S^{-1}F[\tilde{m}_r(p)_{\epsilon\eta}, \tilde{v}_{l,\delta}(q)_{\mu\nu}, \tilde{l}_m(s)_{\kappa\lambda}, \tilde{u}_{n,\gamma}(t)_{\rho\sigma}],$$

of the ring $F[\tilde{m}_r(p)_{\epsilon\eta}, \tilde{v}_{l,\delta}(q)_{\mu\nu}, \tilde{l}_m(s)_{\kappa\lambda}, \tilde{u}_{n,\gamma}(t)_{\rho\sigma}]$ w.r.t. S. Every F-linear derivation $\Delta \in \mathrm{Der}_F(R_{\mathbb{Z}})$ is completely determined by describing freely all the

$$\{\Delta(\tilde{m}_r(p)_{\epsilon\eta}),\ \Delta(\tilde{v}_{l,\delta}(q)_{\mu\nu}), \Delta(\tilde{l}_m(s)_{\kappa\lambda}), \Delta(\tilde{u}_{n,\gamma}(t)_{\rho\sigma})\}.$$

For, the derivation property requires for all $p \in \mathbb{Z}$

$$\Delta(\det(\tilde{m}_{-1}(p))^{-1}) = -\frac{1}{\det(\tilde{m}_{-1}(p))^2}\Delta(\det(\tilde{m}_{-1}(p))).$$

Such a Δ can be constructed by prescribing which matrices in $M_{\mathbb{Z}}(R_{\mathbb{Z}})$ of a similar form as the generating matrices $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}$, the $\tilde{U}_\gamma$ and the $\tilde{V}_\delta$ are the image under Δ of each of these generators.

Keeping the foregoing remark in mind one defines for each element $\tilde{P}_{i\alpha} = \tilde{\mathcal{L}}^i\tilde{U}_\gamma$, $i \geq 0$ and $\alpha \in \{1, \cdots, m_0\}$, a derivation $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ on the generators $\tilde{\mathcal{M}}$ and the $\tilde{V}_\delta$ by

$$\tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{\mathcal{M}}) = [(\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k), \tilde{\mathcal{M}}] \text{ and } \tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{V}_\delta) = [(\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k), \tilde{V}_\delta]. \tag{181}$$

Note that the first commutator in (181) is of order ≥ -1 in Λ^k in UT and the second of positive order in Λ^k in accordance with the fact that the orders of $\tilde{\mathcal{M}}$ resp. $\tilde{V}_\delta$ are minus one resp. zero. Next one defines the action of $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ on the generators $\tilde{\mathcal{L}}$ and the $\tilde{U}_\gamma$ by

$$\tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{\mathcal{L}}) = -[(\tilde{P}_{i\alpha})_{<0}(\Lambda^k), \tilde{\mathcal{L}}] \text{ and } \tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{U}_\gamma) = -[(\tilde{P}_{i\alpha})_{<0}(\Lambda^k), \tilde{U}_\gamma]. \tag{182}$$

Note that the first commutator in (182) is of order ≤ 0 in Λ^k in LT and the second of order < 0 in Λ^k in accordance with the fact that that the leading terms of $\tilde{\mathcal{L}}$ and the $\tilde{U}_\gamma$ are constant and that $\tilde{\mathcal{L}}$ is of order 1 and the $\tilde{U}_\gamma$ are of order zero in LT. This defines a $\mathbb{C}$-linear derivation $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ of $R_{\mathbb{Z}}$.

Similarly one defines for each element $\tilde{Q}_{j\beta} = \tilde{\mathcal{M}}^j\tilde{V}_\beta$, $j \geq 1$ and $\beta \in \{1, \cdots, m_0\}$, the derivation $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ on the generators $\tilde{\mathcal{L}}$ and the $\tilde{U}_\gamma$ by

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{\mathcal{L}}) = [(\tilde{Q}_{j\beta})_{<0}(\Lambda^k), \tilde{\mathcal{L}}] \text{ and } \tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{U}_\gamma) = [(\tilde{Q}_{j\beta})_{<0}(\Lambda^k), \tilde{U}_\gamma]. \tag{183}$$

The first commutator in (183) is of order ≤ 0 in Λ^k in LT and the second of negative order in Λ^k, which agrees with the fact that the leading terms of $\tilde{\mathcal{L}}$ and the $\tilde{U}_\gamma$ are constant and that the orders of $\tilde{\mathcal{L}}$ resp. $\tilde{U}_\gamma$ are one resp. zero. Next one defines the action of $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ on the $\tilde{\mathcal{M}}$ and the $\tilde{V}_\delta$ by

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{\mathcal{M}}) = -[(\tilde{Q}_{j\beta})_{\geq 0}(\Lambda^k), \tilde{\mathcal{M}}] \text{ and } \tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{V}_\delta) = -[(\tilde{Q}_{j\beta})_{\geq 0}(\Lambda^k), \tilde{V}_\delta]. \tag{184}$$

Again the first commutator in (184) is of order ≤ -1 in Λ^k in UT and the second of order ≥ 0 in Λ^k in accordance with the fact that $\tilde{\mathcal{M}}$ is of order -1 and the $\tilde{U}_\gamma$ are of order zero. This defines a $\mathbb{C}$-linear derivation $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ of $R_{\mathbb{Z}}$. Now both the elements of the set of matrices $\tilde{\mathcal{M}}$ and the $\tilde{V}_\delta$ as well as that of the matrices $\tilde{\mathcal{L}}$ and the $\tilde{U}_\gamma$ do not commute among each other. Therefore one sanctions the commutativity relations

$$[\tilde{\mathcal{L}}, \tilde{U}_\gamma] = 0 \text{ and } [\tilde{U}_{\gamma_1}, \tilde{U}_{\gamma_2}] = 0, \text{ for all } \gamma_1 \text{ and } \gamma_2. \tag{185}$$

and

$$[\tilde{\mathcal{M}}, \tilde{V}_\alpha] = 0 \text{ and } [\tilde{V}_{\delta_1}, \tilde{V}_{\delta_2}] = 0, \text{ for all } \delta_1 \text{ and } \delta_2 \tag{186}$$

upon them, meaning that we pass to the quotient ring $\underline{R_{\mathbb{Z}}}$ of $R_{\mathbb{Z}}$ by the ideal $I_{\mathbb{Z}}$ generated by the matrix coefficients of the commutators in the left hand sides of the equations (185) and (186). Note that the derivations $\tilde{\partial}_{\tilde{P}}$ also factorize to derivations of $\underline{R_{\mathbb{Z}}}$, for one has

$$\begin{aligned}
\tilde{\partial}_{\tilde{P}_{i\alpha}}([\tilde{\mathcal{L}}, \tilde{U}_\alpha]) &= [-(\tilde{P}_{i\alpha})_{<0}(\Lambda^k), [\tilde{\mathcal{L}}, \tilde{U}_\alpha]], \\
\tilde{\partial}_{\tilde{P}_{i\alpha}}([\tilde{U}_\alpha, \tilde{U}_\beta]) &= [-(\tilde{P}_{i\alpha})_{<0}(\Lambda^k), [\tilde{U}_\alpha, \tilde{U}_\beta]], \\
\tilde{\partial}_{\tilde{P}_{i\alpha}}([\tilde{\mathcal{M}}, \tilde{V}_\delta]) &= [(\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k), [\tilde{\mathcal{M}}, \tilde{V}_\delta]], \\
\tilde{\partial}_{\tilde{P}_{i\alpha}}([\tilde{V}_\delta, \tilde{V}_\gamma]) &= [(\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k), [\tilde{V}_\delta, \tilde{V}_\gamma]]
\end{aligned} \tag{187}$$

and thus $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ maps the ideal $I_{\mathbb{Z}}$ into itself. As the action of $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ on $\tilde{\mathcal{L}}$ and the $\{\tilde{U}_\alpha\}$ consists also of taking the commutator with a certain matrix and the same is true for its action on $\tilde{\mathcal{M}}$ and the $\{\tilde{V}_\delta\}$, one may conclude that also all the $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ factorize over $\underline{R_{\mathbb{Z}}}$. The same notation will be used for all these induced derivations of $\underline{R_{\mathbb{Z}}}$. Also the natural images in $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$ of the matrices $\tilde{\mathcal{L}}$, $\tilde{\mathcal{M}}$, the $\{\tilde{U}_\alpha\}$, the $\{\tilde{V}_\delta\}$, the $\{\tilde{P}_{i\alpha}\}$ and the $\{\tilde{Q}_{j\beta}\}$ will hold the same notation as their originals.

Since the matrices $\tilde{\mathcal{L}}$ and the $\{\tilde{U}_\alpha\}$ commute among each other in $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$, the equations (182) are equal to

$$\tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{\mathcal{L}}) = [(\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k), \tilde{\mathcal{L}}] \text{ and } \tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{U}_\gamma) = [(\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k), \tilde{U}_\gamma] \tag{188}$$

and in that case one has a uniform expression for $\tilde{\partial}_{\tilde{P}_{i\alpha}}$ as the commutator with the matrix

$$\tilde{B}_{i\alpha} := (\tilde{P}_{i\alpha})_{\geq 0}(\Lambda^k).$$

Now the same holds for the $\tilde{\mathcal{M}}$ and the $\{\tilde{V}_\delta\}$ in $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$ and therefore the equations (184) are equal to

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{\mathcal{M}}) = [(\tilde{Q}_{j\beta})_{<0}(\Lambda^k), \tilde{\mathcal{M}}] \text{ and } \tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{V}_\delta) = [(\tilde{Q}_{j\beta})_{<0}(\Lambda^k), \tilde{V}_\delta] \tag{189}$$

and in that case one has a uniform expression for $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ as the commutator with the matrix

$$\tilde{C}_{j\beta} := (\tilde{Q}_{j\beta})_{<0}(\Lambda^k).$$

Thus the relations (186) and (185) together with the definitions (183), (184), (181) and (182) yield the minimal realization of the $(\Lambda^k, \mathfrak{h})$-hierarchy referred to at the beginning of this section.

One concludes this section with the discussion of some consequences of this minimal realization of the Lax equations of the $(\Lambda^k, \mathfrak{h})$-hierarchy.

Proposition 9. *Let $\tilde{\mathcal{L}}$, $\tilde{\mathcal{M}}$, the $\tilde{U}_\gamma$ and the $\tilde{V}_\delta$ be the matrices in $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$ as introduced above. Then the matrices $\{\tilde{C}_{j\beta}\}$ resp. $\{\tilde{B}_{i\alpha}\}$ satisfy the following so-called zero curvature equations w.r.t. the derivations $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ and $\tilde{\partial}_{\tilde{P}_{i\alpha}}$*

$$\tilde{\partial}_{\tilde{Q}_{j_1\beta_1}}(\tilde{C}_{j_2\beta_2}) - \tilde{\partial}_{\tilde{Q}_{j_2\beta_2}}(\tilde{C}_{j_1\beta_1}) - [\tilde{C}_{j_1\beta_1}, \tilde{C}_{j_2\beta_2}] = 0, \tag{190}$$

$$\tilde{\partial}_{\tilde{P}_{i_1\alpha_1}}(\tilde{B}_{i_2\alpha_2}) - \tilde{\partial}_{\tilde{P}_{i_2\alpha_2}}(\tilde{B}_{i_1\alpha_1}) - [\tilde{B}_{i_1\alpha_1}, \tilde{B}_{i_2\alpha_2}] = 0, \tag{191}$$

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{B}_{i\alpha}) - \tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{C}_{j\beta}) - [\tilde{C}_{j\beta}, \tilde{B}_{i\alpha}] = 0 \tag{192}$$

Proof. The proof of the first equation is given in Proposition 6 in subsection 5.2 and that of the second equality in Proposition 2 in subsection 4.2. To prove the last equation one will show that the left hand side belongs both to $UT_{<0}(\Lambda^k)$ and to $UT_{\geq 0}(\Lambda^k)$ and thus has to be zero.

Because of the Lax equations for $\tilde{\mathcal{L}}$ and the $\tilde{U}_\gamma$ w.r.t. the $\tilde{\partial}_{\tilde{Q}_{j\beta}}$ there holds for their products

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{\mathcal{L}}^i\tilde{U}_\alpha) = [\tilde{C}_{j\beta}, \tilde{\mathcal{L}}^i\tilde{U}_\alpha].$$

By decomposing the $\tilde{\mathcal{L}}^i\tilde{U}_\alpha$ as $(\tilde{\mathcal{L}}^i\tilde{U}_\alpha)_{<0}(\Lambda^k) + \tilde{B}_{i\alpha}$ this equation results in

$$\tilde{\partial}_{\tilde{Q}_{j\beta}}(\tilde{B}_{i\alpha}) - [\tilde{C}_{j\beta}, \tilde{B}_{i\alpha}] = -\tilde{\partial}_{\tilde{Q}_{j\beta}}((\tilde{\mathcal{L}}^i\tilde{U}_\alpha)_{<0}(\Lambda^k)) + [\tilde{C}_{j\beta}, (\tilde{\mathcal{L}}^i\tilde{U}_\alpha)_{<0}(\Lambda^k)]$$

and this shows that the left hand side of equation (192) belongs to $UT_{<0}(\Lambda^k)$.

Likewise the Lax equations for $\tilde{\mathcal{M}}$ and the $\tilde{V}_\beta$ yield for their products

$$\tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{\mathcal{M}}^j\tilde{V}_\beta) = [\tilde{B}_{i\alpha}, \tilde{\mathcal{M}}^j\tilde{V}_\beta].$$

Again one decomposes the $\tilde{\mathcal{M}}^j\tilde{V}_\beta$ as $(\tilde{\mathcal{M}}^j\tilde{V}_\beta)_{\geq 0}(\Lambda^k) + \tilde{C}_{j\beta}$ and now one gets that

$$-\tilde{\partial}_{\tilde{P}_{i\alpha}}(\tilde{C}_{j\beta}) + [\tilde{B}_{i\alpha}, \tilde{C}_{j\beta}] = \tilde{\partial}_{\tilde{P}_{i\alpha}}((\tilde{\mathcal{M}}^j\tilde{V}_\beta)_{\geq 0}(\Lambda^k)) - [\tilde{B}_{i\alpha}, (\tilde{\mathcal{M}}^j\tilde{V}_\beta)_{\geq 0}(\Lambda^k)]$$

and from this equality one sees that the left hand side of equation (192) belongs to $UT_{\geq 0}(\Lambda^k)$ and this proves the proposition. □

The Proposition 9 permits you to show the following property that unites the equations that belong to the same hierarchy

Corollary 5. *The collections of derivations* $\{\tilde{\partial}_{\tilde{Q}_{j\beta}} \mid \beta \in \{1, \cdots, m_0\}, j \geq 1\}$ *and the* $\{\tilde{\partial}_{\tilde{P}_{i\alpha}} \mid i \geq 0$ *and* $\alpha \in \{1, \cdots, m_0\}\}$ *of the algebra* $\underline{R}_{\mathbb{Z}}$ *all commute.*

Proof. The matrix coefficients of the matrices $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}$, the $\tilde{U}_\gamma$ and the $\tilde{\mathcal{V}}_\delta$ generate the algebra $\underline{R}_{\mathbb{Z}}$. Therefore it suffices to show

$$(\partial_1 \circ \partial_2 - \partial_2 \circ \partial_1)(X) = 0 \tag{193}$$

for two derivations ∂_1 and ∂_2 from the sets $\{\tilde{\partial}_{\tilde{Q}_{j\beta}} \mid \beta \in \{1, \cdots, m_0\}, j \geq 1\}$ and the $\{\tilde{\partial}_{\tilde{P}_{i\alpha}} \mid i \geq 0$ and $\alpha \in \{1, \cdots, m_0\}\}$ and for X one of the matrices $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}$, $\tilde{U}_\gamma$ or $\tilde{\mathcal{V}}_\delta$. One gets the desired identities by applying the following property: since ∂_1 and ∂_2 have on the generators the form

$$\partial_i(X) = [D_i, X] \text{ for some } D_i \in M_{\mathbb{Z}}(\underline{R}_{\mathbb{Z}}),$$

where the commutator in the right hand side of equation 194 is well-defined, since the relevant D_i are finite band matrices. Then a straightforward computation shows that

$$(\partial_1 \circ \partial_2 - \partial_2 \circ \partial_1)(X) = [\partial_1(D_2) - \partial_2(D_1) - [D_1, D_2], X] \tag{194}$$

The zero curvature relations from Proposition 9 yield then the desired equalities.

□

6.3 The zero curvature form of the $(\Lambda^k, \mathfrak{h})$-hierarchy

In the last section one has built a minimal context in which the equations (173), (165), (174) and (175) hold. Here one discusses other realizations and the zero curvature form of the hierarchies.

Let $\tilde{\mathcal{L}}$, $\tilde{\mathcal{M}}$, the $\tilde{\mathcal{U}}_\alpha$ and the $\tilde{\mathcal{V}}_\beta$ be the matrices in $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$ introduced in the foregoing section. As in chapter 4 and 5 we will assign appropriate matrices $\mathcal{L}$, $\mathcal{M}$, the $\mathcal{U}_\alpha$ and the $\mathcal{V}_\beta$ to them to get other realizations.

For the lower triangular part of the deformation the matrices $\mathcal{L}$ and the $\mathcal{U}_\alpha$ are determined by assigning to each matrix coefficient $\tilde{l}_m(s)_{\gamma\delta}, m \leq 0$, of $\tilde{\mathcal{L}}$ an element $l_m(s)_{\kappa\lambda}$ in some F-algebra R and to each matrix coefficient $\tilde{u}_{i,\alpha}(t)_{\rho\sigma}, i < 0$, of $\tilde{U}_\alpha$ an element $u_{i,\alpha}(t)_{\rho\sigma}$ in the same F-algebra R.

Similarly, the upper triangular part of the deformation consisting of the matrices $\mathcal{M}$ and $\mathcal{V}_\beta$ is determined by assigning to each matrix coefficient $\tilde{v}_{l,\beta}(t)_{\mu\nu}$, $l \geq 1$, of the $\tilde{\mathcal{V}}_\beta$ an element $v_{l,\beta}(t)_{\mu\nu}$ in the same F-algebra R and by assigning to each matrix coefficient $\tilde{m}_r(p)_{\epsilon\eta}, r \geq -1$, of $\tilde{\mathcal{M}}$ an element $m_r(p)_{\epsilon\eta}$ in R such that all the $m_{-1}(p), p \in \mathbb{Z}$, are invertible in $M_k(R)$. Those matrices $m_{-1}(p), p \in \mathbb{Z}$, determine also the invertible element $w_0 \in \mathcal{D}_k(R)$ by making the same substitution in the element $\tilde{w}_0 \in \mathcal{D}_k(\underline{R_{\mathbb{Z}}})$ defined in the last section. For the leading coefficient m_{-1} of $\mathcal{M}$ this amounts to $m_{-1} = w_0 \Lambda^{-k} w_0^{-1} \Lambda^k$ and the leading coefficient $\mathrm{diag}(v_{0,\beta}(t))$ of $\mathcal{V}_\beta$ is then given by $w_0 i_k(E_\alpha) w_0^{-1}$. In algebraic terms all these assignments correspond to a F -algebra morphism $\nu : \underline{R_{\mathbb{Z}}} \to R$. The morphism ν determines an F-linear map from $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$ to $M_{\mathbb{Z}}(R)$ for which one uses the same notation. The map ν induces F-algebra morphisms from respectively

$UT(\underline{R}_{\mathbb{Z}})$ to $UT(R)$ and $LT(\underline{R}_{\mathbb{Z}})$ to $LT(R)$. Moreover it also furnishes you two sets of commuting of matrices the

$$\mathcal{L} := \nu(\tilde{\mathcal{L}}) \text{ and the } \mathcal{U}_\gamma := \nu(\tilde{\mathcal{U}}_\gamma)$$

and the

$$\mathcal{M} := \nu(\tilde{\mathcal{M}}) \text{ and the } \mathcal{V}_\delta := \nu(\tilde{\mathcal{V}}_\delta)$$

of the right shape for being solutions of the hierarchy.

To keep notations consistent, one writes for all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$,

$$Q_{j\beta} := \mathcal{M}^j \mathcal{V}_\beta \text{ and } C_{j\beta} := (Q_{j\beta})_{<0}(\Lambda^k)$$

and likewise for $i \geq 0$ and all $\alpha \in \{1, \cdots, m_0\}$,

$$P_{i\alpha} := \mathcal{L}^i \mathcal{U}_\alpha \text{ and } B_{i\alpha} := (P_{i\alpha})_{\geq 0}(\Lambda^k).$$

The next step is to transfer the Lax equations to the matrices with coefficients from R and in that light one needs that R is equipped with two sets of F-linear derivations, namely the $\partial_{Q_{j\beta}} : R \to R$ and the $\partial_{P_{i\alpha}} : R \to R$, with $Q_{j\beta}$ and $P_{i\alpha}$ as above. The hope then is that these derivations will be compatible with the minimal ones introduced in the foregoing section, i.e. they should satisfy

$$\partial_{Q_{j\beta}} \circ \nu = \nu \circ \tilde{\partial}_{\tilde{Q}_{j\beta}} \text{ and } \partial_{P_{i\alpha}} \circ \nu = \nu \circ \tilde{\partial}_{\tilde{P}_{i\alpha}}. \tag{195}$$

Clearly, the relations in (195) imply equations for the matrices $\mathcal{M}$, $\mathcal{L}$, the $\mathcal{U}_\gamma$ and the $\mathcal{V}_\delta$ in $M_{\mathbb{Z}}(R)$. Namely they have to satisfy the Lax equations

$$\begin{aligned} \partial_{Q_{j\beta}}(\mathcal{M}) = [C_{j\beta}, \mathcal{M}], \partial_{Q_{j\beta}}(\mathcal{L}) = [C_{j\beta}, \mathcal{L}], \\ \partial_{Q_{j\beta}}(\mathcal{V}_\delta) = [C_{j\beta}, \mathcal{V}_\delta], \partial_{Q_{j\beta}}(\mathcal{U}_\gamma) = [C_{j\beta}, \mathcal{U}_\gamma] \end{aligned} \tag{196}$$

and

$$\begin{aligned} \partial_{P_{i\alpha}}(\mathcal{M}) = [B_{i\alpha}, \mathcal{M}], \partial_{P_{i\alpha}}(\mathcal{L}) = [B_{i\alpha}, \mathcal{L}], \\ \partial_{P_{i\alpha}}(\mathcal{V}_\delta) = [B_{i\alpha}, \mathcal{V}_\delta], \partial_{P_{i\alpha}}(\mathcal{U}_\gamma) = [B_{i\alpha}, \mathcal{U}_\gamma]. \end{aligned} \tag{197}$$

Since the matrix coefficients of $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}$, the $\tilde{\mathcal{V}}_\delta$ and the $\tilde{\mathcal{U}}_\gamma$ generate $\underline{R}_{\mathbb{Z}}$, the equations (196) and (197) are also sufficient in order that relation (195) holds. Thus finding such an R and the sets of derivations $\partial_{Q_{j\beta}}$ and $\partial_{P_{i\alpha}}$ such that relation (195) holds, amounts to finding a solution, namely $\mathcal{L}$, $\mathcal{M}$, the $\mathcal{U}_\gamma$ and the $\mathcal{V}_\alpha$, of the equations of the $(\Lambda^k, \mathfrak{h})$-hierarchy. Since ν is a F-algebra morphism, one may conclude from relation (195) and Proposition 9 that then also the zero curvature relations hold

$$\partial_{Q_{j_1\beta_1}}(C_{j_2\beta_2}) - \partial_{Q_{j_2\beta_2}}(C_{j_1\beta_1}) - [C_{j_1\beta_1}, C_{j_2\beta_2}] = 0. \tag{198}$$

$$\partial_{P_{i_1\alpha_1}}(B_{i_2\alpha_2}) - \partial_{P_{i_2\alpha_2}}(B_{i_1\alpha_1}) - [B_{i_1\alpha_1}, B_{i_2\alpha_2}] = 0. \tag{199}$$

$$\partial_{Q_{j\beta}}(B_{i\alpha}) - \partial_{P_{i\alpha}}(C_{j\beta}) - [C_{j\beta}, B_{i\alpha}] = 0. \tag{200}$$

Next it will be shown that, reversely, the equations (198), (199) and (200) also imply the Lax equations (196) and (197).

Theorem 7. *Let $\mathcal{R}$ be an F -algebra equipped with two collections of F-linear derivations the $\partial_{Q_{j\beta}} : \mathcal{R} \to \mathcal{R}, j \geq 1, \beta \in \{1, \cdots, m_0\}$, and the $\partial_{P_{i\alpha}}, i \geq 0, \alpha \in \{1, \cdots, m_0\}$, and let $\nu : \underline{R_{\mathbb{Z}}} \to \mathcal{R}$ be a F-algebra morphism. Let $\tilde{\mathcal{M}}$, $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{V}}_\delta$ and the $\tilde{\mathcal{U}}_\gamma$ be the matrices in $M_{\mathbb{Z}}(\underline{R_{\mathbb{Z}}})$ as introduced in the foregoing section. Then the Lax equations (196) and (197) for their images $\mathcal{M}$, $\mathcal{L}$, the $\mathcal{U}_\gamma$ and the $\mathcal{V}_\delta$ under ν are equivalent to the zero curvature relations (198), (199) and (200) for the matrices $\{C_{r\delta}\}$ and the $\{B_{n\gamma}\}$.*

Proof. One merely has to prove sufficiency still.

To get the Lax equations for $\mathcal{M}$ and the $\{\mathcal{V}_\delta\}$ w.r.t. the $\partial_{Q_{j\beta}}$, consider for all $m \geq 1$ the matrices

$$\partial_{Q_{j\beta}}(\mathcal{M}^m\mathcal{V}_\delta) - [C_{j\beta}, \mathcal{M}^m\mathcal{V}_\delta].$$

By substituting in it the formula $\mathcal{M}^m\mathcal{V}_\delta = C_{m\delta} + (\mathcal{M}^m\mathcal{V}_\delta)_{\geq 0}(\Lambda^k)$ and by using the zero curvature equations (198), one gets the equality

$$\partial_{Q_{j\beta}}(\mathcal{M}^m\mathcal{V}_\delta) - [C_{j\beta}, \mathcal{M}^m\mathcal{V}_\delta] = \partial_{Q_{j\beta}}((\mathcal{M}^m\mathcal{V}_\delta)_{\geq 0}(\Lambda^k)) +$$
$$\partial_{Q_{m\delta}}(C_{j\beta}) - [C_{j\beta}, (\mathcal{M}^m\mathcal{V}_\delta)_{\geq 0}(\Lambda^k)]$$

From this equality one sees that the order in Λ^k in all the expressions

$$\partial_{Q_{j\beta}}(\mathcal{M}^m\mathcal{V}_\delta)) - [C_{j\beta}, \mathcal{M}^m\mathcal{V}_\delta)]$$

is for all $m \geq 1$ bounded below by $-j$. Consider first the operator $\mathcal{M}$. Suppose that one has

$$\partial_{Q_{j\beta}}(\mathcal{M}) - [C_{j\beta}, \mathcal{M}] = \beta\Lambda^{kl} + \text{ higher order in } \Lambda^k, \tag{201}$$

with $\beta \in \mathcal{D}_k(R)$ nonzero. By induction with respect to m one shows

$$\partial_{Q_{j\beta}}(\mathcal{M}^m) - [C_{j\beta}, \mathcal{M}^m] = \sum_{i=0}^{m-1} \mathcal{M}^i\{\partial_{Q_{j\beta}}(\mathcal{M}) - [C_{j\beta}, \mathcal{M}]\}\mathcal{M}^{m-i-1}. \tag{202}$$

Hence the leading term in Λ^k of the right hand side is

$$\sum_{i=0}^{m-1} (m_{-1}\Lambda^{-k})^i \beta\Lambda^{kl}(m_{-1}\Lambda^{-k})^{-i}(m_{-1}\Lambda^{-k})^{m-1}, \tag{203}$$

which is of order $l+1-m$ in Λ^k. If m tends to infinity this contradicts again the fact that the left hand side belongs to $UT_{\geq -n}(\Lambda^k)$, unless for all sufficiently large m

$$\sum_{i=0}^{m-1} (m_{-1}\Lambda^{-k})^i \beta\Lambda^{kl}(m_{-1}\Lambda^{-k})^{-i} = 0.$$

This in its turn implies that the matrix $\beta\Lambda^{lk}$ is equal to zero and hence that $\beta = 0$. So, the Lax equations w.r.t. $\partial_{Q_{j\beta}}$ have to hold for all $\mathcal{M}^m$, $m \geq 1$. To obtain the Lax equations w.r.t. $\partial_{Q_{j\beta}}$ for the $\mathcal{V}_\delta$ one considers the operator $\mathcal{M}^m\mathcal{V}_\delta$ and applies the Lax equations for $\mathcal{M}^m$ to get

$$\partial_{Q_{j\beta}}(\mathcal{M}^m\mathcal{V}_\delta) - [C_{j\beta}, \mathcal{M}^m\mathcal{V}_\delta)] = 0 + \mathcal{M}^m(\partial_{Q_{j\beta}}(\mathcal{V}_\delta) - [C_{j\beta}, \mathcal{V}_\delta]). \tag{204}$$

If the operator $\mathcal{V}_\delta$ would not satisfy the Lax equation, then the matrix

$$\partial_{Q_{j\beta}}(\mathcal{V}_\delta) - [C_{j\beta}, \mathcal{V}_\delta]$$

has an order s in Λ^k and the right hand side of (208) would be of order $-m+s$ in Λ^k and this contradicts the fact that it was bounded below by $-j$. Therefore the Lax equations w.r.t. $\partial_{Q_{j\beta}}$ hold for the $\mathcal{V}_\delta$ as well.

Next one shows the Lax equations for $\mathcal{L}$ and the $\{\mathcal{U}_\gamma\}$ w.r.t. the $\partial_{Q_{j\beta}}$. Consider for all $t \geq 1$ the matrices

$$\partial_{Q_{j\beta}}(\mathcal{L}^t\mathcal{U}_\gamma) - [C_{j\beta}, \mathcal{L}^t\mathcal{U}_\gamma].$$

By substituting in it the formula $\mathcal{L}^t\mathcal{U}_\gamma = C_{t\gamma} + (\mathcal{M}^m\mathcal{V}_\gamma)_{\geq 0}$ and by using the zero curvature equations (200), one gets the equality

$$\partial_{Q_{j\beta}}(\mathcal{L}^t\mathcal{U}_\gamma) - [C_{j\beta}, \mathcal{L}^t\mathcal{U}_\gamma)] = \partial_{Q_{j\beta}}((\mathcal{L}^t\mathcal{U}_\gamma)_{\geq 0}) + \partial_{Q_{t\gamma}}(C_{j\beta}) - [C_{j\beta}, (\mathcal{L}^t\mathcal{U}_\gamma)_{\geq 0}]$$

From this equality one sees that the order in Λ^k in all the

$$\partial_{Q_{j\beta}}(\mathcal{L}^t\mathcal{U}_\gamma)) - [C_{j\beta}, \mathcal{L}^t\mathcal{U}_\gamma)]$$

is for all $t \geq 1$ bounded above by $-j$. Consider first the operator $\mathcal{L}$. Suppose that one has

$$\partial_{Q_{j\beta}}(\mathcal{L}) - [C_{j\beta}, \mathcal{L}] = \beta\Lambda^{kl} + \text{ lower order in } \Lambda^k, \tag{205}$$

with $\beta \in \mathcal{D}_k(R)$ nonzero. By induction with respect to t one shows

$$\partial_{Q_{j\beta}}(\mathcal{L}^t) - [C_{j\beta}, \mathcal{L}^t] = \sum_{i=0}^{t-1} \mathcal{L}^i\{\partial_{Q_{j\beta}}(\mathcal{L}) - [C_{j\beta}, \mathcal{L}]\}\mathcal{L}^{m-i-1}. \tag{206}$$

Hence the leading term in Λ^k of the right hand side is

$$\sum_{i=0}^{t-1}(\Lambda^k)^i\beta\Lambda^{kl}(\Lambda^k)^{-i}(\Lambda^k)^{m-1}, \tag{207}$$

which is of order $l-1+m$ in Λ^k. If t tends to infinity this contradicts again the fact that the left hand side belongs to $LT_{\leq 0}(\Lambda^k)$, unless for all sufficiently large t

$$\sum_{i=0}^{t-1}(m_{-1}\Lambda^{-k})^i\beta\Lambda^{kl}(m_{-1}\Lambda^{-k})^{-i} = 0.$$

This implies that the matrix $\beta\Lambda^{lk}$ is equal to zero and hence that $\beta = 0$. So, the Lax equations (196) have to hold for all $\mathcal{L}^t$, $t \geq 1$. and all $\beta \in \{1, \cdots, m_0\}$. To obtain the Lax equations w.r.t. for the $\mathcal{U}_\gamma$ one considers the operator $\mathcal{L}^t\mathcal{U}_\gamma$ and applies the Lax equations for $\mathcal{L}^t$ to get

$$\partial_{Q_{j\beta}}(\mathcal{L}^t\mathcal{U}_\gamma) - [C_{j\beta}, \mathcal{L}^t\mathcal{U}_\gamma)] = 0 + \mathcal{L}^t(\partial_{Q_{j\beta}}(\mathcal{U}_\gamma) - [C_{j\beta}, \mathcal{U}_\gamma]). \tag{208}$$

If the operator $\mathcal{U}_\gamma$ would not satisfy the Lax equation, then the matrix

$$\partial_{Q_{j\beta}}(\mathcal{V}_\alpha) - [C_{j\beta}, \mathcal{V}_\alpha]$$

has an order s in Λ^k and the right hand side of (208) would be of order $-m+s$ in Λ^k and this contradicts the fact that it was bounded above by 0. Therefore the Lax equations w.r.t. $\partial_{Q_{j\beta}}$ hold for the $\mathcal{U}_\gamma$ as well.

Finally, the Lax equations w.r.t. the $\partial_{P_{i\alpha}}$ are proved in the same fashion, since the zero curvature equations (200) imply that the order of $\partial_{P_{i\alpha}}(\mathcal{M}^m\mathcal{V}_\delta) - [B_{i\alpha}, \mathcal{M}^m\mathcal{V}_\delta]$ in Λ^k is bounded below by zero and similarly the zero curvature equations (199) imply that the order of $\partial_{P_{i\alpha}}(\mathcal{L}^t\mathcal{U}_\gamma) - [B_{i\alpha}, \mathcal{L}^t\mathcal{U}_\gamma]$ in Λ^k is bounded above by i. This completes the proof of the theorem. □

6.4 Wave matrices for the $(\Lambda^k, \mathfrak{h})$-hierarchy

In view of the zero curvature form of $(\Lambda^k, \mathfrak{h})$-hierarchy one can expect a Cauchy problem to play a role in the hierarchy. We present here an algebraic analogue of a basis of horizontal sections for the connection associated with the formal connection form

$$\omega_{\mathbb{Z}} = \sum_{j=1}^{\infty}\sum_{\beta=1}^{m_0} C_{j\beta} ds_{j\beta} + \sum_{i=0}^{\infty}\sum_{\alpha=1}^{m_0} B_{i\alpha} dt_{i\alpha}.$$

Thereto one starts out with a F-algebra R equiped with two collections

$$\{\partial_{Q_{j\beta}}, j \geq 1, \beta \in \{1, \cdots, m_0\}\} \text{ and } \{\partial_{P_{i\alpha}}, i \geq 0, \alpha \in \{1, \cdots, m_0\}\}$$

of mutually commuting F-linear derivations of R. Further one has the corresponding potential solutions, namely commuting matrices $\mathcal{M}$ and $\mathcal{V}_\delta$ in $UT(R)$ of the form (166) resp. (169) and commuting matrices $\mathcal{L}$ and $\mathcal{U}_\gamma$ in $LT(R)$ of the form (161) resp. (162). The *linearization of the* $(\Lambda^k, \mathfrak{h})$*-hierarchy* consists of two sets of equations. The first set is for $\mathcal{M}$ and the $\mathcal{V}_\delta$ and reads

$$\mathcal{M}\phi = \phi\Lambda^{-k}, V_\delta\phi = \phi i_k(E_\delta), \partial_{Q_{j\beta}}(\phi) = C_{j\beta}\phi \text{ and } \partial_{P_{i\alpha}}(\phi) = B_{i\alpha}\phi. \qquad (209)$$

Here ϕ is a not yet specified object for which all the operations in (209), like multiplying from the left and right with $\mathbb{Z}\times\mathbb{Z}$-matrices and applying all the derivations $\partial_{Q_{j\beta}}$ and $\partial_{P_{i\alpha}}$ make sense. The second set is similar and concerns the matrices $\mathcal{L}$ and $\mathcal{U}_\gamma$

$$\mathcal{L}\psi = \psi\Lambda^{k}, U_\gamma\psi = \psi i_k(E_\gamma), \partial_{Q_{j\beta}}(\psi) = C_{j\beta}\psi \text{ and } \partial_{P_{i\alpha}}(\psi) = B_{i\alpha}\psi. \qquad (210)$$

The ψ in these equations, like the ϕ in the equations (209), needs further specification. Before specifying the ϕ and ψ, we first show how the Lax equations for the matrices $\mathcal{M}$, the $\mathcal{V}_\delta$, $\mathcal{L}$ and the $\mathcal{U}_\gamma$ can be derived from the linearization. In this derivation one assumes that the actions of $\partial_{Q_{j\beta}}$ and $\partial_{P_{i\alpha}}$ on matrices A and B on ϕ are coupled by the Leibnitz rule

$$\partial_{Q_{j\beta}}(A\phi B) = \partial_{Q_{j\beta}}(A)\phi B + A\partial_{Q_{j\beta}}(\phi)B + A\phi\partial_{Q_{j\beta}}(B).$$

To get the Lax equations for $\mathcal{M}$ one applies the derivation $\partial_{Q_{j\beta}}$ resp. $\partial_{P_{i\alpha}}$ to the first equation in (209) and substitutes the one before last resp. the last one. This leads to the following manipulations

$$\begin{aligned}\partial_{Q_{j\beta}}(\mathcal{M}\phi - \phi\Lambda^{-k}) = \partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}(\partial_{Q_{j\beta}}(\phi)) - (\partial_{Q_{j\beta}}(\phi))\Lambda^{-k} = \\ \partial_{Q_{j\beta}}(\mathcal{M})\phi + \mathcal{M}C_{j\beta}\phi - C_{j\beta}\phi\Lambda^{-k} = \{\partial_{Q_{j\beta}}(\mathcal{M}) - [C_{j\beta}, \mathcal{M}\,]\}\phi = 0\end{aligned} \tag{211}$$

and similarly

$$\partial_{P_{i\alpha}}(\mathcal{M}\phi - \phi\Lambda^{-k}) = \{\partial_{P_{i\alpha}}(\mathcal{M}) - [B_{i\alpha}, \mathcal{M}\,]\}\phi = 0. \tag{212}$$

Hence, if it is allowed to eliminate the function ϕ from the foregoing equations, one obtains the Lax equations for $\mathcal{M}$. For the operator $\mathcal{V}_\delta$ one applies $\partial_{Q_{j\beta}}$ resp. $\partial_{P_{i\alpha}}$ to the second equation in (209) and substitutes the one before last resp. the last. Thus one gets

$$\begin{aligned}&\partial_{Q_{j\beta}}(\mathcal{V}_\delta\phi - \phi i_k(E_\delta)) = \partial_{Q_{j\beta}}(\mathcal{V}_\delta)\phi + \mathcal{V}_\delta(\partial_{Q_{j\beta}}(\phi)) - \\ &- (\partial_{Q_{j\beta}}(\phi))i_k(E_\delta) = \partial_{Q_{j\beta}}(\mathcal{V}_\delta)\phi + \mathcal{V}_\delta C_{j\beta}\phi - C_{j\beta}\phi i_k(E_\delta) = \\ &= \{\partial_{Q_{j\beta}}(\mathcal{V}_\delta) - [C_{j\beta}, \mathcal{V}_\delta\,]\}\phi = 0.\end{aligned} \tag{213}$$

resp.

$$\partial_{P_{i\alpha}}(\mathcal{V}_\delta\phi - \phi i_k(E_\delta)) = \{\partial_{P_{i\alpha}}(\mathcal{V}_\delta) - [B_{i\alpha}, \mathcal{V}_\delta\,]\}\phi = 0. \tag{214}$$

If one can leave out ϕ again, this yields the Lax equations for $\mathcal{V}_\delta$. In the same way one gets the Lax equations for $\mathcal{L}$ and the $\mathcal{U}_\gamma$ by applying the derivations to the first and the second equation in (210).

For the equations (209) one needs a left action of $\mathcal{M}$, the $\mathcal{V}_\delta$ and all the $C_{j\beta}$ on the functions ϕ and for the equations (210) a left action of all the matrices $\mathcal{L}$, the $\mathcal{U}_\gamma$ and the $B_{i\alpha}$ on the functions ψ. To realize this, one builds a left $UT(R)$-module for the functions ϕ and a left $LT(R)$-module for the functions ψ. The actual form of the elements in each module is guided by the trivial solution $\mathcal{M} = \Lambda^{-k}$, $\mathcal{L} = \Lambda^k$, $\mathcal{U}_\gamma = i_k(E_\gamma)$ and $\mathcal{V}_\delta = i_k(E_\delta)$ of the hierarchy. In that case the equations (209) resp. (210) become

$$\Lambda^{-k}\phi = \phi\Lambda^{-k}, i_k(E_\delta)\phi = \phi i_k(E_\delta) \text{ and} \qquad (215)$$
$$\partial_{Q_{j\beta}}(\phi) = \Lambda^{-kj} i_k(E_\beta)\phi, \partial_{P_{i\alpha}}(\phi) = (\Lambda^{ik} i_k(E_\alpha)\phi.$$

resp.

$$\Lambda^k\psi = \phi\Lambda^k, i_k(E_\delta)\psi = \psi i_k(E_\delta) \text{ and} \qquad (216)$$
$$\partial_{Q_{j\beta}}(\psi) = \Lambda^{-kj} i_k(E_\beta)\psi, \partial_{P_{i\alpha}}(\psi) = \Lambda^{ik} i_k(E_\alpha)\psi.$$

Note that the action of $\partial_{Q_{j\beta}}$ on ϕ and ψ is multiplying from the left with the constant matrix $\Lambda^{-kj} i_k(E_\beta)$ and that of $\partial_{P_{i\alpha}}$ is multiplying from the left with the constant matrix $\Lambda^{ik} i_k(E_\alpha)$. Let $s_{j\beta}$ be the local parameter corresponding to $\partial_{Q_{j\beta}}$ so that $\partial_{Q_{j\beta}}$ acts as taking the partial derivative $\frac{\partial}{\partial s_{j\beta}} =: \partial_{s_{j\beta}}$ w.r.t. the parameter $s_{j\beta}$ and let $t_{i\alpha}$ be the local parameter corresponding to $\partial_{P_{i\alpha}}$ so that $\partial_{P_{i\alpha}}$ acts as taking the partial derivative $\frac{\partial}{\partial t_{i\alpha}} =: \partial_{t_{i\alpha}}$ w.r.t. the parameter $t_{i\alpha}$. The equations (215) and (216) can then formally be integrated simultaneously. For, consider the formal series

$$\phi_{\mathbb{Z}} := \exp(\sum_{j=1}^{\infty}\sum_{\beta=1}^{m_0} s_{j\beta} i_k(E_\beta)\Lambda^{-kj}) \exp(\sum_{i=0}^{\infty}\sum_{\alpha=1}^{m_0} t_{i\alpha} i_k(E_\alpha)\Lambda^{kj}) =: \psi_{\mathbb{Z}}. \qquad (217)$$

Note that $\phi_{\mathbb{Z}}$ is a formal power series in the variables $\{s_{j\beta}\} \cup \{t_{i\alpha}\}$ with finite band matrices in $M_{\mathbb{Z}}(\mathbb{Z})$ as coefficients and $\phi_{\mathbb{Z}}$ satisfies the equations (215) and (216). . Another solution to the linearization for the trivial solution of the hierarchy is the formal product of $\phi_{\mathbb{Z}}$ and an invertible $\mathbb{Z} \times \mathbb{Z}$-matrix δ that commutes with Λ^k and the $i_k(E_\beta)$ and that is constant for all the flows , i.e. it satisfies

$$\partial_{Q_{j\beta}}(\delta) = \partial_{P_{i\alpha}}(\delta) = 0, \text{ for all } j \geq 1, i \geq 0, \beta \text{ and } \alpha \in \{1, \cdots, m_0\}.$$

The module for the linearization equations for the matrices $\mathcal{M}$ and the $\mathcal{V}_\delta$ will consist of perturbations in $UT(R)$ of these trivial solutions. Consider namely the collection $M^{(0)}(\Lambda^k, \delta)$ consisting of the formal products

$$\{\sum_{j=N}^{\infty} d_j \Lambda^{kj}\}\delta\phi_{\mathbb{Z}}, \text{ where } d_j \in \mathcal{D}_k(R). \tag{218}$$

The elements of $M^{(0)}(\Lambda^k, \delta)$ are called *oscillating matrices of type δ at zero* as the perturbation part has, as series in Λ^k, utmost a pole at zero . In general these formal products do not give a well-defined element of $M_{\mathbb{Z}}(R)$. Nevertheless there is a well-defined left action of $UT(R)$ on it. For all u_1 and $u_2 \in UT(R)$ one puts namely

$$u_1\{u_2\}\delta\phi_{\mathbb{Z}} = \{u_1u_2\}\delta\phi_{\mathbb{Z}}. \tag{219}$$

Also the right multiplication with Λ^{-k} and $i_k(E_\beta)$ is well-defined on elements of $M^{(0)}(\Lambda^k, \delta)$:

$$\{u\}\delta\phi_{\mathbb{Z}}\Lambda^{-k} = \{u\Lambda^{-k}\}\delta\phi_{\mathbb{Z}} \text{ and } \{u\}\delta\phi_{\mathbb{Z}} i_k(E_\beta) = \{u i_k(E_\beta)\}\delta\phi_{\mathbb{Z}}.$$

The action of the derivations $\partial_{Q_{j\beta}}$ and $\partial_{P_{i\alpha}}$ on $M^{(0)}(\Lambda^k, \delta)$ is defined as if the product in the elements of the module $M^{(0)}(\Lambda^k, \delta)$ is a real one

$$\partial_{Q_{j\beta}}\left(\{\sum_{t=N}^{\infty} d_t\Lambda^{tk}\}\delta\phi_{\mathbb{Z}}\right) = \{\sum_{t=N}^{\infty} \partial_{Q_{j\beta}}(d_t)\Lambda^{kt} + \sum_{t=N}^{\infty} d_t\Lambda^{kt}\Lambda^{-jk} i_k(E_\beta)\}\delta\phi_{\mathbb{Z}}.$$

and

$$\partial_{P_{i\alpha}}\left(\{\sum_{t=N}^{\infty} d_t\Lambda^{tk}\}\delta\phi_{\mathbb{Z}}\right) = \{\sum_{t=N}^{\infty} \partial_{P_{i\alpha}}(d_t)\Lambda^{kt} + \sum_{t=N}^{\infty} d_t\Lambda^{kt}\Lambda^{ik} i_k(E_\alpha)\}\delta\phi_{\mathbb{Z}}.$$

Similarly, the module for the linearization equations for the matrices $\mathcal{L}$ and the $\mathcal{U}_\gamma$ will consist of perturbations in $LT(R)$ of the trivial solutions $\delta\psi_{\mathbb{Z}}$. Consider namely the collection $M^{(\infty)}(\Lambda^k, \delta)$ consisting of the formal products

$$\{\sum_{j=-\infty}^{N} d_j\Lambda^{kj}\}\delta\psi_{\mathbb{Z}}, \text{ where } d_j \in \mathcal{D}_k(R).$$

The elements of $M^{(\infty)}(\Lambda^k, \delta)$ are called *oscillating matrices of type δ at infinity* as the perturbation part has, as a series in Λ^k, maximally a pole at infinity.

In general these formal products do not give a well-defined element of $M_{\mathbb{Z}}(R)$. Nevertheless there is a well-defined left action of $LT(R)$ on it. For all l_1 and $l_2 \in LT(R)$ one puts namely

$$l_1\{l_2\}\delta\phi_{\mathbb{Z}} = \{l_1 l_2\}\delta\phi_{\mathbb{Z}}. \tag{220}$$

Also the right multiplication with the matrices Λ^k and $i_k(E_\alpha)$ is defined on elements of $M^{(\infty)}(\Lambda^k, \delta)$ as it was done on $M^{(0)}(\Lambda^k, \delta)$ for Λ^{-k} and $i_k(E_\alpha)$. The action of the derivations $\partial_{Q_{j\beta}}$ and $\partial_{P_{i\alpha}}$ on $M^{((\infty)}(\Lambda^k, \delta)$ is again defined as if the product in the elements of the module $M^{(\infty)}(\Lambda^k, \delta)$ is a real one

$$\partial_{Q_{j\beta}}\left(\{\sum_{t=-\infty}^{N} d_t\Lambda^{tk}\}\delta\phi_{\mathbb{Z}}\right) = \{\sum_{t=-\infty}^{N} \partial_{Q_{j\beta}}(d_t)\Lambda^{kt} + \sum_{t=-\infty}^{N} d_t\Lambda^{kt}\Lambda^{-jk}i_k(E_\beta)\}\delta\phi_{\mathbb{Z}}.$$

and

$$\partial_{P_{i\alpha}}\left(\{\sum_{t=-\infty}^{N} d_t\Lambda^{tk}\}\delta\phi_{\mathbb{Z}}\right) = \{\sum_{t=-\infty}^{N} \partial_{P_{i\alpha}}(d_t)\Lambda^{kt} + \sum_{t=-\infty}^{N} d_t\Lambda^{kt}\Lambda^{ik}i_k(E_\alpha)\}\delta\phi_{\mathbb{Z}}.$$

So, the ϕ occurring in the equations (209) will be an element from $M^{(0)}(\Lambda^k, \delta)$ and the ψ in the equations (210) will belong to $M^{(\infty)}(\Lambda^k, \delta)$.

In general the formal products in both spaces $M^{(0)}(\Lambda^k, \delta)$ and $M^{(\infty)}(\Lambda^k, \delta)$ do not give a well-defined element of $M_{\mathbb{Z}}(R)$. We have already met a setting where one can make sense of them. Choose R equal to $F[[t_{i\alpha}, s_{j\beta}]]$ as in Example 5 and take for δ an element from $LT(F) \cap UT(F)$ that commutes with the basic directions, a condition that is satisfied in the relevant cases, see section 6.6. Recall that $\phi_{\mathbb{Z}}$ is a formal power series in the variables $\{s_{j\beta}\} \cup \{t_{i\alpha}\}$ with finite band matrices in $M_{\mathbb{Z}}(F)$ as coefficients. Then each $\phi \in M^{(\infty)}(\Lambda^k, \delta)$ is a formal power series in the same variables with coefficients from $LT(F)$ and each $\psi \in M^{(0)}(\Lambda^k, \delta)$ is likewise a formal power series in the same variables but with coefficients from $UT(F)$.

Note that each of the exponential factors in (217) defines a well-defined matrix in $M_{\mathbb{Z}}(F[s_{j\beta}, t_{i\alpha}])$. To make sense out of the formal products in both $M^{(0)}(\Lambda^k, \delta)$ and $M^{(\infty)}(\Lambda^k, \delta)$ as a well-defined matrix in $M_{\mathbb{Z}}(R)$ in a more general setting, requires first of all an embedding of the polynomial ring

$$F[s_{j\beta}, t_{i\alpha}], j \geq 1, i \geq 0, \beta \text{ and } \alpha \in \{1, \cdots, m_0\},$$

as a subalgebra of R. Further one needs that that the derivations $\partial_{Q_{j\beta}}$ resp. $\partial_{P_{i\alpha}}$ are extensions to R of the partial derivative $\partial_{s_{j\beta}}$ resp. the partial derivative $\partial_{t_{i\alpha}}$ and last but not least convergence considerations. Such a setting is given in section (6.6).

All the actions and elements occurring in the linearization have been introduced now. Note that $M^{(0)}(\Lambda^k, \delta)$ is a free $UT(R)$-module with generator $\delta\phi_{\mathbb{Z}}$ and $M^{(\infty)}(\Lambda^k, \delta)$ is a free $LT(R)$-module with generator $\delta\psi_{\mathbb{Z}}$. In particular scratching ϕ from the equations (211), (212), (213), and (214) is permitted as soon as one knows that $\phi = \hat{\phi}\delta\phi_0$ with $\hat{\phi} \in UT(R)$ invertible. In this last case the first equation from (209) $\mathcal{M}\phi = \phi\Lambda^{-k}$ implies that

$$\mathcal{M} = \mathcal{M}(\hat{\phi}) := \hat{\phi}\delta\Lambda^{-k}\delta^{-1}\hat{\phi}^{-1} = \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$$

and the equation $\mathcal{V}_\alpha\phi = \phi i_k(E_\alpha)$ implies that

$$\mathcal{V}_\alpha = \mathcal{V}_\alpha(\hat{\phi}) := \hat{\phi}\delta i_k(E_\alpha)\delta^{-1}\hat{\phi}^{-1} = \hat{\phi} i_k(E_\alpha)\hat{\phi}^{-1}.$$

Likewise, if $\psi \in M^{(\infty)}(\Lambda^k, \delta)$ has the form $\psi = \hat{\psi}\delta\psi_{\mathbb{Z}}$ with $\hat{\psi} = \mathrm{Id} + \sum_{i<0} d_i\Lambda^{ki}$, then the equations $\mathcal{L}\psi = \psi\Lambda^k$ and $U_\gamma\psi = \psi i_k(E_\gamma)$ imply resp. that

$$\mathcal{L} = \mathcal{L}(\hat{\psi}) := \hat{\psi}\Lambda^k\hat{\psi}^{-1} \text{ and } U_\gamma = U_\gamma(\hat{\psi}) := \hat{\psi} i_k(E_\gamma)\hat{\psi}^{-1}$$

and ψ is a generator of $M^{(\infty)}(\Lambda^k, \delta)$. In particular, the last two equations in (210) imply the Lax equations for this $\mathcal{L}$ and these U_γ.

Consider pairs $(\phi, \psi) \in M^{(0)}(\Lambda^k, \delta) \times M^{(\infty)}(\Lambda^k, \delta)$ where $\phi = \hat{\phi}\delta\phi_{\mathbb{Z}}$, with $\hat{\phi} = \sum_{i=0}^{\infty} d_i\Lambda^{ki}$ and d_0 invertible and the second component ψ has the form $\psi = \hat{\psi}\delta\psi_{\mathbb{Z}}$ with $\hat{\psi} = \mathrm{Id} + \sum_{i<0} d_i\Lambda^{ki}$. Such a pair (ϕ, ψ) is called *a pair of wave matrices of type* δ, if the equations (209) and (210) hold for the matrices $\mathcal{M}(\hat{\phi})$, $\mathcal{L}(\hat{\psi})$, the $\mathcal{V}_\delta(\hat{\phi})$ and the $\mathcal{U}_\gamma(\hat{\psi})$ just defined. The component ϕ of such a pair is called the *wave matrix at zero* of type δ and the ψ the *wave matrix at infinity* of type δ. Since the manipulations to get the Lax equations are well-defined on such a pair (ϕ, ψ) of wave matrices, the matrices $\mathcal{M}(\hat{\phi})$, $\mathcal{L}(\hat{\psi})$, the $\mathcal{V}_\delta(\hat{\phi})$ and the $\mathcal{U}_\gamma(\hat{\psi})$ form a solution of the $(\Lambda^k, \mathfrak{h})$-hierarchy .

Remark 20. If R and the derivations are as in Example 5 and (ϕ, ψ) is a pair of wave matrices for the matrices $\mathcal{M}$, $\mathcal{L}$, the $\mathcal{V}_\delta$ and the $\mathcal{U}_\gamma$ from $M_{\mathbb{Z}}(R)$, then we have seen that both ϕ and ψ are well-defined power series in the variables $\{s_{j\beta}\} \cup \{t_{i\alpha}\}$ with coefficients from $UT(F)$ resp. $LT(F)$. Because of the last two equations in the linearizations (209) and (210) one sees that they are fundamental matrices for the Cauchy problem corresponding to the solutions $\{C_{j\beta}\}$ and the $\{B_{i\alpha}\}$ of the zero curvature equations. In other words the notion of wave matrices is a generalization of that of a fundamental matrix for a Cauchy problem.

Different pairs of wave matrices of type δ may lead to the same solution of the $(\Lambda^k, \mathfrak{h})$-hierarchy. Assume namely that

$$\mathcal{M} = \hat{\phi}_1 \Lambda^{-k} \hat{\phi}_1^{-1} = \hat{\phi}_2 \Lambda^{-k} \hat{\phi}_2^{-1}, \mathcal{V}_\delta = \hat{\phi}_1 i_k(E_\delta) \hat{\phi}_1^{-1} = \hat{\phi}_2 i_k(E_\delta) \hat{\phi}_2^{-1}$$

and

$$\mathcal{L} = \hat{\psi}_1 \Lambda^{k} \hat{\psi}_1^{-1} = \hat{\psi}_2 \Lambda^{k} \hat{\psi}_2^{-1}, \mathcal{U}_\gamma = \hat{\psi}_1 i_k(E_\gamma) \hat{\psi}_1^{-1} = \hat{\psi}_2 i_k(E_\gamma) \hat{\psi}_2^{-1},$$

where both pairs (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are wave matrices of type δ. Then one has first of all that both $\hat{\phi}_1^{-1} \hat{\phi}_2$ and $\hat{\psi}_1^{-1} \hat{\psi}_2$ commute with Λ^k and the $i_k(E_\beta)$ and thus have to have the form

$$\hat{\phi}_1^{-1} \hat{\phi}_2 = \sum_{r \geq 0} v_r \Lambda^{kr} \text{ and } \hat{\psi}_1^{-1} \hat{\psi}_2 = \sum_{a \leq 0} w_a \Lambda^{ka}$$

where $v_r, w_a \in i_k(M_k(R))$ commute with the $i_k(E_\beta)$. One has seen in the proof of Proposition 10 that for all $j \geq 1$, $i \geq 0$ and all $\beta, \alpha \in \{1, \cdots, m_0\}$ and $t = 1, 2$,

$$\partial_{s_{j\beta}}(\hat{\phi}_t) = C_{j\beta} \hat{\phi}_t - \hat{\phi}_t \Lambda^{-jk} i_k(E_\beta) \text{ and } \partial_{t_{i\alpha}}(\hat{\phi}_t) = B_{i\alpha} \hat{\phi}_t - \hat{\phi}_t \Lambda^{ik} i_k(E_\alpha).$$

Now, if one applies the operator $\partial_{Q_{j\beta}}$ to the equality $\hat{\phi}_2 = \hat{\phi}_1 \sum_r v_r \Lambda^{kr}$, then one obtains

$$\partial_{Q_{j\beta}}(\hat{\phi}_2) = \partial_{Q_{j\beta}}(\hat{\phi}_1)(\sum_{r \geq 0} v_r \Lambda^{kr}) + \hat{\phi}_1 \sum_{r \geq 0} \partial_{Q_{j\beta}}(v_r) \Lambda^{kr} = \quad (221)$$

$$(C_{j\beta} \hat{\phi}_1 - \hat{\phi}_1 \Lambda^{-kj} i_k(E_\beta))(\sum_{r \geq 0} v_r \Lambda^{kr}) + \hat{\phi}_1 \sum_{r \geq 0} \partial_{Q_{j\beta}}(v_r) \Lambda^{kr} = \quad (222)$$

$$(C_{j\beta} \hat{\phi}_2 - \hat{\phi}_2 \Lambda^{-kj} i_k(E_\beta)) + \hat{\phi}_1 \sum_{r \geq 0} \partial_{s_{j\beta}}(v_r) \Lambda^{kr}. \quad (223)$$

Hence one must have for all $r \geq 0$, all $j \geq 1$ and all $\beta \in \{1, \cdots, m_0\}$ that $\partial_{Q_{j\beta}}(v_i) = 0$. By applying $\partial_{P_{i\alpha}}$ instead of $\partial_{Q_{j\beta}}$ to the same identity, one gets in the same way that for all $i \geq 0$, $r \geq 0$ and all $\alpha \in \{1, \cdots, m_0\}$ that $\partial_{P_{i\alpha}}(v_r) = 0$. Similar results hold for the components in $M^{(\infty)}(\Lambda^k, \delta)$. For completeness sake, this result is resumed in a corollary

Corollary 6. *If the pairs (ϕ_1, ψ_1) and (ϕ_2, ψ_2) are wave matrices of type δ w.r.t. the same operators $\mathcal{M}$, $\mathcal{L}$, the $\mathcal{V}_\delta$ and the $\mathcal{U}_\gamma$, then there holds*

$$\phi_2 = \phi_1 \sum_{r \geq 0} v_r \Lambda^{kr} \text{ and } \psi_2 = \psi_1 \sum_{a \leq 0} w_a \Lambda^{ka},$$

where all the v_r and all the w_a are constant for all the derivations $\partial_{Q_{j\beta}}$ and $\partial_{P_{i\alpha}}$, i.e.

$$\partial_{Q_{j\beta}}(v_r) = 0 = \partial_{Q_{j\beta}}(w_a) \text{ and } \partial_{P_{i\alpha}}(v_r) = 0 = \partial_{P_{i\alpha}}(w_a)$$

6.5 The bilinear form

In this section each derivation $\partial_{Q_{j\beta}}$ will be the partial differentiation $\partial_{s_{j\beta}}$ and likewise each $\partial_{P_{i\alpha}}$ will be the partial differentiation $\partial_{t_{i\alpha}}$. Related to the linearization there is another form of the hierarchy: the bilinear form. It will be discussed in this section.

It starts with the following observation: if (ϕ, ψ) is a pair of wave matrices for the $(\Lambda^k, \mathfrak{h})$-hierarchy, then there hold the equations

$$\partial_{s_{j\beta}}(\psi)\psi^{-1} = \partial_{s_{j\beta}}(\phi)\phi^{-1} \tag{224}$$

$$\partial_{t_{i\alpha}}(\psi)\psi^{-1} = \partial_{t_{i\alpha}}(\phi)\phi^{-1} \tag{225}$$

Now, if one wants to prove for a pair $(\phi, \psi) \in M^{(0)}(\Lambda^k, \delta) \times M^{(\infty)}(\Lambda^k, \delta)$ of the right form the equations (209) and (210), then it suffices to prove a weaker result, for there holds

Proposition 10. *Let ϕ be an oscillating matrix of type δ at zero of the form $\phi = \hat{\phi}\delta\phi_{\mathbb{Z}}$, with $\hat{\phi} - d_0 \in UT_{\geq 1}(\Lambda^k)$ and $d_0 \in \mathcal{D}_k(R)$ invertible. Similarly, assume that ψ is $\psi = \hat{\psi}\delta\psi_{\mathbb{Z}}$, with $\hat{\psi} - \mathrm{Id} \in LT_{\leq -1}(\Lambda^k)$ is. If they satisfy for all $j \geq 1$,*

$i \geq 0$ and all β and $\alpha \in \{1, \cdots, m_0\}$

$$\partial_{s_{j\beta}}(\phi) = G_{j\beta}\phi, \partial_{s_{j\beta}}(\psi) = G_{j\beta}\psi, \text{ with } G_{j\beta} \in UT(R)_{<0}(\Lambda^k)$$
$$\partial_{t_{i\alpha}}(\psi) = H_{i\alpha}\psi, \partial_{t_{i\alpha}}(\phi) = H_{i\alpha}\phi, \text{ with } H_{i\alpha} \in LT(R)_{\geq 0}(\Lambda^k).$$

then the matrix $G_{j\beta} = (\mathcal{M}(\hat{\phi})^j \mathcal{V}_\beta(\hat{\phi}))_{<0}(\Lambda^k)$ with $\mathcal{M}(\hat{\phi}) := \hat{\phi}\Lambda^{-k}\hat{\phi}^{-1}$ and $\mathcal{V}_\beta(\hat{\phi}) := \hat{\phi} i_k(E_\beta)\hat{\phi}^{-1}$, and the matrix $H_{i\alpha} = (\mathcal{L}(\hat{\psi})^i \mathcal{U}_\alpha(\hat{\psi}))_{\geq 0}(\Lambda^k)$, where $\mathcal{L}(\hat{\psi}) := \hat{\psi}\Lambda^k\hat{\psi}^{-1}$ and $\mathcal{U}_\alpha(\hat{\psi}) := \hat{\psi} i_k(E_\alpha)\hat{\psi}^{-1}$. The pair (ϕ, ψ) is then a pair of wave matrices for the $(\Lambda^k, \mathfrak{h})$-hierarchy and the matrices $\mathcal{L}(\hat{\psi})$, $\mathcal{M}(\hat{\phi})$, the $\mathcal{U}_\alpha(\hat{\psi})$ and the $\mathcal{V}_\beta(\hat{\phi})$ are a solution of this hierarchy.

Proof. From the definition of the action of $\partial_{Q_{j\beta}}$ on $M^{(0)}(\Lambda^k, \delta)$ and the fact that $M^{(0)}(\Lambda^k, \delta)$ is a free $UT(R)$-module with generator $\delta\phi_{\mathbb{Z}}$, one gets the matrix equation

$$\partial_{s_{j\beta}}(\hat{\phi}) + \hat{\phi}\Lambda^{-kj} i_k(E_\beta) = G_{j\beta}\hat{\phi}. \tag{226}$$

Multiplying this equation from the right with $\hat{\phi}^{-1}$ and taking the strict lower triangular part "< 0" in Λ^k gives the first equality. Similarly, $M^{(\infty)}(\Lambda^k, \delta)$ is a free $LT(R)$-module with generator $\delta\psi_{\mathbb{Z}}$ and then the equality $\partial_{t_{i\alpha}}(\psi) = H_{i\alpha}\psi$ amounts to

$$\partial_{t_{i\alpha}}(\hat{\psi}) + \hat{\psi}\Lambda^{ki} i_k(E_\alpha) = H_{i\alpha}\hat{\psi}. \tag{227}$$

Then multiplying this equation from the right with $\hat{\psi}^{-1}$ and taking the upper triangular part "≥ 0" in Λ^k yields the second equality. The final conclusion follows directly from these equalities. □

This proposition applies in the following situation:

Corollary 7. *Consider a pair $(\phi, \psi) \in M^{(0)}(\Lambda^k, \delta) \times M^{(\infty)}(\Lambda^k, \delta)$ of the form $\phi = \hat{\phi}\delta\phi_{\mathbb{Z}}$, with $\hat{\phi} - d_0 \in UT_{\geq 1}(\Lambda^k)$ and $d_0 \in \mathcal{D}_k(R)$ invertible, and $\psi = \hat{\psi}\delta\psi_{\mathbb{Z}}$, with $\hat{\psi} - \mathrm{Id} \in LT_{\leq -1}(\Lambda^k)$. If they satisfy the relations (224) and (225), then (ϕ, ψ) is a pair of wave matrices for the $(\Lambda^k, \mathfrak{h})$-hierarchy.*

Proof. For any such a pair (ϕ, ψ) the equations (224) resp. (225) yield

$$\partial_{s_{j\beta}}(\hat{\psi})\hat{\psi}^{-1} + \hat{\psi}\Lambda^{-kj} i_k(E_\beta)\hat{\psi}^{-1} = \partial_{s_{j\beta}}(\hat{\phi})\hat{\phi}^{-1} + \hat{\phi}\Lambda^{-kj} i_k(E_\beta)\hat{\phi}^{-1} \tag{228}$$
$$\partial_{t_{i\alpha}}(\hat{\psi})\hat{\psi}^{-1} + \hat{\psi}\Lambda^{ki} i_k(E_\alpha)\hat{\psi}^{-1} = \partial_{t_{i\alpha}}(\hat{\phi})\hat{\phi}^{-1} + \hat{\phi}\Lambda^{ki} i_k(E_\alpha)\hat{\phi}^{-1} \tag{229}$$

As the left hand side of equation (228) has strict negative degree in Λ^k and the right hand side of (229) has positive degree in Λ^k, the conditions of Proposition 10 are fulfilled. □

The conditions (224) and (225) extend to any constant coefficient differential operator in the partial derivatives w.r.t. the parameters of the flows

Corollary 8. *Let P be any differential operator in the $\partial_{s_{j\beta}}$ and the $\partial_{t_{i\alpha}}$ with constant coefficients. Then there holds for a pair of wave matrices (ϕ, ψ) for the $(\Lambda^k, \mathfrak{h})$-hierarchy that*

$$P(\psi)\psi^{-1} = P(\phi)\phi^{-1}. \tag{230}$$

Proof. Thanks to the equations (224) and (225) the statement is correct for first order operators and then one proceeds by induction. By induction it suffices to show that if the relation holds for P, then it also holds for $\partial_{s_{j\beta}}P$ and $\partial_{t_{i\alpha}}P$. Applying $\partial_{s_{j\beta}}$ to equation (230) gives

$$\partial_{s_{j\beta}}P(\psi)\psi^{-1} - P(\psi)\psi^{-1}\partial_{s_{j\beta}}(\psi)\psi^{-1} = \partial_{s_{j\beta}}P(\psi)\psi^{-1} - P(\psi)\psi^{-1}C_{j\beta} =$$
$$\partial_{s_{j\beta}}P(\phi)\phi^{-1} - P(\phi)\phi^{-1}\partial_{s_{j\beta}}(\phi)\phi^{-1} = \partial_{s_{j\beta}}P(\phi)\phi^{-1} - P(\phi)\phi^{-1}C_{j\beta}.$$

and this implies the desired equality for $\partial_{s_{j\beta}}P$. The proof for $\partial_{t_{i\alpha}}P$ is the same. □

The identity (230) has the following interpretation: consider the variations

$$\tilde{s}_{j\beta} = s_{j\beta} + g_{j\beta} \text{ and } \tilde{t}_{i\alpha} = t_{i\alpha} + h_{i\alpha}$$

of the parameters $s_{j\beta}$ and $t_{i\alpha}$ and consider the two formal products

$$\psi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha})\psi(s_{j\beta}, t_{i\alpha})^{-1} = \hat{\psi}(s_{j\beta} + g_{j\beta}, t_{i\alpha} + h_{i\alpha})\psi_{\mathbb{Z}}(g_{j\beta}, h_{i\alpha})\hat{\psi}(s_{j\beta}, t_{i\alpha})^{-1}$$

and

$$\phi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha})\phi(s_{j\beta}, t_{i\alpha})^{-1} = \hat{\phi}(s_{j\beta} + g_{j\beta}, t_{i\alpha} + h_{i\alpha})\phi_{\mathbb{Z}}(g_{j\beta}, h_{i\alpha})\hat{\phi}(s_{j\beta}, t_{i\alpha})^{-1}.$$

According to (230) their Taylor coefficients w.r.t. the $g_{j\beta}$ and the $h_{i\alpha}$ are the same and this one denotes by the bilinear identity

$$\psi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha})\psi(s_{j\beta}, t_{i\alpha})^{-1} = \phi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha})\phi(s_{j\beta}, t_{i\alpha})^{-1}.$$

Multiplying on the left with the inverse of $\psi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha})$ and on the right with the matrix $\phi(s_{j\beta}, t_{i\alpha})$ results in the identity

$$\psi(s_{j\beta}, t_{i\alpha})^{-1}\phi(s_{j\beta}, t_{i\alpha}) = \psi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha})^{-1}\phi(\tilde{s}_{j\beta}, \tilde{t}_{i\alpha}). \tag{231}$$

In other words, if $\psi^{-1}\phi$ is a well-defined product, then this matrix is constant w.r.t. the parameters $s_{j\beta}$ and $t_{i\alpha}$. This can also be seen directly

$$\begin{aligned}&\partial_{s_{j\beta}}(\psi^{-1}\phi) = -\psi^{-1}\partial_{s_{j\beta}}(\psi)\psi^{-1}\phi + \psi^{-1}\partial_{s_{j\beta}}(\phi) =\\ &-\psi^{-1}C_{j\beta}\psi\psi^{-1}\phi + \psi^{-1}C_{j\beta}\phi = 0\end{aligned}$$

and similarly for the differentiation w.r.t. $t_{i\alpha}$. The identity (231) lies at the heart of the construction of the solutions.

Remark 21. To get real solutions for the hierarchy, we will construct from a geometric setting pairs of oscillating functions of the right form in which the formal products are convergent. This will be done in the final section.

6.6 A construction of solutions of the $(\Lambda^k, \mathfrak{h})$-hierarchy

In this subsection we limit ourselves to the construction of complex solutions of the hierarchy and thus one takes from now on the field F equal to $\mathbb{C}$.

First one discusses the group that is the starting point for the construction of the pairs of wave matrices for the $(\Lambda^k, \mathfrak{h})$-hierarchy. As with the foregoing two classes of hierarchies one has also now some freedom in the choice of this group and that is illustrated by its dependence of a class of compact operators. Recall that for a Hilbert space H and any integer $r \geq 1$ the Schatten class S_r consists of the bounded operators $A : H \mapsto H$ such that

$$||A||_r^r := \text{trace}((A^*A)^{\frac{r}{2}}) = \text{trace}(|A|^r) < \infty.$$

Thanks to the fact that the Schatten classes are two-sided ideals of compact operators , one can introduce for each such a r a group $G(r)$ by

$$G(r) = \left\{ g = (g_{ij}) \in \text{GL}(H) \;\middle|\; g - \text{Id} \in S_r \right\}.$$

It is a direct verification that the space S_r is the Lie algebra $\mathcal{G}(r)$ of $G(r)$. The Lie algebra $\mathcal{G}(r)$ is the sum of the Lie subalgebras

$$\mathcal{P}(r) := \left\{ p = (p_{ij}) \in \mathcal{G}(r) \;\middle|\; p_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$\mathcal{U}_-(r) := \left\{ u = (u_{ij}) \in \mathcal{G}(r) \;\middle|\; u_{ij} = 0 \text{ for all } i \leq j \right\}.$$

Their corresponding Lie groups are

$$P(r) := \left\{ p = (p_{ij}) \in G(r) \;\middle|\; p_{ij} = 0 \text{ for all } i > j \right\}$$

and

$$U_-(r) := \left\{ u = (u_{ij}) \in G(r) \;\middle|\; \begin{array}{c} u_{ij} = 0 \text{ for all } i < j \\ u_{ii} = \mathrm{Id} \text{ for all } i \in \mathbb{Z} \end{array} \right\}.$$

The collections of $\mathbb{Z} \times \mathbb{Z}$-matrices of the groups $P(r)$ and $U_-(r)$ are denoted by $[P(r)]$ resp. $[U_-(r)]$. Also in the present situation there is a dense open subset that plays a central role in the construction. Consider the map from $\mathcal{U}_-(r) \times \mathcal{P}(r)$ to $G(r)$ defined by

$$(u, p) \mapsto \exp(u) \exp(p).$$

As it is a local diffeomorphism at $(0, 0)$, the set $U_-(r)P(r)$ is an open subset of $G(r)$. It can be characterized in a similar fashion as in the finite dimensional case

Proposition 11. *Let $\Omega \subset G(r)$ be the collection of all $g \in G(r)$ such that $g_{--}(i)$ is invertible for all $i \in \mathbb{Z}$. Then Ω is equal to $U_-(r)P(r)$ and is called the big cell in $G(r)$ w.r.t. these subgroups.*

The basic commuting directions from BD are the generators of the commuting flows that are relevant for the $(\Lambda^k, \mathfrak{h})$-hierarchy. As we saw in 4.6 this group $\Gamma(\mathfrak{h})$ of commuting flows associated with $\mathfrak{h}$ embeds into $\mathrm{GL}(H)$, but none of them can occur as a $\mathbb{Z} \times \mathbb{Z}$-matrix of an operator in $G(r)$. From Proposition 5 in subsection 4.5 one sees that each $\gamma \in \Gamma(\mathfrak{h})$ splits uniquely as $\gamma = \gamma_{\geq 0}\delta(\gamma)\gamma_{<0}$, where $\gamma_{\geq 0} \in \Gamma(\mathfrak{h})_{\geq 0}$, $\delta(\gamma) \in \Delta(\mathfrak{h})$ and $\gamma_{<0} \in \Gamma(\mathfrak{h})_{<0}$. Since the Schatten-class operators S_r form a two-sided ideal in the ring of bounded operators on H, the group $\Gamma(\mathfrak{h})$ acts by conjugation on the group $G(r)$. For each $g \in G(r)$, consider the open subset of $\Gamma(\mathfrak{h})$ defined by

$$\Gamma(g, \mathfrak{h}) = \{\gamma \in \Gamma(\mathfrak{h}) \mid \gamma g \gamma^{-1} \in \Omega\}.$$

By definition, one has for all $\gamma \in \Gamma(g, \mathfrak{h})$

$$[\gamma g \gamma^{-1}] = \hat{\Phi}_-(\gamma, g)^{-1} \hat{\Phi}_+(\gamma, g), \tag{232}$$

with $\hat{\Phi}_-(\gamma, g) \in [U_-(r)]$ and $\hat{\Phi}_+(\gamma, g) \in [P(r)]$. Now one takes for the ring R the collection of holomorphic functions on $\Gamma(g, \mathfrak{h})$, then $\hat{\Phi}_-(\gamma, g) \in LT_0(R)$ and $\hat{\Phi}_+(\gamma, g) \in UT_0(R)$. Consider the oscillating functions of type $[\delta(\gamma)]$

$$\Phi_+(\gamma, g) = \hat{\Phi}_+(\gamma, g)[\gamma] \in M^{(0)}(\Lambda^k, [\delta(\gamma)])$$

and

$$\Phi_-(\gamma, g) = \hat{\Phi}_-(\gamma, g)[\gamma] \in M^{(\infty)}(\Lambda^k, [\delta(\gamma)]).$$

They have the right shape for candidate wave functions and are related through

$$\hat{\Phi}_-(\gamma, g)[\gamma][g] = \hat{\Phi}_+(\gamma, g)[\gamma], \tag{233}$$

The claim is now that the pair (Φ_+, Φ_-) is a pair of wave matrices of type $[\delta(\gamma)]$ for the $(\Lambda^k, \mathfrak{h})$-hierarchy. Differentiating the left hand side of equation (233) w.r.t. $t_{i\alpha}$ leads to

$$\{\partial_{t_{i\alpha}}(\hat{\Phi}_-(\gamma, g)) + \hat{\Phi}_-(\gamma, g)\Lambda^{ik} i_k(E_\alpha)\}[\gamma][g] =$$
$$\{\partial_{t_{i\alpha}}(\hat{\Phi}_-(\gamma, g))\hat{\Phi}_-(\gamma, g)^{-1} + \mathcal{L}(\hat{\Phi}_-)^i U_\alpha(\hat{\Phi}_-)\}\Phi_-(\gamma, g)[g].$$

Applying $\partial_{t_{i\alpha}}$ to the right hand side yields

$$\begin{aligned} &\partial_{t_{i\alpha}}(\hat{\Phi}_+(\gamma, g)) + \hat{\Phi}_+(\gamma, g)\Lambda^{ik} i_k(E_\alpha))[\gamma] = \\ &= \{\partial_{t_{i\alpha}}(\hat{\Phi}_+(\gamma, g))\hat{\Phi}_+(\gamma, g)^{-1} + \hat{\Phi}_+(\gamma, g)\Lambda^{ik} i_k(E_\alpha)\hat{\Phi}_+(\gamma, g)^{-1}\}\Phi_+(\gamma, g) \end{aligned}$$

and the combination of both expressions gives

$$\begin{aligned} &\partial_{t_{i\alpha}}(\hat{\Phi}_-(\gamma, g))\hat{\Phi}_-(\gamma, g)^{-1} + \mathcal{L}(\hat{\Phi}_-)^i \mathcal{U}_\alpha(\hat{\Phi}_-) = \\ &= \partial_{t_{i\alpha}}(\hat{\Phi}_+(\gamma, g))\hat{\Phi}_+(\gamma, g)^{-1} + \hat{\Phi}_+(\gamma, g)\Lambda^{ik} i_k(E_\alpha)\hat{\Phi}_+(\gamma, g)^{-1}. \end{aligned} \tag{234}$$

Now the matrix $\hat{\Phi}_-(\gamma, g) \in LT_{\leq 0}(R)$ has the identity as its leading coefficient and this implies that $\partial_{t_{i\alpha}}(\hat{\Phi}_-(\gamma, g))\hat{\Phi}_-(\gamma, g)^{-1} \in LT_{\leq -1}(R)$ and as the right hand side of equation (234) belongs to $UT_{\geq 0}(R)$, one can conclude from equation (234) that the following relations hold

$$(\mathcal{L}(\hat{\Phi}_-)^i \mathcal{U}_\alpha(\hat{\Phi}_-))_{\geq 0} = \partial_{t_{i\alpha}}(\hat{\Phi}_+(\gamma, g))\hat{\Phi}_+(\gamma, g)^{-1}$$
$$\partial_{t_{i\alpha}}(\hat{\Phi}_-(\gamma, g))\hat{\Phi}_-(\gamma, g)^{-1} = (\mathcal{L}(\hat{\Phi}_-)^i \mathcal{U}_\alpha(\hat{\Phi}_-))_{<0}.$$

In particular this implies that the linearization equations w.r.t. the $t_{i\alpha}$ hold for Φ_+ and Φ_-

$$\partial_{t_{i\alpha}}(\Phi_+) = (\mathcal{L}(\hat{\Phi}_-)^i \mathcal{U}_\alpha(\hat{\Phi}_-))_+(\Phi_+) \text{ and } \partial_{t_{i\alpha}}(\Phi_-) = (\mathcal{L}(\hat{\Phi}_-)^i \mathcal{U}_\alpha(\hat{\Phi}_-))_+(\Phi_-).$$

For the parameters $\{s_{j\beta}\}$ one proceeds similarly: differentiating both sides of equation (233) results in respectively

$$(\mathcal{M}(\hat{\Phi}_+)^j \mathcal{V}_\beta(\hat{\Phi}_+))_{<0}(\Lambda^k) = \partial_{s_{j\beta}}(\hat{\Phi}_-(\gamma, g))\hat{\Phi}_-(\gamma, g)^{-1},$$
$$(\mathcal{M}(\hat{\Phi}_+)^j \mathcal{V}_\beta(\hat{\Phi}_+))_{\geq 0}(\Lambda^k) = \partial_{s_{j\beta}}(\hat{\Phi}_+(\gamma, g))\hat{\Phi}_+(\gamma, g)^{-1}$$

and this implies the remaining equations for Φ_+ and Φ_-

$$\partial_{s_{j\beta}}(\Phi_+) = (\mathcal{M}(\hat{\Phi}_+)^j \mathcal{V}_\beta(\hat{\Phi}_+))_{<0}(\Lambda^k)\Phi_+ \text{ and}$$
$$\partial_{s_{j\beta}}(\Phi_-) = (\mathcal{M}(\hat{\Phi}_+)^i \mathcal{V}_\beta(\hat{\Phi}_+))_{<0}(\Lambda^k)\Phi_-.$$

This final result is resumed in

Theorem 8. *Consider for $g \in G(r)$ the ring R of holomorphic functions on $\Gamma(g, \mathfrak{h})$. The coefficients of Φ_+ and Φ_- in $[\gamma g \gamma^{-1}] = \hat{\Phi}_-^{-1}\hat{\Phi}_+$ belong to R and the matrices*

$$\mathcal{L}(\hat{\Phi}_-), \{\mathcal{U}_\alpha(\hat{\Phi}_-)\}, \mathcal{M}(\hat{\Phi}_+) \text{ and the } \{\mathcal{V}_\beta(\hat{\Phi}_+)\}$$

are a solution of the $(\Lambda^k, \mathfrak{h})$-hierarchy.

Perspectives: This concludes the discussion of the algebraic and geometric structure of various hierarchies in the $\mathbb{Z} \times \mathbb{Z}$-matrices. Natural next steps will be the treatment of the so-called τ-functions, the corresponding geometry and last but not least the application of the present theory to multi-matrix models, see [21].

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Summary

In this work we start from various basic sets of commuting directions in the $\mathbb{Z} \times \mathbb{Z}$-matrices that are the generators of a group of commuting flows. The main topic in this thesis form deformations of these original directions w.r.t. the parameters of these flows and their evolution equations. The combination of the evolutions of the perturbed matrices w.r.t. all directions yields so-called hierarchies of nonlinear differential and difference equations. Specific examples of these equations that one gets in this way, are the equations satisfied by infinite Toda lattices.

We study three types of deformations: the first deforms the basic directions into the lower triangular matrices with the leading term equal to the basic direction. The second type of deformation is into the upper triangular matrices, where one drops the preservation of the leading term. In the third case one deforms roughly speaking half of the directions according to the first category and the other half according to the second one. In all three instances we study the algebraic structure of the equations and show the equivalence of the various forms in which they occur. One of them, the zero curvature form, is an indication that there is a relation with an infinite dimensional Cauchy problem. We show the uniqueness of the solvability of this Cauchy problem in the formal power series setting. Finally, the discussion of each type of deformation is concluded with a geometric construction of solutions of the hierarchies. This furnishes new illustrative examples of infinite dimensional varieties that play a central role at integrable hierarchies.

Samenvatting

In dit proefschrift beginnen we met diverse basiscollecties commuterende $\mathbb{Z} \times \mathbb{Z}$-matrices die de voortbrengers zijn van een groep commuterende stromingen. Het centrale thema hier zijn deformaties van deze basisrichtingen met betrekking tot de parameters van deze stromingen en de evolutievergelijkingen waaraan ze aan dienen te voldoen. Alle evoluties van de gedeformeerde basisrichtingen combineert men in zogenaamde hierarchiën van niet lineaire differentie- en differentiaalvergelijkingen. Specifieke voorbeelden van deze vergelijkingen zijn de vergelijkingen waaraan de zogenaamde oneindige Toda roosters voldoen.

We bestuderen drie types deformaties: bij het eerste type worden de basisrichtingen zo verbogen in de benedendriehoeksmatrices dat de leidende term behouden blijft. Het tweede type deformatie is in de bovendriehoeksmatrices maar nu zonder behoud van de leidende term. In het derde geval verbuigt men grofweg de helft van de richtingen volgens het eerste type en en de andere helft op de tweede manier. In alle drie de gevallen bestuderen we de algebraische structuur en we laten zien dat de diverse vormen waarin ze zich manifesteren, equivalent zijn. Een van deze, de "kromming nul vorm", is een vingerwijzing dat er een relatie is met een oneindig dimensionaal Cauchy probleem. We laten zien dat dit Cauchy probleem een unieke oplossing heeft in de context van de formele machtreeksen. Tenslotte wordt de discussie van elk type besloten met een meetkundige constructie van oplossingen. Dit geeft nieuwe illustratieve voorbeelden van oneindig dimensionale variëteiten die een centrale rol spelen bij integreerbare hierarchiën.

Curriculum Vitae

A.V. Opimakh was born in Orenburg, Russia, where he also spent his childhood. He went to the secondary school number 19 in that city and attended High School there. The last two years of his High School he followed the intensive program in mathematics. In 1994 he became a student in mathematics and physics at the Pedagogical University of Orenburg. He finished there with distinction the master program with a thesis on heat conductivity of the sphere written under the supervision of M.A. Shleinikova, who was in her turn a student of the well-known Academician N. Ya. Vilenkin. He continued his scientific career as a lecturer at Tambov State University in the group of V. F. Molchanov. There he was teaching Calculus, Mathematical Analysis and Probability Theory for students in mathematics. He participated in summer schools on representation theory in Poland, France and Denmark. By joint projects of the Molchanov group with the Netherlands he met G.F. Helminck and they started a collaboration, which resulted in the present thesis. Currently he is a senior lecturer at the Pedagogical University of Orenburg. He is a keen guitar player and writer of cabaret texts.

Acknowledgements

At the occasion of his defense the author likes to express his gratitude to a number of people. First of all, he wants to thank his promotor Prof. Dr. N.Y. Reshitikhin for accepting him as his PhD-student. Next the members of the committee should be thanked for their careful reading of the manuscript, their remarks and suggestions for improvement of the text. He is very grateful to his friend and colleague Svetlana Polenkova for her moral and practical support during all these years. Finally he likes to thank Gerard Helminck for introducing him to this topic, for his patience at explaining him the necessary ingredients and his perseverence in beleiving that the author could bring this project to a good end.