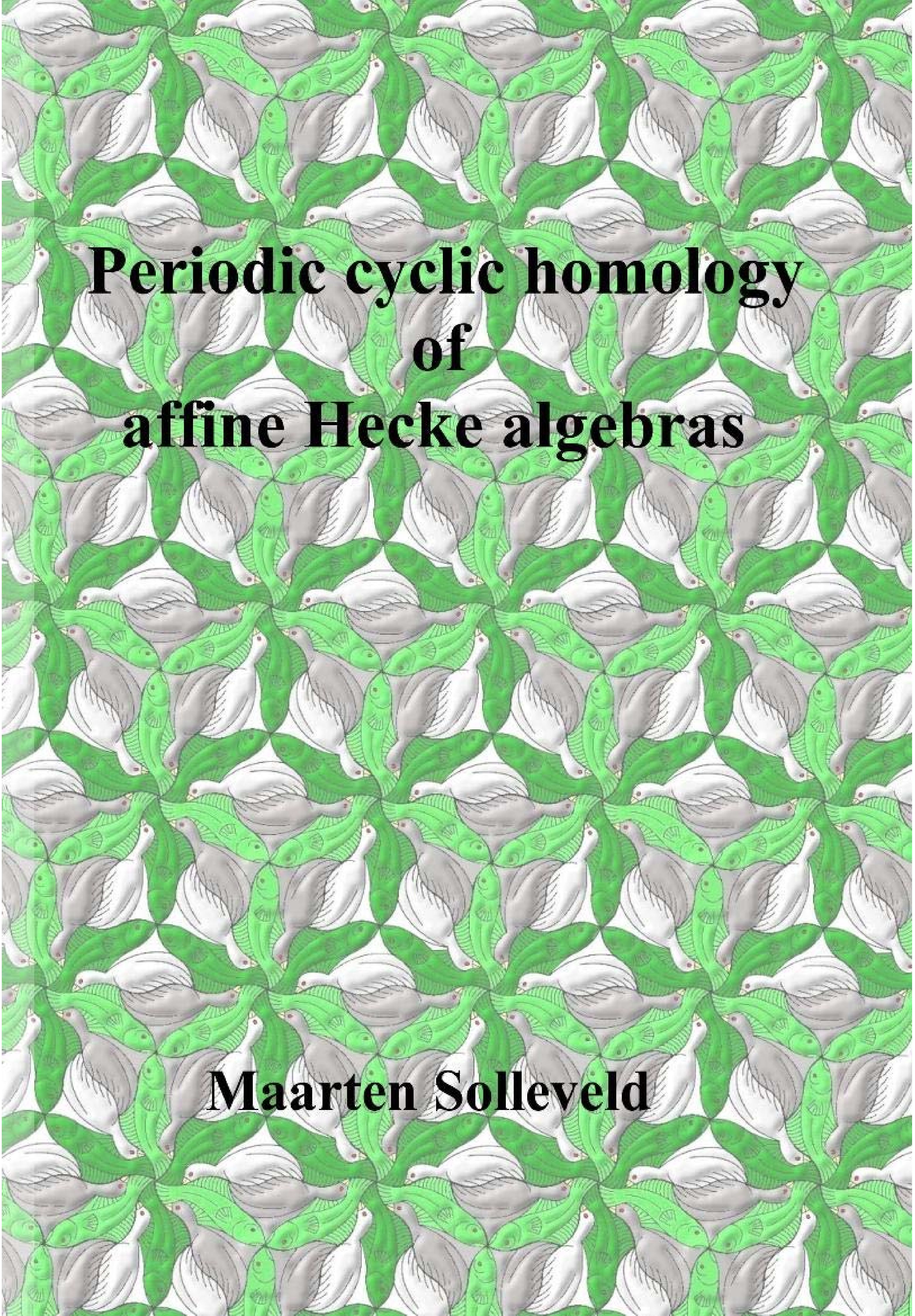

Periodic cyclic homology of affine Hecke algebras

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NWO

THOMAS STIELTJES INSTITUTE
FOR MATHEMATICS

Preface

The book you've just opened is the result of four years of research at the Korteweg-de Vries Institute for Mathematics of the Universiteit van Amsterdam. With this thesis I hope to obtain the degree of doctor.

Due to the highly specialized nature of my research, quite some mathematical background is required to read this book. If you want to get an idea of what it's about, then you may find it useful to start with the Dutch summary.

I would like to use this opportunity to express my gratitude to all those who supported me in this work, intentionally or not.

First of course my advisor, Eric Opdam, without whom almost my entire research would have been impossible. Over the years we had many pleasant conversations, not only about mathematical issues, but also about chess, movies, education, Japan, and many other things. The intensity of our contact fluctuated a lot. Sometimes we didn't talk for weeks, while in other periods we spoke many hours, discussing our new findings every day.

Vividly I recall one particular evening. After a prolonged discussion I had finally returned home and was preparing dinner. Then unexpectedly Eric called to tell me about some further calculations. Watching the boiling rice with one eye and trying to visualize an affine Hecke algebra with the other, I quickly decided that I had to opt for dinner this time. Nevertheless that phonecall had a profound influence on Section 6.6. I admire Eric's deep mathematical insight, with which he managed to put me on the right track quite a few times.

Also I would like to thank all the members of the promotion committee for the time and effort they made to read the manuscript carefully. Niels Kowalzig was so kind to read and comment on the second chapter. For the summary I am indebted to Klaas Slooten. His thesis was a source of inspiration, even though I didn't use many theorems from it.

The KdVI is such a nice place that I come there for more than nine years already. Everybody who worked there is responsible for that, but in particular Erdal, Fokko, Geertje, Harmen, Misja, Peter, Rogier and Simon.

Both my ushers, Mariska Bertholée-de Mie and Ionica Smeets, are very dear to me. For being such good friends, for having faith in me, for lending me an ear, for cheering me up when things did not go as I would have liked. I am very happy that they will support me during the defence of this thesis.

Furthermore I thank all my friends at US badminton for lots of (sporting) pleasure. Especially Paul den Hertog, who also took care of printing this book.

To Karel van der Weide I am grateful for sharing his laconic yet hilarious views on the chess world, on internetdating and on life in general.

Bill Wenger was very generous in granting me permission to use his artwork on the cover of this book.

But above all I thank Lieske Tibbe, for being my mother, and everything that naturally comes with that. If there is anybody who gave me the right scientific attitude to complete this thesis, it's her.

Amsterdam, January 2007

Contents

Chapter 1

Introduction

This thesis is about a very interesting kind of algebras, Hecke algebras. They appear in various fields of mathematics, for example knot theory, harmonic analysis, special functions and noncommutative geometry. The motivation for the research presented here lies mainly in the harmonic analysis of reductive p-adic groups. The description and classification of smooth representations of such groups is a long standing problem. This is motivated by number theoretic investigations.

The category of smooth representations is divided in certain blocks called Bernstein components. It is known that in many cases Bernstein components can be described with the representation theory of certain affine Hecke algebras. This translation is a step forward, since in a sense affine Hecke algebras are much smaller and easier to handle than reductive p-adic groups.

Hence it is desirable to obtain a good description of all irreducible representations of an affine Hecke algebra. Such an algebra $\mathcal{H}(W, q)$ can be considered as a deformation of the (complex) group algebra of an affine Weyl group W, which involves a few parameters $q_i \in \mathbb{C}^\times$. Let us briefly mention what is known about the classification of its spectrum in various cases.

1) All parameters q_i equal to 1.
In this special case $\mathcal{H}(W, q)$ is just the group algebra of W. The representations of groups like W have been known explicitly for a long time, already from the work of Clifford [31].

2) All parameters q_i equal to the same complex number, not a root of unity.
With the use of equivariant K-theory, Kazhdan and Lusztig [76] gave a complete classification of the irreducible representations of $\mathcal{H}(W, q)$. It turns out that they are in bijection with the irreducible representations of W. This bijection can be made explicit with Lusztig's asymptotic Hecke algebra [84].

3) Exceptional cases.
These may occur for example if there exist integers n_i such that $\prod_i q_i^{n_i/2} \neq 1$ is a root of unity, cf. page 132. The affine Hecke algebras for such parameters

may differ significantly from those in the other cases, so we will not study them here.

4) General positive parameters.
Quite strong partial classifications are available, mainly from the work of Delorme and Opdam [39, 40].

5) General unequal parameters.
These algebras have been studied in particular by Lusztig [88]. Recently Kato [74] parametrized the representations of certain affine Hecke algebras with three independent parameters, extending ideas that were used in case 2).

The affine Hecke algebras that arise from reductive p-adic groups have rather special parameters: they are all powers of the cardinality of the residue field of the p-adic field. Sometimes they are all equal, and sometimes they are not, so these algebras are in case 4).

Lusztig [88, Chapter 14] conjectured that

Conjecture 1.1 *For general unequal parameters there also exists an asymptotic Hecke algebra. It yields a natural bijection between the irreducible representations of* $\mathcal{H}(W, q)$ *and those of* W*.*

We are mainly concerned with a somewhat weaker version:

Conjecture 1.2 *For positive parameters there is an isomorphism between the Grothendieck groups of finite dimensional* $\mathcal{H}(W, q)$*-modules and of finite dimensional* W*-modules.*

In principle the verification of this conjecture would involve two steps

a) Assign a W-representation to each (irreducible) $\mathcal{H}(W, q)$-representation, in some natural way (or the other way round).

b) Prove that this induces an isomorphism on the above Grothendieck groups.

In our study we make use of a technique that is obviously not available for p-adic groups, we deform the parameters continuously. We would like to do this in the context of topological algebras, preferably operator algebras. For this reason, and to avoid the exceptional cases 3), we assume throughout most of the book that all q_i are positive real numbers. It was shown in [98] that for such parameters there is a nice natural way to complete $\mathcal{H}(W, q)$ to a Schwartz algebra $\mathcal{S}(W, q)$ (these notations are preliminary). We will compare this algebra to the Schwartz algebra $\mathcal{S}(W)$ of the group W. Using the explicit description of $\mathcal{S}(W, q)$ given by Delorme and Opdam [39], in Section 5.3 we construct a Fréchet algebra homomorphism

$$\phi_0 : \mathcal{S}(W) \to \mathcal{S}(W, q) \tag{1.1}$$

with good properties. This provides a map from $\mathcal{S}(W,q)$-representations to $\mathcal{S}(W)$-representations. Together with the Langlands classification [40, Section 6] for representations of $\mathcal{H}(W,q)$ and of W this takes care of a).

To reformulate b) in more manageable terms we turn to noncommutative geometry. There are (at least) three functors which are suited to deal with such problems: periodic cyclic homology (HP), in either the algebraic or the topological sense, and topological K-theory.

Conjecture 1.3 *There are natural isomorphisms*

1. $HP_*(\mathcal{H}(W,q)) \cong HP_*(\mathbb{C}[W])$
2. $HP_*(\mathcal{S}(W,q)) \cong HP_*(\mathcal{S}(W))$
3. $K_*(\mathcal{S}(W,q)) \cong K_*(\mathcal{S}(W))$

The relation with Conjecture 1.2 is as follows. Although HP_* and K_* are functors on general noncommutative algebras, in our setting they depend essentially only on the spectra of the algebras that we are interested in. These spectra are ill-behaved as topological spaces: the spectrum $\widehat{\mathcal{H}(W,q)}$ is a non-separated algebraic variety, while $\widehat{\mathcal{S}(W,q)}$ is a compact non-Hausdorff space. Nevertheless topological K-theory and periodic cyclic homology can be considered as cohomology theories on such spaces. With this interpretation Conjecture 1.3 asserts that the "cohomology groups" of $\widehat{\mathcal{H}(W,q)}$ and of $\widehat{\mathcal{S}(W,q)}$ are invariant under the deformations in the parameters q_i. Contrarily to what one would expect from the results on page 9, from this noncommutative geometric point of view the algebras $\mathcal{S}(W,q)$ actually become easier to understand when the parameters q_i have less relations among themselves.

For equal parameters Conjecture 1.3 has been around for a while. Part 3 already appeared in the important paper [5], while part 1 was proven by Baum and Nistor [8]. The proof relies on the aforementioned results of Kazhdan and Lusztig. In the unequal parameter case Conjecture 1.3.3 was formulated independently by Opdam [98, Section 1.0.1].

In this thesis we make the following progress concerning these conjectures. In Section 3.4 we prove that there are natural isomorphisms

$$HP_*(\mathcal{H}(W,q)) \cong HP_*(\mathcal{S}(W,q)) \cong K_*(\mathcal{S}(W,q)) \otimes_{\mathbb{Z}} \mathbb{C} \tag{1.2}$$

Hence parts 1 and 2 of Conjecture 1.3 are equivalent, and both are weaker than part 3. Moreover in Section 5.4 we show that the Conjectures 1.2 and 1.3.3 are equivalent.

Guided by these considerations we propose the following refined conjecture, which has also been presented by Opdam [99, Section 7.3]:

Conjecture 1.4 *The natural map*

$$K_*(\phi_0) : K_*(\mathcal{S}(W)) \to K_*(\mathcal{S}(W,q))$$

is an isomorphism for all positive parameters q_i.

Now let us give a brief overview of this book. More detailed outlines can be found at the beginning of each chapter.

Chapter 2 deals with noncommutative geometry. This chapter does not depend on the rest of the book, we do not mention any Hecke algebras. We provide a solid foundations for the philosophy that K-theory and periodic cyclic homology can be considered as cohomology theories for certain non-Hausdorff spaces. Among others we prove comparison theorems like (1.2) for more general classes of algebras, all derived from so-called finite type algebras [77].

In the next chapter we introduce affine Hecke algebras. A large part of the material presented here relies on the work of Opdam, in collaboration with Delorme, Heckman, Reeder and Slooten. We study the representation theory of affine Hecke algebras, which provides a clear picture of their spectra as topological spaces. We are especially interested in the image of $\mathcal{S}(W, q)$ under the Fourier transform, as this turns out to be an algebra of the type that we studied in Chapter 2. We conclude with the above isomorphisms (1.2).

These are also interesting because they can be generalized to algebras associated with reductive p-adic groups, which we will do in Chapter 4. Given a reductive p-adic group G we recall the constructions of its Hecke algebra $\mathcal{H}(G)$, its Schwartz algebra $\mathcal{S}(G)$ and its reduced C^*-algebra $C_r^*(G)$. The main new results in this chapter are natural isomorphisms

$$HP_*(\mathcal{H}(G)) \cong HP_*(\mathcal{S}(G)) \cong K_*(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \tag{1.3}$$

These have some consequences in relation with the Baum-Connes conjecture for G.

In Chapter 5 we really delve into the study of deformations of affine Hecke algebras. The Fréchet space underlying $\mathcal{S}(W, q)$ is independent of q, and we show that all the (topological) algebra operations in $\mathcal{S}(W, q)$ depend continously on q. After that we focus on parameter deformations of the form $q \to q^\epsilon$ with $\epsilon \in [0, 1]$. For positive ϵ we construct isomorphisms

$$\phi_\epsilon : S(W, q^\epsilon) \to S(W, q) \tag{1.4}$$

that depend continuously on ϵ. The limit $\lim_{e \downarrow 0} \phi_\epsilon$ is well-defined and indeed is (1.1). Furthermore we elaborate on the conjectures mentioned in this introduction.

In support of the these conjectures, and to show what the techniques we developed are up to, we dedicate Chapter 6 to calculations for affine Hecke algebras of classical type. We verify Conjecture 1.4 in some low-dimensional cases and for types GL_n and A_n.

We conclude the purely scientific part of the book with a short appendix. It contains some rather elementary results on crossed product algebras that are used at various places.

Chapter 2

K-theory and cyclic type homology theories

This chapter is of a more general nature than the rest of this book. We start with the study of some important covariant functors on the category of complex algebras. These are Hochschild homology, cyclic homology and periodic cyclic homology. Contravariant versions of these functors also exist, but we will leave these aside. All these functors go together by the name of cyclic theory.

It is well-known that cyclic homology is related to K-theory by a natural transformation of functors called the Chern character. We are not satisfied with K-theory for Banach algebras, but instead study its extension to the larger categories of Fréchet algebras or even m-algebras. From these abstract considerations we will see that there are three functors which share almost identical properties:

a) periodic cyclic homology, purely algebraically

b) periodic cyclic homology, with the completed projective tensor product

c) K-theory for Fréchet algebras

These functors can be regarded as noncommutative analogues of

1) De Rham cohomology in the algebraic sense, for complex affine varieties

2) De Rham cohomology in the differential geometric sense, for smooth manifolds

3) K-theory for topological spaces

By a comparison theorem of Deligne and Grothendieck 1) and 2) agree for a complex affine variety. For smooth manifolds 2) and 3) (with real coefficients) give the same result, essentially because both are generalized cohomology theories. This is also the reason that both can be computed as

4) Čech cohomology of a constant sheaf

A noncommutative analogue of 4) does not appear to exist, so we develop it. It will be a sheaf that depends only on the spectrum of an algebra. Then we can also consider

d) Čech cohomology of this sheaf

The main goal of this chapter is to generalize the isomorphisms between 1) - 4) to the setting of noncommutative algebras. So far this has been done only for b) and c).

Let us also give a more concrete overview of this chapter. We start by recalling the definitions and properties of cyclic theory in the purely algebraic sense. Then we specialize to finite type algebras, mainly following [77]. For such algebras we define a sheaf which provides the isomorphism between a) and d).

After that we move on to topological algebras, especially Fréchet algebras. Most of the properties of algebraic cyclic theory have been carried over to this topological setting, but unfortunately these results have been scattered throughout the literature. We hope that bringing them together will serve the reader. We also recall several results concerning K-theory for Fréchet algebras, which are mostly due to Cuntz [33] and Phillips [102].

In the final section we have to decide for what kind of topological algebras we want to compare b), c) and d). Natural candidates are algebras that are finitely generated as modules over their center. For finitely generated (non-topological) algebras this condition leads to the aforementioned finite type algebras. Their spectrum has the structure of a non-separated complex affine variety.

In the topological setting we need to impose more conditions. Cyclic homology works best if there is a kind of smooth structure, so our topological analogue of a finite type algebra is of the form

$$C^\infty(X; M_N(\mathbb{C}))^G \tag{2.1}$$

where X is a smooth manifold and G a finite group. The action of G is a combination of an action on X and conjugation by certain matrices.

The comparisons between 1) - 4) all rely on triangulations and Mayer-Vietoris sequences. We will apply these techniques to X in a suitable way. This will enable us to define d) and prove that it gives the same results as b) and c). Moreover we prove that if X happens to be a complex affine variety, then there is a natural isomorphism

$$HP_*\big(\mathcal{O}(X; M_N(\mathbb{C}))^G\big) \xrightarrow{\sim} HP_*\big(C^\infty(X; M_N(\mathbb{C}))^G\big) \tag{2.2}$$

The $\mathcal{O}$ stands for algebraic functions, so the left hand side corresponds to a) and 1) above.

2.1 Algebraic cyclic theory

We give the definitions and most important properties of Hochschild homology, cyclic homology and periodic cyclic homology. We do this only in the category of algebras over $\mathbb{C}$, although quite a big deal of the theory is also valid over arbitrary fields. We will be rather concise, referring to Loday's monograph [81] for more background and proofs.

Cyclic homology was discovered more or less independently by several people, confer the work of Connes [32], Loday and Quillen [82] and Tsygan [129]. We will define it with the so-called cyclic bicomplex. So, let $n \in \mathbb{N}$ let A be any complex algebra and let $A^{\otimes n}$ be the n-fold tensor product of A over $\mathbb{C}$. We abbreviate an elementary tensor $a_1 \otimes \cdots \otimes a_n$ to $(a_1, \ldots, a_n)$ and we define linear operators $b, b' : A^{\otimes n+1} \to A^{\otimes n}$ and $\lambda, N : A^{\otimes n} \to A^{\otimes n}$ by the following formulas:

$$\begin{aligned} b'(a_0, a_1, \ldots, a_n) &= \sum_{i=0}^{n-1} (-1)^i (a_0, \ldots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \ldots, a_n) \\ b(a_0, a_1, \ldots, a_n) &= b'(a_0, a_1, \ldots, a_n) + (-1)^n (a_n a_0, a_1, \ldots, a_{n-1}) \\ \lambda(a_1, \ldots, a_n) &= (-1)^{n-1}(a_n, a_1, \ldots, a_{n-1}) \\ N &= 1 + \lambda + \cdots + \lambda^{n-1} \end{aligned} \tag{2.3}$$

For unital A we also define $s, B : A^{\otimes n} \to A^{\otimes n+1}$:

$$\begin{aligned} s(a_1, \ldots, a_n) &= (1, a_1, \ldots, a_n) \\ B &= (1 - \lambda) s N \end{aligned} \tag{2.4}$$

These are the ingredients of a bicomplex $CC^{per}(A)$:

$$\begin{array}{cccccccccccc} & & & \cdots & & \cdots & & \cdots & & \cdots & & \\ & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 2 & \cdots & \leftarrow & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-\lambda} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \leftarrow & \cdots \\ & & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \downarrow b & & \\ 1 & \cdots & \leftarrow & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-\lambda} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \leftarrow & \cdots \\ & & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \downarrow b & & \\ 0 & \cdots & \leftarrow & A & \xleftarrow{N} & A & \xleftarrow{1-\lambda} & A & \xleftarrow{N} & A & \leftarrow & \cdots \\ & & & & & & & & & & & \\ & & & -1 & & 0 & & 1 & & 2 & & \end{array} \tag{2.5}$$

The indicated grading means that $CC^{per}_{p,q}(A) = A^{\otimes q+1}$.

Consider also the subcomplexes $CC(A)$, consisting of all the columns labelled by $p \geq 0$, and $CC^{\{2\}}(A)$, which consists only of the columns numbered 0 and 1.

With these bicomplexes we associate differential complexes with a single grading. Their spaces in degree n are

$$\begin{aligned} CC_n^{\{2\}}(A) &= CC_{0,n}^{per}(A) \oplus CC_{1,n-1}^{per}(A) &&= A^{\otimes n+1} \oplus A^{\otimes n} \\ CC_n(A) &= \bigoplus_{p=0}^{n} CC_{p,n-p}^{per}(A) &&= A^{\otimes n+1} \oplus A^{\otimes n} \oplus \cdots \oplus A \\ CC_n^{per}(A) &= \prod_{p+q=n} CC_{p,q}^{per}(A) &&= \prod_{q\geq 0} A^{\otimes q+1} \end{aligned} \tag{2.6}$$

This enables us to define the Hochschild homology $HH_n(A)$, the cyclic homology $HC_n(A)$ and the periodic cyclic homology $HP_n(A)$:

$$\begin{aligned} HH_n(A) &= H_n(CC_*^{\{2\}}(A)) \\ HC_n(A) &= H_n(CC_*(A)) \\ HP_n(A) &= H_n(CC_*^{per}(A)) \end{aligned} \tag{2.7}$$

Since all the above complexes are functorial in A these homology theories are indeed covariant functors. The definitions we gave are neither the simplest possible ones, nor the best for explicit computations, but they do have the advantage that they work for every algebra, unital or not.

By the way, we can always form the unitization A^+. This is the vector space $\mathbb{C} \oplus A$ with multiplication

$$(z_1, a_1)(z_2, a_2) = (z_1 z_2, z_1 a_2 + z_2 a_1 + a_1 a_2) \tag{2.8}$$

Clearly every algebra morphism $\phi : A \to B$ gives a unital algebra morphism $\phi^+ : A^+ \to B^+$. There are natural isomorphisms

$$HH_n(A) \cong \operatorname{coker}\big(HH_n(\mathbb{C}) \to HH_n(A^+)\big) \cong \ker\big(HH_n(A^+) \to HH_n(\mathbb{C})\big) \tag{2.9}$$

and similarly for HC_n and HP_n.

Often we shall want to consider all degrees at the same time, and for this purpose we write

$$\begin{aligned} HH_*(A) &= \bigoplus_{n\geq 0} HH_n(A) \\ HC_*(A) &= \bigoplus_{n\geq 0} HC_n(A) \end{aligned}$$

The map $S : CC_{p,q}^{per}(A) \to CC_{p-2,q}^{per}(A)$ simply shifting everything two columns to the left is clearly an automorphism of $CC^{per}(A)$. Moreover it decreases the degree by two, so it induces a natural isomorphism

$$HP_n(A) \xrightarrow{\sim} HP_{n-2}(A) \tag{2.10}$$

Thus we may consider periodic cyclic homology as a $\mathbb{Z}/2\mathbb{Z}$-graded functor, or we may restrict n to $\{0,1\}$. In particular we shall write

$$HP_*(A) = HP_0(A) \oplus HP_1(A)$$

Regarding $CC(A)$ as a quotient of $CC^{per}(A)$, we get an induced map $\bar{S} : CC(A) \to CC(A)$. This map is surjective, and its kernel is exactly $CC^{\{2\}}(A)$. This leads to Connes' periodicity exact sequence :

$$\cdots \to HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \to \cdots \tag{2.11}$$

Here I comes from the inclusion of $CC^{\{2\}}(A)$ in $CC(A)$ and B is induced by the map from (2.4). In combination with the five lemma this is a very useful tool; it enables one to prove easily that many functorial properties of Hochschild homology also hold for cyclic homology.

Furthermore we notice that the bicomplex $CC^{per}(A)$ is the inverse limit of its subcomplexes $S^r(CC(A))$. In many cases this gives an isomorphism between $HP_n(A)$ and $\varprojlim HC_{n+2r}(A)$. In general however it only leads to a short exact sequence

$$0 \to \lim_{\infty \leftarrow r}{}^1 HC_{n+1+2r}(A) \to HP_n(A) \to \lim_{\infty \leftarrow r} HC_{n+2r}(A) \to 0 \tag{2.12}$$

Here $\varprojlim^1$ is the first derived functor of $\varprojlim$, see [81, Propostion 5.1.9].

Next we state some well-known features of the functors under consideration.

1. Additivity. If A_m $(m \in \mathbb{N})$ are algebras then

$$HH_n\left(\bigoplus_{m=1}^{\infty} A_m\right) \cong \bigoplus_{m=1}^{\infty} HH_n(A_m)$$
$$HH_n\left(\prod_{m=1}^{\infty} A_m\right) \cong \prod_{m=1}^{\infty} HH_n(A_m)$$

   and similarly for HC_n and HP_n.

2. Stability.

$$HH_n(M_m(A)) \cong HH_n(A)$$

   More generally, if B and C are unital and Morita-equivalent, then

$$HH_n(B) \cong HH_n(C)$$

   These statements hold also for HC_n and HP_n.

3. Continuity. If $A = \lim_{m \to \infty} A_m$ is an inductive limit then

$$HH_n(A) \cong \lim_{m\to\infty} HH_n(A_m)$$
$$HC_n(A) \cong \lim_{m\to\infty} HC_n(A_m)$$

However, HP_* is not continuous in general. A sufficient condition for continuity can be found in [16, Theorem 3] : there exists a $N \in \mathbb{N}$ such that $HH_n(A_m) = 0 \ \forall n > N \, \forall m$.

To an extension of algebras

$$0 \to A \to B \to C \to 0$$

we would like to associate long exact sequences of homology groups. This is no problem for unital algebras, but in general it is not always possible. However if B is nonunital then so is C, and we can take their unitizations instead, see (2.8). The sequence

$$0 \to A \to B^+ \to C^+ \to 0$$

is still exact, so by (2.9) we may, without loss of generality, assume that B and C are unital. It was discovered by Wodzicki that what we need for A is not so much unitality, but a weaker notion called homological unitality, or H-unitality for short. It is easily seen that for unital algebras the map s defines a contracting homotopy for the complex $(A^{\otimes n}, b')$, and in fact with some slight modifications this construction also applies to algebras that have left or right local units. Thus, we call a complex algebra A H-unital if the homology of the complex $(A^{\otimes n}, b')$ is 0. Now Wodzicki's excision theorem [136] says

Theorem 2.1 *Let $0 \to A \to B \to C \to 0$ be an extension of algebras, with A H-unital. There exist long exact sequences*

$$\begin{array}{ccccccccccc} \cdots & \to & HH_n(A) & \to & HH_n(B) & \to & HH_n(C) & \to & HH_{n-1}(A) & \to & \cdots \\ \cdots & \to & HC_n(A) & \to & HC_n(B) & \to & HC_n(C) & \to & HC_{n-1}(A) & \to & \cdots \\ \cdots & \to & HP_n(A) & \to & HP_n(B) & \to & HP_n(C) & \to & HP_{n-1}(A) & \to & \cdots \end{array}$$

It turns out [36] that for HP_* it is not necessary to require H-unitality. Due to the 2-periodicity of this functor we get, for any extension of algebras, an exact hexagon

$$\begin{array}{ccccc} HP_0(A) & \to & HP_0(B) & \to & HP_0(C) \\ \uparrow & & & & \downarrow \\ HP_1(C) & \leftarrow & HP_1(B) & \leftarrow & HP_1(A) \end{array} \tag{2.13}$$

It will be very useful to combine the excision property with the five lemma:

Lemma 2.2 *Suppose we have a commutative diagram of abelian groups, with exact rows:*

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \downarrow f_4 & & \downarrow f_5 \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

If f_1 is surjective, f_2 and f_4 are isomorphisms and f_5 is injective, then f_3 is an isomorphism.

Because we intend to apply the next result to several different functors, we formulate it very abstractly.

Lemma 2.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be categories of algebras, and $\mathcal{AG}$ the category of abelian groups. Suppose that $F_* : \mathcal{A} \to \mathcal{AG}$ and $G_* : \mathcal{B} \to \mathcal{AG}$ are $\mathbb{Z}$-graded, covariant functors satisfying excision, and that $T_* : F_* \to G_*$ is a natural transformation of such functors. Consider two sequences of ideals*

$$\begin{array}{ccccccccccc} 0 & = & I_0 & \subset & I_1 & \subset \cdots \subset & I_n & \subset & I_{n+1} & = & A \\ 0 & = & J_0 & \subset & J_1 & \subset \cdots \subset & J_n & \subset & J_{n+1} & = & B \end{array} \tag{2.14}$$

in $\mathcal{A}$ and $\mathcal{B}$ respectively. If we have an algebra homomorphism $\phi : A \to B$ such that $\phi(I_m) \subset J_m$ and

$$T(J_m/J_{m+1})F(\phi) = G(\phi)T(I_m/I_{m+1}) : F(I_m/I_{m+1}) \to G(J_m/J_{m+1})$$

is an isomorphism $\forall m \leq n$, then

$$T(\phi) := T(B)F(\phi) = G(\phi)T(A) : F(A) \to G(B)$$

is an isomorphism. Similarly, consider two exact sequences

$$\begin{array}{ccccccccccc} 0 & \to & A_1 & \to & A_2 & \to \cdots \to & A_n & \to & 0 \\ 0 & \to & B_1 & \to & B_2 & \to \cdots \to & B_n & \to & 0 \end{array} \tag{2.15}$$

in $\mathcal{A}$ and $\mathcal{B}$. Suppose that we have a morphism of exact sequences $\psi = (\psi_m)_{m=1}^n$, such that

$$T(\psi_m) : F(A_m) \to G(B_m)$$

is an isomorphism for all but one m. Then it is an isomorphism for all m.

Proof. Consider the short exact sequences

$$\begin{array}{ccccccccc} 0 & \to & I_{m-1} & \to & I_m & \to & I_m/I_{m-1} & \to & 0 \\ 0 & \to & J_{m-1} & \to & J_m & \to & J_m/J_{m-1} & \to & 0 \\ 0 & \to & \mathrm{im}\,(A_{m-1} \to A_m) & \to & A_m & \to & \mathrm{im}\,(A_m \to A_{m+1}) & \to & 0 \\ 0 & \to & \mathrm{im}\,(B_{m-1} \to B_m) & \to & B_m & \to & \mathrm{im}\,(B_m \to B_{m+1}) & \to & 0 \end{array}$$

They degenerate for $m = 1$, so with induction we reduce the entire lemma to the statement for exact sequences, with $m = 3$. Now we consider only the case where $T(\psi_m)$ is an isomorphism for $m = 1$ and $m = 3$, since the other cases are very similar. For any $k \in \mathbb{Z}$ we see from the commutative diagram

$$\begin{array}{ccccccccc} F_{k+1}(A_3) & \to & F_k(A_1) & \to & F_k(A_2) & \to & F_k(A_3) & \to & F_{k-1}(A_1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G_{k+1}(B_3) & \to & G_k(B_1) & \to & G_k(B_2) & \to & G_k(B_3) & \to & G_{k-1}(B_1) \end{array}$$

and Lemma 2.2 that $T_k(\psi_2)$ is an isomorphism. □

Having elaborated a little on the functorial properties of HH_*, HC_* and HP_*, we will show now what they look like on some nice algebras. First we fix the notations of some well-known objects from algebraic geometry.

Assume for the rest of this section that A is a commutative, unital complex algebra. The A-module of Kähler differentials $\Omega^1(A)$ is generated by the symbols da, subject to the following relations, for any $a, b \in A, z \in \mathbb{C}$:

$$\begin{aligned} d(za) &= z\, da \\ d(a+b) &= da + db \\ d(ab) &= a\, db + b\, da \end{aligned} \tag{2.16}$$

The A-module of differential n-forms is the n-fold exterior product over A :

$$\Omega^n(A) = \bigwedge\nolimits_A^n \Omega^1(A) \tag{2.17}$$

and, just to be sure, we decree that $\Omega^0(A) = A$. The formal operator d defines a differential $\Omega^n(A) \to \Omega^{n+1}(A)$ by

$$d(a_0 da_1 \wedge \cdots \wedge da_n) = da_0 \wedge da_1 \wedge \cdots \wedge da_n \tag{2.18}$$

The De Rham homology of A is defined as

$$H_n^{DR}(A) = H_n(\Omega^*(A), d) \tag{2.19}$$

If $A = \mathcal{O}(V)$ is the ring of regular functions on an affine complex algebraic variety V, not necessarily irreducible, then we also write

$$\Omega^n(V) = \Omega^n(A) \quad \text{and} \quad H^n_{DR}(V) = H_n^{DR}(A)$$

One can check that the following formulas define natural maps:

$$\begin{array}{llll} HH_n(A) \to & \Omega^n(A) & : & (a_0, a_1, \ldots, a_n) \quad \to a_0 da_1 \wedge \cdots \wedge da_n \\ \Omega^n(A) & \to HH_n(A): & & a_0 da_1 \wedge \cdots \wedge da_n \to \sum\limits_{\sigma \in S_n} \epsilon(\sigma)(a_0, a_{\sigma(1)}, \ldots, a_{\sigma(n)}) \end{array} \tag{2.20}$$

The celebrated Hochschild-Kostant-Rosenberg theorem [60] says that these maps are isomorphisms if A is a smooth algebra. Yet the author believes that a precise definition of smoothness would digress too much, so we only mention that a typical example is $\mathcal{O}(V)$ with V nonsingular, and that all the details can be found in [81, Appendix E]. Anyway, under (2.20) the differential d corresponds to the map B from (2.4) and therefore the Hochschild-Kostant-Rosenberg theorem also gives the (periodic) cyclic homology of smooth algebras:

$$HH_n(A) \cong \Omega^n(A) \tag{2.21}$$

$$HC_n(A) \cong \Omega^n(A)/d\Omega^{n-1}(A) \oplus H_{n-2}^{DR}(A) \oplus H_{n-4}^{DR}(A) \oplus \cdots \tag{2.22}$$

$$HP_n(A) \cong \prod_{m \in \mathbb{Z}} H_{n+2m}^{DR}(A) \tag{2.23}$$

2.2 Periodic cyclic homology of finite type algebras

The theory of finite type algebras was built by Baum, Kazhdan, Nistor and Schneider [8, 77]. This turns to be a pleasant playground for cyclic theory, culminating roughly speaking in the statement "the periodic cyclic homology of finite type algebra is an invariant of its spectrum." We discuss this result, and some of its background. We also add one new ingredient to support this point of view, namely a sheaf, depending only on the spectrum of A, whose Čech cohomology is isomorphic to $HP_*(A)$.

All this is made possible by several extra features that HP_* possesses, compared to HH_* and HC_*. Recall that an extension of algebras

$$0 \to I \to A \to A/I \to 0 \tag{2.24}$$

is called nilpotent if the ideal I is nilpotent, i.e. if $I^n = 0$ for some $n \in \mathbb{N}$.

An algebraic homotopy between two algebra homomorphisms $f, g : A \to B$ is a collection $\phi_t : A \to B$ of morphisms, depending polynomially on t, such that $\phi_0 = f$ and $\phi_1 = g$. This is equivalent to the existence of a morphism $\phi : A \to B \otimes \mathbb{C}[t]$ such that $f = \mathrm{ev}_0 \circ \phi$ and $g = \mathrm{ev}_1 \circ \phi$.

Goodwillie [49, Corollary II.4.4 and Theorem II.5.1] established two closely related features:

Theorem 2.4 *The functor HP_* is homotopy invariant and turns nilpotent extensions into isomorphisms. Thus, with the above notation,*

$$HP_*(f) = HP_*(g)$$
$$HP_*(I) = 0$$

and $HP_*(A) \xrightarrow{\sim} HP_*(A/I)$ is an isomorphism.

Homotopy invariance can be regarded as a special case of the Künneth theorem, which holds for periodic cyclic homology under some mild conditions.

Theorem 2.5 *Suppose that A is a unital algebra such that*

- *the $\varprojlim^1$-term in (2.12) vanishes, i.e. $HP_n(A) \cong \lim_{\infty \leftarrow r} HC_{n+2r}(A)$*
- *$HP_*(A)$ has finite dimension*

Then the Künneth theorem holds for $HP_*(A)$. This means that for any unital algebra B there is a natural isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces

$$HP_*(A) \otimes HP_*(B) \xrightarrow{\sim} HP_*(A \otimes B)$$

Proof. See [70, Theorem 3.10] and [44, Theorem 4.2]. □

Reconsider the Hochschild-Kostant-Rosenberg theorem (2.23) for the periodic cyclic homology of the ring of regular functions on a nonsingular affine complex variety. It gives an isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces

$$HP_*(\mathcal{O}(V)) \cong H^*_{DR}(V) \tag{2.25}$$

Now let V^{an} be the set V endowed with its natural analytic topology. By a famous theorem of Grothendieck and Deligne (cf. [52] and [57, Theorem IV.1.1]) the algebraic De Rham cohomology of V is naturally isomorphic to the analytic De Rham cohomology of V^{an}:

$$H^*_{DR}(V) \cong H^*_{DR}(V^{an};\mathbb{C}) \tag{2.26}$$

As is well-known, all classical cohomology theories agree on the category of smooth manifolds, for instance

$$H^*_{DR}(V^{an};\mathbb{C}) \cong \check{H}^*(V^{an};\mathbb{C}) \tag{2.27}$$

the latter denoting Čech cohomology with coefficients in $\mathbb{C}$. Because of the similar functorial properties, it is not surprising that the composite isomorphism of (2.25) - (2.27) holds in greater generality. This was confirmed in [77, Theorem 9] :

Theorem 2.6 *Let X be an affine complex variety, $I \subset \mathcal{O}(X)$ an ideal and $Y \subset X$ the subvariety defined by I. It is neither assumed that X is nonsingular or irreducible, nor that I is prime. There is a natural isomorphism*

$$HP_n(I) \cong \check{H}^{[n]}(X^{an},Y^{an};\mathbb{C}) := \prod_{m\in\mathbb{Z}} \check{H}^{n+2m}(X^{an},Y^{an};\mathbb{C})$$

Recall that a primitive ideal in a complex algebra is the kernel of a (nonzero) irreducible representation of A. The primitive ideal spectrum $\mathrm{Prim}(A)$ is the set of all primitive ideals of A, and the Jacobson radical $\mathrm{Jac}(A)$ is the intersection of all these primitive ideals. Note that every nilpotent ideal is contained in $\mathrm{Jac}(A)$. We endow $\mathrm{Prim}(A)$ with the Jacobson topology, which means that all closed subsets are of the form

$$\overline{S} := \{I \in \mathrm{Prim}(A) : I \supset S\} \tag{2.28}$$

for some subset S of A. Denote by d_I the dimension of an irreducible representation with kernel $I \in \mathrm{Prim}(A)$. If $d_I < \infty\, \forall I$ then $\mathrm{Prim}(A)$ is a T_1-space, but in general it is only a T_0-space.

For commutative A the primitive ideals are precisely the maximal ideals, and $\mathrm{Prim}(A)$ is an algebraic variety. In this case there also is a natural topology on the set $\mathrm{Prim}(A)$ that makes it into an analytic variety, see [116, Section 5].

If $\phi : A \to B$ is an algebra homomorphism and $J \in \mathrm{Prim}(B)$ then $\phi^{-1}(J)$ is an ideal, but it is not necessarily primitive. So Prim is not a functor, it only induces a map $J \to \overline{\phi^{-1}(J)}$ from Prim(B) to the power set of Prim(A). However, if for every $J \in \text{Prim}(B)$ there exists exactly one $I \in \text{Prim}(A)$ containing $\phi^{-1}(J)$, then ϕ does induce a continuous map $\text{Prim}(B) \to \text{Prim}(A)$ and we call ϕ spectrum preserving.

Now we give the definition of a finite type algebra. Let $\mathbf{k}$ be a finitely generated commutative unital complex algebra, i.e. the ring of regular functions on some affine complex variety. A $\mathbf{k}$-algebra is a (nonunital) algebra A together with a unital morphism from $\mathbf{k}$ to $Z(\mathcal{M}(A))$, the center of the multiplier algebra of A. An algebra B is of finite type if there exists a $\mathbf{k}$ such that B is $\mathbf{k}$-algebra which is finitely generated as a $\mathbf{k}$-module. An algebra morphism $\phi : A \to B$ is a morphism of finite type algebras if it is $\mathbf{k}$-linear, for some $\mathbf{k}$ over which both A and B are of finite type.

As announced, the most important theorem in this category says that HP_* is determined by Prim, see [8, Theorems 3 and 8].

Theorem 2.7 *Let $\phi : A \to B$ be a spectrum preserving morphism of finite type algebras. Then* $\text{Prim}(B) \to \text{Prim}(A)$ *is a homeomorphism and*

$$HP_*(\phi) : HP_*(A) \to HP_*(B)$$

is an isomorphism.

More generally, we might have ideals like in (2.15), such that the induced maps $I_m/I_{m+1} \to J_m/J_{m+1}$ are all spectrum preserving, but $\phi : A \to B$ is not. In that case ϕ is called weakly spectrum preserving. By Theorem 2.7 and Lemma 2.3 such maps also induce isomorphisms on periodic cyclic homology.

To understand this better we zoom in on the spectrum, relying heavily on [77, Section 1]. Until further notice we assume that A is a unital finite type algebra. The central character map

$$\Theta : \text{Prim}(A) \to \text{Prim}(Z(A)) : I \to I \cap Z(A) \tag{2.29}$$

is a finite-to-one continuous surjection. For $k, p \in \mathbb{N}$ we write

$$\begin{array}{lcl} \text{Prim}_k(A) & = & \{I \in \text{Prim}(A) : d_I = k\} \\ \text{Prim}_{\leq p}(A) & = & \bigcup_{k=1}^{p} \text{Prim}_k(A) \end{array} \tag{2.30}$$

The sets $\text{Prim}_{\leq p}(A)$ are all closed and, as the frequent occurence of the word "finite" already suggests, there exists a $N_A \in \mathbb{N}$ such that $\text{Prim}_{\leq N_A}(A) = \text{Prim}(A)$. This leads to the so-called standard filtration of A :

$$\begin{array}{l} A = I_0^{st} \supset I_1^{st} \supset \cdots \supset I_{N_A-1}^{st} \supset I_{N_A}^{st} = \text{Jac}(A) \\ I_p^{st} \;=\; \bigcap_{d_I \leq p} I = \{a \in A : \pi(a) = 0 \text{ if } \pi \text{ is a representation with } \dim \pi \leq p\} \end{array} \tag{2.31}$$

Observe that

$$\text{Prim}\left(I_q^{st}/I_p^{st}\right) = \bigcup_{k=q+1}^{p} \text{Prim}_k(A) \tag{2.32}$$

From (2.31) we also get a filtration of the cyclic bicomplex :

$$\begin{aligned} & CC_*^{per}(A) = CC_*^{per}(A)_0 \supset CC_*^{per}(A)_1 \supset \cdots \supset CC_*^{per}(A)_{N_A-1} \supset CC_*^{per}(A)_{N_A} \\ & CC_*^{per}(A)_p = \ker\left(CC_*^{per}(A) \to CC_*^{per}\left(A/I_p^{st}\right)\right) \end{aligned} \tag{2.33}$$

Using standard (but involved) techniques from homological algebra we construct a spectral sequence $E_r^{p,q}$ with

$$E_0^{p,q} = CC_*^{per}(A)_{p-1}/CC_*^{per}(A)_p \cong \ker\left(CC_{-p-q}^{per}\left(A/I_p^{st}\right) \to CC_{-p-q}^{per}\left(A/I_{p-1}^{st}\right)\right) \tag{2.34}$$

$$E_1^{p,q} = HP_{-p-q}\left(I_{p-1}^{st}/I_p^{st}\right) \tag{2.35}$$

$$E_\infty^{p,q} = HP_{-p-q}\left(I_{p-1}^{st}\right)/HP_{-p-q}\left(I_p^{st}\right) \tag{2.36}$$

Moreover $d_0^E : E_0^{p,q} \to E_1^{p,q+1}$ comes directly from the differential in the cyclic bicomplex and $d_1^E : E_1^{p,q} \to E_1^{p+1,q}$ is the composition

$$HP_{-p-q}\left(I_{p-1}^{st}/I_p^{st}\right) \to HP_{-p-q}\left(A/I_p^{st}\right) \to HP_{-p-q-1}\left(I_p^{st}/I_{p+1}^{st}\right)$$

of the map induced by the inclusion $I_{p-1}^{st}/I_p^{st} \to A/I_p^{st}$ and the connecting map of the extension

$$0 \to I_p^{st}/I_{p+1}^{st} \to A/I_{p+1}^{st} \to A/I_p^{st} \to 0$$

A most pleasant property of the standard filtration (2.31) is that the quotients I_{p-1}^{st}/I_p^{st} behave like commutative algebras. More precisely, consider the analytic space X_p associated to $\mathrm{Prim}\big(Z\left(A/I_p^{st}\right)\big)$, and its subvariety

$$Y_p = \left\{I \in X_p : Z\left(A/I_p^{st}\right) \cap I_{p-1}/I_p \subset I\right\} \tag{2.37}$$

The central character map for A/I_p^{st} defines a bijection

$$\mathrm{Prim}_p(A) = \mathrm{Prim}\left(I_{p-1}^{st}/I_p^{st}\right) \to X_p \setminus Y_p \tag{2.38}$$

and according to [77, Theorem 1] there is a natural isomorphism

$$E_1^{p,q} = HP_{-p-q}\left(I_{p-1}^{st}/I_p^{st}\right) \cong \check{H}^{[p+q]}(X_p, Y_p; \mathbb{C}) \tag{2.39}$$

Comparing this with Theorem 2.6 we see that I_{p-1}^{st}/I_p^{st} is indeed "close to commutative" in the sense that its periodic cyclic homology can be computed as the Čech cohomology of a constant sheaf over its spectrum.

We seek to generalize this to "less commutative", nonunital finite type algebras. Let X be the set $\mathrm{Prim}(\mathbf{k})$ with the analytic topology, and $V(A)$ the set $\mathrm{Prim}(A)$ with the weakest topology that makes $\Theta : \mathrm{Prim}(A) \to X$ continuous and is stronger than the Jacobson topology. This topology depends only on the fact that A is a finite type algebra, and not on the particular choice of $\mathbf{k}$. So if A is unital we may just as well assume that $\mathbf{k} = Z(A)$ and $X = X_0$.

We will construct a sheaf $\mathfrak{A}$ over X whose stalk at x is the (finite dimensional) complex vector space with basis $\Theta^{-1}(x)$. By definition all continuous sections of this collection of stalks are constructed from local sections of $\Theta : V(A) \to X$. More precisely, given an open $Y \subset X$ we call a section s of $\prod_{x\in Y} \mathfrak{A}(x) \to Y$ continuous at $y \in Y$ if there exist

- a neighborhood U of y in Y
- connected components $C_1, \ldots, C_n$ of $\Theta^{-1}(U)$, not necessarily different
- for every i a section s_i of the quotient map from C_i to its Hausdorffization C_i^H
- complex numbers $z_1, \ldots, z_n$

such that $\forall x \in U$

$$s(x) = \sum_{i=1}^{n} z_i \left(s_i(C_i^H) \cap \Theta^{-1}(x) \right) \tag{2.40}$$

For example if X' is a closed subvariety of X and $A = \{f \in \mathcal{O}(X) : f(X') = 0\}$ then $\mathfrak{A}$ is the direct image of the constant sheaf (with stalk $\mathbb{C}$) on $X \setminus X'$.

Notice that $\mathfrak{A}$ is functorial in A. If $\phi : A \to B$ is a morphism of finite type $\mathbf{k}$-algebras and V is a left A-module, then $B \otimes_A V$ is a B-module. If we consider only the semisimple forms of these modules, then we get a homomorphism

$$\mathbb{Z}[\mathrm{Prim}(A)] \to \mathbb{Z}[\mathrm{Prim}(B)]$$

which extends naturally to a morphism $\mathfrak{A} \to \mathfrak{B}$ of sheaves over X.

The motivation for this sheaf comes from topological K-theory: the local sections s_i are supposed to model "local" idempotents in A. The classes of these things should generate $HP_*(A)$, leading to

Theorem 2.8 *There is an unnatural isomorphism of finite dimensional vector spaces*

$$HP_*(A) \cong \check{H}^*(X; \mathfrak{A})$$

Proof. Assume first that A is unital. Let $\mathfrak{A}_p$ be the sheaf (over X) constructed from A/I_p^{st} in the same way as we constructed $\mathfrak{A}$ from A; it has stalks

$$\mathfrak{A}_p(x) = \mathbb{C}\{\Theta^{-1}(x) \cap \mathrm{Prim}_{\leq p}(A)\} \tag{2.41}$$

Since $\mathrm{Prim}_{\leq p}(A)$ is closed in $\mathrm{Prim}(A)$ there is a natural surjection $\mathfrak{A} \to \mathfrak{A}_p$, which comes down to forgetting all primitive ideals I with $d_I > p$. Thus we get filtrations of the (pre)sheaf $\mathfrak{A}$:

$$\begin{aligned} &\mathfrak{A} = \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \cdots \supset \mathfrak{I}_{N_A - 1} \supset \mathfrak{I}_{N_A} = 0 \\ &\mathfrak{I}_p = \ker(\mathfrak{A} \to \mathfrak{A}_p) \end{aligned} \tag{2.42}$$

and of the Čech complex $\check{C}^*(X;\mathfrak{A})$ (this is a pretty complicated object, see [47, §5.8]) :

$$\begin{aligned}
&\check{C}^*(X;\mathfrak{A}) = \check{C}^*(X;\mathfrak{A})_0 \supset \check{C}^*(X;\mathfrak{A})_1 \supset \cdots \supset \check{C}^*(X;\mathfrak{A})_{N_A-1} \supset \check{C}^*(X;\mathfrak{A})_{N_A} = 0 \\
&\check{C}^*(X;\mathfrak{A})_p = \ker\left(\check{C}^*(X;\mathfrak{A}) \to \check{C}^*(X;\mathfrak{A}_p)\right) \cong \check{C}^*(X;\mathfrak{I}_p)
\end{aligned} \tag{2.43}$$

The presheaf $\mathfrak{B}_p := \ker(\mathfrak{A}_p \to \mathfrak{A}_{p-1})$ is actually a sheaf, and it has stalks

$$\mathfrak{B}_p(x) = \mathbb{C}\{\Theta^{-1}(x) \cap \mathrm{Prim}_p(A)\} \tag{2.44}$$

From these data we construct a spectral sequence $F_r^{p,q}$ with terms

$$F_0^{p,q} = \check{C}^{p+q}(X;\mathfrak{A})_{p-1}/\check{C}^{p+q}(X;\mathfrak{A})_p \cong \check{C}^{p+q}(X;\mathfrak{B}_p) \tag{2.45}$$

$$F_1^{p,q} = \check{H}_{p+q}(X;\mathfrak{B}_p) \tag{2.46}$$

$$F_\infty^{p,q} = \check{H}_{p+q}(X;\mathfrak{I}_{p-1})/\check{H}_{p+q}(X;\mathfrak{I}_p) \tag{2.47}$$

In this sequence $d_0^F : F_0^{p,q} \to F_0^{p,q+1}$ is the normal Čech differential, while $d_1^F : F_1^{p,q} \to F_1^{p+1,q}$ is induced by the inclusion $\mathfrak{B}_p \to \mathfrak{A}_p$ and the connecting map associated to the short exact sequence

$$0 \to \mathfrak{B}_{p+1} \to \mathfrak{A}_{p+1} \to \mathfrak{A}_p \to 0$$

From (2.38) and the local nature of the continuity condition for $\mathfrak{B}_p$ we see that there are natural isomorphisms

$$\begin{aligned}
\check{C}^{p+q}(X;\mathfrak{B}_p) &\cong \check{C}^{p+q}(X_p, Y_p;\mathbb{C}) \\
\check{H}^{p+q}(X;\mathfrak{B}_p) &\cong \check{H}^{p+q}(X_p, Y_p;\mathbb{C})
\end{aligned}$$

Clearly, all this was set up to compare the spectral sequences $E_r^{p,q}$ and $F_r^{p,q}$. On the first level we have a diagram

$$\begin{array}{ccc}
E_1^{p,q} & \xrightarrow{d_1^E} & E_1^{p+1,q} \\
\cong & & \cong \\
\check{H}^{[p+q]}(X_p, Y_p;\mathbb{C}) & & \check{H}^{[p+q+1]}(X_{p+1}, Y_{p+1};\mathbb{C}) \\
\cong & & \cong \\
\prod_{n\in\mathbb{Z}} F_1^{p,q+2n} & \xrightarrow{d_1^F} & \prod_{n\in\mathbb{Z}} F_1^{p+1,q+2n}
\end{array} \tag{2.48}$$

Since d_1^E (d_1^F) is natural with respect to filtration-preserving morphisms of **k**-algebras (of presheaves over X), these differentials must commute with the natural isomorphisms in the diagram (2.48). This yields natural isomorphisms

$$E_r^{p,q} \cong \prod_{n\in\mathbb{Z}} F_r^{p,q+2n}$$

for all $r \geq 1$. For $r = \infty$ we see that there exist filtrations of finite length on $HP_*(A)$ and $\check{H}^*(X;\mathfrak{A})$, such that the associated graded objects are isomorphic. Hence $HP_*(A)$ and $\check{H}^*(X;\mathfrak{A})$, being vector spaces, are unnaturally isomorphic.

Moreover they have finite dimension since every term $\check{H}^{[p+q]}(X_p, Y_p; \mathbb{C})$, being the cohomology of the affine algebraic variety $X_p \setminus Y_p$, has finite dimension by [57, Theorems 4.6 and 6.1].

This proves the theorem for unital finite type algebras, so let us now assume that J is an nonunital finite type **k**-algebra. By stability $HP_*(M_2(J)) \cong HP_*(J)$ and the sheaves corresponding to $M_2(J)$ and J are isomorphic, so we may assume that J has no one-dimensional representations. Consider now the unital finite type algebra $A = \mathbf{k} + J$, with multiplication

$$(f_1, b_1)(f_2, b_2) = (f_1 f_2, f_1 b_2 + f_2 b_1 + b_1 b_2) \tag{2.49}$$

Its standard filtration is

$$A = I_0^{st} \supset J = I_1^{st} \supset \cdots \supset I_{n_A-1}^{st} \supset I_{n_A}^{st} = \mathrm{Jac}(A) = \mathrm{Jac}(J) \tag{2.50}$$

The above considerations show that, as vector spaces,

$$HP_{-m}(J) = HP_{-m}(I_1^{st}) \cong \prod_{p=2}^{m} E_\infty^{p,m-p} \cong \prod_{p=2}^{m} \prod_{n \in \mathbb{Z}} F_\infty^{p,2n+m-p} \cong \check{H}^{[m]}(X; \mathfrak{I}_1) \tag{2.51}$$

It only remains to see that $\mathfrak{I}_1$ is isomorphic to the sheaf constructed from J, but this is clear from looking at the stalks. □

So we managed to describe the periodic cyclic homology of a finite type **k**-algebra using only the following data:

- the spectrum $\mathrm{Prim}(A)$ with a natural topology that makes it a non-Hausdorff manifold
- the complex analytic variety X
- the continuous map $\Theta : \mathrm{Prim}(A) \to X$

For some time the author believed that this construction on page 25 could be extended to a cohomology theory on the category of non-Hausdorff manifolds, but now it seems to him that it only gives good results under rather restrictive conditions. Apparently we need the following implication of (2.38) : there exists a stratification of $\mathrm{Prim}(A)$ such that at every level the set of non-Hausdorff points in a component is either the whole component, or a submanifold of lower dimension.

2.3 Topological cyclic theory

We would like to discuss the topological counterpart of the algebraic cyclic theory of Section 2.1. To prepare for this, and to fix certain notations, we start by recalling some general results for m-algebras and topological tensor products. When studying the literature, it quickly becomes clear that this topological setting is significantly more tricky than the purely algebraic setting, for several reasons. Firstly, the category of topological vector spaces is not abelian, i.e. not every closed subspace has a closed complement. Secondly, the tensor product of two topological vector spaces is not unique, and the functor "$\otimes_t A$" (for some unambiguous choice of a topological tensor product) is in general not exact. And finally, although the appropriate results are all known to experts, there does not appear to be an overview available.

A topological algebra A (over $\mathbb{C}$) is an algebra with a topology such that addition and scalar multiplication are jointly continuous, while multiplication is separately continuous. When we talk about the spectrum of A, we usually mean the set $\mathrm{Prim}(A)$ of all closed primitive ideals of A. The closed subsets of $\mathrm{Prim}(A)$ are as in (2.28).

A seminorm on A is a map $p : A \to [0, \infty)$ with the properties

- $p(\lambda a) = |\lambda| p(a)$
- $p(a + b) \leq p(a) + p(b)$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$. Moreover p is called submultiplicative if

- $p(ab) \leq p(a)p(b)$

We say that p' dominates p if $p'(a) \geq p(a)$ $\forall a \in A$. If $\{p_i\}_{i\in I}$ is a collection of seminorms, then there is a coarsest topology on A making all the p_i continuous. The sets

$$\{a \in A : p_i(a - b) < 1/n\} \qquad b \in A, n \in \mathbb{N}, i \in I$$

form a subbasis for this topology. If it agrees with the original topology, then we call A a locally convex algebra and say that it has the topology defined by the family of seminorms $\{p_i\}_{i\in I}$. Notice that the p_i may have nontrivial nullspaces N_i and that A is Hausdorff if and only if $\cap_{i\in I} N_i = \{0\}$. Furthermore the multiplication in A is jointly continuous if, but not only if, all the p_i are submultiplicative.

Two families of seminorms are equivalent if every member of either family is dominated by a finite linear combination of seminorms from the other family. Two families of seminorms define the same topologies if and only if they are equivalent.

A locally convex algebra is metrizable if and only if its topology can be defined by a countable family of seminorms $\{p_i\}_{i=1}^{\infty}$ with $\cap_{i=1}^{\infty} N_i = \{0\}$. In that case a metric is given by

$$d(a, b) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(a - b)}{1 + p_i(a - b)} \tag{2.52}$$

Clearly this implies a notion of completeness for such algebras, and it can be generalized to all locally convex algebras by means of Cauchy filters on uniform spaces. For sequences this comes down to calling a sequence $\{a_n\}_{n=1}^{\infty}$ in A Cauchy if and only if for every $i \in I$ the sequence $\{x_n + N_i\}_{n=1}^{\infty}$ is Cauchy in the normed space A/N_i.

Combining all these notions, an m-algebra is a complete Hausdorff locally convex algebra A whose topology can be defined by a family of submultiplicative seminorms. We call A Fréchet if it is metrizable on top of that. If B is a topological algebra such that $GL_1(B^+)$ is open in B^+, then we call B a Q-algebra. Every Banach algebra, but not every Fréchet algebra, is a Q-algebra.

Since m-algebras are not so well-known we state some important properties. Let A be a unital m-algebra and $A^\times = GL_1(A)$ the set of invertible elements in A. Recall that the spectrum of an element $a \in A$ is

$$\mathrm{sp}(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin A^\times\}$$

Contrarily to the Banach algebra case, $\mathrm{sp}(a)$ is in general not compact.

Theorem 2.9 *1. M-algebras are precisely the projective limits of Banach algebras.*

2. *Inverting is a continous map from $A^\times$ to A.*
3. *Suppose that $U \subset \mathbb{C}$ is an open neighborhood of sp(a), and let $C^{an}(U)$ be the algebra of holomorphic functions on U. There exists a unique continuous algebra homomorphism, the holomorphic functional calculus*

$$C^{an}(U) \to A : f \to f(a)$$

   such that $1 \to 1$ and $\mathrm{id}_U \to a$.
4. *If Γ is a positively oriented smooth simple closed contour, around sp(a) and in U, then*

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda - a)^{-1} d\lambda \qquad \forall f \in C^{an}(U)$$

Proof. 1 and 2 were proved by Michael [92, Theorems 5.1 and 5.2]. 3 and 4 are well-known for Banach algebras, see e.g. [125, Proposition 2.7]. Using 1 they can be extended to m-algebras, as was noticed in [102, Lemma 1.3]. □

For some typical examples, consider a C^k-manifold X, with $k \in \{0, 1, 2, \ldots, \infty\}$. We shall always assume that our manifolds are σ-compact, hence in particular paracompact. Let $U \subset \mathbb{R}^d$ be an open set and $\phi : U \to X$ a chart. For a multi-index α with $|\alpha| = n$ and $g \in C^n(U)$ let

$$\partial^\alpha g = \frac{\partial^n g}{\partial y_{\alpha_1} \cdots \partial y_{\alpha_n}} \in C^0(U) \tag{2.53}$$

be the derivative of g with respect to the standard coordinates $y_1, \ldots, y_d$ of $\mathbb{R}^d$. For $K \subset U$ compact and $n \in \mathbb{N}_{\leq k}$ we define a seminorm $\nu_{n,\phi,K}$ on $C^k(X)$ by

$$\nu_{n,\phi,K}(f) = \sup_{y \in K} \sum_{|\alpha| \leq n} \frac{|\partial^\alpha (f \circ \phi)(y)|}{|\alpha|!} \tag{2.54}$$

Straighforward estimates show that every $\nu_{n,\phi,K}$ is submultiplicative and that $C^k(X)$ is complete with respect to the family of such seminorms. Moreover, because X is σ-compact, we can cover it by countably many sets $\phi_i(K_i)$.

$$\{\nu_{n,\phi_i,K_i} : i, n \in \mathbb{N}, n \leq k\} \tag{2.55}$$

is a countable collection of seminorms defining the topology of $C^k(X)$, which therefore is a Fréchet algebra.

Finally, if X is compact and $k \in \mathbb{N}$ then $C^k(X)$ is a Banach algebra. Indeed, if we cover X by finitely many sets $\phi_i(K_i)$ then

$$\|f\| = \sum_i \nu_{n,\phi_i,K_i}(f) \tag{2.56}$$

is an appropriate norm.

We now give a quick survey of topological tensor products, completely due to Grothendieck [50]. To fix the notation, we agree that by tena@$\otimes$ without any sub- or superscript we always mean the algebraic tensor product. By default we take it over $\mathbb{C}$ if both factors are complex vector spaces, and over $\mathbb{Z}$ if there is no field over which both factors are vector spaces.

The algebraic tensor product of two vector spaces V and W solves the universal problem for bilinear maps. This means that every bilinear map from $V \times W$ to some vector space Z factors as

$$\begin{array}{ccc} V \times W & \longrightarrow & Z \\ & \searrow \quad \nearrow & \\ & V \otimes W & \end{array}$$

resulting in a bijection between $\text{Bil}(V \times W, Z)$ and $\text{Lin}(V \otimes W, Z)$. This procedure can be extended in several ways to the category of locally convex spaces, corresponding to different classes of bilinear maps and different topologies on $V \otimes W$.

For example we have the projective tensor product $V \otimes_\pi W$ [50, Subsection I.1.1], called so because it commutes with projective limits. It is $V \otimes W$ with the topology solving the universal problem for jointly continuous bilinear maps $V \times W \to Z$. If $\{p_i\}_{i \in I}$ and $\{q_j\}_{j \in J}$ are defining families of seminorms for V and W, then this topology is defined by the family of seminorms

$$\gamma_{ij}(x) = \inf \left\{ \sum_{k=1}^n p_i(v_k) q_j(w_k) : x = \sum_{k=1}^n v_k \otimes w_k \right\} \qquad i \in I, j \in J \tag{2.57}$$

The completion $V \widehat{\otimes} W$ of $V \otimes W$ for the associated uniform structure is called the completed projective tensor product.

Similarly the inductive tensor product $V \otimes_i W$ [50, Subsection I.3.1] solves the universal problem for separately continuous bilinear maps, and it commutes with inductive limits. The topology of $V \otimes_i W$ is finer than that of $V \otimes_\pi W$, and the associated completion is denoted by $V \overline{\otimes} W$. Typically, for a C^k-manifold X and a complete vector space V we have

$$C^k(X) \overline{\otimes} V \cong C^k(X; V) \tag{2.58}$$

There exists also more subtle structures on $V \otimes W$, such as the injective tensor product $V \otimes_\epsilon W$ [50, p. I.89], which in a certain sense has the weakest reasonable topology.

If V satisfies $V \otimes_\epsilon Z = V \otimes_\pi Z$ for every Z then it is called nuclear, and if both V and W are nuclear, then so are $V \otimes_\pi W$ and $V \widehat{\otimes} W$. On the other hand, if V and W are both Fréchet spaces, then $V \otimes_i W = V \otimes_\pi W$ [50, p. I.74] and its completion $V \overline{\otimes} W = V \widehat{\otimes} W$ is again a Fréchet space [50, Théorème II.2.2.9].

Consequently the tensor powers of a nuclear Fréchet space can be defined unambiguously. For example if X and Y are smooth manifolds, then $C^\infty(X)$ and $C^\infty(Y)$ are nuclear Fréchet spaces and

$$C^\infty(X) \widehat{\otimes} C^\infty(Y) \cong C^\infty(X \times Y) \tag{2.59}$$

Now that we have come this far, it is logical to spend a few words on topological tensor products over rings. So let A be an m-algebra, V a right A-module and W a left A-module. We assume that V and W are complete Hausdorff locally convex spaces and that the module operations are jointly continuous. Then the completed projective tensor product $V \widehat{\otimes}_A W$ is the completion of $V \otimes_A W$ for the topology solving the universal problem for jointly continuous A-bilinear maps from $V \times W$ to some A-module Z. Just as over $\mathbb{C}$, this topology is defined by the family of seminorms (2.57).

Let us return to homology of algebras. In any category of locally convex algebras with a topological tensor product $\otimes_t$ we can form the bicomplex $CC^{per}(A, \otimes_t)$ with spaces

$$CC^{per}_{p,q}(A, \otimes_t) = A^{\otimes_t q+1}$$

The maps from (2.3) and (2.4) are continuous because they use only the algebra operations of A. This, and the subcomplexes $CC(A, \otimes_t)$ and $CC^{\{2\}}(A, \otimes_t)$, lead to functors $HH_n(A, \otimes_t)$, $HC_n(A, \otimes_t)$ and $HP_n(A, \otimes_t)$. They are related by Connes' periodicity exact sequence, but to get more nice features it is imperative that we use only completed tensor products and place ourselves in one of the following categories:

- $\mathcal{CLA}$: complete Hausdorff locally convex algebras
- $\mathcal{MA}$: m-algebras
- $\mathcal{FA}$: Fréchet algebras
- $\mathcal{BA}$: Banach algebras

Although the objects of none these categories form a set, we allow ourselves to use $\in$ to indicate with what kind of algebra we are dealing.

By Theorem 2.9 the completed projective tensor product of two m-algebras is again an m-algebra, so we use $\widehat{\otimes}$ as our default and $\overline{\otimes}$ as a reserve. Just as in Section 2.1 we are going to study the functorial properties of the resulting homology theories. Let $A, A_m \in \mathcal{CLA}$, $m \in \mathbb{N}$.

Notice that the topological cyclic bicomplexes under consideration contain the algebraic cyclic bicomplexes. This yields natural transformations from the algebraic cyclic theories to their topological counterparts. Therefore any homomorphism from a complex algebra B to A induces maps on homology groups like

$$HH_n(B) \to HH_n(A, \widehat{\otimes})$$

These maps are compatible with all the properties below.

1. Additivity.

$$HH_n\left(\bigoplus_{m=1}^{\infty} A_m, \overline{\otimes}\right) \cong \bigoplus_{m=1}^{\infty} HH_n(A_m, \overline{\otimes})$$
$$HH_n\left(\prod_{m=1}^{\infty} A_m, \widehat{\otimes}\right) \cong \prod_{m=1}^{\infty} HH_n(A_m, \widehat{\otimes})$$

   and similarly for HC_n and HP_n. The corresponding isomorphisms for $\overline{\otimes}$ and $\prod$ hold if $A_m \in \mathcal{FA}\ \forall m$ and the isomorphisms for $\widehat{\otimes}$ and $\bigoplus$ are valid if $A_m \in \mathcal{BA}\ \forall m$.

   The proof of all these statements can be reduced to that of the algebraic case, by using [50, Propositions I.1.3.6 and I.3.1.14].

2. Stability.

$$HH_n(M_m(A), \widehat{\otimes}) \cong HH_n(A, \widehat{\otimes})$$

   and similarly with HC_n, HP_n and $\overline{\otimes}$.

   This follows from the algebraic case, since all topological tensor products of A with a finite dimensional vector space (such as $M_m(\mathbb{C})$) are the same, and essentially equal to the algebraic tensor product.

   It is not known to the author whether HH_n and HC_n are Morita-invariant in a more general sense, but for HP_n we will soon return to this point.

3. Continuity. Here great concessions to the algebraic case must be made. Assume that all the A_m are nuclear Fréchet algebras and that $A = \lim_{m\to\infty} A_m$ is a strict inductive limit. (Strict means that all the maps $A_m \to A_{m+1}$ are injective and have closed range.) In this setting Brodzki and Plymen showed [16, Theorem 2] that

$$HH_n(A, \overline{\otimes}) \cong \lim_{m\to\infty} HH_n(A_m, \overline{\otimes})$$
$$HC_n(A, \overline{\otimes}) \cong \lim_{m\to\infty} HC_n(A_m, \overline{\otimes})$$

To make HP_n continuous we need even more conditions. For example if $\exists N \in \mathbb{N}$ such that $HH_n(A_m, \overline{\otimes}) = 0 \; \forall n > N, \forall m$, then by [16, Theorem 3]

$$HP_n(A, \overline{\otimes}) \cong \lim_{m\to\infty} HP_n(A_m, \overline{\otimes})$$

The author knows of no continuity results for $\widehat{\otimes}$, which is not surprising, considering the bad compatibility of projective tensor products with inductive limits.

Excision is also pretty subtle for topological algebras. Let $\mathcal{A}$ be one of the four categories from page 31. Extending Wodzicki's terminology, we call $A \in \mathcal{A}$ strongly H-unital if, for every $V \in \mathcal{A}$, the homology of the differential graded complex $(A^{\widehat{\otimes} n} \widehat{\otimes} V \, , \, b' \widehat{\otimes} \, \mathrm{id}_V)$ is 0.

It follows from [66, Section 1] that every Banach algebra with a left or right bounded approximate identity (e.g. a C^*-algebra) is strongly H-unital, and in [15, Section 3] it is claimed that this also holds in $\mathcal{FA}$.

Recall that an extension of topological vector spaces $0 \to Y \to Z \to W \to 0$ is admissible if it has a continuous linear splitting. This implies in particular that Y (or more precisely, its image) has a closed complement in Z. Furthermore we call an extension

$$0 \to A \to B \to C \to 0 \tag{2.60}$$

in $\mathcal{A}$ topologically pure if, for every $V \in \mathcal{A}$,

$$0 \to A\widehat{\otimes}V \to B\widehat{\otimes}V \to C\widehat{\otimes}V \to 0 \tag{2.61}$$

is exact. According to [15, Section 4] the following types of extensions are topologically pure in $\mathcal{FA}$:

1. admissible extensions
2. extensions (2.60) such that A has a bounded left or right approximate identity
3. extensions of nuclear Fréchet algebras

With this terminology, the following is proved in [15, Theorems 2 and 4] :

Theorem 2.10 *Let $0 \to A \to B \to C$ be a topologically pure extension of Fréchet algebras, with A strongly H-unital. Then there are long exact sequences*

$$\begin{array}{ccccccccc}
\to & HH_n(A,\widehat{\otimes}) & \to & HH_n(B,\widehat{\otimes}) & \to & HH_n(C,\widehat{\otimes}) & \to & HH_{n-1}(A,\widehat{\otimes}) & \to \\
\to & HC_n(A,\widehat{\otimes}) & \to & HC_n(B,\widehat{\otimes}) & \to & HC_n(C,\widehat{\otimes}) & \to & HC_{n-1}(A,\widehat{\otimes}) & \to \\
\to & HP_n(A,\widehat{\otimes}) & \to & HP_n(B,\widehat{\otimes}) & \to & HP_n(C,\widehat{\otimes}) & \to & HP_{n-1}(A,\widehat{\otimes}) & \to
\end{array}$$

With the help of Theorem 2.9, all these results on excision (except 3.) can be extended to the category of m-algebras.

Actually HP_* has much more features than those listed above. Let $f, g : A \to B$ be morphisms in $\mathcal{CLA}$. We say that they are homotopic if there exists a morphism $\phi : A \to C([0,1], B)$ such that $f = \mathrm{ev}_0 \circ \phi$ and $g = \mathrm{ev}_1 \circ \phi$. They are called diffeotopic if there exists a morphism

$$\phi : A \to C^\infty([0,1]) \widehat{\otimes} B \cong C^\infty([0,1]; B)$$

with these properties.

Theorem 2.11 *In the category $\mathcal{MA}$ the functor $HP_*(\,\cdot\,, \widehat{\otimes})$ has the following properties:*

1. *If f, g are diffeotopic, then $HP_*(f) = HP_*(g)$.*
2. *Let E and F be linear subspaces of an m-algebra A, and let $A(EF)$, respectively $A(FE)$, be the subalgebra generated by all the products ef, respectively fe, with $e \in E$, $f \in F$. Then $HP_*(A(EF)) \cong HP_*(A(FE))$.*
3. *Every admissible extension* (2.60) *gives rise to an exact hexagon*

$$\begin{array}{ccccc} HP_0(A, \widehat{\otimes}) & \to & HP_0(B, \widehat{\otimes}) & \to & HP_0(C, \widehat{\otimes}) \\ \uparrow & & & & \downarrow \\ HP_1(C, \widehat{\otimes}) & \leftarrow & HP_1(B, \widehat{\otimes}) & \leftarrow & HP_1(A, \widehat{\otimes}) \end{array}$$

Proof. 1 comes from [32, p. 125] 2 from [35] and 3 from [34]. □

A clear omission at this point is a Künneth theorem for topological periodic cyclic homology. It certainly exists, but the author does not know in what generality. Fortunately, for all the algebras that we use there is an ad hoc argument available to prove the Künneth isomorphism.

What happens to differential forms in the presence of a topology? If A is a commutative unital m-algebra, then the definition of $\Omega^1(A)$ must be modified to retain completeness. So, identifying the Kähler differential $a\,db$ with the elementary tensor $a \otimes b$, we define $\Omega^1(A, \widehat{\otimes})$ to be the quotient of $A \widehat{\otimes} A$ by the closed A-submodule generated by the relations (2.16). Furthermore let V_n be closed subspace of $\left(\Omega^1(A, \widehat{\otimes})\right)^{\widehat{\otimes}_A n}$ generated by all the n-forms $\omega_1 \wedge \cdots \wedge \omega_n$ for which there exist $i \neq j$ with $\omega_i = \omega_j$. Then

$$\Omega^n(A, \widehat{\otimes}) = \bigwedge\nolimits_A^n \Omega^1(A, \widehat{\otimes}) := \left(\Omega^1(A, \widehat{\otimes})\right)^{\widehat{\otimes}_A n} / V_n \tag{2.62}$$

Thus, finally, we have the topological De Rham homology

$$H_n^{DR}(A, \widehat{\otimes}) = H_n\left(\Omega^*(A, \widehat{\otimes}), d\right) \tag{2.63}$$

Let us consider the topological counterpart of a smooth algebra. It is not exactly clear with that should be, but obviously it should be related to algebras of smooth functions. Nuclearity is also an advantage. So let $C^\infty(X)$ be the (nuclear Fréchet) algebra of infinitely often differentiable complex valued functions on a smooth real manifold X. It is well known that in this case we have natural isomorphisms

$$\Omega^n(C^\infty(X), \widehat{\otimes}) \cong \Omega^n(X) \tag{2.64}$$

$$H_n^{DR}(C^\infty(X), \widehat{\otimes}) \cong H^n_{DR}(X) \tag{2.65}$$

These hold both with real and with complex coefficients, but we are mostly interested in the latter. Furthermore the maps (2.20) and the Hochschild-Kostant-Rosenberg theorem can be extended to this topological situation [32, 126, 134], so there are natural isomorphisms

$$HH_n(C^\infty(X), \widehat{\otimes}) \cong \Omega^n(X) \tag{2.66}$$

$$HC_n(C^\infty(X), \widehat{\otimes}) \cong \Omega^n(X)/d\Omega^{n-1}(X) \oplus H^{n-2}_{DR}(X) \oplus H^{n-4}_{DR}(X) \oplus \cdots \tag{2.67}$$

$$HP_n(C^\infty(X), \widehat{\otimes}) \cong \prod_{m\in\mathbb{Z}} H^{n+2m}_{DR}(X) \tag{2.68}$$

We conclude the section with a warning. An algebra may be "too big" for cyclic theory to work properly. In fact the results are pretty noninformative for most Banach algebras. Let A be an amenable Banach algebra [66], for example $C(Y)$ with Y a compact Hausdorff space or $L^1(G)$ with G a locally compact amenable group. Then we have

$$HH_n(A, \widehat{\otimes}) = \begin{cases} A/[A,A] & \text{if} \quad n = 0 \\ 0 & \text{if} \quad n > 0 \end{cases} \tag{2.69}$$

where $[A, A]$ is the subspace of A spanned by all commutators $[a, b] = ab - ba$. Thus $\forall n \geq 0$

$$\begin{array}{lllll} HC_{2n}(A, \widehat{\otimes}) & = & HP_{2n}(A, \widehat{\otimes}) & = & A/[A,A] \\ HC_{2n+1}(A, \widehat{\otimes}) & = & HP_{2n+1}(A, \widehat{\otimes}) & = & 0 \end{array} \tag{2.70}$$

2.4 Topological K-theory and the Chern character

Topological K-theory is at the very heart of noncommutative geometry. For a compact topological space it is defined roughly speaking as the Grothendieck group of equivalence classes of vector bundles over X. By the Gelfand-Naĭmark and Serre-Swan theorems it can be transferred to (commutative) C^*-algebras, and there it becomes the Grothendieck group of isomorphism classes of finitely generated projective modules. This in turn can be extended to Banach algebras, and on that category $K_*(A)$ is something like the Grothendieck group of homotopy classes of idempotents or invertibles in the stabilization of A.

In the present section we study the K-functor on even larger categories of topological algebras. We collect some important theorems, focussing especially on those results that are fit to compare K-theory with topological periodic cyclic homology.

Of course there also exists a purely algebraic K-theory, which is a natural companion of the algebraic cyclic theory of Section 2.1. However, since these algebraic K-groups are notoriously difficult to compute, and since they contain more number-theoretic than geometric information, we will not study them here.

The most general construction of a topological K-functor is due to Cuntz [33], and it realizes K_* as the covariant half of a bivariant functor on the category of m-algebras. As Cuntz's construction is rather complicated, we will not elaborate on it. Instead we recall the definition of Phillips [102], which works for Fréchet algebras and is similar to that for Banach algebras.

Let $\mathfrak{K}$ be the nuclear Fréchet algebra of infinite matrices with rapidly decreasing coefficients. It is also referred to as the algebra of smooth compact operators, because it is a holomorphically closed dense *-subalgebra of the usual algebra of compact operators, and it is isomorphic, as a nuclear Fréchet space, to the algebra of smooth functions $C^\infty(\mathbb{T}^2)$ on the two-dimensional torus.

For any Fréchet algebra A, let $(\mathfrak{K}\widehat{\otimes}A)^+$ be the unitization of $\mathfrak{K}\widehat{\otimes}A$, and consider the Fréchet algebra $M_2\left((\mathfrak{K}\widehat{\otimes}A)^+\right)$. Define $\bar{P}(A)$ to be the set of all idempotents e in this algebra satisfying

$$e - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\mathfrak{K}\widehat{\otimes}A)$$

Similarly $\bar{U}(A)$ is the set of all invertible elements $u \in M_2\left((\mathfrak{K}\widehat{\otimes}A)^+\right)$ for which

$$u - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathfrak{K}\widehat{\otimes}A)$$

Following [102, Definition 3.2] we put

$$K_0(A) = \pi_0\left(\bar{P}(A)\right) \tag{2.71}$$

$$K_1(A) = \pi_0\left(\bar{U}(A)\right) \tag{2.72}$$

With the multiplication defined by the direct sum of matrices, these turn out to be abelian groups with unit elements

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \quad \text{and} \quad \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

Later we shall want to pick "nice" representants of K-theory classes, so now we try to discover how much is possible in this respect. Let A be unital, $e \in M_n(A)$

idempotent and $u \in GL_n(A)$. Pick a rank one projector $p \in \mathfrak{K}$ and an isomorphism $M_n(\mathfrak{K}) \to \mathfrak{K}$ and extend it to

$$\lambda_n : M_n(\mathfrak{K}\widehat{\otimes}A) \xrightarrow{\sim} \mathfrak{K}\widehat{\otimes}A$$

Now consider the elements

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda_n(pep) \end{pmatrix} \in \bar{P}(A) \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & \lambda_n(1-p+pup) \end{pmatrix} \in \bar{U}(A) \tag{2.73}$$

The resulting classes in $K_*(A)$ do not depend on p and λ_n, and are simply denoted $[e]$ and $[u]$. The natural inclusion $u \to u \oplus 1$ of $GL_n(A)$ in $GL_{n+1}(A)$ enables us to construct the inductive limit group

$$\lim_{n\to\infty} \pi_0\big(GL_n(A)\big)$$

Similarly, the inclusion $e \to e \oplus 0$ of $M_n(A)$ in $M_{n+1}(A)$ leads to the inductive limit space

$$K_0^+(A) := \lim_{n\to\infty} \pi_0\big(\mathrm{Idem}\ M_n(A)\big)$$

Actually this is an abelian semigroup with unit element 0. By [102, Lemma 7.4] it is naturally isomorphic to the monoid of equivalence classes of finitely generated projective A-modules. In this notation [102, Theorem 7.7] becomes

Theorem 2.12 *Let A be a unital Fréchet Q-algebra. The assignments $e \to [e]$ and $u \to [u]$ extend to natural isomorphisms*

$$G\left(K_0^+(A)\right) \xrightarrow{\sim} K_0(A)$$
$$\lim_{n\to\infty} \pi_0\big(GL_n(A)\big) \xrightarrow{\sim} K_1(A)$$

where the G stands for Grothendieck group.

In particular $K_0(A)$ has a natural ordering, for which $K_0^+(A)$ is precisely the semigroup of positive elements. These construction are especially important in connection with density theorem for K-theory [12, Théorème A.2.1] :

Theorem 2.13 *Let A and B be Fréchet Q-algebras, and $\phi : A \to B$ a morphism with dense range. Suppose that $a \in A^+$ is invertible whenever $\phi^+(a) \in B^+$ is invertible. Then for any $n \in \mathbb{N}$ the induced maps*

$$\begin{array}{lcl} \mathrm{Idem}\ M_n(A^+) & \to & \mathrm{Idem}\ M_n(B^+) \\ GL_n(A^+) & \to & GL_n(B^+) \end{array}$$

are homotopy equivalences, and $K_*(\phi) : K_*(A) \to K_*(B)$ is an isomorphism.

The conditions are typically satisfied if B is a unital Banach algebra, A is a dense unital subalgebra which is Fréchet in its own, finer, topology, and $A \cap B^\times = A^\times$.

If we are working in m*-algebras then everywhere in the above discussion we may replace invertibles by unitaries, and idempotents by projections. This is a consequence of the following elementary result.

Lemma 2.14 *Let A be a unital m*-algebra. The set of unitaries in A is a deformation retract of the set of invertibles in A. Likewise, the set of projections in A is a deformation retract of the set of idempotents in A.*

Proof. Using Theorem 2.9.3 write $|z| = (z^*z)^{1/2}$. Then $z|z|^{-1}$ is unitary for every $z \in A^\times$ and

$$[0,1] \times A^\times \to A^\times : (t,z) \to z|z|^{-t}$$

is the desired deformation retraction. Similarly, there is a natural path from an idempotent to its associated Kaplansky projector, see e.g. [10, Proposition 4.6.2]. □

Quite often it is possible to find a bound on the size n of matrices that we need to construct all K_1-classes. To measure this we recall the notion of topological stable rank. Given a unital topological algebra A define

$$Lg_n(A) := \{(a_1, \ldots, a_n) \in A^n : Aa_1 + \cdots + Aa_n = A\} \tag{2.74}$$

$$tsr(A) := \inf \{n : Lg_n(A) \text{ is dense in } A^n\} \tag{2.75}$$

Rieffel [106, 107] showed that this is useful for K-theory of C^*-algebras. The most general result in this direction is [107, Theorem 2.10] :

Theorem 2.15 *Let A be a unital C^*-algebra. For any $n \geq tsr(A)$ we have*

$$\pi_0\big(GL_n(A)\big) \cong K_1(A)$$

To bound the topological stable rank of algebras that are not too far from commutative we use the following tools, cf. [106, Propostion 1.7] and [100, Theorem 2.4] :

Proposition 2.16 *Let X be a compact Hausdorff space and dim X its covering dimension. Also let $A \subset B$ be an inclusion of unital C^*-algebras, such that B is a left A-module of rank n. Then*

$$tsr(C(X)) = 1 + \lfloor \dim X/2 \rfloor$$
$$tsr(B) \leq n \, tsr(A)$$

Together with Theorems 2.12 - 2.15 this will allow us to realize the K_1-group of certain C^*-algebras entirely by invertible matrices, of a certain bounded size, with coefficients in a dense subalgebra.

Now we return to the study of the more abstract features of the K-functor.

1. Additivity. For any m-algebras A_m $(m \in \mathbb{N})$

$$K_n\left(\prod_{m=1}^{\infty} A_m\right) \cong \prod_{m=1}^{\infty} K_n(A_m)$$

2. Stability.
$$K_n(\mathfrak{K}\widehat{\otimes}A) \cong K_n(M_m(A)) \cong K_n(A)$$

3. Continuity. If A_m ($m \in \mathbb{N}$) are Banach algebras and $A = \lim_{m\to\infty} A_m$ is their Banach inductive limit, then
$$K_n(A) \cong \lim_{m\to\infty} K_n(A_m)$$

4. Excision. Let $0 \to A \to B \to C \to 0$ be an extension of m-algebras, admissible if not all the algebras are Fréchet. There exists an exact hexagon
$$\begin{array}{ccccc} K_0(A) & \to & K_0(B) & \to & K_0(C) \\ \uparrow & & & & \downarrow \\ K_1(C) & \leftarrow & K_1(B) & \leftarrow & K_1(A) \end{array}$$

5. Diffeotopy invariance. Let $f, g : A \to B$ be diffeotopic morphisms of m-algebras, or homotopic morphisms of Fréchet algebras. Then
$$K_*(f) = K_*(g)$$

In this list 3 is classical, but the author does not know of any extension to Fréchet algebras. Proofs of 1, 2, 4 and 5 can be found in [33] and [102].

Obviously we will compare the features of K_* with those of HP_* given in Section 2.3. Since topological K-theory is built with the completed projective tensor product, it only makes sense to compare it with the cyclic theory with the same topological tensor product. Hence, from now on $HP_*(A)$ will mean $HP_*(A, \widehat{\otimes})$ for any m-algebra A, unless explicitly specified otherwise.

First we deduce from the excision and diffeotopy properties that K_* and HP_* react in the same way on suspending an algebra. This is essentially a manifestation of Bott periodicity. By definition the smooth suspension of A is
$$S_\infty(A) = \{f \in C^\infty(S^1; A) : f(1) = 0\}$$

Lemma 2.17 *For $i = 0, 1$ there are natural isomorphisms*
$$\begin{array}{lcl} K_{1-i}(A) & \xrightarrow{\sim} & K_i(S_\infty(A)) \\ K_0(A) \oplus K_1(A) & \xrightarrow{\sim} & K_i(C^\infty(S^1; A)) \end{array}$$

The same holds for periodic cyclic homology.

Proof. Let
$$C_\infty(A) := \{f \in C^\infty([0,1]; A) : f(0) = 0, f^{(n)}(0) = f^{(n)}(1)\ \forall n > 0\}$$

be the smooth cone of A. Consider the admissible extension
$$0 \to S_\infty(A) \to C_\infty(A) \to A \to 0 \tag{2.76}$$

where the third map is evaluation at 0 and the second arrow comes from composing a function with the surjection

$$[0,1] \to S^1 : x \to e^{2\pi i x}$$

The boundary maps from the exact hexagon associated with (2.76) are the desired maps $K_{1-i}(A) \to K_i(S_\infty(A))$. To prove that these are isomorphisms we will show that $C_\infty(A)$ is diffeotopy equivalent to the algebra 0. Let $r : [0,1] \to [0,1]$ be a bijective diffeomorphism with the properties

- $r'(x) > 0 \; \forall x \in (0,1)$
- $r^{(n)}(0) = r^{(n)}(1) = 0 \; \forall n > 0$

We have m-algebra homomorphisms

$$C_\infty(A) \to C_0^\infty([0,1],\{0\};A) : f \to f$$
$$C_0^\infty([0,1],\{0\};A) \to C_\infty(A) : f \to f \circ r$$

Since r is diffeotopic to $\mathrm{id}_{[0,1]}$, both compositions of these algebra homomorphisms are diffeotopic to the respective identity homomorphisms. Hence we get natural isomorphisms

$$K_*(C_\infty(A)) \xrightarrow{\sim} K_*(C_0^\infty([0,1],\{0\};A))$$

However, $C_0^\infty([0,1],\{0\};A)$ is diffeotopy equivalent to 0 by means of the homomorphisms

$$\phi_t : C_0^\infty([0,1],\{0\};A) \to C_0^\infty([0,1],\{0\};A)$$
$$\phi_t(f)(s) = f(ts)$$

and therefore its K-theory vanishes.

Similarly there is an admissible extension

$$0 \to S_\infty(A) \to C^\infty(S^1;A) \to A \to 0 \tag{2.77}$$

But this extension splits, just send $a \in A$ to the element $\tilde{a} \in C^\infty(S^1;A)$ with $\tilde{a}(t) = a \; \forall t \in S^1$. Applying K_i we get a split exact sequence of abelian groups

$$0 \to K_i(S_\infty(A)) \to K_i(C^\infty(S^1;A)) \to K_i(A) \to 0$$

Combined with the above this yields natural isomorphisms

$$K_{1-i}(A) \oplus K_i(A) \xrightarrow{\sim} K_i(S_\infty(A)) \oplus K_i(A) \xrightarrow{\sim} K_i(C^\infty(S^1;A))$$

The same proof applies with periodic cyclic homology. □

Continuing our comparison, we see from Theorem 2.11.2 that HP_* is also $\mathfrak{K}$-stable. Namely, we may take for E all the matrices whose only nonzero entries are in the first column, and for F all the matrices which have only zeros outside the first row. The isomorphism

$$HP_*(A) \xrightarrow{\sim} HP_*(\mathfrak{K}\widehat{\otimes}A) \tag{2.78}$$

is induced by the algebra morphism $a \to pap$, where $p \in \mathfrak{K}$ is an arbitrary rank one projector. Its inverse

$$HP_*(\mathfrak{K}\widehat{\otimes}A) \xrightarrow{\sim} HP_*(A) \tag{2.79}$$

is a little more tricky, since it is not given by an algebra morphism, but a morphism of bicomplexes, the so-called generalized trace map. This is the linear map

$$tr : \left(\mathfrak{K}\widehat{\otimes}A\right)^{\widehat{\otimes}n} \to A^{\widehat{\otimes}n} \tag{2.80}$$

defined on elementary tensors by

$$tr(k_1 a_1 \otimes \cdots \otimes k_n a_n) = tr(k_1 \cdots k_n)\, a_1 \otimes \cdots \otimes a_n \tag{2.81}$$

Note that this works equally well if $\mathfrak{K}$ is replaced by a finite dimensional matrix algebra $M_m(\mathbb{C})$, cf. [81, Section 1.2].

So now we know that HP_* is halfexact, diffeotopy invariant and $\mathfrak{K}$-stable. On the other hand, by [33, Section 6] K_* is the universal halfexact, diffeotopy invariant, $\mathfrak{K}$-stable covariant functor from m-algebras to abelian groups. This implies the existence of a unique natural transformation of functors

$$ch : K_* \to HP_* \tag{2.82}$$

respecting these features. It is called the Chern character, because it is a far-reaching generalization of the classical Chern character

$$Ch : K^*(X) \to \check{H}^*(X; \mathbb{Q}) \tag{2.83}$$

that assigns to a complex vector bundle over a paracompact Hausdorff space X a class in the even Čech cohomology. Indeed, we can get (2.83) for smooth manifolds by applying (2.82) to $C^\infty(X)$ and using the isomorphism (2.68).

The Chern character is compatible with the countable additivity of K_* and HP_*, and also with excision, as was shown by Nistor [95, Theorem 1.6] :

Theorem 2.18 *Let $0 \to A \to B \to C \to 0$ be an extension of m-algebras. The various Chern characters make a commutative diagram*

$$\begin{array}{ccccccccccc} K_1(A) & \to & K_1(B) & \to & K_1(C) & \to & K_0(A) & \to & K_0(B) & \to & K_0(C) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ HP_1(A) & \to & HP_1(B) & \to & HP_1(C) & \to & HP_0(A) & \to & HP_0(B) & \to & HP_0(C) \end{array}$$

Moreover, if the extension is admissible and $\eta : K_0(C) \to K_1(A)$ and $\partial : HP_0(C) \to HP_1(A)$ denote the connecting maps, then $ch \circ \eta = 2\pi i\, \partial \circ ch$.

Explicit formulas for the Chern character, from Phillips' picture to the cyclic bicomplex, were first given by Karoubi [69, Chapitre II]. In the setting of (2.73) we may replace A by $M_n(A)$ to achieve that $e, u \in A$. Define

$$\begin{aligned} c_{2m+1}(e) &= (-1)^m \frac{(2m)!}{m!}(e, \ldots, e) &&\in A^{\otimes 2m+1} \\ c_{2m}(e) &= (-1)^{m-1} \frac{(2m)!}{2(m!)}(e, \ldots, e) &&\in A^{\otimes 2m} \\ c_{2m}(u) &= (m-1)!\,(u^{-1}, u, \ldots, u^{-1}, u) &&\in A^{\otimes 2m} \\ c_{2m+1}(u) &= m!\,(2, u^{-1}, u, \ldots, u^{-1}, u) &&\in A^{\otimes 2m+1} \end{aligned} \tag{2.84}$$

and place $c_n(e)$ in $CC^{per}_{1-n,n-1}(A, \widehat{\otimes}) \cong A^{\widehat{\otimes} n}$ and $c_n(u)$ in $CC^{per}_{2-n,n-1}(A, \widehat{\otimes}) \cong A^{\widehat{\otimes} n}$. By Lemma 2.1.6, Theorem 8.3.4 and Proposition 8.4.9 of [81] we have

$$ch\,[e] = [(c_n(e))_{n=1}^{\infty}] \in HP_0(A) \tag{2.85}$$

$$ch\,[u] = [(c_n(u))_{n=1}^{\infty}] \in HP_1(A) \tag{2.86}$$

With the density theorem and the homotopy invariance of K-theory we can compute it for many Fréchet algebras, in particular commutative ones. The maximal ideal space of a commutative m-algebra A is defined like in algebraic geometry : it is the collection $\mathrm{Max}(A)$ of all closed maximal ideals of A, endowed with the coarsest topology that makes all elements of A into continuous functions on $\mathrm{Max}(A)$. This is called the Gelfand topology, and we denote it by $\mathcal{T}_G$.

Contrarily to the C^*-algebra case, there may be several commutative Fréchet algebras with the same maximal ideal space. The spectrum of a commutative Fréchet algebra is Hausdorff, σ-compact and paracompact, but it need not be locally compact. Therefore we also consider the compactly generated topology $\mathcal{T}_c$ on $\mathrm{Max}(A)$. This means that we call $U \subset \mathrm{Max}(A)$ open in $\mathcal{T}_c$ if and only if $U \cap C$ is open in C, for any compact C with the relative topology from $(\mathrm{Max}(A), \mathcal{T}_G)$. If A^+ is the unitization of A, then $\mathrm{Max}(A^+) = \mathrm{Max}(A) \cup \{A\}$ and we put

$$C_A := \{ f \in C(\mathrm{Max}(A^+), \mathcal{T}_c) : f(A) = 0 \} \tag{2.87}$$

Theorem 2.19 *For any commutative Fréchet algebra A there are natural isomorphisms*

$$K_*(A) \cong K_*(C_A) \cong K^*\big(\mathrm{Max}(A^+), \{A\}\big)$$

$$K_*(A) \otimes \mathbb{Q} \cong \check{H}^*(\mathrm{Max}(A^+), \{A\}; \mathbb{Q})$$

Proof. The isomorphisms with integral coefficients are due to Phillips [102, Theorem 7.15]. Here K^* means representable K-theory of topological spaces, in the sense of Karoubi [68]. He represents

$$K^n(X) := [X, \mathfrak{F}U^n] \tag{2.88}$$

as the set of homotopy classes of continuous maps from a paracompact Hausdorff space X to some classifying space $\mathfrak{F}U^n$. This agrees with the usual definition if

X is compact, but in general it yields a generalized cohomology theory without compact supports. It has been known since the beginning of topological K-theory that (2.83) gives an isomorphism

$$Ch \otimes \mathrm{id}_{\mathbb{Q}} : K^*(X) \otimes \mathbb{Q} \xrightarrow{\sim} \check{H}^*(X; \mathbb{Q}) \tag{2.89}$$

if X is a finite CW-complex [2, Section 2.4]. With spectral sequences, as in [114], one can extend this to all paracompact Hausdorff spaces, since these are homotopy equivalent to CW-complexes. □

This theorem can be considered as the counterpart in topological K-theory of Theorem 2.6. If we apply it to a smooth manifold we get

$$K_*(C^\infty(X)) \otimes \mathbb{C} \cong \check{H}^*(X; \mathbb{C}) \tag{2.90}$$

Since Čech cohomology agrees with De Rham cohomology (both with complex coefficients) on the category of smooth manifolds, we deduce from (2.68), (2.90) and the naturality of the Chern character that

$$ch \otimes \mathrm{id}_{\mathbb{C}} : K_*(C^\infty(X)) \otimes \mathbb{C} \xrightarrow{\sim} HP_*(C^\infty(X)) \tag{2.91}$$

is an isomorphism. If we think a little more about this, it becomes clear that such an isomorphism should hold for many more algebras, even noncommutative ones. To make this precise, we introduce yet another category of topological algebras, denoted $\mathcal{CIA}$. It is a full subcategory of the category of m-algebras $\mathcal{MA}$, and its objects are those $A \in \mathcal{MA}$ for which the Chern character induces an isomorphism

$$ch \otimes \mathrm{id}_{\mathbb{C}} : K_*(A) \otimes \mathbb{C} \xrightarrow{\sim} HP_*(A) \tag{2.92}$$

Proposition 2.20 *The category $\mathcal{CIA}$ is closed under the following operations:*

1. *countable direct products*
2. *tensoring with $M_m(\mathbb{C})$ or $\mathfrak{K}$*
3. *diffeotopy equivalences*
4. *admissible extensions, quotients and ideals*

Proof. 1,2 and 3 follow directly from the features of K_* and HP_* on pages 32 and 39. As concerns 4, by Theorem 2.18 we can apply Lemma 2.3 to the functors $K_*(\,\cdot\,) \otimes \mathbb{C}$ and $HP_*(\,\cdot\,, \widehat{\otimes})$ on the category $\mathcal{MA}$ with admissible morphisms. The factor $2\pi i$ in Theorem 2.18 is inessential. □

In view of these similarities, it is logical to try to extend the material from Section 2.2 to topological K-theory. However, this is somewhat problematic, as general compact Hausdorff spaces are much less easy to handle then algebraic varieties. In the next section we will avoid these difficulties by considering only smooth manifolds with a finite group action. Right now we will prove a coarse analogue of Theorem 2.7, which applies to semisimple algebras which "live" on finite simplicial complexes. We formulate it in terms of C^*-algebras, but with Theorem 2.13 it can easily be generalized to certain Fréchet algebras.

Proposition 2.21 *Let Σ be a finite simplicial complex and $\phi : A \to B$ a homomorphism of C^*-algebras. Suppose that*

- *there are unital homomorphisms from $C(\Sigma)$ to the centers of the multiplier algebras $\mathcal{M}(A)$ and $\mathcal{M}(B)$*
- *for every simplex σ of Σ there are finite dimensional C^*-algebras A_σ and B_σ such that*

$$AC_0(\sigma, \delta\sigma) \cong C_0(\sigma, \delta\sigma; A_\sigma) \quad \text{and} \quad BC_0(\sigma, \delta\sigma) \cong C_0(\sigma, \delta\sigma; B_\sigma)$$

- *ϕ is $C(\Sigma)$-linear*
- *for every $x_\sigma \in \sigma \setminus \delta\sigma$ the localization $\phi(x_\sigma) : A_\sigma \to B_\sigma$ induces an isomorphism on K-theory*

Then

$$K_*(\phi) : K_*(A) \xrightarrow{\sim} K_*(B)$$

is an isomorphism.

Proof. Let Σ^n be the n-skeleton of Σ and consider the ideals

$$\begin{aligned} C(\Sigma) &= I_0 \supset I_1 \supset \cdots \supset I_n \supset \cdots \\ I_n &= C_0(\Sigma, \Sigma^n) \end{aligned} \tag{2.93}$$

They give rise to ideals $A_n = AI_n$ and $B_n = BI_n$. Because Σ is finite all these ideals are 0 for large n. We can identify

$$A_{n-1}/A_n \cong AC_0(\Sigma^n, \Sigma^{n-1}) \cong \bigoplus_{\sigma\in\Sigma\,:\,\dim\sigma=n} AC_0(\sigma, \delta\sigma) := \bigoplus_{\sigma\in\Sigma\,:\,\dim\sigma=n} C_0(\sigma, \delta\sigma; A_\sigma)$$

and similarly for B. Because ϕ is $C(\Sigma)$-linear, it induces homomorphisms

$$\phi(\sigma) : C_0(\sigma, \delta\sigma; A_\sigma) \to C_0(\sigma, \delta\sigma; B_\sigma)$$

By Lemma 2.3 and the additivity of K-theory it suffices to show that every $\phi(\sigma)$ induces an isomorphism on K-theory. Let x_σ be any interior point of σ. Because $\sigma \setminus \delta\sigma$ is contractible, ϕ_σ is homotopic to $\mathrm{id}_{C_0(\sigma,\delta\sigma)} \otimes \phi(x_\sigma)$. By assumption the latter map induces an isomorphism on K-theory. With the homotopy invariance of K-theory it follows that $K_*(\phi(\sigma)) \otimes \mathrm{id}_{\mathbb{Q}}$ is an isomorphism. □

Note that this proof applies equally well to the functor $K_*(\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.5 Equivariant cohomology and algebras of invariants

This section is all about algebras that carry an action of a finite group, and their subalgebras of invariant elements. To place things in all classical context we first recall some beautiful theorems on equivariant topological K-theory and cyclic theory for crossed product algebras.

All the results after that are due to the author, but some of them already appeared in [123]. We broaden our view, to algebras of the form

$$A^G = C^\infty(X; M_N(\mathbb{C}))^G$$

with X a smooth manifold. We relate K-theory of such algebras to a G-equivariant cohomology theory due to Bredon [14], which can also be described as the Čech cohomology of a certain sheaf over the orbifold X/G. This depends on the existence of a G-equivariant triangulation of X. Using the same Mayer-Vietoris type of arguments we also prove that the Chern character for A^G becomes an isomorphism after tensoring with $\mathbb{C}$. Finally, if X happens to be a complex affine variety, then we show that the "polynomial" subalgebra A^G_{alg} has the same periodic cyclic homology as A^G.

Let G be a topological group acting continuously on a topological space X. Then G also acts continuously on the closed subspace

$$\widetilde{X} := \{(g, x) \in G \times X : gx = x\} \tag{2.94}$$

of $G \times X$ by

$$g(g', x) = (gg'g^{-1}, gx) \tag{2.95}$$

and $\widetilde{X}/G$ is called the extended quotient of X by G. In the literature one often encounters the notation $\hat{X}$ for $\widetilde{X}$, but we avoid this because it might be confused with the spectrum of a topological group.

Let $\langle G \rangle$ be the set of conjugacy classes in G, and denote the class containing g by $\langle g \rangle$. We have a decomposition

$$\widetilde{X}/G \cong \bigsqcup_{\langle g \rangle \in \langle G \rangle} (g, X^g / Z_G(g)) \cong \bigsqcup_{\langle g \rangle \in \langle G \rangle} X^g / Z_G(g) \tag{2.96}$$

where $Z_G(g)$ is the centralizer of g in G and

$$X^g = \{x \in X : gx = x\} \tag{2.97}$$

Notice that the components of this partition are always closed, and they are open if G is finite.

A G-vector bundle over X is a vector bundle $p : V \to X$ together with an action of G on V, such that $\forall v \in V, x \in X, g \in G$

- $p(gv) = gp(v)$
- $g : p^{-1}(x) \to p^{-1}(gx)$ is linear

If X and G are both compact Hausdorff, then it makes sense to consider the Grothendieck group of equivalence classes of complex G-vector bundles. This group was first studied by Atiyah [1], and it is denoted by $K^0_G(X)$. By the same suspension procedure as in the nonequivariant case this leads to a sequence of functors K^n_G, together called equivariant K-theory. This is a equivariant cohomology theory which shares most of the properties of ordinary topological K-theory.

For any $g \in G$, the restriction of a complex G-bundle $p : V \to X$ to X^g is a vector bundle on which g acts linearly in every fiber. So we can decompose it canonically into to its g-eigenspaces :

$$V\big|_{X^g} = \bigoplus_i V_i := \bigoplus_i \big\{v \in p^{-1}(X^g) : gv = \lambda_i v\big\} \tag{2.98}$$

By the continuity of the action and the compactness of X^g there are only finitely many $\lambda_i \in \mathbb{C}$ for which V_i is nonzero. From this decompostion we cook a canonical map

$$\rho_g : K^*_G(X) \to K^*(X^g) \otimes \mathbb{C} \tag{2.99}$$

sending $[V]$ to $\sum_i \lambda_i [V_i]$. All these ρ_g's together combine to a map that classifies G-bundles over X in terms of ordinary vector bundles over the extended quotient $\widetilde{X}/G$. Indeed, for finite G the identification

$$K^*(\widetilde{X}) \cong \bigoplus_{\langle g\rangle \in \langle G\rangle} K^*(X^g) \tag{2.100}$$

gives a map

$$\rho := \sum_{g\in G} \rho_g : K^*_G(X) \to K^*(\widetilde{X}) \otimes \mathbb{C} \tag{2.101}$$

It is easy to see that the image of ρ is contained in the subspace of G-invariants, so if we compose it with the Chern character for $\widetilde{X}$ we land in $\check{H}^*(\widetilde{X}; \mathbb{C})^G$, which by [51, Corollaire 5.2.3] is naturally isomorphic to $\check{H}^*(\widetilde{X}/G; \mathbb{C})$. By the way, this composition

$$Ch_G := (Ch \otimes \mathrm{id}_{\mathbb{C}}) \circ \rho \tag{2.102}$$

is called the equivariant Chern character. The punchline is of course [4, Theorem 1.19] :

Theorem 2.22 *For any finite group G acting on a compact Hausdorff space X there are natural isomorphisms*

$$\rho \otimes \mathrm{id}_{\mathbb{C}} : K^*_G(X) \otimes \mathbb{C} \xrightarrow{\sim} \Big(K^*(\widetilde{X}) \otimes \mathbb{C}\Big)^G$$
$$Ch_G \otimes \mathrm{id}_{\mathbb{C}} : K^*_G(X) \otimes \mathbb{C} \xrightarrow{\sim} \Big(\check{H}^*(\widetilde{X}; \mathbb{C})\Big)^G \cong \check{H}^*(\widetilde{X}/G; \mathbb{C})$$

We switch back to a more algebraic point of view. Suppose that the compact group G acts on by $*$-automorphisms on a C^*-algebra A. The above leads us to consider finitely generated projective A-modules M with a G-action satisfying $g(am) = (ga)(gm)$. The Grothendieck group of equivalence classes of such modules is the equivariant K-theory $K_0^G(A)$. This gives rise to sequence of functors K_n^G which by the equivariant Serre-Swan theorem [101, Theorem 2.3.1] are related to the above homonymous functors as

$$K_*^G(C(X)) \cong K_G^*(X) \tag{2.103}$$

We already managed to describe G-bundles in terms of ordinary vector bundles over a related space, and the same is possible over C^*-algebras. Namely, Julg [67] showed that there is a natural identification

$$K_*^G(A) \cong K_*(A \rtimes G) \tag{2.104}$$

Combining (2.103) and (2.104) with Theorem 2.22 we see that

$$K_*(C(X) \rtimes G) \otimes \mathbb{C} \cong \check{H}^*(\widetilde{X}; \mathbb{C})^G \cong \check{H}^*(\widetilde{X}/G; \mathbb{C}) \tag{2.105}$$

This is important for our purposes, since it shows that the homology of a crossed product algebra can be described in geometric terms. This can even be refined in cyclic theory. Let X be either a nonsingular complex affine variety or a smooth real manifold, not necessarily compact, and A the algebra of either regular or smooth functions on X. Suppose that the action of G preserves this structure, so that the partially invariant subspaces X^g are of the same type as X. Brylinski [21, 22] proved that

$$HH_n(A \rtimes G) \cong \bigoplus_{\langle g \rangle \in \langle G \rangle} \Omega^n \left(X^g\right)^{Z_G(g)} \cong \Omega^n \big(\widetilde{X}\big)^G \tag{2.106}$$

$$HP_n(A \rtimes G) \cong \bigoplus_{\langle g \rangle \in \langle G \rangle} \prod_{m \in \mathbb{Z}} H_{DR}^{n+2m} \left(X^g\right)^{Z_G(g)} \cong \prod_{m \in \mathbb{Z}} H_{DR}^{n+2m} \big(\widetilde{X}\big)^G \tag{2.107}$$

$$\begin{aligned} HC_n(A \rtimes G) &\cong \bigoplus_{\langle g \rangle \in \langle G \rangle} \Big(\Omega^n(X^g)/d\Omega^{n-1}(X^g) \oplus H_{DR}^{n-2}(X^g) \oplus H_{DR}^{n-4}(X^g) \oplus \cdots\Big)^{Z_G(g)} \\ &\cong \Big(\Omega^n\big(\widetilde{X}\big)/d\Omega^{n-1}\big(\widetilde{X}\big) \oplus H_{DR}^{n-2}\big(\widetilde{X}\big) \oplus H_{DR}^{n-4}\big(\widetilde{X}\big) \oplus \cdots\Big)^G \end{aligned} \tag{2.108}$$

Moreover Nistor [96, Theorem 2.11] constructed an explicit map

$$HH_n(A \rtimes G) \to \bigoplus_{\langle g \rangle \in \langle G \rangle} \Omega^n(X^g) \tag{2.109}$$

(not the naive restriction!) and showed that it induces these isomorphisms.

When we combine (2.105),(2.107) and Theorem 2.13 with the naturality of the Chern character, we arrive at

Theorem 2.23 *Suppose that a finite group G acts by diffeomorphisms on a compact smooth manifold X. Then the Chern character gives an isomorphism*

$$ch \otimes \mathrm{id}_{\mathbb{C}} : K_*(C^\infty(X) \rtimes G) \otimes \mathbb{C} \xrightarrow{\sim} HP_*(C^\infty(X) \rtimes G)$$

Later we will see that the compactness assumption in this theorem is not necessary, so we drop it now, at least for the rest of this section.

Interestingly, the crossed product $C^\infty(X) \rtimes G$ can also be realized as an algebra of invariants :

$$C^\infty(X) \rtimes G \cong C^\infty\big(X; \mathrm{End}(\mathbb{C}[G])\big)^G \tag{2.110}$$

where the group algebra $\mathbb{C}[G]$ carries the right regular representation of G, see Lemma A.3. We propose to study such algebras also with other G-representations instead of $\mathbb{C}[G]$. Then $\mathbb{C}[G]$ is universal in the sense that it contains every irreducible G-representation. At the other extreme we have the trivial representation, which leads us to the Fréchet algebra

$$C^\infty(X/G) := C^\infty(X)^G \tag{2.111}$$

of smooth functions on the orbifold X/G, cf. [110]. Wassermann [134, Section IV] showed that the periodic cyclic homology of this algebra equals, as one would expect, the Čech cohomology of X/G:

$$HP_*\left(C^\infty(X)^G\right) \cong H^*_{DR}(X)^G \cong \check{H}^*(X/G; \mathbb{C}) \tag{2.112}$$

But he also noticed that Hochschild homology does not behave so well in this case, as

$$HH_*\left(C^\infty(X)^G\right) \quad \text{and} \quad \Omega^*(X)^G$$

are not isomorphic in general.

Let $Z \subset Y$ be arbitrary subsets of $\mathbb{R}^n$, and V a Fréchet space. To include manifolds with boundary in our studies we adhere to the following conventions :

$$\begin{aligned} C^\infty(Y) &:= \left\{ f\big|_Y : f \in C^\infty(U) \text{ for some open } U \text{ with } Y \subset U \subset \mathbb{R}^n \right\} \\ C_0^\infty(Y,Z) &:= \left\{ f \in C^\infty(Y) : f\big|_Z = 0 \right\} \\ C_0^\infty(Y,Z;V) &:= C_0^\infty(Y,Z) \widehat{\otimes} V \end{aligned} \tag{2.113}$$

Unfortunately this slightly ambiguous for orbifolds embedded in $\mathbb{R}^n$. For example if $Y = \mathbb{R}/\{\pm 1\}$, identified as a topological space with $[0,\infty)$, then

$$C^\infty\big([0,\infty)\big) \supsetneq C^\infty(\mathbb{R})^{\{\pm 1\}}$$

since the right hand side contains only functions whose odd derivatives vanish at 0. However, the difference is not too big, since both algebras are diffeotopy equivalent to $\mathbb{C} \oplus C_0^\infty\big(\mathbb{R}, (-\infty,0]\big)$ via $f \to f \circ \phi$, where $\phi \in C_0^\infty\big(\mathbb{R}, (-\infty,0]\big)$ is an automorphism of $[0,\infty)$ which is diffeotopic to the identity of $[0,\infty)$. In such situations we shall usually give priority to the orbifold structure and use (2.111) as a definition, at least locally.

Now let X be a smooth manifold with boundary, still σ-compact, and consider the Fréchet algebra

$$A := C^\infty(X; M_N(\mathbb{C})) \cong M_N(C^\infty(X)) \tag{2.114}$$

We assume we have elements $u_g \in A^\times$ and diffeomorphisms α_g of X such that

$$ga(x) = u_g(x) a(\alpha_g^{-1} x) u_g^{-1}(x) \tag{2.115}$$

defines an action of G on A. Although this implies that $g \to \alpha_g$ is a group homomorphism, $g \to u_g$ need not be one. The algebra

$$A^G = C^\infty(X; M_N(\mathbb{C}))^G \tag{2.116}$$

will be our Fréchet version of a finite type algebra. Clearly A^G is finitely generated as a module over $C^\infty(X)^G$. It follows from a classical theorem of Newman that the set of points of X whose G-stabilizer equals ker α is open and everywhere dense in X, see [42, Theorem 1]. Hence, if $\ker \alpha = \{e\}$ then $Z(A^G) = C^\infty(X)^G$.

To compute its K-theory we will use an " equivariant cohomology theory with a local coefficient system", as defined by Bredon [14]. This theory can be combined with the ideas of Segal [115] and Słomińska [120] to describe $K_*\left(A^G\right)$ in sheaf-cohomological terms.

First we recall some of Bredon's constructions, referring to [14] for more precise information. Let Σ be a countable, locally finite and finite dimensional G-CW complex. Assume that all cells are oriented and that the action of G preserves these orientations.

We define a category $\mathcal{K}$ whose objects are the finite subcomplexes of Σ. The morphisms from K to K' are the maps $K \to K' : x \to gx$ for $g \in G$ such that $gK \subset K'$. Now a local coefficient system $\mathfrak{L}$ on Σ is a covariant functor from $\mathcal{K}$ to the category of abelian groups, and the group $C^q(\Sigma; \mathfrak{L})$ of q-cochains is the set of all functions f on the q-cells of Σ with the property that $f(\tau) \in \mathfrak{L}(\tau)\, \forall \tau$. Furthermore we define a coboundary map $\partial : C^q(\Sigma; \mathfrak{L}) \to C^{q+1}(\Sigma; \mathfrak{L})$ by

$$(\partial f)(\sigma) = \sum_\tau [\tau : \sigma] \mathfrak{L}(\tau \to \sigma) f(\tau) \tag{2.117}$$

where the sum runs over all q-cells τ and the incidence number $[\tau : \sigma]$ is the degree of the attaching map from $\partial\sigma$ (the boundary of σ in the standard topological sense) to $\tau/\partial\tau$. The group G acts naturally on this complex by cochain maps, so, for any $K \subset \Sigma$, $\left(C^*(K; \mathfrak{L})^G, \partial\right)$ is a differential complex and we can define the equivariant cohomology of K with coefficients in $\mathfrak{L}$ as

$$H^q_G(K; \mathfrak{L}) := H^q\left(C^*(K; \mathfrak{L})^G, \partial\right) \tag{2.118}$$

More generally for $K' \subset K$, $C^*(K, K'; \mathfrak{L})$ is the kernel of the restriction map $C^*(K; \mathfrak{L}) \to C^*(K'; \mathfrak{L})$ and

$$H^q_G(K, K'; \mathfrak{L}) = H^q\left(C^*(K, K'; \mathfrak{L})^G, \partial\right) \tag{2.119}$$

By construction there exists a local coefficient system $\mathfrak{L}^G$ (more or less consisting of the G-invariant elements of $\mathfrak{L}$) on the CW-complex Σ/G such that the differential complexes $(C^*(K,K';\mathfrak{L})^G,\partial)$ and $(C^*(K/G,K'/G;\mathfrak{L}^G),\partial)$ are isomorphic. Notice that $\mathfrak{L}^G$ defines a sheaf over Σ/G (with the cells as cover), so that

$$H^q_G(K,K';\mathfrak{L}) \cong \check{H}^q\left(K/G,K'/G;\mathfrak{L}^G\right) \tag{2.120}$$

Let Σ^p be the p-skeleton of Σ. We capture all the above things in a spectral sequence $(E_r^{p,q})_{r\geq 1}$, degenerating already for $r\geq 2$, as follows :

$$E_1^{p,q} = H_G^{p+q}(\Sigma^p,\Sigma^{p-1};\mathfrak{L}) = \begin{cases} C^p(\Sigma;\mathfrak{L})^G & \text{if} \quad q=0 \\ 0 & \text{if} \quad q>0 \end{cases} \tag{2.121}$$

$$E_2^{p,q} = \begin{cases} H_G^p(\Sigma;\mathfrak{L}) & \text{if} \quad q=0 \\ 0 & \text{if} \quad q>0 \end{cases} \tag{2.122}$$

The differential d_1^E is the composition

$$E_1^{p,q} \to H_G^{p+q}(\Sigma^p;\mathfrak{L}) \to E_1^{p+1,q} \tag{2.123}$$

of the maps induced by the inclusion $(\Sigma^p,\emptyset)\to(\Sigma^p,\Sigma^{p-1})$ and the coboundary ∂.

Now let B^G be an algebra like (2.116), but without the differentiable structure. We will see later that this algebra has the same K-theory as A^G, for a suitable triangulation of X. So we put

$$B = C(\Sigma;M_N(\mathbb{C})) = M_N(C(\Sigma)) \tag{2.124}$$

and we assume that we have $u_g \in B^\times$ such that

$$gb(x) = u_g(x)b(g^{-1}x)u_g^{-1}(x) \tag{2.125}$$

defines an action of G on B. To associate a local coefficient system $\mathfrak{L}_u$ to this algebra we first assume that K is connected. In that case we let

$$G_K := \{g\in G : gx = x \quad \forall x\in K\} \tag{2.126}$$

be the isotropy group of K and we define $\mathfrak{L}_u(K)$ to be the free abelian group on the (equivalence classes of) irreducible G_K-representations contained in $(\pi_x,\mathbb{C}^N)$, where $\pi_x(g) = u_g(x)$ for $g\in G_K, x\in K$. By the continuity of the u_g we get the same group for any $x\in K$. If K is not connected, then we let $\{K_i\}_i$ be its connected components, and we define

$$\mathfrak{L}_u(K) = \prod_i \mathfrak{L}_u(K_i) \tag{2.127}$$

Suppose that $gK\subset K'$ and that ρ is a projective G_K-representation. Then we define a projective $G_{K'}$-representation by

$$\mathfrak{L}_u(g:K\to K')\rho(g') = \rho(g^{-1}g'g) \quad g'\in G_{K'} \tag{2.128}$$

If $h \in G$ gives the same map from K to K' as g then $h^{-1}g \in G_K$ and

$$\mathfrak{L}_u(h : K \to K')\rho(g') = \rho(h^{-1}g'h) = \rho(h^{-1}g)\rho(g^{-1}g'g)\rho(g^{-1}h) \tag{2.129}$$

so $\mathfrak{L}_u(h : K \to K')\rho$ is isomorphic to $\mathfrak{L}_u(g : K \to K')\rho$ as a projective representation. This makes $\mathfrak{L}_u$ into a functor.

Suppose for example that $u_g(x) = 1\,\forall x \in \Sigma, g \in G$. Then $\mathfrak{L}_u$ and $\mathfrak{L}_u^G$ are the constant sheaves $\mathbb{Z}$ over Σ and Σ/G respectively, and

$$H_G^*(\Sigma; \mathfrak{L}_u) \cong \check{H}^*(\Sigma/G; \mathbb{Z}) \tag{2.130}$$

is the ordinary cellular cohomology of Σ/G. Moreover $B^G \cong C(\Sigma/G; M_N(\mathbb{C}))$, so $K_*\left(B^G\right) \cong K^*(\Sigma/G)$, which is isomorphic to $\check{H}^*(\Sigma/G; \mathbb{Z})$ modulo torsion.

Of most interest is the case where $B^G \cong C(\Sigma) \rtimes G$ is the crossed product, as in Lemma A.3. Then we compare $\mathfrak{L}_u^G$ to the direct image of the constant sheaf $\mathbb{Z}$ on $\widetilde{\Sigma}$, under the canonical map $p : \widetilde{\Sigma}/G \to \Sigma/G$. Although they may not always be isomorphic, their Čech complexes are identical, for any cover that refines the cell structure. Since p is finite to one we can deduce, using even more spectral sequences [47, Chapitre 5], that

$$H_G^*(\Sigma; \mathfrak{L}_u \otimes \mathbb{Q}) \cong \check{H}^*\left(\widetilde{\Sigma}/G; \mathbb{Q}\right) \tag{2.131}$$

and by Theorem 2.23 this is the same as $K_*\left(B^G\right) \otimes \mathbb{Q}$ if Σ is compact, i.e. if it is a finite CW-complex.

It turns out that this close relation between $K_*\left(B^G\right)$ and the Čech cohomology $H^*\left(\Sigma/G; \mathfrak{L}_u^G\right)$ is valid in general. Consider the following analogue of (2.33)

$$\begin{aligned}
&K_*^0\left(B^G\right) = K_*^0\left(B^G\right) \supset K_*^1\left(B^G\right) \supset \cdots \supset K_*^{\dim\Sigma}\left(B^G\right) \supset K_*^{1+\dim\Sigma}\left(B^G\right) = 0\\
&K_*^p\left(B^G\right) := \ker\left(K_*\left(C(\Sigma; M_N(\mathbb{C}))^G\right) \to K_*\left(C(\Sigma^{p-1}; M_N(\mathbb{C}))^G\right)\right)
\end{aligned} \tag{2.132}$$

Theorem 2.24 *The graded group associated with the filtration* (2.132) *is isomorphic to*
$H^*\left(\Sigma/G; \mathfrak{L}_u^G\right)$. *In particular there is an (unnatural) isomorphism*

$$K_*\left(B^G\right) \otimes \mathbb{Q} \cong \check{H}^*\left(\Sigma/G; \mathfrak{L}_u^G \otimes \mathbb{Q}\right) \tag{2.133}$$

and

$$K_*\left(B^G\right) \cong \check{H}^*\left(\Sigma/G; \mathfrak{L}_u^G\right)$$

if one of both sides is torsion free.

Proof. Using [26, Section XV.7] we construct a spectral sequence $(F_r^{p,q})_{r\geq 1}^{q\in\mathbb{Z}/2\mathbb{Z}}$ with the following terms:

$$\begin{aligned}
F_1^{p,q} &= K_{p+q}\left(C_0(\Sigma^p/\Sigma^{p-1}; M_N(\mathbb{C}))^G\right)\\
F_2^{p,q} &= H^p(\Sigma/G; \mathcal{K}_u^q)\\
F_\infty^{p,q} &= K_{p+q}^p\left(A^G\right)/K_{p+q}^{p+1}\left(B^G\right)
\end{aligned} \tag{2.134}$$

where $\mathcal{K}_u^q(\sigma) = K_* \left(C(G\sigma; M_N(\mathbb{C}))^G \right)$. Now replace $\mathfrak{L}$ in (2.121) by $\mathfrak{L}_u$ and sum over all q. If we compare the result with $F_1^p = F_1^{1,0} \oplus F_1^{1,1}$ we see that $E_1^p \cong F_1^p$. So we get a diagram like (2.48) :

$$\begin{array}{ccc} F_1^{p,q} & \xrightarrow{d_1^F} & F_1^{p+1,q} \\ \cong & & \cong \\ \prod_{n \in \mathbb{Z}} E_1^{p,q+2n} & \xrightarrow{d_1^E} & \prod_{n \in \mathbb{Z}} E_1^{p+1,q+2n} \end{array} \tag{2.135}$$

The differential d_1^F for F_1 is induced from the construction of a mapping cone of a Puppe sequence in the category of C^*-algebras. This is the noncommutative counterpart of the construction of the differential in cellular homology, so by naturality d_1^F corresponds to d_1^E under the above isomorphism. Therefore the spectral sequences E_r^p and F_r^p are isomorphic, and in particular F_r degenerates for $r \geq 2$. Now the isomorphism (2.133) follows from (2.120).

If either $K_* \left(B^G \right)$ or $\check{H}^* \left(\Sigma / G; \mathfrak{L}_u^G \right)$ is torsion free, then every term $E_\infty^{p,q} \cong F_\infty^{p,q}$ must be torsion free. Hence in this case both $K_* \left(B^G \right)$ and $\check{H}^* \left(\Sigma / G; \mathfrak{L}_u^G \right)$ are free abelian groups, of the same rank. $\square$

The main use of this theorem is really to compute $K_* \left(B^G \right)$, now the extensive machinery of Čech cohomology becomes available.

The requirements (2.126) - (2.128) allow us to construct the sheaves $\mathfrak{L}_u$ and $\mathfrak{L}_u^G$ without reference to the cellular structure of Σ. If we do this for the algebra A^G of (2.116) then the stalk of $\mathfrak{L}_u$ over $x \in X$ is the free abelian group on the (equivalence classes of) irreducible G_x-representations contained in $(\pi_x, \mathbb{C}^N)$ and the G-action on $\mathfrak{L}_u$ is determined by (2.128). A section s is continuous at x if there exists a neighborhood U of x such that $\forall y \in U$:

- $G_y \subset G_x$
- $s(x) = s(y)$ as virtual projective G_y-representations

This $\mathfrak{L}_u$ is a generalization of the sheaf constructed in [4, §2]. Clearly, the subsheaf $\mathfrak{L}_u^G$ of G-invariant continuous sections descends to a sheaf on X/G.

To relate this sheaf to the K-theory and periodic cyclic homology of A^G we need two preparatory results. The first is a weak version of Theorem 2.19, which however does include HP_*.

Lemma 2.25 *Let $U \subset \mathbb{R}^n$ be an open bounded star-shaped set. The Fréchet algebra*
$C_0^\infty(\mathbb{R}^n, \mathbb{R}^n \setminus U)$ belongs to $\mathcal{CIA}$ and

$$K_*(C_0^\infty(\mathbb{R}^n, \mathbb{R}^n \setminus U)) \cong \check{H}^*(\mathbb{R}^n, \mathbb{R}^n \setminus U; \mathbb{Z}) \cong \mathbb{Z}$$

is concentrated in degree n.

Proof. Clearly we may assume that 0 is the center of U. Let P be the point of the n-sphere corresponding to infinity under the stereographic projection $S^n \to \mathbb{R}^n$. By assumption $C_0^\infty(\mathbb{R}^n, \mathbb{R}^n \setminus U) \cong C_0^\infty(S^n, Y)$ for some closed neighborhood Y of P, and we will show that the latter algebra is diffeotopy equivalent to $C_0^\infty(S^n, P)$. Let $(r_t)_{t\in[0,1]}$ be a diffeotopy of smooth maps $S^n \to S^n$ such that

- $\forall t\ : r_t(P) = P$ and $r_t(Y) \subset Y$
- a neighborhood of $-P$ is fixed pointwise by all r_t
- $r_1 = \mathrm{id}_{S^n}$ and $r_0(Y) = P$

To construct such maps, we can require that r_t stabilizes every geodesic from $-P$ to P and declare that furthermore $r_t(Q)$ depends only on t and on the distance from Q to P. Then we only have to pick a suitable smooth function of t and this distance. Given this, consider the Fréchet algebra homomorphisms

$$\begin{aligned} &\phi : C_0^\infty(S^n, P) \to C_0^\infty(S^n, Y) \\ &\phi(f) = f \circ r_0 \\ &i : C_0^\infty(S^n, Y) \to C_0^\infty(S^n, P) \\ &i(f) = f \end{aligned} \tag{2.136}$$

By construction $\phi \circ i$ and $i \circ \phi$ are diffeotopic to the respective identity maps on $C_0^\infty(S^n, Y)$ and $C_0^\infty(S^n, P)$, so these algebras are indeed diffeotopy equivalent. Thus we reduced our task to calculating the K-groups and periodic cyclic homology of $C_0^\infty(S^n, P)$. Fortunately there is an obvious split extension

$$0 \to C_0^\infty(S^n, P) \to C^\infty(S^n) \to \mathbb{C} \to 0 \tag{2.137}$$

which by Proposition 2.20 consists entirely of Fréchet algebras in the category $\mathcal{CIA}$. It is well known that

$$K_*(C^\infty(S^n)) \cong K^*(S^n) \cong \check{H}^*(S^n; \mathbb{Z}) \cong \mathbb{Z}^2 \tag{2.138}$$

with one copy of $\mathbb{Z}$ in degree 0 and the other in degree n. Since $K_*(\mathbb{C}) = K_0(\mathbb{C}) \cong \mathbb{Z}$ the lemma follows from the excision property of the K-functor. $\Box$

Next we prove an equivariant version of the Poincaré lemma.

Lemma 2.26 *Let X, A, G and A^G be as in* (2.116)*, and suppose that X is G-equivariantly contractible to a point $x_0 \in X$. Then A^G is diffeotopy equivalent to its fiber $\mathrm{End}_G\left(\mathbb{C}^N\right)$ over x_0. In particular $K_*\left(A^G\right) = K_0\left(A^G\right)$ is a free abelian group of finite rank, and $A^G \in \mathcal{CIA}$.*

Proof. Our main task is to adjust the u_g suitably. Since X is contractible we can find for every $g \in G$ a function $f_g \in C^\infty(X)$ such that $f_g^{-N} = \det(u_g)$. The G-action does not change when we replace u_g with $f_g u_g$, so we may assume that

$\det(u_g) \equiv 1$, $\forall g \in G$. The premise that (2.115) is a group action guarantees that there is a smooth function $\lambda : G \times G \times X \to \mathbb{C}$ such that

$$u_g(x) u_h(\alpha_g^{-1} x) = \lambda(g, h, x) u_{gh}(x) \tag{2.139}$$

Taking determinants we see that in fact $\lambda(g, h, x)^N \equiv 1$, so λ does not depend on $x \in X$. All the elements of $\alpha(G)$ fix x_0, so the fiber $V_0 = \mathbb{C}^N$ over that point carries a projective G-representation π_0. Thus we are in a position to apply Schur's theorem [113], which says that there exists a finite central extension G^* of G such that π_0 lifts to a representation of G^*. This lift only involves scalar multiples of the $u_g(x_0)$, so it immediately extends to X. Then (2.139) becomes the cocycle relation

$$u_{gh}(x) = u_g(x) u_h(\alpha_g^{-1} x) \tag{2.140}$$

Notice that still $A^{G^*} = A^G$, so without loss of generality we can replace G by G^*.

Now we want to make the $u_g(x)$ independent of $x \in X$. Wassermann [133] indicated how this can be done in the continuous case, and his argument can easily be adapted to our smooth setting. The crucial observation, first made by Rosenberg [109], is that A^G can be rewritten as the image of an idempotent in a larger algebra. This idempotent can then be deformed to one independent of x.

Indeed, let $A \rtimes_\alpha G$ be the crossed product of A and G with respect to the action α of G on X, and $(r_t)_{t \in [0,1]}$ a smooth G-equivariant contraction from X to x_0. (For smooth manifolds the existence of a continuous contraction implies the existence of a smooth one.) Define

$$p_t(x) := |G|^{-1} \sum_{g \in G} u_g(r_t x) g \tag{2.141}$$

Then $p_t \in A \rtimes_\alpha G$ is an idempotent by (2.140), and by Lemma A.2

$$\begin{aligned} \phi_1 &: A^G \to p_1 (A \rtimes_\alpha G) p_1 \\ \phi_1(\sigma) &= p_1 \sigma p_1 \end{aligned} \tag{2.142}$$

is an isomorphism of Fréchet algebras. Clearly the idempotents p_t are all homotopic, so they are conjugate in the completion $C(X; M_N(\mathbb{C})) \rtimes_\alpha G$ of $A \rtimes_\alpha G$, which is a Banach algebra if X is compact. Moreover the standard argument for this, as for example in [10, Proposition 4.3.2], shows that p_0 and p_1 are conjugate by an element of $A \rtimes_\alpha G$. Alternatively we can use the stronger result that homotopic idempotents in Fréchet algebras are conjugate, but this statement is vastly more difficult to prove than its Banach algebra version, cf. [102, Lemmas 1.12 and 1.15]. In any case, we have

$$A^G \cong p_1 (A \rtimes_\alpha G) p_1 \cong p_0 (A \rtimes_\alpha G) p_0 \cong C^\infty(X; \mathrm{End}_{\mathbb{C}}(V_0))^G \tag{2.143}$$

To this last algebra we can apply the obvious diffeotopy $\sigma \to \sigma \circ r_t$. This shows that A^G is diffeotopy equivalent to its fiber $\mathrm{End}_G(V_0)$ over x_0, and the remaining statements on $K_* \left(A^G\right)$ and $HP_* \left(A^G\right)$ follow from the semisimplicity of the finite dimensional algebra $\mathrm{End}_G(V_0)$. □

Now we can prove the main result of this section, which extends Lemma 2.26 to general X.

Theorem 2.27 *Let X, A, G and $A^G = C^\infty(X; M_N(\mathbb{C}))^G$ be as in* (2.116)*, and let $\mathfrak{L}_u^G$ be the sheaf over X/G constructed on page 52. Then there exists a filtration on $K_*\left(A^G\right)$ whose associated graded group is isomorphic to $\check{H}^*\left(X/G; \mathfrak{L}_u^G\right)$, and the Chern character induces an isomorphism*

$$K_*\left(A^G\right) \otimes \mathbb{C} \xrightarrow{\sim} HP_*\left(A^G\right)$$

Moreover $K_*\left(A^G\right)$ is a finitely generated abelian group whenever X is compact.

Proof. All our arguments will depend on the existence of a specific cover of X. To construct it we use a theorem of Illman [62], which states that X has a smooth equivariant triangulation. In slightly more down-to-earth language this means (among others) that there exists a countable, locally finite simplicial complex Σ in a finite dimensional orthogonal representation space V of G, and a G-equivariant homeomorphism $\psi : \Sigma \to X$. Moreover ψ is smooth as a map from a subset of V to X, and its restriction to any simplex σ of Σ is an embedding. In particular Σ is a G-CW complex, so the assertion on the Čech cohomology of $\mathfrak{L}_u^G$ follows from Theorems 2.13 and 2.24.

For a simplex σ we put

$$U'_\sigma := \{v \in \Sigma : d(v, \sigma) \leq r_\sigma\} \tag{2.144}$$

where d is the Euclidean distance in V. We require that the radius r_σ depends only on the G-orbit of σ and that $r_\tau > r_\sigma > 0$ if τ is a face of σ. The orthogonality of the action of G on V guarantees that

$$gU'_\sigma = U'_{g\sigma} \quad \text{and} \quad U'_\sigma \cap U'_\tau \subset U'_{\sigma\cap\tau}$$

if we take our radii small enough. Let D'_σ be the union, over all faces τ of σ, of the U'_τ, and G_σ the stabilizer of σ in G. From the above we deduce that $U'_\sigma \setminus D'_\sigma$ is G_σ-equivariantly retractible to $\sigma \setminus D'_\sigma$.

Now we abbreviate $U_\sigma := \psi(U'_\sigma)$ and $D_\sigma := \psi(D'_\sigma)$, so that $\{U_\sigma : \sigma \text{ simplex of } \Sigma\}$ is a closed G-equivariant cover of X. Let X_m be the union of all those U_σ for which $m + \dim \sigma \leq \dim X$. It is a closed subvariety (with boundary and corners) of X and it is stable under the action of G. Define the following G-stable ideals of A:

$$I_m := \{a \in A : a\big|_{X_m} = 0\} \tag{2.145}$$

By [128, Théorème IX.4.3]

$$0 \to I_m \to A = C^\infty(X; M_N(\mathbb{C})) \to C^\infty(X_m; M_N(\mathbb{C})) \to 0 \tag{2.146}$$

is an admissible extension of Fréchet algebras. Using the finiteness of G we see that I_m^G is an admissible ideal in I_{m+1}^G and that

$$I_{m+1}^G / I_m^G \cong (I_{m+1}/I_m)^G \cong C_0^\infty(X_m, X_{m+1}; M_N(\mathbb{C}))^G \tag{2.147}$$

In order to apply Lemma 2.3 to the sequence

$$0 = I_0^G \subset I_1^G \subset \cdots \subset I_{\dim X}^G \subset I_{1+\dim X}^G = A^G \tag{2.148}$$

we only have to show that the algebras in (2.147) are in the category $\mathcal{CIA}$. In fact, since $\overline{U_\sigma \setminus D_\sigma} \cap \overline{U_\tau \setminus D_\tau} = \emptyset$ if $\dim\sigma = \dim\tau$ and $\sigma \neq \tau$, we have an isomorphism

$$I_{m+1}/I_m \cong \prod_{m+\dim\sigma=\dim X} C_0^\infty(U_\sigma, D_\sigma; M_N(\mathbb{C})) \tag{2.149}$$

Now G permutes the simplices in this product, so

$$I_{m+1}^G / I_m^G \cong \prod_{\sigma\in L_m} C_0^\infty(U_\sigma, D_\sigma; M_N(\mathbb{C}))^{G_\sigma} \tag{2.150}$$

where L_m is a set of representatives of the simplices of dimension $\dim X - m$ modulo the action of G. Invoking the additivity of K_* and HP_* we reduce our task to verifying that every factor of (2.150) belongs to $\mathcal{CIA}$.

If $m = \dim X$ then D_σ is empty and we see from Lemma 2.26 that $C^\infty(U_\sigma; M_N(\mathbb{C}))^{G_\sigma}$ has the required property.

For smaller m there also exists (for every σ) a G_σ-equivariant contraction of U_σ to a point $x_\sigma \in \psi(\sigma)$. Thus we can follow the proof of Lemma 2.26 up to equation (2.143), where we find that the factor of (2.150) corresponding to σ is isomorphic to $C_0^\infty(U_\sigma, D_\sigma; \mathrm{End}_\mathbb{C}(V_\sigma))^{G_\sigma}$. Here (π_σ, V_σ) denotes the projective G_σ-representation over the point x_σ. Using the G_σ-equivariant retraction of $U_\sigma \setminus D_\sigma$ to $\psi(\sigma \setminus D'_\sigma)$ we see that this algebra is diffeotopy equivalent to $C_0^\infty(\sigma, \sigma\cap D'_\sigma) \otimes \mathrm{End}_{G_\sigma}(V_\sigma)$. The right hand side of this tensor product has finite dimension and is semisimple, so by the stability of $\mathcal{CIA}$ it presents no problems. Seen from its barycenter $\sigma \setminus D'_\sigma$ is star-shaped, hence by Lemma 2.25 the left hand side is also in the category $\mathcal{CIA}$.

We conclude that all the algebras in (2.147) and (2.150) are indeed objects of $\mathcal{CIA}$, so Lemma 2.3 can be applied to (2.148) to prove that $A^G \in \mathcal{CIA}$.

Note that the simplicial complex Σ has only finitely many vertices if X is compact, so then all the above direct products are in fact finite and $K_*\left(A^G\right)$ is finitely generated. □

It is clear from the proofs of Lemma 2.26 and Theorem 2.27 that many similar Fréchet algebras are also in $\mathcal{CIA}$. For example if Y is a closed submanifold of X then the algebra

$$B = \{f \in C^\infty(X; M_2(\mathbb{C})) : f(y) \text{ diagonal } \forall y \in Y\} \tag{2.151}$$

is in $\mathcal{CIA}$, as can be seen from the admissible extension

$$0 \to C_0^\infty(X, Y; M_2(\mathbb{C})) \to B \to C^\infty(Y)^2 \to 0 \tag{2.152}$$

One might even study arbitrary Fréchet algebras that are finitely generated over $C^\infty(Y)$, with Y an orbifold. Although it is not unlikely that these are all in the category $\mathcal{CIA}$, it seems that a substantial generalization of Lemma 2.26 is needed to show this.

Since the periodic cyclic homology of A^G is finite dimensional and can be computed in terms of Čech cohomology, it is not surprising that the Künneth formula is an isomorphism for such algebras

Corollary 2.28 *Let* (X, A, G, u) *and* (X', A', G', u') *both be sets of data like we used in* (2.116). *There is a natural isomorphism of graded vector spaces*

$$HP_*\big(A^G\big) \otimes HP_*\big(A'^{G'}\big) \xrightarrow{\sim} HP_*\big(A^G \widehat{\otimes} A'^{G'}\big)$$

Proof. By Theorem 2.27 we have

$$\begin{aligned}
HP_*\big(A^G\big) &\cong \check H^*\big(X/G; \mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}\big) \\
HP_*\big(A'^{G'}\big) &\cong \check H^*\big(X'/G'; \mathfrak{L}_{u'}^{G'} \otimes_{\mathbb{Z}} \mathbb{C}\big) \\
HP_*\big(A^G \widehat{\otimes} A'^{G'}\big) &\cong \check H^*\Big((X \times X')/(G \times G'); (\mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathfrak{L}_{u'}^{G'})^{G \times G'} \otimes_{\mathbb{Z}} \mathbb{C}\Big) \\
&\cong \check H^*\Big(X/G \times X'/G'; (\mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}) \otimes_{\mathbb{C}} (\mathfrak{L}_{u'}^{G'} \otimes_{\mathbb{Z}} \mathbb{C})\Big)
\end{aligned} \tag{2.153}$$

According to [47, §6.3] there is a natural map of Čech complexes

$$\begin{aligned}
C^*\big(X/G; \mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}\big) \otimes_{\mathbb{C}} C^*\big(X'/G'; \mathfrak{L}_{u'}^{G'} \otimes_{\mathbb{Z}} \mathbb{C}\big) \quad &\longrightarrow \\
\check C^*\Big(X/G \times X'/G'; (\mathfrak{L}_u^G \otimes_{\mathbb{Z}} \mathbb{C}) \otimes_{\mathbb{C}} (\mathfrak{L}_{u'}^{G'} \otimes_{\mathbb{Z}} \mathbb{C})\Big)&
\end{aligned}$$

Because all three cohomology groups are finite dimensional vector spaces the abstract Künneth theorem [26, Theorem VI.3.1] tells us that

$$HP_*\big(A^G\big) \otimes HP_*\big(A'^{G'}\big) \cong HP_*\big(A^G \widehat{\otimes} A'^{G'}\big)$$

The construction of the map Θ in [44, p. 211] is also possible in the topological setting, and yields a natural map

$$\Theta : HP_*\big(A^G\big) \otimes HP_*\big(A'^{G'}\big) \to HP_*\big(A^G \widehat{\otimes} A'^{G'}\big)$$

Although the isomorphisms (2.153) are not natural, they come from certain filtrations of the underlying topological spaces, which is enough to ensure that Θ is also an isomorphism. $\Box$

Notice also the similarity between $\mathfrak{L}_u^G$ and the sheaf $\mathfrak{A}$ contructed on page 25. Suppose that our manifold X has the underlying structure of a complex nonsingular affine variety X_{alg}, that the α_g are automorphisms of X_{alg} and that the u_g are invertibles in

$$A_{alg} := \mathcal{O}(X_{alg}) \otimes M_N(\mathbb{C}) = M_N(\mathcal{O}(X_{alg})) \tag{2.154}$$

Then

$$ga(x) = u_g(x)a(\alpha_g^{-1}x)u_g^{-1}(x) \tag{2.155}$$

defines an action of G on A_{alg}, and A_{alg}^G has the same irreducible representations as A^G. Applying the recipe on page 25 to the finite type algebra A_{alg}^G, we see that the bases of the stalks $\mathfrak{L}_u^G(Gx)$ and $\mathfrak{A}(Gx)$ can be identified, and that the requirements for continuity of sections on page 25 reduce to those on page 52. Therefore

$$\mathfrak{A} = \mathfrak{L}_u^G \otimes \mathbb{C} \tag{2.156}$$

This insight, together with Theorems 2.8 and 2.27, was the inspiration for an explicit comparison theorem between algebraic and topological periodic cyclic homology, analogous to (2.26).

Theorem 2.29 *The inclusion $A_{alg}^G \to A^G$ induces an isomorphism of finite dimensional vector spaces*

$$HP_* \left(A_{alg}^G\right) \xrightarrow{\sim} HP_* \left(A^G\right)$$

Proof. After noticing that by Theorem 2.8 the left hand side has finite dimension, we introduce some notations. Let Y be any complex algebraic variety, Z a subvariety and V a complex vector space. Like for smooth functions we write

$$\mathcal{O}_0(Y, Z) = \{f \in \mathcal{O}(Y) : f\big|_Z = 0\} \tag{2.157}$$

$$\mathcal{O}_0(Y, Z; V) = \mathcal{O}_0(Y, Z) \otimes V \tag{2.158}$$

Start with the finite collection $\mathcal{L}$ of all irreducible components of the X_{alg}^g, as g runs over G. Extend this to a collection $\{V_j\}_j$ of subvarieties of X_{alg} by including all irreducible components of the intersection of any subset of $\mathcal{L}$. Notice that

$$\dim \left(X^g \cap X^h\right) < \max \left\{\dim X^g ,\, \dim X^h\right\} \tag{2.159}$$

if $\alpha_g \neq \alpha_h$. Define G-stable subvarieties

$$X_p = \bigcup_{j:\dim V_j \leq p} V_j \tag{2.160}$$

and construct the ideals

$$\begin{array}{lllll} I_p & := & \{a \in A_{alg}^G : a(X_p) = 0\} & \cong & \mathcal{O}_0(X, X_p; M_N(\mathbb{C}))^G \\ J_p & := & \{a \in A^G : a(X_p) = 0\} & \cong & C_0^\infty(X, X_p; M_N(\mathbb{C}))^G \end{array} \tag{2.161}$$

From (2.146) and (2.147) we see that all the ideals J_p are admissible in A^G, and by Theorem 2.27 they are in $\mathcal{CIA}$. By Lemma 2.3 it suffices to show that for every p the inclusion

$$\mathcal{O}_0(X_p, X_{p-1}; M_N(\mathbb{C}))^G \cong I_{p-1}/I_p \to C_0^\infty(X_p, X_{p-1}; M_N(\mathbb{C}))^G \cong J_{p-1}/J_p \tag{2.162}$$

induces an isomorphism on periodic cyclic homology. These algebras have the same primitive ideal spectrum, namely $Y_p \setminus Z_p$, where

$$\begin{aligned} Y_p &= \text{Prim}\left(A^G/J_p\right) &= \text{Prim}\left(A^G_{alg}/I_p\right) \\ Z_p &= \text{Prim}\left(A^G/J_{p-1}\right) &= \text{Prim}\left(A^G_{alg}/I_{p-1}\right) \end{aligned} \tag{2.163}$$

Because all the representations $(\pi_x, \mathbb{C}^N)$ are completely reducible

$$A^G_{alg} = I_0 \supset I_1 \supset \cdots \supset I_N = 0 \tag{2.164}$$

is an abelian filtration of A^G_{alg}, in the sense of [77, Definition 3], and

$$Z\left(A^G_{alg}/I_p\right) \cap I_{p-1}/I_p = Z\left(I_{p-1}/I_p\right) \cong \mathcal{O}_0(Y_p, Z_p) \tag{2.165}$$

This gives alternative descriptions

$$Y_p = \text{Max}\left(Z\left(A^G_{alg}/I_p\right)\right) \tag{2.166}$$

$$Z_p = \{I \in Y_p : Z\left(I_{p-1}/I_p\right) \subset I\} \tag{2.167}$$

and the proof of [77, Theorem 10] shows that there are natural isomorphisms

$$HP_*\left(I_{p-1}/I_p\right) \cong HP_*\left(Z\left(I_{p-1}/I_p\right)\right) \cong \check{H}^*(Y_p, Z_p; \mathbb{C}) \tag{2.168}$$

To get something similar on the topological side we turn to K-theory, knowing already that $J_{p-1}/J_p \in \mathcal{CIA}$. Moreover this algebra is dense in $C_0(X_p, X_{p-1}; \mathbb{C})^G$, so from theorem 2.13 we get

$$HP_*(J_{p-1}/J_p) \cong K_*\left(C_0(X_p, X_{p-1}; \mathbb{C})^G\right) \otimes \mathbb{C} \tag{2.169}$$

Since $u_g \in A_{alg}$ the type of $(\pi_x, \mathbb{C}^N)$ as a projective G_x-representation cannot change along the (connected or irreducible) components of $X^{G_x} \cap X_p \setminus X_{p-1}$. It follows from this, (2.159) and Theorem 2.24 that the inclusions

$$C_0^\infty(Y_p, Z_p) \to C_0(Y_p, Z_p) \cong Z\left(C_0(X_p, X_{p-1}; \mathbb{C})^G\right) \to C_0(X_p, X_{p-1}; \mathbb{C})^G \tag{2.170}$$

induce isomorphisms on K-theory with rational coefficients. Moreover

$$K_*\left(C_0^\infty(Y_p, Z_p)\right) \otimes \mathbb{C} \cong HP_*\left(C_0^\infty(Y_p, Z_p)\right) \cong \check{H}^*(Y_p, Z_p; \mathbb{C}) \tag{2.171}$$

We put all the above in a diagram

$$\begin{array}{ccccc} HP_*(I_{p-1}/I_p) & \leftarrow & HP_*(\mathcal{O}_0(Y_p, Z_p)) & \rightarrow & \check{H}^*(Y_p, Z_p; \mathbb{C}) \\ \downarrow & & \downarrow & & \| \\ HP_*(J_{p-1}/J_p) & \cong & HP_*\left(C_0^\infty(Y_p, Z_p)\right) & \rightarrow & \check{H}^*(Y_p, Z_p; \mathbb{C}) \end{array} \tag{2.172}$$

The horizontal arrows are all natural isomorphisms, so the diagram commutes and the vertical arrows are isomorphisms as well. $\square$

Chapter 3

Affine Hecke algebras

Here we commence our study of the main subjects of this thesis, affine Hecke algebras. They first appeared in the representation theory of certain topological groups, but that will discussed only in the next chapter. Instead we consider Hecke algebras as deformations of the group algebra of a Weyl group. More precisely, Iwahori-Hecke algebras are deformations of Coxeter groups, while affine Hecke algebras are deformations of affine Weyl groups. This deformation is achieved as follows. Let s be a typical generator of a Coxeter group W. There is a relation

$$(T_s - 1)(T_s + 1) = 0$$

in $\mathbb{Z}[W]$. We replace this relation by

$$(T_s - q(s))(T_s + 1) = 0$$

where the label $q(s)$ can be any element of a commutative ring. In general it is possible to have different labels for different generators. Section 3.1 is mainly dedicated to making this precise, by providing the definitions of root data, label functions and related objects.

The affine Hecke algebra associated with these data will be denoted by $\mathcal{H}(\mathcal{R}, q)$. If the labels are all positive then one can complete this to a C^*-algebra $C_r^*(\mathcal{R}, q)$ or, more subtly, to a Schwartz algebra $\mathcal{S}(\mathcal{R}, q)$.

We have two main goals in this chapter. On one hand we want to prepare everything for a careful study of deformations in the parameters q, which we will undertake in Chapter 5. This dictates that we should provide explicit formulas whenever possible.

On the other hand we would like to apply the ideas developed in Chapter 2 to affine Hecke algebras. Therefore it is imperative that we get a clear picture of representations and the spectrum of $\mathcal{H}(\mathcal{R}, q)$. This is provided by the work of Opdam [98] on the Plancherel measure and the Fourier transform for affine Hecke algebras. It turns out that $\text{Prim}(\mathcal{H}(\mathcal{R}, q))$ is a non-separated variety lying over a complex torus modulo a finite group, see Theorems 3.24 and 3.25. Similarly

$\mathrm{Prim}(\mathcal{S}(\mathcal{R},q))$ is a non-Hausdorff orbifold. On these spaces $\mathcal{H}(\mathcal{R},q)$ is related to polynomial functions, $\mathcal{S}(\mathcal{R},q)$ to smooth functions and $C_r^*(\mathcal{R},q)$ to continuous functions. In fact $\mathcal{S}(\mathcal{R},q)$ is isomorphic to an algebra of the type that we studied in Section 2.5

However, this is not enough, we also need the Langlands classification for $\mathcal{H}(\mathcal{R},q)$. In Section 3.2 we explain that it says essentially that $\mathrm{Prim}(\mathcal{S}(\mathcal{R},q))$ is a deformation retract of $\mathrm{Prim}(\mathcal{H}(\mathcal{R},q))$. The final form in which we will actually apply this is the rather technical parametrization of irreducible $\mathcal{H}(\mathcal{R},q)$-representations Theorem 3.31. With all these preparations, and the interpretation of periodic cyclic homology as a cohomology theory on primitive ideal spectra, we can prove the main theorem of this chapter. It says that there are natural isomorphisms

$$HP_*(\mathcal{H}(\mathcal{R},q)) \cong HP_*(\mathcal{S}(\mathcal{R},q)) \cong K_*(C_r^*(\mathcal{R},q)) \otimes \mathbb{C} \tag{3.1}$$

3.1 Definitions of Hecke algebras

We give precise definitions of (most of) the objects needed to construct Hecke algebras. We do this both for Iwahori-Hecke algebras associated to Coxeter groups and for affine Hecke algebras associated to root data.

A Coxeter system (W,S) consists of a finite set S such that W is the group generated by S, subject only to the relations

$$(s_i s_j)^{m_{ij}} = e$$

for certain $m_{ij} \in \{1,2,\ldots,\infty\}$ such that

- $m_{ij} = 1$ if and only if $s_i = s_j$
- $m_{ji} = m_{ij}$

Because the most important examples are Weyl groups, we denote the Coxeter group by W and call the elements of S simple reflections. This simple definition still imposes a lot of structure, and indeed Coxeter groups have been studied deeply. The most relevant results for us can be found for example in [61].

A Coxeter system is completely determined by its Coxeter graph. This is a graph whose vertices correspond to elements of S. There is an edge between s_i and s_j if and only if $m_{ij} \geq 3$, and it is labelled by this number m_{ij}. A Coxeter system is called irreducible if its Coxeter graph is connected.

Just as for any finitely generated group, there is a natural length function ℓ on W, which assigns length 1 to any $s \in S$. To define it, pick $w \in W$ and write it as

$$w = s_1 \cdots s_r$$

If $r \geq 0$ is as small as possible, then this is called a reduced expression for w and $\ell(w) = r$. Notice that $\ell(w^{-1}) = \ell(w)$ since all the simple reflections have order 2.

Depending on the numbers m_{ij}, (W, ℓ) can be finite, of polynomial growth or of exponential growth.

If $P \subset S$ then $W_P := \langle P \rangle$ is very special kind of subgroup of W, a standard parabolic subgroup. In general a parabolic subgroup of W is conjugate to some W_P. The pair (W_P, P) is a Coxeter system in its own right, and its length function agrees with the restriction of ℓ to W_P. Every right coset wW_P contains a unique element of minimal length, so there is a canonical set of representatives W^P for W/W_P. Moreover, if $\{P_j\}_j$ are the connected components of the Coxeter graph of (W, S) then $W = \bigoplus_j W_{P_j}$. Hence one can learn a lot about Coxeter systems by studying only irreducible ones.

Let $q : W \to \mathbf{k}$ be a function from W to a commutative unital ring $\mathbf{k}$ which is length-multiplicative, i.e.

$$q(wv) = q(w)q(v) \quad \text{if} \quad \ell(wv) = \ell(w) + \ell(v) \tag{3.2}$$

This is equivalent to giving a map $q : S \to \mathbf{k}$ such that $q(s_i) = q(s_j)$ whenever m_{ij} is odd.

We say that q is an equal label function in the special case that $q(s_i) = q(s_j)$ for all $s_i, s_j \in S$. In this situation it is customary to denote $q(s_i)$ simply by q, so that

$$q(w) = q^{\ell(w)}$$

The Iwahori-Hecke algebra $\mathcal{H}(W, q) = \mathcal{H}_{\mathbf{k}}(W, q)$ associated to the Coxeter group W (S is usually suppressed from the notation) and the label function q is an associative $\mathbf{k}$-algebra which is a free $\mathbf{k}$-module with bases $\{T_w : w \in W\}$ and fulfills the multiplication rules

$$T_w T_v = T_{vw} \qquad \text{if} \quad \ell(wv) = \ell(w) + \ell(v) \tag{3.3}$$

$$T_s T_s = (q(s) - 1)T_s + q(s)T_e \qquad \text{if} \quad s \in S \tag{3.4}$$

It is proved in [61, Section 7.1] that such an object exists and is uniquely determined by these conditions.

Notice that if $q(s) = 1 \, \forall s \in S$, then $\mathcal{H}(W, q) = \mathbf{k}[W]$ is the group algebra of W over $\mathbf{k}$. Any standard parabolic subgroup V gives rise to a parabolic subalgebra $\mathcal{H}\left(V, q\big|_V\right)$, which as an A-module has bases $\{T_v : v \in V\}$.

Two choices of $\mathbf{k}$ are especially important. The first is simply $\mathbf{k} = \mathbb{C}$. For the second, write $q_i = q(s_i)$ if $s_i \in S$. Let $q_i^{1/2}$ be indeterminates satisfying $\left(q_i^{1/2}\right)^2 = q(s_i)$, and put

$$\mathbf{k} = \mathbb{Z}\left[\left\{q_i^{1/2}, q_i^{-1/2}\right\}_{s_i \in S}\right] \tag{3.5}$$

In this case we have

$$T_s^{-1} = q(s)^{-1}T_s + (q(s)^{-1} - 1)T_e \quad s \in S \tag{3.6}$$

So T_w is invertible in $\mathcal{H}(W,q)$ for any $w \in W$. Indeed, if $w = s_1 \cdots s_r$ is a reduced expression, then

$$T_w^{-1} = T_{s_r}^{-1} \cdots T_{s_1}^{-1}$$

To explain the introduction of the square roots we write

$$N_w = q(w)^{-1/2} T_w \tag{3.7}$$

These elements form again a bases of $\mathcal{H}(W,q)$, while (3.3) and (3.4) become somewhat more manageable:

$$N_w N_v = N_{vw} \qquad \text{if} \quad \ell(wv) = \ell(w) + \ell(v) \tag{3.8}$$

$$\big(N_{s_i} - q_i^{1/2}\big)\big(N_{s_i} + q_i^{-1/2}\big) = 0 \qquad \text{if} \quad s_i \in S \tag{3.9}$$

We generalize the notion of an Iwahori-Hecke algebra as follows. Let Ω be a group acting on W, and consider the semidirect product $W \rtimes \Omega$. Assume that $\forall w \in W, \omega \in \Omega$

$$\ell(\omega w \omega^{-1}) = \ell(w) \quad \text{and} \quad q(\omega w \omega^{-1}) = q(w)$$

so that we can extend ℓ and q to $W \rtimes \Omega$ by

$$\ell(w\omega) := \ell(w) \quad \text{and} \quad q(w\omega) := q(w)$$

In this situation the rules (3.3) and (3.4) again define a unique associative **k**-algebra with basis $\{T_g : g \in W \rtimes \Omega\}$. Such algebras are called extended Iwahori-Hecke algebras. Note that this is extremely general, since Ω can be nearly any group. If we do not want to get too far away from proper Iwahori-Hecke algebras we have to impose some restrictions on this group.

We do this in the setting of an important class of such algebras, namely affine Hecke algebras. We will mainly follow the notation of [98], which implies that sometimes we attach a subscript 0 to a finite object, to distinguish it from its affine counterpart.

First we introduce root data, for which we need the following objects.

- X and Y are free abelian groups of the same finite rank, and $\langle \cdot\,,\cdot \rangle : X \times Y \to \mathbb{Z}$ is a perfect pairing between them
- $R_0 \subset X$ and $R_0^\vee \subset Y$ are finite subsets with a given bijection $\alpha \to \alpha^\vee$

The elements of R_0 are called roots and the elements and those of $R_0^\vee$ are called coroots. Define endomorphisms s_α of X and $s_\alpha^\vee$ of Y by

$$s_\alpha(x) = x - \langle x\,, \alpha^\vee \rangle \alpha \tag{3.10}$$

$$s_\alpha^\vee(y) = y - \langle \alpha\,, y \rangle \alpha^\vee \tag{3.11}$$

For every $\alpha \in R_0$ we impose the conditions

- $\langle \alpha\,, \alpha^\vee \rangle = 2$

- $s_\alpha\big(R_0\big) = R_0$
- $s_\alpha^\vee\big(R_0^\vee\big) = R_0^\vee$

A quadruple $\mathcal{R} = (X, Y, R_0, R_0^\vee)$ with these properties is a root datum. Furthermore it is

- reduced if $\mathbb{Z}\alpha \cap R_0 = \{\alpha, -\alpha\}\ \forall \alpha \in R_0$
- semisimple if $R_0^\perp = \{0\} \subset Y$

Let $Q := \mathbb{Z}R_0 \subset X$ and $Q^\vee := \mathbb{Z}R_0^\vee \subset Y$ be the root lattice and the coroot lattice. The weight lattice is $\mathrm{Hom}_\mathbb{Z}(Q^\vee, \mathbb{Z}) \supset Q$. If $\mathcal{R}$ is semisimple then it contains X.

These lattices do not necessarily span $\mathfrak{t}^* := X \otimes \mathbb{R}$ and its linear dual $\mathfrak{t} := Y \otimes \mathbb{R}$, but except for that R_0 and $R_0^\vee$ are root systems in the classical sense, and they are dual to each other.

Recall that a basis of R_0 is a linearly independent subset F_0 such that every $\alpha \in R_0$ can be written as

$$\alpha = \sum_{\beta \in F_0} n_\beta \beta$$

where either

$$n_\beta \in \mathbb{Z}_{\geq 0}\ \forall \beta \in F_0 \quad \text{or} \quad n_\beta \in \mathbb{Z}_{\leq 0}\ \forall \beta \in F_0$$

This gives a partition $R_0 = R_0^+ \cup R_0^-$. We call the roots in F_0 simple, those in R_0^+ positive and those in R_0^- negative. Bases always exist, and we will assume that one is given with the root datum, which we will henceforth write as

$$\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0) \tag{3.12}$$

There are several ways to construct new root data from a given one. Firstly, we can take the direct product of two root data:

$$\mathcal{R} \times \mathcal{R}' := (X \times X', Y \times Y', R_0 \cup R_0', R_0^\vee \cup {R_0'}^\vee, F_0 \cup F_0') \tag{3.13}$$

In particular we always have the "trivial one dimensional extension"

$$\mathcal{R} \times \mathbb{Z} := (X \times \mathbb{Z}, Y \times \mathbb{Z}, R_0, R_0^\vee, F_0) \tag{3.14}$$

Alternatively one can simply exchange the roles of X and Y. Then

$$\mathcal{R}^\vee = (Y, X, R_0^\vee, R_0, F_0^\vee) \tag{3.15}$$

is the dual root datum of $\mathcal{R}$. Furthermore, for $P \subset F_0$ write

$$R_P = \mathbb{Q}P \cap R_0 \quad \text{and} \quad R_P^\vee = \mathbb{Q}P^\vee \cap R_0^\vee$$

This R_P is a parabolic root subsystem of R_0, and with it we associate the root datum

$$\mathcal{R}^P := \big(X, Y, R_P, R_P^\vee, P\big) \tag{3.16}$$

Finally, define

$$
\begin{aligned}
X_P &= X/\big(X \cap (P^\vee)^\perp\big) & Y_P &= Y \cap \mathbb{Q}P^\vee \\
X^P &= X/\left(X \cap \mathbb{Q}P\right) & Y^P &= Y \cap P^\perp \\
\mathcal{R}_P &= \big(X_P, Y_P, R_P, R_P^\vee, P\big)
\end{aligned}
\tag{3.17}
$$

Let us have a look at the various Weyl groups associated to $\mathcal{R}$. Clearly every s_α induces a reflection of $\mathfrak{t}$, and they generate a finite Coxeter group W_0, the Weyl group of R_0. In accordance with the terminology for roots, we can take

$$S_0 = \{s_\alpha : \alpha \in F_0\} \tag{3.18}$$

as the set of simple reflections. The action of W_0 on X is by group homomorphisms, we can construct the semidirect product

$$W = W(\mathcal{R}) := X \rtimes W_0 \tag{3.19}$$

By identifying $x \in X$ with the translation t_x, we can regard W as a group of affine linear translations of X. This W is the Weyl group of $\mathcal{R}$, and in the same way we construct its normal subgroup

$$W_{\text{aff}} := Q \rtimes W_0 \tag{3.20}$$

which we call the affine Weyl group of either $\mathcal{R}$ or R_0. Although W may be isomorphic to $\mathbb{Z}^n$ (if $R_0 = \emptyset$), W_{aff} is always a Coxeter group, and it is possible to extend S_0 to a set of Coxeter generators for W_{aff}. To do so, observe that pairing with F_0 defines a partial ordering on Y, and let $F_m^\vee$ be the set of maximal elements of $R_0^\vee$ for this ordering. For every $\alpha^\vee \in F_m^\vee$ we consider the affine reflection

$$t_\alpha s_\alpha : x \to x - \langle x\,,\,\alpha^\vee\rangle \alpha + \alpha \tag{3.21}$$

in the hyperplane $\{x \in X : \langle x\,,\,\alpha^\vee\rangle = 1\}$. The desired set of generators is

$$S_{\text{aff}} := S_0 \cup \ \{t_\alpha s_\alpha : \alpha^\vee \in F_m^\vee\} \tag{3.22}$$

The Coxeter graph of $\mathcal{R}$ is that of $(W_{\text{aff}}, S_{\text{aff}})$. It is obtained from the Coxeter graph of $\big(W_0, S_0\big)$ by suitably adding exactly one vertex to every connected component. Hence a standard parabolic subgroup $W_P \subset W_{\text{aff}}$ is infinite whenever $|P| > |F_0|$. If $(W_{\text{aff}}, S_{\text{aff}})$ is irreducible, then any proper parabolic subgroup of W_{aff} is finite.

We have a length function ℓ on W_{aff}, which however does not immediately extend to W. To achieve that we need a different characterization of ℓ. It is well known that in the finite Weyl group W_0 the length of w is the number of positive roots made negative by w :

$$\ell(w) = \#\left\{\alpha \in R_0^+ : w(\alpha) \in R_0^-\right\} = \left|R_0^+ \cap w^{-1}\big(R_0^-\big)\right| \tag{3.23}$$

The same holds in W_{aff}, one only has to replace R_0 by an affine root system. [65, Propostion 1.23] says

Proposition 3.1 *The following formula defines a natural extension of ℓ to W:*

$$\ell(wt_x) = \sum_{\alpha \in R_0^+ \cap w^{-1}(R_0^-)} |\langle x\,,\,\alpha^\vee\rangle + 1| + \sum_{\alpha \in R_0^+ \cap w^{-1}(R_0^+)} |\langle x\,,\,\alpha^\vee\rangle| \qquad w \in W_0, x \in X$$

Moreover the set

$$\Omega := \{\omega \in W : \ell(\omega) = 0\}$$

is a subgroup of W, complementary to W_{aff}:

$$W = W_{\text{aff}} \rtimes \Omega$$

Hence we can say that reduced expression for $w \in W$ is a decomposition $w = v\omega$ or $w = \omega v$, where $\omega \in \Omega$ and $v \in W_{\text{aff}}$ is written in a reduced way. Define

$$\begin{aligned} X^+ &:= \{x \in X : \langle x\,,\,\alpha^\vee\rangle \geq 0\ \forall \alpha \in F_0\} \\ X^- &:= \{x \in X : \langle x\,,\,\alpha^\vee\rangle \leq 0\ \forall \alpha \in F_0\} = -X^+ \end{aligned} \tag{3.24}$$

It is well known that

$$\begin{aligned} Z(W) &= X^+ \cap X^- \\ W_0 X^+ &= X \end{aligned} \tag{3.25}$$

It follows immediately from Proposition 3.1 that for $x \in X^+$

$$\ell(t_x) = \sum_{\alpha \in R_0^+} \langle x\,,\,\alpha^\vee\rangle \tag{3.26}$$

so ℓ is additive on the abelian semigroup $X^+ \subset W$.

Our definition of a label function is more restrictive than that for an extended Iwahori-Hecke algebra. It is a function $q : W \to \mathbb{C}^\times$ such that $\forall w, v \in W$, $\omega \in \Omega$

- $q(\omega) = 1$
- $q(wv) = q(w)q(v) \quad \text{if} \quad \ell(wv) = \ell(w) + \ell(v)$

The set of label functions is in bijection with the set of functions $q : S_{\text{aff}} \to \mathbb{C}^\times$ such that $q(s_i) = q(s_j)$ whenever s_i and s_j are conjugate in W.

One can also describe q in terms of a function on a root system associated to $\mathcal{R}$. Assume for simplicity that R_0 is reduced, and define a non-reduced root sytem in X :

$$R_{nr} := R_0 \cup \{2\alpha : \alpha^\vee \in 2Y\} \tag{3.27}$$

Obviously we write $(2\alpha)^\vee = \alpha^\vee/2$, and we let R_1 be the root system of long roots in R_{nr} :

$$R_1 := \{\alpha \in R_{nr} : \alpha^\vee \notin 2Y\} \tag{3.28}$$

The set of simple long roots is

$$F_1 := \{\alpha \in R_1 : \alpha \in F_0 \text{ or } \alpha/2 \in F_0\} \tag{3.29}$$

For $\alpha \in R_0 \setminus R_1$ and $\beta \in R_0 \cap R_1$ we put

$$q_{\alpha^\vee} = q(t_\alpha s_\alpha) \qquad q_{\alpha^\vee/2} = q(s_\alpha) q(t_\alpha s_\alpha)^{-1} \tag{3.30}$$

$$q_{\beta^\vee} = q(s_\beta) = q(t_\beta s_\beta) \qquad q_{\beta^\vee/2} = q_{2\beta^\vee} = 1 \tag{3.31}$$

Since $\beta^\vee/2 \notin Y$ and $2\beta^\vee \notin R_{nr}^\vee$, we can ignore them here. However we include these conventions because they will simplify some future notations. Clearly $q : R_{nr}^\vee \to \mathbb{C}^\times$ is W_0-invariant, and conversely every W_0-invariant function $R_{nr}^\vee \to \mathbb{C}^\times$ determines a unique label function $W \to \mathbb{C}^\times$ as above. This correspondence implies the following formulas [97, Corollaries 1.3 and 1.5]

Corollary 3.2 *For $w \in W_0$ and $x \in X^+$*

$$q(w) = \prod_{\alpha \in R_{nr}^+ \cap w^{-1}(R_{nr}^-)} q_{\alpha^\vee} \quad \text{and} \quad q(t_x) = \prod_{\alpha \in R_{nr}^+} q_{\alpha^\vee}^{\langle x \, , \, \alpha^\vee \rangle}$$

Given a reduced root datum $\mathcal{R} = (X, Y, R_0, R_0^\vee, F_0)$ and a label function q, we define the affine Hecke algebra $\mathcal{H}(\mathcal{R}, q)$ as the unique associative $\mathbb{C}$-algebra which has basis $\{T_w : w \in W\}$ and satisfies the multiplication rules

$$\begin{aligned} T_w T_v &= T_{wv} && \text{if} \quad \ell(wv) = \ell(w) + \ell(v) \\ T_s T_s &= (q(s) - 1) T_s + q(s) T_e && \text{if} \quad s \in S_{\text{aff}} \end{aligned} \tag{3.32}$$

This algebra is canonically isomorphic to the crossed product of an Iwahori-Hecke algebra and the group of elements of length 0 in W:

$$\mathcal{H}(\mathcal{R}, q) \cong \mathcal{H}(W_{\text{aff}}, q) \rtimes \Omega \tag{3.33}$$

Our affine Hecke algebra has a large commutative subalgebra $\mathcal{A}$, isomorphic to the group algebra of X. We will regard this also as $\mathcal{O}(T)$, where

$$T = \text{Hom}_{\mathbb{Z}} \big(X, \mathbb{C}^\times \big) \cong \text{Prim}\big(\mathbb{C}[X] \big) \tag{3.34}$$

The action of W_0 on X induces an action on T by

$$(w \cdot t)(x) = t(w^{-1} x) \tag{3.35}$$

To identify this algebra $\mathcal{A}$, let $q^{1/2} : W \to \mathbb{C}$ be a label function such that $q^{1/2}(w)^2 = q(w) \; \forall w \in W$. Abbreviate

$$q^{1/2}(w)^{-1} = q^{-1/2}(w) \quad \text{and} \quad q^{1/2}(s_i) - q^{-1/2}(s_i) = \eta_i \tag{3.36}$$

In terms of the new bases

$$\Big\{ N_w = q^{-1/2}(w) T_w : w \in W \Big\}$$

the multiplication rules for $\mathcal{H}(\mathcal{R}, q)$ become

$$\begin{aligned} N_w N_v &= N_{wv} && \text{if} \quad \ell(wv) = \ell(w) + \ell(v) \\ \big(N_s - q^{1/2}(s) \big) \big(N_s + q^{-1/2}(s) \big) &= 0 && \text{if} \quad s \in S_{\text{aff}} \end{aligned} \tag{3.37}$$

Furthermore every N_w is invertible, and

$$N_{s_i}^{-1} = N_{s_i} - \eta_i N_e \quad s_i \in S_{\text{aff}} \tag{3.38}$$

Because ℓ is additive on X^+, the map

$$X^+ \to \mathcal{H}(\mathcal{R}, q)^\times : x \to N_{t_x} \tag{3.39}$$

is a monomorphism of semigroups. Extend it to a group homomorphism $\theta : X \to \mathcal{H}(\mathcal{R}, q)^\times$ by defining

$$\theta_x = N_{t_y} N_{t_z}^{-1} = N_{t_z}^{-1} N_{t_y} \tag{3.40}$$

if $x = y - z$ with $y, z \in X^+$. This independent of the choice of y and z.

The following theorem is due to Bernstein, Lusztig and Zelevinski, see [86, Section 3]. It describes what is also known as the Bernstein presentation of $\mathcal{H}(\mathcal{R}, q)$.

Theorem 3.3 *1. The sets $\{N_w \theta_x : w \in W_0, x \in X\}$ and $\{\theta_x N_w : w \in W_0, x \in X\}$ are both bases of $\mathcal{H}(\mathcal{R}, q)$.*

2. The subalgebra $\mathcal{A} := \text{span}\{\theta_x : x \in X\}$ is naturally isomorphic to $\mathbb{C}[X]$. The Weyl group W_0 acts on $\mathcal{A}$ by $w \cdot \theta_x = \theta_{wx}$, or equivalently

$$(w \cdot a)(t) = a^w(t) := a(w^{-1}t) \qquad t \in T, a \in \mathcal{A} \cong \mathcal{O}(T)$$

3. The center of $\mathcal{H}(\mathcal{R}, q)$ is

$$Z(\mathcal{H}(\mathcal{R}, q)) = \mathcal{A}^{W_0} \cong \mathcal{O}(T)^{W_0} \cong \mathcal{O}(T/W_0)$$

4. Take $a \in \mathcal{A}$, $\alpha_i \in F_0$ and let s_0 be the unique vertex of the Coxeter graph of $\mathcal{R}$ which is connected to $s_i = s_{\alpha_i}$ but does not lie in S_0. Then

$$aN_{s_i} - N_{s_i} a^{s_i} = \begin{cases} \eta_i (a - a^{s_i})(\theta_0 - \theta_{-\alpha_i})^{-1} & \text{if} \quad \alpha_i^\vee \notin 2Y \\ (\eta_i + \eta_0\, \theta_{-\alpha_i})(a - a^{s_i})(\theta_0 - \theta_{-2\alpha_i})^{-1} & \text{if} \quad \alpha_i^\vee \in 2Y \end{cases} \tag{3.41}$$

Equations (3.41) are also known as the Bernstein-Lusztig-Zelevinski relations. Since $\mathcal{A}$ is of finite rank over $\mathcal{A}^{W_0}$, it follows that $\mathcal{H}(\mathcal{R}, q)$ is of finite rank as a module over its center. In particular it is a finite type algebra in the sense of Section 2.2.

For $P \subset F_0$ we can use the above to define label functions q^P and q_P on the root data $\mathcal{R}^P$ and $\mathcal{R}_P$. For q^P, use (3.30) and (3.31) as a definition for all $\alpha, \beta \in R_P$, and extend this to $q^P : W\left(\mathcal{R}^P\right) \to \mathbb{C}^\times$. Notice that $q^P(t_x) = 1$ whenever $x \perp P^\vee$. Now $q_P : W(\mathcal{R}_P) \to \mathbb{C}^\times$ is simply the map induced by q^P. The affine Hecke algebra $\mathcal{H}\left(\mathcal{R}^P, q^P\right)$ can be identified with the parabolic subalgebra of $\mathcal{H}(\mathcal{R}, q)$ generated by $\mathcal{A}$ and $\mathcal{H}\big(W_P, q\big|_{W_P}\big)$. Furthermore $\mathcal{H}(\mathcal{R}_P, q_P)$ is naturally a quotient of $\mathcal{H}\left(\mathcal{R}^P, q^P\right)$.

Notice also that if $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$, then q restricts to label functions on the Weyl groups associated with $\mathcal{R}_1$ and $\mathcal{R}_2$, and there is a canonical identification

$$\mathcal{H}(\mathcal{R}, q) \cong \mathcal{H}(\mathcal{R}_1, q) \otimes \mathcal{H}(\mathcal{R}_2, q) \tag{3.42}$$

To study the above algebras we introduce the complex tori

$$\begin{array}{lllll} T_P & = & \mathrm{Hom}_{\mathbb{Z}}\left(X_P, \mathbb{C}^\times\right) & = & \left\{ t \in T : t(x) = 1 \text{ if } x \perp P^\vee \right\} \\ T^P & = & \mathrm{Hom}_{\mathbb{Z}}\left(X^P, \mathbb{C}^\times\right) & = & \left\{ t \in T : t(x) = 1 \text{ if } x \in \mathbb{Q}P \right\} \end{array} \tag{3.43}$$

They decompose into a unitary and a real split part:

$$\begin{array}{lllll} T_P & = & T_{P,u} \times T_{P,rs} & = & \mathrm{Hom}_{\mathbb{Z}}\left(X_P, S^1\right) \times \mathrm{Hom}_{\mathbb{Z}}\left(X_P, \mathbb{R}^+\right) \\ T^P & = & T^P_u \times T^P_{rs} & = & \mathrm{Hom}_{\mathbb{Z}}\left(X^P, S^1\right) \times \mathrm{Hom}_{\mathbb{Z}}\left(X^P, \mathbb{R}^+\right) \end{array} \tag{3.44}$$

Notice that

$$K_P = T^P \cap T_P = T^P_u \cap T_{P,u}$$

is a finite group, not necessarily equal to $\{1\}$. We make the identifications

$$\begin{array}{lllll} \mathrm{Lie}\left(T^P_{rs}\right) & = & \mathfrak{t}^P & = & Y^P \otimes_{\mathbb{Z}} \mathbb{R} \\ \mathrm{Lie}\left(T^P\right) & = & \mathfrak{t}^P \otimes_{\mathbb{R}} \mathbb{C} & = & Y^P \otimes_{\mathbb{Z}} \mathbb{C} \\ \mathrm{Lie}\left(T^P_u\right) & = & i\mathfrak{t}^P & = & iY^P \otimes_{\mathbb{Z}} \mathbb{R} \end{array} \tag{3.45}$$

Also define the positive parts

$$\begin{array}{lll} \mathfrak{t}^{P,+} & = & \left\{ \lambda \in \mathfrak{t}^P : \langle \alpha \, , \, \lambda \rangle > 0 \; \forall \alpha \in F_0 \setminus P \right\} \\ T^{P,+}_{rs} & = & \left\{ t \in T^P_{rs} : t(\alpha) > 1 \; \forall \alpha \in F_0 \setminus P \right\} \; = \; \exp\left(\mathfrak{t}^{P,+}\right) \end{array} \tag{3.46}$$

3.2 Representation theory

This section is meant as an introduction to the representation theory of affine Hecke algebras. None of the results presented here are original, varying from classical (Theorem 3.15) to very recent (Proposition 3.10).

Since an affine Hecke algebra is of finite type over its center, all its irreducible representations have finite dimension. Therefore we mainly study finite dimensional representations. We give two partial classifications of $\mathcal{H}(\mathcal{R}, q)$-representations. One is in terms of their restrictions to subalgebras associated with a finite Coxeter group. The other is more important, and in the spirit of Langlands. It shows that the study of $\mathcal{H}(\mathcal{R}, q)$-representations can be reduced to the study of so-called tempered representations. After that we define the C^* and Schwartz completions $C^*_r(\mathcal{R}, q)$ and $\mathcal{S}(\mathcal{R}, q)$ of an affine Hecke algebra. It turns out that $\mathcal{S}(\mathcal{R}, q)$-representations are characterized among $\mathcal{H}(\mathcal{R}, q)$-representations by the requirement that they are tempered.

Let $\mathcal{H} = \mathcal{H}(\mathcal{R}, q)$ be an affine Hecke algebra and $\mathrm{Rep}(\mathcal{H})$ its category of finite dimensional representations. Since $\mathcal{A} \cong \mathbb{C}[X]$ is commutative, every irreducible

$\mathcal{A}$-module is onedimensional, of the form $\mathbb{C}_t$ for a character $t \in T$. For $(\pi, V) \in \mathrm{Rep}(\mathcal{H})$ and $t \in T$ we put

$$V_t = \{v \in V : \exists n \in \mathbb{N} : (\pi(a) - a(t))^n v = 0 \; \forall a \in \mathcal{A}\} \tag{3.47}$$

If $V_t \neq 0$ then there exists an eigenvector $v \in V$ with $\pi(a)v = a(t)v \; \forall a \in \mathcal{A}$. In this case t is called an $\mathcal{A}$-weight of V, and V_t a generalized weight space. As an $\mathcal{A}$-module V is the direct sum of the nonzero V_t.

Let us consider principal series representations. By definition they are the representations

$$I_t = \mathrm{Ind}_{\mathcal{A}}^{\mathcal{H}}(\mathbb{C}_t) \qquad \text{for } t \in T \tag{3.48}$$

They have dimension $|W_0|$, and we realize them all on the vector space

$$\mathcal{H}(W_0, q) \cong \mathcal{H}/\langle\{a - a(t) : a \in \mathcal{A}\}\rangle \tag{3.49}$$

The importance of principal series representations is already clear from the following well-known result.

Lemma 3.4 *1. Every irreducible $\mathcal{H}$-representation (π, V) is a quotient of some I_t*

2. If $I_t(h) = 0$ for all t in a Zariski-dense subset of T, then $h = 0$

3. $Jac(\mathcal{H}) = 0$

Proof. 1 was first proven in [89]. Take an $\mathcal{A}$-weight t of V and a corresponding eigenvector $v \in V$. Define an $\mathcal{H}$-module homomorphism

$$I_t \to V : h \to \pi(h)v \tag{3.50}$$

This is surjective because V is irreducible.
2 is based upon Theorem 3.3. The function

$$T \to \mathrm{End}_{\mathbb{C}}(\mathcal{H}(W_0, q)) : t \to I_t(h) \tag{3.51}$$

is polynomial, and since it is 0 on a Zariski-dense subset it vanishes identically. Write $h = \sum_{w \in W_0} a_w T_w$ with $a_w \in \mathcal{A}$ and suppose that $h \neq 0$. Then we can find $w' \in W_0$ such that $a_{w'} \neq 0$ and $\ell(w')$ is maximal with respect to this property. From (3.41) we see that

$$I_t(h)(T_e) = \sum_{w \in W_0} b_w(t) T_w \quad \text{with} \quad b_{w'} = a_{w'}$$

Therefore (3.51) is not identically 0, contradicting our assumption $h \neq 0$.
3. By [90, (3.4.5)] or [71, Theorem 2.2] there is a nonempty Zariski-open subset T' of T such that I_t is irreducible $\forall t \in T'$. So if $h \in \mathrm{Jac}(\mathcal{H})$ then $h = 0$ by part 2. $\square$

Similarly we can define the $\mathcal{Z}$-weights of V, where

$$\mathcal{Z} = Z\big(\mathcal{H}(\mathcal{R}, q)\big)$$

is the center of $\mathcal{H}$. Using 3.3.2 we identify $\text{Prim}(\mathcal{Z})$ with T/W_0. Assume that $\mathcal{Z}$ acts by scalars on V, which by Schur's lemma is the case if V is irreducible. Then the central character of V is the unique orbit $W_0 t \in T/W_0$ such that

$$\pi(a)v = a(t)v \quad \forall v \in V, a \in \mathcal{Z}$$

For any W_0-stable $U \subset T$ let $\text{Rep}_U(\mathcal{H}(\mathcal{R}, q))$ be the category of all finite dimensional $\mathcal{H}$-modules whose $\mathcal{Z}$-weights are contained in U/W_0. There are a few ways to "localize" the algebra $\mathcal{H}$ at U, i.e. to construct an algebra whose representations are precisely $\text{Rep}_U(\mathcal{H}(\mathcal{R}, q))$. One way, suitable for open U, is by tensoring (over $\mathcal{A}$) with analytic functions on U, as we shall see in (3.116). Another way, which works best if U is a closed subvariety of T, is by completing $\mathcal{H}$ with respect to the ideal

$$J_U = \{h \in \mathcal{H} : I_t(h) = 0 \; \forall t \in U\}$$

Notice that J_U is generated by $\{a \in \mathcal{Z} : a\big|_U = 0\}$. Explicitly, we get the modules

$$\begin{aligned} V_U &= V \otimes_{\mathcal{H}} \mathcal{H}_U \\ \mathcal{H}_U &= \lim_{n \leftarrow \infty} \mathcal{H}/J_U^n \end{aligned} \tag{3.52}$$

For $U = W_0 t$ with $t \in T$ this was done in [86, Section 7], and it is consistent with (3.47) in the sense that

$$V_{W_0 t} = \sum_{w \in W_0} V_{wt} \tag{3.53}$$

Finally, we can vary on this construction in less subtle fashion, by replacing $\mathcal{H}_U$ in (3.52) with $\mathcal{H}/J_U$. This has the advantage that we reduce things to modules over finite dimensional algebras.

Now we start the preparations for the Langlands classification, which can be found in [40]. We say that V is a tempered $\mathcal{H}$-module if $|x(t)| \leq 1$ for every $x \in X^+$ and every $\mathcal{A}$-weight of V. The explanation will follow in Lemma 3.14. Contrarily, we call V anti-tempered if $|x(t)| \geq 1$ for every such x and t. And less restrictively we say that V is essentially tempered if $|x(t)| \leq 1$ for every $\mathcal{A}$-weight t of V and every $x \in \mathbb{Z}R_0 \cap X^+$. The only difference with tempered is that $t\big|_{Z(W)}$ need not be a unitary character.

Lemma 3.5 *Let (π, V) be an essentially tempered $\mathcal{H}$-representation which admits a central character $W_0 rt$, with $t \in T_{rs}^{F_0}$ and $|r| \in T_{F_0,rs}$. There exists an automorphism ψ_t of $\mathcal{H}$ such that $(\pi \circ \psi_t^{-1}, V)$ is a tempered $\mathcal{H}$-representation with central character $W_0 r$.*

Proof. Define

$$\psi_t(N_w \theta_x) = t(x) N_w \theta_x \tag{3.54}$$

This is an automorphism because t is 1 on $\mathbb{Z}R_0$. Let $t_1, \dots, t_d$ be the $\mathcal{A}$-weights of (π, V). Clearly, the $\mathcal{A}$-weights of $(\pi \circ \psi_t^{-1}, V)$ are $t_1 t^{-1}, \dots, t_d t^{-1}$. For $x \in \mathbb{Z}R_0 \cap X^+$ we have

$$|t_i t^{-1}(x)| = |t_i(x)| \leq 1$$

because π is essentially tempered, and for $x \in X^+ \cap X^-$ we have $|t_i t^{-1}(x)| = 1$ because $|t_i| \in W_0|r| \subset T_{F_0,rs}$. Hence $(\pi \circ \psi_t^{-1}, V)$ is tempered with central character $W_0 t_i t^{-1} = W_0 r$. $\square$

In general, let (σ, V_σ) be any finite dimensional representation of $\mathcal{H}^P = \mathcal{H}(\mathcal{R}^P, q^P)$. Recall [61, Proposition 1.10] that

$$W^P = \{w \in W_0 : w(P) \subset R_0^+\} \tag{3.55}$$

is the set of minimal length representatives of W_0/W_P. Construct the vector space

$$\mathcal{H}\big(W^P\big) = \text{span}\big\{N_w : w \in W^P\big\} \subset \mathcal{H}(W_0, q) \tag{3.56}$$

The $\mathcal{H}$-representation

$$\pi(P, \sigma) := \text{Ind}_{\mathcal{H}^P}^{\mathcal{H}}(\sigma) \tag{3.57}$$

can be realized on $\mathcal{H}\big(W^P\big) \otimes_{\mathbb{C}} V_\sigma$. From the proof of [98, Proposition 4.20] we can see what the weights of this representation are:

Lemma 3.6 *The $\mathcal{A}$-weights of $\pi(P, \sigma)$ are precisely the elements $w(t)$, where t is an $\mathcal{A}$-weight of (σ, V_σ) and $w \in W^P$.*

If σ is irreducible than it has a central character $W_P r \in T/W_P$. Since

$$T_{rs} = T_{rs}^P \times T_{P,rs} \tag{3.58}$$

and W_P acts trivially on T_{rs}^P, there is a unique $r_\sigma \in T_{rs}^P$ such that

$$W_P|r| = W_P r' r_\sigma \text{ for some } r' \in T_{P,rs}$$

Let Λ be the set of all pairs (P, σ), where σ is an irreducible essentially tempered representation of $\mathcal{H}^P$. The set of Langlands data is

$$\Lambda^+ = \big\{(P, \sigma) \in \Lambda : r_\sigma \in T_{rs}^{P,+}\big\} \tag{3.59}$$

Now we can state the Langlands classification for affine Hecke algebras:

Theorem 3.7 *Let $(P, \sigma) \in \Lambda^+$. The $\mathcal{H}$-module $\pi(P, \sigma)$ is indecomposable and has a unique irreducible quotient $L(P, \sigma)$. For every irreducible $\mathcal{H}$-representation π there is precisely one Langlands datum $(P, \sigma) \in \Lambda^+$ such that π is equivalent to $L(P, \sigma)$.*

Proof. This is entirely analogous to the corresponding statements for graded Hecke algebras, which were proved by Evens [45, Theorem 2.1]. See also [40, Section 6]. □

Another way to study $\mathcal{H}$-representations is by their restrictions to simpler subalgebras. We do this by constructing a nice (projective) resolution of an $\mathcal{H}$-module, which stems from joint work of Opdam and Reeder, cf. [99, Section 8].

Number the $s_i \in S_{\text{aff}}$ such that the elements corresponding to one connected component of the Coxeter graph of $(W_{\text{aff}}, S_{\text{aff}})$ are numbered successively. For $I \subset \{1, 2, \ldots, |S_{\text{aff}}|\}$ let $\mathbb{C}[I]$ be the vector space with basis $\{e_i : i \in I\}$ and W_I the standard parabolic subgroup of W_{aff} generated by $\{s_i : i \in I\}$. Recall that the "length 0" subgroup Ω of W acts on S_{aff} by conjugation. Transfer this to an action of Ω on the indices i and put

$$\Omega_I := \{\omega \in \Omega : \omega(I) = I\}$$

By definition Ω_I acts on W_I, so the extended Iwahori-Hecke algebra

$$\mathcal{H}(\mathcal{R}, I, q) = \mathcal{H}(W_I, q) \rtimes \Omega_I$$

is well-defined. Note that W_I can be either finite or infinite, but that we always have $X^+ \cap X^- = Z(W) \subset \Omega_I$. If $\mathcal{R}$ is semisimple and W_I is finite, then $\mathcal{H}(\mathcal{R}, I, q)$ has finite dimension.

Because we want to define a conjugate-linear, anti-multiplicative involution on $\mathcal{H}(\mathcal{R}, q)$, from now we will we assume the following.

Condition 3.8 *The label function of an affine Hecke algebra only takes values in* $(0, \infty)$

Lemma 3.9 *Suppose that W_I is finite and let χ be a character of $Z(W)$. Then*

$$\mathcal{H}(\mathcal{R}, I, q)_\chi := \mathcal{H}(\mathcal{R}, I, q) / \ker \left(\text{Ind}_{\mathbb{C}[Z(W)]}^{\mathcal{H}(\mathcal{R}, I, q)} \mathbb{C}_\chi \right)$$

is a finite dimensional semisimple algebra.

Proof. As vector spaces we may identify

$$\mathcal{H}(\mathcal{R}, I, q)_\chi = \text{Ind}_{\mathbb{C}[Z(W)]}^{\mathcal{H}(\mathcal{R}, I, q)} \mathbb{C}_\chi = \mathcal{H}(W_I, q) \otimes \mathbb{C}[\Omega_I / Z(W)]$$

We can extend $|\chi|$ canonically to $X \otimes \mathbb{R}$, making it 1 on R_0. Using this extension we define an involution $*_\chi$ on $\mathcal{H}(\mathcal{R}, I, q)$ by

$$(h_w T_w)^{*_\chi} = \overline{h_w}\, |\chi|(2w(0)) T_{w^{-1}}$$

This map is antimultiplicative by Condition 3.8. The associated bilinear form is

$$\langle h\,,\, h' \rangle_\chi = x_e \quad \text{if} \quad h^{*_\chi} \cdot h' = \sum_{w \in W} x_w N_w$$

By construction $\mathrm{Ind}_{\mathbb{C}[Z(W)]}^{\mathcal{H}(\mathcal{R},I,q)} \mathbb{C}_\chi$ is now a unitary representation. This makes $\mathcal{H}(\mathcal{R}, I, q)_\chi$ into a finite dimensional Hilbert algebra, so in particular it is semisimple. □

For $(\pi, V) \in \mathrm{Rep}(\mathcal{H})$ and $n \in \mathbb{N}$ consider the $\mathcal{H}$-module

$$P_n(V) = \bigoplus_{|I|=n, |W_I|<\infty} \mathcal{H} \otimes_{\mathcal{H}(W_I,q) \rtimes Z(W)} V\big|_{\mathcal{H}(W_I,q)\rtimes Z(W)} \otimes_{\mathbb{C}} \bigwedge^n \mathbb{C}[I] \tag{3.60}$$

Here $\rtimes Z(W)$ is just an abbreviaton of $\otimes \mathbb{C}[Z(W)]$. By definition $P_{|F_0|+1}(V) = V$ and $P_n(V) = 0$ if $n < 0$ or $n \geq |F_0| + 2$. Define $\mathcal{H}$-module homomorphisms

$$\begin{aligned} &d_n : P_n(V) \to P_{n+1}(V) \\ &d_n(h \otimes_{\mathcal{H}(W_I,q)\rtimes Z(W)} v \otimes \lambda) = \bigoplus_{j : |W_{I\cup\{j\}}|<\infty} h \otimes_{\mathcal{H}(W_{I\cup\{j\}},q)\rtimes Z(W)} v \otimes \lambda \wedge e_j \end{aligned} \tag{3.61}$$

Notice that actually the sum runs only over $j \notin I$, for otherwise $\lambda \wedge e_j = 0$. To construct a fitting map $d_{|F_0|}$ we need to exert ourselves in a little more. We introduce a sign function by

$$\mathrm{sign}(e_{n_1} \wedge \cdots \wedge e_{n_k}) = \begin{cases} 1 & \text{if} \quad e_{n_1} \wedge \cdots \wedge e_{n_k} \wedge \eta = e_1 \wedge e_2 \wedge \cdots \wedge e_{|S_{\mathrm{aff}}|} \\ -1 & \text{if} \quad e_{n_1} \wedge \cdots \wedge e_{n_k} \wedge \eta = -e_1 \wedge e_2 \wedge \cdots \wedge e_{|S_{\mathrm{aff}}|} \end{cases}$$

where η is the wedge product of the e_i with $1 \leq i \leq |S_{\mathrm{aff}}|$ and $i \neq n_j$, in standard order. Using this convention we define

$$\begin{aligned} &d_{|F_0|} : P_{|F_0|}(V) \to V \\ &d_{|F_0|}(h \otimes_{\mathcal{H}(W_I,q)\rtimes Z(W)} v \otimes \lambda) = \mathrm{sign}(\lambda)\pi(h)v \end{aligned}$$

Looking at the $\wedge$-terms we see that $d_{n+1} \circ d_n = 0$, so $(P_*(V), d_*)$ is an augmented differential complex. The group Ω acts naturally on this complex by

$$\omega(h \otimes_{\mathcal{H}(W_I,q)\rtimes Z(W)} v \otimes \lambda) = hT_\omega^{-1} \otimes_{\mathcal{H}(W_{\omega(I)},q)\rtimes Z(W)} \pi(T_\omega)v \otimes \omega(\lambda) \tag{3.62}$$

This action commutes with the action of $\mathcal{H}$ and with the differentials d_n, so $\big(P_*(V)^\Omega, d_*\big)$ is again a differential complex.

Proposition 3.10

$$0 \longrightarrow P_0(V)^\Omega \xrightarrow{d_0} P_1(V)^\Omega \xrightarrow{d_1} \cdots \longrightarrow P_{|F_0|}(V)^\Omega \xrightarrow{d_{|F_0|}} V \to 0$$

is a natural resolution of V by finitely generated modules. If $\mathcal{X}$ is the set of $Z(W)$-weights of V, then every module $P_n(V)^\Omega$ is projective in the category of all $\mathcal{H}$-modules whose $Z(W)$-weights lie in $\mathcal{X}$.

Proof. This result a generalization of [99, Proposition 8.1] to root data that are not semisimple. The proof is based upon [73, Section 1], as indicated by Opdam and Reeder.

First we consider the case $\Omega = Z(W) = \{e\}$, $W = W_{\text{aff}}$. There is a linear bijection

$$\begin{aligned} \phi : \mathbb{C}[W] \otimes_{\mathbb{C}} V &\to \mathcal{H} \otimes_{\mathbb{C}} V \\ \phi(w \otimes v) &= T_w \otimes \pi(T_w)^{-1} v \end{aligned} \tag{3.63}$$

For $s_i \in S_{\text{aff}}$ we write

$$\begin{array}{rclcl} L_i & := & \text{span}\{hT_{s_i} \otimes \pi(T_{s_i})^{-1}v - h \otimes v : h \in \mathcal{H},\, v \in V\} & \subset & \mathcal{H} \otimes_{\mathbb{C}} V \\ \mathbb{C}[W]_i & := & \big\{ \sum_{w \in W} x_w w : x_{ws_i} = -x_w \;\forall w \in W \big\} & \subset & \mathbb{C}[W] \end{array} \tag{3.64}$$

This L_i is interesting because

$$\mathcal{H} \otimes_{\mathcal{H}(W_I,q)} V = \big(\mathcal{H} \otimes_{\mathbb{C}} V\big) / \sum_{i \in I} L_i \tag{3.65}$$

Let $w \in W$ be such that $\ell(ws_i) > \ell(w)$.

$$\begin{aligned} \phi((ws_i - w) \otimes v) &= T_{ws_i} \otimes \pi(T_{ws_i})^{-1} v - T_w \otimes \pi(T_w)^{-1} v \\ &= T_w T_{s_i} \otimes \pi(T_{s_i})^{-1} \pi(T_w)^{-1} v - T_w \otimes \pi(T_w)^{-1} v \in L_i \end{aligned} \tag{3.66}$$

so $\phi(\mathbb{C}[W]_i \otimes V) \subset L_i$. On the other hand, L_i is spanned by elements as in (3.64) with $h = T_w$ or $h = T_{ws_i}$.

$$\begin{array}{ll} \phi^{-1}(T_{ws_i} T_{s_i} \otimes \pi(T_{s_i})^{-1} v - T_{ws_i} \otimes v) & = \\ \phi^{-1}(q_i T_w + (q_i - 1) T_{ws_i} \otimes \pi(T_{s_i})^{-1} v) - ws_i \otimes \pi(T_{ws_i}) v & = \\ q_i w \otimes \pi(T_w) \pi(T_{s_i})^{-1} v + (q_i - 1) ws_i \otimes \pi(T_{ws_i}) \pi(T_{s_i})^{-1} v - ws_i \otimes \pi(T_{ws_i}) v & = \\ q_i (w - ws_i) \otimes \pi(T_w T_{s_i}^{-1}) v + ws_i \otimes \pi\big(q_i T_w T_{s_i}^{-1} + (q_i - 1) T_{ws_i} T_{s_i}^{-1} - T_{ws_i}\big) v & = \\ (w - ws_i) \otimes \pi(T_w q_i T_{s_i}^{-1}) v + ws_i \otimes \pi\big(T_w (T_{s_i} + 1 - q_i) + (q_i - 1) T_w - T_w T_{s_i}\big) v & = \\ (w - ws_i) \otimes \pi\big(T_w (T_{s_i} + 1 - q_i)\big) v \qquad \in \qquad \mathbb{C}[W]_i \otimes V & \end{array} \tag{3.67}$$

We conclude that $\phi^{-1}(L_i) = \mathbb{C}[W]_i \otimes V$. Now we bring the linear bijections

$$\mathbb{C}[W] / \sum_{i \in I} \mathbb{C}[W]_i \to \mathbb{C}[W/W_I] : w \to wW_I \tag{3.68}$$

into play. Under these identifications our differential complex becomes

$$0 \to \mathbb{C}[W] \otimes V \otimes \mathbb{C} \to \cdots \to \bigoplus_{|I|=n, |W_I|<\infty} \mathbb{C}[W/W_I] \otimes V \otimes \bigwedge^n \mathbb{C}[I] \to \cdots \to V \to 0$$

We have to show that the cohomology of this complex vanishes in all degrees. Because $d_{|F_0|}$ is surjective it suffices to show that the complex

$$\begin{array}{rcl} C'_n & = & \bigoplus_{|I|=n, |W_I|<\infty} \mathbb{C}[W/W_I] \otimes \bigwedge^n \mathbb{C}[I] \\ d'_n(wW_I \otimes \lambda) & = & \sum_{j : |W_{I \cup \{j\}}| < \infty} wW_{I \cup \{j\}} \otimes \lambda \wedge e_j \end{array} \tag{3.69}$$

has cohomology

$$H'_n = \begin{cases} \mathbb{C} & \text{if} \quad n = |F_0| \\ 0 & \text{if} \quad n \neq |F_0| \end{cases} \tag{3.70}$$

This is best seen by a geometrical interpretation. It is well-known that the alcove

$$C_\emptyset := \big\{ x \in Q \otimes_{\mathbb{Z}} \mathbb{R} : \langle x\,,\,\alpha^\vee \rangle \geq 0 \;\forall \alpha^\vee \in F_0^\vee ,\, \langle x\,,\,\beta^\vee \rangle \leq 1 \;\forall \beta^\vee \in F_m^\vee \big\}$$

is a fundamental domain for the action of W on $Q \otimes_{\mathbb{Z}} \mathbb{R}$. The finite groups W_I are naturally identified with the stabilizers of the faces C_I of $C_\emptyset$. Thus every coset wW_I corresponds to a polysimplex wC_I of dimension $|F_0| - |I|$. It follows that (C'_*, d'_*) is the complex that computes $H_{|F_0|-*}(Q \otimes_{\mathbb{Z}} \mathbb{R}; \mathbb{C})$ by means of a (polysimplicial) triangulation. Together with the Poincaré Lemma this leads to (3.70), proving the proposition in the special case $\Omega = \{e\}$.

Now the general case. Recall that $Z(W)$ is the lattice $X^+ \cap X^-$. Since $\mathbb{C}[Z(W)]$ is contained in $Z(\mathcal{H})$, there is a direct sum decomposition $V = \bigoplus_{\chi \in \mathcal{X}} V_\chi$. This persists to all the modules $P_n(V)^\Omega$, so we may and will assume that V admits a single $Z(W)$-character χ. Then $\pi\big|_{\mathcal{H}(\mathcal{R},I,q)}$ factors through $\mathcal{H}(\mathcal{R},I,q)_\chi$ and by Lemma 3.9 it is projective as a $\mathcal{H}(\mathcal{R},I,q)_\chi$-representation. This implies that $P_n(V)$ is projective in the category of $\mathcal{H}$-representations with $Z(W)$-character χ. Moreover $P_n(V)$ is finitely generated because V has finite dimension and because there are only finitely many I's involved. Since the action of Ω on S_{aff} factors through a finite group we can construct a Reynolds operator

$$R_\Omega := [\Omega : Z(W)]^{-1} \sum_{\omega \in \Omega / Z(W)} \omega \quad \in \quad \mathrm{End}_{\mathcal{H}}\big(P_n(V)\big)$$

It follows that $P_n(V)^\Omega = R_\Omega \cdot P_n(V)$ is a direct summand of $P_n(V)$, and hence also projective in the specified sense.

We generalize (3.63) to a bijection

$$\begin{aligned} &\phi : \mathbb{C}[W/Z(W)] \otimes_{\mathbb{C}} V \to \mathcal{H} \otimes_{\mathbb{C}[Z(W)]} V \\ &\phi(w \otimes v) = T_w \otimes \pi(T_w)^{-1} v \end{aligned} \tag{3.71}$$

Just as above this leads to bijections

$$\bigoplus_{|I|=n, |W_I| < \infty} \mathbb{C}[W/(W_I \times Z(W))] \otimes V \otimes \bigwedge^n \mathbb{C}[I] \to P_n(V) \tag{3.72}$$

Since both sides are free $\Omega/Z(W)$-modules we see that

$$\bigoplus_{|I|=n, |W_I| < \infty} \mathbb{C}[W_{\text{aff}}/W_I] \otimes V \otimes \bigwedge^n \mathbb{C}[I] \cong P_n(V)^\Omega \tag{3.73}$$

Now the above geometrical argument shows that the $P_n(V)^\Omega$ do indeed form a resolution of V. $\square$

Corollary 3.11 *Suppose that $V, V' \in \mathrm{Rep}(\mathcal{H})$ are such that $V\big|_{\mathcal{H}(\mathcal{R},I,q)}$ and $V'\big|_{\mathcal{H}(\mathcal{R},I,q)}$ are equivalent whenever W_I is finite. Then V and V' define the same class in the Grothendieck group of finitely generated $\mathcal{H}$-modules.*

Proof. By assumption we can find a collection of $\mathcal{H}(\mathcal{R}, I, q)$-module isomorphisms $\alpha_I : V \to V'$ such that

$$\pi'(T_\omega)\alpha_I = \alpha_{\omega(I)}\pi(T_\omega) \qquad \forall \omega \in \Omega$$

These combine to $\mathcal{H}$-module isomorphisms

$$\begin{aligned} &\alpha_n : P_n(V)^\Omega \to P_n(V')^\Omega \\ &\alpha_n(h \otimes_{\mathcal{H}(W_I,q)\rtimes Z(W)} v \otimes \lambda) = h \otimes_{\mathcal{H}(W_I,q)\rtimes Z(W)} \alpha_I(v) \otimes \lambda \end{aligned}$$

Hence by Proposition 3.10 V and V' have equivalent finitely generated resolutions. $\square$

Unfortunately this result is not very strong, as the class of V is nearly always 0. This certainly is the case if V is of the form $\pi(P, \sigma)$ with $P \neq F_0$.

Let $G(\mathcal{H})$ be the Grothendieck group of finite dimensional $\mathcal{H}$-modules, and $K_0(\mathcal{H})$ the Grothendieck group of finitely generated projective $\mathcal{H}$-modules. The Euler-Poincaré pairing [112, Section III.4] on $G(\mathcal{H})$ is defined as

$$EP\,(V, V') = \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_{\mathcal{H}}(V, V') \tag{3.74}$$

where $\mathrm{Ext}^n_{\mathcal{H}}$ is the higher derived functor of $\mathrm{Hom}_{\mathcal{H}}$.

Let us recall some relevant observations from [99, Section 8]. Assume that $\mathcal{R}$ is semisimple. By Proposition 3.10 the Euler characteristic can be defined for finite dimensional $\mathcal{H}$-modules by

$$\begin{aligned} &\mathrm{Eul} : G(\mathcal{H}) \to K_0(\mathcal{H}) \\ &\mathrm{Eul}\,[V] = \sum_{n=0}^{|F_0|} (-1)^{|F_0|-n} \left[P_n(V)^\Omega\right] \end{aligned} \tag{3.75}$$

Moreover there is a natural pairing between $G(\mathcal{H})$ and $K_0(\mathcal{H})$. Given a representation π of $\mathcal{H}$ and an idempotent $p \in M_n(\mathbb{C}) \otimes \mathcal{H}$ we put

$$\big[\,[p], [\pi]\,\big] = \mathrm{rank}\,(\mathrm{id} \otimes \pi)(p) \in \mathbb{Z} \tag{3.76}$$

Because $(P_*(V)^\Omega, d_*)$ is a projective resolution of V we have the equalities

$$EP\,(V, V') = \sum_{n=0}^{|F_0|} (-1)^{|F_0|-n} \dim \mathrm{Hom}_{\mathcal{H}}\left(P_n(V)^\Omega, V'\right) = \big[\,\mathrm{Eul}\,[V], [V']\,\big] \tag{3.77}$$

But the modules $P_n(V)$ are induced from semisimple subalgebras of finite dimension, so this can be expressed more explicitly. By Frobenius reciprocity we have

$$\dim \mathrm{Hom}_{\mathcal{H}}(P_n(V), V') = \sum_{|I|=n, |W_I|<\infty} \dim \mathrm{Hom}_{\mathcal{H}(W_I,q)\rtimes Z(W)}(V, V') \tag{3.78}$$

Let ϵ_I be the character of the Ω_I-representation $\bigwedge^{|I|} \mathbb{C}[I]$. Taking the Ω-invariants of $P_n(V)$ in (3.78) we find

$$EP(V, V') = \sum_{I\subset S_{\mathrm{aff}}, |W_I|<\infty} \frac{(-1)^{|F_0|-|I|}}{[\Omega : \Omega_I]} \dim \mathrm{Hom}_{\mathcal{H}(\mathcal{R},I,q)}(V \otimes \epsilon_I, V') \tag{3.79}$$

Thus the Euler-Poincaré pairing of two finite dimensional $\mathcal{H}$-modules is completely determined by their restrictions to the finite dimensional semisimple subalgebras $\mathcal{H}(\mathcal{R}, I, q)$. Because such algebras have only finitely many inequivalent irreducible modules, this pairing is symmetric and invariant under continuous deformations of its input. However, from (3.74) we quickly deduce that modules with different central characters are orthogonal for EP. We will see in (3.92) that every module of the form $\pi(P, \sigma)$ with $P \subsetneq F_0$ admits continuous deformations of its central character. Therefore all such modules are in the radical of EP.

So far for the purely algebraic properties of affine Hecke algebras, we are also interested in their analytic structure. Condition 3.8 gives us a canonical way to define the label function $q^{1/2}$ and the basis elements N_w. Let $x = \sum_{w\in W} x_w N_w$ be an element of $\mathcal{H}(\mathcal{R}, q)$ and define its adjoint and its trace by

$$x^* := \sum_{w\in W} \overline{x_w} N_{w^{-1}} \quad \text{and} \quad \tau(x) = x_e \tag{3.80}$$

Condition (3.8) assures that indeed τ is positive and that $(xy)^* = y^* x^*$. This leads to a bitrace or Hermitian inner product

$$\langle x\, ,\, y\rangle := \tau(x^* y) \quad x, y \in \mathcal{H}(\mathcal{R}, q) \tag{3.81}$$

and a norm

$$\|x\|_\tau := \sqrt{\langle x\, ,\, x\rangle} = \sqrt{\tau(x^* x)} \tag{3.82}$$

By a simple calculation one can show that $\{N_w : w \in W\}$ is an orthonormal bases of $\mathcal{H}(\mathcal{R}, q)$ for this inner product. The bitrace $\langle \cdot\, ,\, \cdot\rangle$ gives $\mathcal{H}(\mathcal{R}, q)$ the structure of a Hilbert algebra, in the sense of [41, Appendice A 54]. Let $\mathfrak{H}(\mathcal{R}, q)$ be its Hilbert space completion, and $B(\mathfrak{H}(\mathcal{R}, q))$ the associated C^*-algebra of bounded operators. Consider the multiplication maps

$$\begin{aligned} \lambda(x) &: y \to xy \\ \rho(x) &: y \to yx \end{aligned}$$

According to [98, Lemma 2.3] these maps extend to bounded operators on $\mathfrak{H}(\mathcal{R}, q)$ of the same norm, which we denote by

$$\|x\|_o := \|\lambda(x)\|_{B(\mathfrak{H}(\mathcal{R},q))} = \|\rho(x)\|_{B(\mathfrak{H}(\mathcal{R},q))} \tag{3.83}$$

Thus, $\mathcal{H}(\mathcal{R}, q)$ being a *-subalgebra of $B(\mathfrak{H}(\mathcal{R}, q))$, we can take its closure $C_r^*(\mathcal{R}, q)$ in the norm topology. By definition this is a separable unital C^*-algebra, and we call it the reduced C^*-algebra of $\mathcal{H}(\mathcal{R}, q)$. This is analogous to the reduced C^*-algebra of a locally compact group G, which is the completion of $C_0(G)$ for the topology coming from the regular representation of G.

There is also a "smoother" way to complete an affine Hecke algebra. As a topological vector space, it will consist of rapidly decreasing functions on W with respect to some length function. In this respect it is a little unsatisfactory that ℓ is 0 on the subgroup $Z(W) = X^+ \cap X^-$, since this can be the whole of W. To overcome this inconvenience, let $f : X \otimes \mathbb{R} \to [0, \infty)$ be a function such that

- $f(X) \subset \mathbb{Z}$
- $f(x + q) = f(x) \quad \forall x \in X \otimes \mathbb{R},\ q \in Q \otimes \mathbb{R}$
- f induces a norm on $X \otimes \mathbb{R}/Q \otimes \mathbb{R} \cong Z(W) \otimes \mathbb{R}$

Now we define for $w \in W$

$$\mathcal{N}(w) := \ell(w) + f(w(0)) \tag{3.84}$$

so that for any $w, v \in W$, $u \in W_{\mathrm{aff}}, \omega \in \Omega$

$$\mathcal{N}(u\omega) = \mathcal{N}(\omega u) = \ell(u) + \mathcal{N}(\omega) \tag{3.85}$$

$$\mathcal{N}(vw) \leq \mathcal{N}(v) + \mathcal{N}(w) \tag{3.86}$$

Because $Z(W) + Q$ is of finite index in X, the set $\Omega' = \{w \in W : \mathcal{N}(w) = 0\}$ is finite. Moreover, since W is the semidirect product of a finite group and an abelian group, it is of polynomial growth, and different choices of f lead to equivalent length functions $\mathcal{N}$. Now we can define for any $n \in \mathbb{N}$ the norm

$$p_n\left(\sum_{w \in W} x_w N_w\right) := \sup_{w \in W} |x_w| (\mathcal{N}(w) + 1)^n \tag{3.87}$$

The completion $\mathcal{S}(\mathcal{R}, q)$ of $\mathcal{H}(\mathcal{R}, q)$ with respect to the family of norms $\{p_n\}_{n \in \mathbb{N}}$ is a nuclear Fréchet space. It consists of all possible infinite sums $x = \sum_{w \in W} x_w N_w$ such that $p_n(x) < \infty \ \forall n \in \mathbb{N}$. Opdam [98, Section 6.2] proved that these norms behave reasonably with respect to multiplication:

Theorem 3.12 *There exist $C_q > 0$, $d \in \mathbb{N}$ such that $\forall x, y \in \mathcal{S}(\mathcal{R}, q), n \in \mathbb{N}$*

$$\|x\|_o \leq C_q p_d(x) \tag{3.88}$$

$$p_n(xy) \leq C_q p_{n+d}(x) p_{n+d}(y) \tag{3.89}$$

*In particular $\mathcal{S}(\mathcal{R}, q)$ is a unital locally convex *-algebra, and it is contained in $C_r^*(\mathcal{R}, q)$.*

The proof of this theorem uses heavy machinery, namely the spectral decomposition of the trace τ, which we will state in Theorem 3.24.4. Closer examination of that proof shows that we can take $d = \mathrm{rk}(X) + |W_0|^2 + 1$. In Section 5.2 we will use more elementary tools to prove a generalization of this theorem, resulting in a smaller d.

We call $\mathcal{S}(\mathcal{R}, q)$ the Schwartz algebra of $\mathcal{H}(\mathcal{R}, q)$. In Section 4.2 we will see that this construction is analogous to the Schwartz algebra of a reductive p-adic group. Considering Schwartz completions of parabolic subalgebras, we see that $\mathcal{S}(\mathcal{R}_P, q_P)$ is still a quotient of $\mathcal{S}(\mathcal{R}^P, q^P)$, but that the latter algebra is in general no longer contained in $\mathcal{S}(\mathcal{R}, q)$. The same holds for the respective reduced C^*-algebras.

From the work of Casselman [29, §4.4] one can find necessary and sufficient conditions under which $\mathcal{H}$-representations extend continuously to certain completions. Indeed it is shown in [98, Lemma 2.20] that

Lemma 3.13 *For an irreducible $\mathcal{H}$-representation (π, V) the following conditions are equivalent, and summarized by saying that π belongs to the discrete series:*

- *(π, V) is a subrepresentation of the left regular representation $(\lambda, \mathfrak{H}(\mathcal{R}, q))$*
- *all matrix coefficients of (π, V) are in $\mathfrak{H}(\mathcal{R}, q)$*
- *$|x(t)| < 1$ for every $\mathcal{A}$-weight t of V and every $x \in X^+ \setminus 0$*

By definition a discrete series representation is unitary, and it extends to the reduced C^*-algebra $C_r^*(\mathcal{R}, q)$. Because this is a Hilbert algebra, a suitable version of [41, Proposition 18.4.2] shows that π is an isolated point in its spectrum $\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)\big)$. Moreover, since $C_r^*(\mathcal{R}, q)$ is unital, its spectrum is compact [41, Proposition 3.18], so there can be only finitely many inequivalent discrete series representations. On the other hand, they can only exist if $X^+ \cap X^- = 0$, i.e. if the root datum $\mathcal{R}$ is semisimple.

Let us mention two important examples. If $W = W_{\mathrm{aff}}$ and $q(s) > 1 \; \forall s \in S_{\mathrm{aff}}$ then

$$\pi_{St} : N_w \to (-1)^{\ell(w)} q(w)^{-1/2}$$

defines a discrete series representation. This is called the Steinberg representation of $\mathcal{H}(\mathcal{R}, q)$. Contrarily, if $q(s) < 1 \; \forall s \in S_{\mathrm{aff}}$ then the representation

$$\pi_{triv} : N_w \to q(w)^{1/2}$$

is discrete series. This is known as the trivial representation of $\mathcal{H}(\mathcal{R}, q)$, because it is a deformation of the trivial representation of W.

A linear functional $f : \mathcal{H} \to \mathbb{C}$ is tempered if there exist $C, N \in (0, \infty)$ such that for all $w \in W$

$$|f(N_w)| \leq C(1 + \mathcal{N}(w))^N$$

The collection of all tempered functionals is the linear dual of $\mathcal{S}(\mathcal{R}, q)$. From [98, Lemma 2.20] we get the following characterization:

Lemma 3.14 *For $(\pi, V) \in \mathrm{Rep}(\mathcal{H})$ the following are equivalent:*

- *π extends continuously to $\mathcal{S}(\mathcal{R}, q)$*
- *every matrix coefficient of (π, V) is a tempered functional*
- *V is a tempered $\mathcal{H}$-representation*

Suppose that our root datum is a product $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$. It is clear from (3.87) that the Schwartz completion respects the decomposition (3.42), so

$$\mathcal{S}(\mathcal{R}, q) \cong \mathcal{S}(\mathcal{R}_1, q) \,\widehat{\otimes}\, \mathcal{S}(\mathcal{R}_2, q) \tag{3.90}$$

An isomorphism like

$$C_r^*(\mathcal{R}, q) \cong C_r^*(\mathcal{R}_1, q) \otimes_t C_r^*(\mathcal{R}_2, q)$$

should also hold, but here the particular topological tensor product $\otimes_t$ is probably dictated by the very isomorphism.

Let q^0 be the label function that is identically 1 and T_u the unitary or real compact part of T. For any $\mathcal{R}$ the above constructions reduce to the well-known algebras

$$\begin{array}{lllllll} \mathcal{H}(\mathcal{R}, q^0) & = & \mathbb{C}[W] & = & \mathbb{C}[X] \rtimes W_0 & \cong & \mathcal{O}(T) \rtimes W_0 \\ \mathcal{S}(\mathcal{R}, q^0) & = & \mathcal{S}(W) & \cong & \mathcal{S}(X) \rtimes W_0 & \cong & C^\infty(T_u) \rtimes W_0 \\ C_r^*(\mathcal{R}, q^0) & = & C_r^*(W) & \cong & C_r^*(X) \rtimes W_0 & \cong & C(T_u) \rtimes W_0 \end{array} \tag{3.91}$$

where $\mathcal{S}(X)$ denotes the space of Schwartz functions on X. The representation theory of these algebras is not very complicated, and can be obtained from classical results which go back to Frobenius and Clifford [31].

Theorem 3.15 *1. The W-module $I_t = \mathrm{Ind}_X^W(\mathbb{C}_t)$ is completely reducible for any $t \in T$.*

2. I_t is unitary if and only if $t \in T_u$, in which case it is also tempered and anti-tempered.

3. The number of inequivalent irreducible summands of I_t is exactly the number of conjugacy classes in the isotropy group $W_{0,t}$.

Proof. 1. As an X-representation

$$I_t \cong \bigoplus_{w \in W_0} \mathbb{C}_{wt}$$

and the T_w act on I_t by permuting these onedimensional subspaces. Hence there is a natural bijection between subrepresentations of the left regular representation of $W_{0,t}$ and subrepresentations of I_t. It is given explicitly by $V \to \mathrm{Ind}_{W_{0,t}}^{W_0} V$, with X acting on the subspace $T_w V$ by the character wt. Since $\mathbb{C}[W_{0,t}]$ is completely reducible, so is I_t.
2 will be a special case of Proposition 3.17.
3 follows from counting the irreducible representations of the finite group $W_{0,t}$. □

3.3 The Fourier transform

Now we really delve into the representation theory of affine Hecke algebras. This is a very complicated subject and we will barely prove anything. In fact this section is more like an overview of some of the work of Delorme and Opdam [39, 40, 98]. In principle we mention only those things that we use later in some way or another, but this turns out to be a lot. The highlights are undoubtedly the concise Theorems 3.24 and 3.25, which describe the Fourier transform of affine Hecke algebras and their completions. Most of the other things are technicalities that can be skipped on first reading.

For any $t \in T^P$ there is a surjective algebra homomorphism

$$\phi_t : \mathcal{H}^P \to \mathcal{H}_P$$

which is the identity on $\mathcal{H}(W_0, q)$ and sends θ_x to $t(x)\theta_{x_P}$, where x_P is the image of x in $X_P = X/\big(X \cap (P^\vee)^\perp\big)$. If $\sigma \in \mathrm{Rep}(\mathcal{H}_P)$ then we can construct the $\mathcal{H}$-representation

$$\pi(P, \sigma, t) = \mathrm{Ind}_{\mathcal{H}^P}^{\mathcal{H}} (\sigma \circ \phi_t) \tag{3.92}$$

Because $\mathcal{A}$ is in general not a *-subalgebra of $\mathcal{H}$, it is not immediately clear whether this procedure preserves unitarity or temperedness of representations. Recall from [97, Propostion 1.12] that

Lemma 3.16 *Let w_0 be the longest element of W_0. For $x \in X$ we have*

$$\theta_x^* = T_{w_0} \theta_{-w_0 x} T_{w_0}^{-1} = N_{w_0} \theta_{-w_0 x} N_{w_0}^{-1}$$

In particular for $x \in X \cap \big(P^\vee\big)^\perp$ we have $\theta_x^* = \theta_{-x}$ in $\mathcal{H}^P$, so ϕ_t preserves the * if and only if $t \in T_u^P$. Similarly the inclusion $\mathcal{H}^P \to \mathcal{H}$ is in general not *-preserving. Nevertheless

Proposition 3.17 *Let σ be an irreducible $\mathcal{H}_P$-representation and $t \in T^P$.*

1. *$\pi(P, \sigma, t)$ is unitary if and only if σ is unitary and $t \in T_u^P$*
2. *$\pi(P, \sigma, t)$ is (anti-)tempered if and only if σ is (anti-)tempered and $t \in T_u^P$*

Proof. The "if" statements are [98, Propositions 4.19 and 4.20], so we prove the "only if" parts.
1. Clearly $\sigma \circ \phi_t$ can only be unitary if σ is. If now $t \in T^P \setminus T_u^P$ then there is an $x \in X \cap \big(P^\vee\big)^\perp$ with $|t(x)| \neq 1$. Hence

$$\sigma \circ \phi_t(\theta_x^*) = \sigma \circ \phi_t(\theta_{-x}) = \sigma(t(-x)) = t(x)^{-1} \neq \overline{t(x)} = \sigma(t(x))^* = \big(\sigma \circ \phi_t(\theta_x)\big)^*$$

2. Again it is obvious that σ needs to be tempered for $\sigma \circ \phi_t$ to be so. If t and x are as above, then either $|t(x)| > 1$ or $|t(-x)| > 1$, so by Lemma 3.14 $\sigma \circ \phi_t$ is not

tempered. This argument also applies in the anti-tempered case. □

Let Δ_P be the finite set of equivalence classes of discrete series representations of $\mathcal{H}_P = \mathcal{H}(\mathcal{R}_P, q_P)$. We pick a representative in every class and confuse Δ_P with this set of representations. Write

$$\Delta = \{(P, \delta) : P \subset F_0, \, \delta \in \Delta_P\}$$

and denote by Ξ the complex algebraic variety consisting of all triples (P, δ, t) with $P \subset F_0$, $\delta \in \Delta_P$ and $t \in T^P$. Similarly let Ξ_u be the compact smooth manifold of all triples $(P, \delta, t) \in \Xi$ with $t \in T_u^P$.

Let V_δ be the representation space of δ, endowed with the inner product $\langle \cdot \, , \, \cdot \rangle_\delta$, and define an inner product on $\mathcal{H}\big(W^P\big) \otimes_{\mathbb{C}} V_\delta$ by

$$\langle x \otimes u \, , \, y \otimes v \rangle = \tau(x^* y) \langle u \, , \, v \rangle_\delta \qquad x, y \in \mathcal{H}\big(W^P\big) \quad u, v \in V_\delta \tag{3.93}$$

With $\xi = (P, \delta, t) \in \Xi$ we associate the parabolically induced representation

$$\pi(\xi) = \pi(P, \delta, t) = \mathrm{Ind}_{\mathcal{H}^P}^{\mathcal{H}} (\delta \circ \phi_t) \tag{3.94}$$

We realize it on $\mathcal{H}\big(W^P\big) \otimes_{\mathbb{C}} V_\delta$ with the inner product (3.93).

For $P = \emptyset$ we have $\mathcal{H}_P \cong \mathbb{C}$. We denote its unique discrete series representation by $\delta_\emptyset$. Note that $\pi(\emptyset, \delta_\emptyset, t)$ is just the principal series representation I_t. .

We gather all parabolically induced representations in the following vector bundle over Ξ :

$$\mathcal{V}_\Xi = \bigcup_{(P,\delta) \in \Delta} T^P \times \mathcal{H}\big(W^P\big) \otimes V_\delta \tag{3.95}$$

Sometimes we will write $\xi \in \Xi$ as $(P, W_P r, \delta, t)$ to indicate that $W_P r \in T_P / W_P$ is the central character of δ. Then the central character of $\pi(\xi)$ is $W_0 r t \in T/W_0$.

If we let t run over T^P, we obtain a coset rT^P of the subtorus T^P in T. The collection of all such cosets, for different $(P, \delta) \in \Delta$ and different representatives of their central characters, has some special properties. It can be described combinatorially with the help of Macdonald's c-function, whose construction we recall now. For a long root $\alpha \in R_1$ we put

$$c_\alpha = \Big(1 + q_{\alpha^\vee}^{-1/2} \theta_{-\alpha/2}\Big) \Big(1 - q_{\alpha^\vee}^{-1/2} q_{2\alpha^\vee}^{-1} \theta_{-\alpha/2}\Big) \big(1 - \theta_{-\alpha}\big)^{-1} \qquad \alpha \in R_1 \tag{3.96}$$

This is a rational function on T, i.e. an element of the quotient field $Q(\mathcal{A})$ of $\mathcal{A}$. Strictly speaking, for $\alpha \in R_0 \cap R_1$ this formula is ill-defined, because there is no such thing as $\theta_{-\alpha/2}$. However in that case $q_{2\alpha^\vee} = 1$ by our convention (3.31), so we can interpret the above formula as

$$c_\alpha = \big(1 - q_{\alpha^\vee}^{-1} \theta_{-\alpha}\big) \big(1 - \theta_{-\alpha}\big)^{-1} \qquad \alpha \in R_0 \cap R_1 \tag{3.97}$$

Macdonald's c-function is the product

$$c(t) = \prod_{\alpha \in R_1^+} c_\alpha(t) \tag{3.98}$$

Suppose that $L \subset T$ is a coset of a subtorus T^L, and let

$$R_L = \{\alpha \in R_1 : \alpha(T^L) = 1\} \tag{3.99}$$

be the set of long roots that are constant on T^L. Then

$$T^L = \{t \in T : x(t) = 1 \text{ if } x \in \mathbb{Q}R_L \cap X\} = \mathrm{Hom}_{\mathbb{Z}}\big(X/(\mathbb{Q}R_L \cap X), \mathbb{C}^\times\big) \tag{3.100}$$

Hence $\theta_{-\alpha/2}(L)$ (or $\theta_{-\alpha}(L)$ if $\alpha \notin R_0$) is well-defined. Put

$$\begin{aligned} R_L^p &= \left\{\alpha \in R_L : \left(1 + q_{\alpha^\vee}^{-1/2}\theta_{-\alpha/2}(L)\right)\left(1 - q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}\theta_{-\alpha/2}(L)\right) = 0\right\} \\ R_L^z &= \{\alpha \in R_L : 1 - \theta_{-\alpha}(L) = 0\} \end{aligned} \tag{3.101}$$

By construction the pole order of the rational function

$$t \to c^{-1}(t)c^{-1}(t^{-1}) \tag{3.102}$$

along L is $|R_L^p| - |R_L^z|$. We say that L is a residual coset if

$$|R_L^p| - |R_L^z| = \dim(T) - \dim(L) \tag{3.103}$$

A residual coset of dimension 0 is also called a residual point. More or less everything there is to know about residual cosets can be found in [58, Section 4] and [98, Appendix A]. The collection of residual cosets is finite and W_0-invariant. They have been classified completely, and it turns out that a coset of a subtorus is residual if and only if its pole order for (3.102) is maximal, given its dimension.

From (3.101) we see that the residual cosets depend algebraically on q. In particular the number of residual cosets is maximal for all q in a certain Zariski open subset of the parameter space of all possible q's. Hence it makes sense to call q generic if and only if the number of residual cosets attains its maximum at q.

Define

$$T_L = \left\{t \in T : t(x) = 1 \text{ if } x \in (R_L^\vee)^\perp \cap X\right\} = \mathrm{Hom}_{\mathbb{Z}}\left(X \big/ \left((R_L^\vee)^\perp \cap X\right), \mathbb{C}^\times\right) \tag{3.104}$$

This is a subtorus of T and $K_L = T_L \cap T^L$ is a finite group. Notice that if L is a coset of T^P for some $P \subset F_0$, then $R_L = R_P$, $T_L = T_P$ and $K_L = K_P$. Pick any $r_L \in T_L \cap L$ and consider

$$r_L|r_L|^{-1} \in T_{L,u} \cap L$$

Since r_L is unique up to $K_L \cap T_u^L$, it follows that the tempered form

$$L^{\text{temp}} = r_L|r_L|^{-1}T_u^L \tag{3.105}$$

is independent of the choice of r_L. The tempered forms of residual cosets tend to be disjoint, cf. [98, Theorem A.18] :

Theorem 3.18 *Let L_1 and L_2 be residual cosets. If $L_1^{\text{temp}} \cap L_2^{\text{temp}} \neq \emptyset$ then $L_1 = w(L_2)$ for some $w \in W_0$.*

Our reason for studying residual cosets is a particular consequence of [98, Theorem 4.23] :

Theorem 3.19 *The collection of residual cosets is precisely*

$$\left\{ rT^P : (P, W_P r, \delta, t) \in \Xi \right\}$$

The tempered form of $L = rT^P$ is $L^{\text{temp}} = rT_u^P$.

This theorem implies that for every W_0-orbit of residual points there is at least one discrete series representation with that central character. In many cases there is in fact exactly one such representation:

Proposition 3.20 *1. Let $t \in T$ be a residual point such that the orbit $W_0 t$ contains exactly $|W_0|$ points. Then there is, up to equivalence, a unique discrete series representation with this central character.*

2. If the root system R_0 is of type A_n then 1. applies, and the representation in question is onedimensional.

Proof. For 1 see [121, Corollary 1.2.11]. For 2 we use the classification of the residual points for type A_n in [58, Proposition 4.1]. This shows that we can apply 1. To show that the resulting representation has dimension one, we just construct it. Since all simple roots are conjugate, we necessarily have an equal label function. If $W = W_{\text{aff}}$ then either π_{St} or π_{triv} is a onedimensional discrete series representation. Which one depends on whether $q(s) > 1$ or $q(s) < 1$ for any $s \in S_{\text{aff}}$, see page 81. If $W \neq W_{\text{aff}}$ then, given the central character, there is unique way to extend this representation to $C_r^*(\mathcal{R}, q)$. □

However, in general there may be more than one discrete series representation with a given central character. This is known to happen already for R_0 of type B_2, for certain label functions, see Section 6.4.

Following [98, Definition 3.24] we define a radical and a residual algebra at $t \in T$:

$$\begin{aligned} \overline{\mathcal{H}^t} &= \mathcal{H}(\mathcal{R}, q)/\text{Rad}_t \\ \text{Rad}_t &= \cap_\pi \ker \pi \end{aligned} \tag{3.106}$$

where π runs over the irreducible representations of $\mathcal{H}(\mathcal{R}, q)$ with the following two properties :

- the central character of π is $W_0 t$
- π extends to $C_r^*(\mathcal{R}, q)$

Clearly Rad_t contains all elements of $\mathcal{Z}$ that vanish at t, so $\overline{\mathcal{H}^t}$ is a finite dimensional C^*-algebra whose irreducible representations correspond to the irreducible representations of $C_r^*(\mathcal{R}, q)$ with central character $W_0 t$. By Theorem 3.19 the collection of discrete series representations of $\mathcal{H}(\mathcal{R}, q)$ is precisely the union, over all residual points t, of the irreducible representations of $\overline{\mathcal{H}^t}$.

As one would expect, two parabolically induced representations can be related if they have the same central character. Let try to clarify when and how this occurs. For $P, Q \subset F_0$ write

$$W(P,Q) = \{n \in W_0 : n(P) = Q\} \tag{3.107}$$

For such n there is a *-algebra isomorphism $\psi_n : \mathcal{H}^P \to \mathcal{H}^Q$, defined by

$$\psi_n(N_w \theta_x) = N_{nwn^{-1}} \theta_{nx} \qquad w \in W_P,\ x \in X \tag{3.108}$$

Similarly for $k \in K_P$ there is a *-algebra automorphism $\psi_k : \mathcal{H}^P \to \mathcal{H}^P$, defined by

$$\psi_k(N_w \theta_x) = k(x) N_w \theta_x \qquad w \in W_P,\ x \in X \tag{3.109}$$

These maps descend to isomorphisms $\psi_n : \mathcal{H}_P \to \mathcal{H}_Q$ and $\psi_k : \mathcal{H}_P \to \mathcal{H}_P$.

Let $\mathcal{W}$ be the finite groupoid, over the power set of F_0, with

$$\mathcal{W}_{PQ} = K_Q \times W(P,Q) \tag{3.110}$$

$$(k_1,n_1)(k_2,n_2) = (k_1 n_1(k_2), n_1 n_2) \qquad (k_1,n_1) \in \mathcal{W}_{PQ},\ (k_2 n_2) \in \mathcal{W}_{P'P} \tag{3.111}$$

For $kn \in \mathcal{W}_{PQ}$ and $\delta \in \Delta_P$, let $\Psi_{kn}(\delta)$ be the equivalence class of the discrete series representation $\delta \circ \psi_n^{-1} \circ \psi_k^{-1}$ of $\mathcal{H}_Q$. Then there exists a unitary isomorphism $\tilde{\delta}_{kn} : V_\delta \to V_{\Psi_{kn}(\delta)}$ such that

$$\tilde{\delta}_{kn} \circ \delta(\psi_n^{-1} \psi_k^{-1} h) = \Psi_{kn}(\delta)(h) \circ \tilde{\delta}_{kn} \qquad h \in \mathcal{H}_Q \tag{3.112}$$

By Schur's lemma $\tilde{\delta}_{kn}$ is unique up to a complex number of absolute value 1, but whether or not there is a canonical way to choose these scalars is an open problem. This is a subtle point to which we will return later.

The above maps induce an action of $\mathcal{W}$ on Ξ :

$$kn(P, W_P r, \delta, t) = \big(Q, W_Q n(r), \Psi_{kn}(\delta), kn(t)\big) \tag{3.113}$$

We would like to lift $\tilde{\delta}_g$ to an intertwiner $\pi(g,\xi)$ between $\pi(\xi)$ and $\pi(g\xi)$. For $k \in K_p$ this is easy, we can simply define

$$\pi(k,\xi) = \mathrm{id}_{\mathcal{H}(W^P)} \otimes \tilde{\delta}_k : \mathcal{H} \otimes_{\mathcal{H}^P} V_\delta \to \mathcal{H} \otimes_{\mathcal{H}^P} V_\delta \tag{3.114}$$

This is a well-defined intertwiner since, for all $h \in \mathcal{H}$, $h' \in \mathcal{H}^P$, $v \in V$:

$$\begin{aligned} \pi(k,\xi)(hh' \otimes v) &= hh' \otimes \tilde{\delta}_k(v) \\ &= h \otimes \Psi_k(\delta)(\phi_{kt} h')\big(\tilde{\delta}_k v\big) \\ &= h \otimes \Psi_k(\delta)\big(\psi_k(\phi_t h')\big)\big(\tilde{\delta}_k v\big) \\ &= h \otimes \tilde{\delta}_k\big(\delta(\phi_t h')v\big) \\ &= \pi(k,\xi)\big(h \otimes \delta(\phi_t h')v\big) \end{aligned} \tag{3.115}$$

On the other hand, for $w \in W_0$ it is a tricky business, which is taken care of in [98, §4]. We shall recall the parts of the construction that we use later on.

Let U be a W_0-invariant subset of T, open in the analytic topology, and let $C^{an}(U)$ and $\mathcal{Z}^{an}(U)$ be the algebras of, respectively, holomorphic functions on U and W_0-invariant holomorphic functions on U. There are natural injections $\mathcal{A} \to C^{an}(U)$ and $\mathcal{Z} \to \mathcal{Z}^{an}(U)$, so we can construct

$$\mathcal{H}^{an}(U) = \mathcal{Z}^{an}(U) \otimes_{\mathcal{Z}} \mathcal{H} = C^{an}(U) \otimes_{\mathcal{A}} \mathcal{H} \tag{3.116}$$

Similarly we define the algebras $C^{me}(U)$ and $\mathcal{Z}^{me}(U)$ of (W_0-invariant) meromorphic functions on U, and

$$\mathcal{H}^{me}(U) = \mathcal{Z}^{me}(U) \otimes_{\mathcal{Z}} \mathcal{H} = C^{me}(U) \otimes_{\mathcal{A}} \mathcal{H} \tag{3.117}$$

It follows from Theorem 3.3.1 that $\mathcal{H}^{an}(U)$ and $\mathcal{H}^{me}(U)$ are free modules over respectively $C^{an}(U)$ and $C^{me}(U)$, with bases $\{N_w : w \in W_0\}$. We define a * on these algebras by

$$f^* = T_{w_0} f^{-w_0} T_{w_0}^{-1} = N_{w_0} f^{-w_0} N_{w_0}^{-1} \qquad f \in C^{me}(U) \tag{3.118}$$

By Lemma 3.16 this extends the * on $\mathcal{H}(\mathcal{R}, q)$.

A finite dimensional $\mathcal{H}$-representation extends to $\mathcal{H}^{an}(U)$ if and only if all its $\mathcal{Z}$-weights are in U/W_0. Applying this to $\mathcal{H}^P$, we see that $\delta \circ \phi_t$ extends to $\mathcal{H}^{P,an}(U)$ if and only if $W_P rt \in U/W_P$. The proof of these claims can be found on [98, p. 582].

Pick $t \in T$ and consider a set $B \subset \text{Lie}(T) = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ which satisfies the following conditions:

Condition 3.21 *1. B is an open ball centred around $0 \in \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$*

2. $\forall \alpha \in R_{nr}, b \in \overline{B} :\ |\Im(\alpha(b))| < \pi$ (so $\exp : B \to \exp(B)$ is a diffeomorphism)

3. Write $U_t = t \exp(B)$. If $w \in W_0$ and $w\overline{U_t} \cap \overline{U_t} \neq \emptyset$, then $wt = t$

4. $\forall u \in \overline{U_t}$ we have

$$R^p_{\{u\}} \cup R^z_{\{u\}} \subset R^p_{\{t\}} \cup R^z_{\{t\}}$$

Recall that $R^p_{\{t\}} \cup R^z_{\{t\}}$ is the set of roots $\alpha \in R_1$ for which t is a critical point of the rational function c_α. Notice that these conditions are almost the same as [98, Conditions 4.9]. In fact the only difference is that we replaced some statements with B by slightly stronger versions with $\overline{B}$. This is not an essential difference, it only makes it easier in Section 5.3 to see that certain functions are bounded.

By [98, Proposition 4.7] Condition 3.21 can always be fulfilled. This yields an alternative description of the representation space of $\pi(P, W_P r, \delta, t)$:

$$\mathcal{H}(W^P) \otimes_{\mathbb{C}} V_\delta = \mathcal{H} \otimes_{\mathcal{H}^P} V_\delta = \mathcal{H}^{an}(W_0 U_{rt}) \otimes_{\mathcal{H}^{P,an}(W_P U_{rt})} V_\delta \tag{3.119}$$

Let $s \in S_0$ be the simple reflection corresponding to $\alpha \in F_1$. Consider the following element of $Q(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} = Q(\mathcal{Z}) \otimes_{\mathcal{Z}} \mathcal{H}$:

$$\begin{aligned}
\imath_s^0 &= \left(q_{\alpha^\vee}^{1/2} + \theta_{-\alpha/2}\right)^{-1} \left(q_{\alpha^\vee}^{1/2} q_{2\alpha^\vee} - \theta_{-\alpha/2}\right)^{-1} \\
&\quad \left((1-\theta_{-\alpha})T_s + (q_{\alpha^\vee} q_{2\alpha^\vee} - 1) + q_{\alpha^\vee}^{1/2}(q_{2\alpha^\vee} - 1)\theta_{-\alpha/2}\right) \\
&= \left(T_s(1-\theta_\alpha) + (q_{\alpha^\vee} q_{2\alpha^\vee} - 1)\theta_\alpha + q_{\alpha^\vee}^{1/2}(q_{2\alpha^\vee} - 1)\theta_{\alpha/2}\right) \\
&\quad \left(q_{\alpha^\vee}^{1/2} + \theta_{\alpha/2}\right)^{-1} \left(q_{\alpha^\vee}^{1/2} q_{2\alpha^\vee} - \theta_{\alpha/2}\right)^{-1}
\end{aligned} \tag{3.120}$$

For $\alpha \in F_1 \cap R_0$ this simplifies to

$$\imath_s^0 = (q(s)-\theta_{-\alpha})^{-1}((1-\theta_{-\alpha})T_s+q(s)-1) = (T_s(1-\theta_\alpha)+(q(s)-1)\theta_\alpha)(q(s)-\theta_\alpha)^{-1}$$

Important properties of such elements come from [86, Proposition 5.5] and [98, Lemma 4.1] :

Theorem 3.22 *The map*

$$S_0 \to Q(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} : s \to \imath_s^0$$

extends to a group homomorphism

$$W_0 \to \left(Q(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}\right)^\times$$

For all $w \in W_0, f \in Q(\mathcal{A})$ we have

$$\imath_w^0 f \imath_{w^{-1}}^0 = f^w = f \circ w^{-1} \in Q(\mathcal{A}) \tag{3.121}$$

As $Q(\mathcal{A})$-modules, we can identify

$$Q(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} = \bigoplus_{w \in W_0} \imath_w^0 Q(\mathcal{A}) = \bigoplus_{w \in W_0} Q(\mathcal{A}) \imath_w^0$$

If $t \in T$ and $c_\alpha^{-1}(t) = 0$ then by definition $\imath_{s_\alpha}^o(t) = 1$. If we combine this with (3.121) we see that it is not specific for simple reflections:

$$\beta \in R_0,\, c_\beta^{-1}(t) = 0 \Rightarrow \imath_{s_\beta}^o = 1 \tag{3.122}$$

With a straightforward but tedious computation one may show that

$$\left(\imath_w^0\right)^* = T_{w_0} \prod_{\alpha \in R_1^+ \cap w' R_1^-} \left(\frac{c_\alpha}{c_{-\alpha}}\right) \imath_{w'}^0 T_{w_0}^{-1} \tag{3.123}$$

where $w' = w_0 w^{-1} w_0$. It follows more readily from the Bernstein-Lusztig-Zelevinski relations (3.41) that, for $n \in W(P,Q)$ and $w \in W_P$,

$$\imath_{n^{-1}}^0 N_w \imath_n^0 = N_{nwn^{-1}} \tag{3.124}$$

Hence we have the following equality in $Q(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}$:

$$\imath_{n^{-1}}^0 h \imath_n^0 = \psi_n(h) \qquad h \in \mathcal{H}^P \tag{3.125}$$

For $\xi = (P, W_P r, \delta, t) \in \Xi$ we define

$$\begin{aligned} &\pi(n,\xi) : \mathcal{H}^{an}(W_0 U_{rt}) \otimes_{\mathcal{H}^{P,an}(W_P U_{rt})} V_\delta \to \mathcal{H}^{an}(W_0 U_{rt}) \otimes_{\mathcal{H}^{Q,an}(W_Q U_{n(rt)})} V_{\Psi_n(\delta)} \\ &\pi(n,\xi)(h \otimes v) = h\, \imath_{n^{-1}}^0 \otimes \tilde{\delta}_n(v) \qquad h \in \mathcal{H}, v \in V_\delta \end{aligned} \tag{3.126}$$

With (3.119) in mind, this map intertwines the $\mathcal{H}$-representations $\pi(\xi)$ and $\pi(n\xi)$ since,
for $h \in \mathcal{H}$, $h' \in \mathcal{H}^P$, $v \in V_\delta$:

$$\begin{aligned} \pi(n,\xi)(hh' \otimes v) &= hh' \imath_{n^{-1}}^0 \otimes \tilde{\delta}_n(v) \\ &= h\imath_{n^{-1}}^0 (\imath_n^0 h' \imath_{n^{-1}}^0) \otimes \tilde{\delta}_n(v) \\ &= h\, \imath_{n^{-1}}^0 \otimes \Psi_n(\delta)\big(\phi_{n(t)}(\imath_n^0 h' \imath_{n^{-1}}^0)\big)\big(\tilde{\delta}_n v\big) \\ &= h\, \imath_{n^{-1}}^0 \otimes \Psi_n(\delta)\big(\phi_{n(t)}(\psi_n h')\big)\big(\tilde{\delta}_n v\big) \\ &= h\, \imath_{n^{-1}}^0 \otimes \Psi_n(\delta)\big(\psi_n(\phi_t h')\big)\big(\tilde{\delta}_n v\big) \\ &= h\, \imath_{n^{-1}}^0 \otimes \tilde{\delta}_n\big(\delta(\phi_t h')v\big) \\ &= \pi(n,\xi)\big(h \otimes \delta(\phi_t h')v\big) \end{aligned} \tag{3.127}$$

For $g = kn \in \mathcal{W}$ we define the intertwiner

$$\pi(g,\xi) = \pi(k, n\xi)\pi(n,\xi) \tag{3.128}$$

Because every $\tilde{\delta}_g$ is unique up to a scalar, there is a $c(g_1, g_2, \delta) \in \mathbb{T}$ such that

$$\pi(g_1, g_2\xi)\pi(g_2,\xi) = c(g_1, g_2, \delta)\pi(g_1 g_2, \xi) \tag{3.129}$$

Things would simplify considerably if we could arrange that all $c(g_1, g_2, \delta)$ would be 1. In many cases this is indeed possible, but there are some examples for which it cannot be done.

A priori $\pi(kn, \xi)$ is well-defined only on the Zariski-open subset of (P, δ, T^P) where $\imath_n^0$ and $\imath_{n^{-1}}^0$ are regular. In a worst case scanerio this set could even be empty, but fortunately is assured by [98, Theorem 4.33 and Corollary 4.34] that

Theorem 3.23 *For any $g \in \mathcal{W}_{PQ}$ intertwining map*

$$\pi(g,\xi) : \mathcal{H}\big(W^P\big) \otimes V_\delta \to \mathcal{H}\big(W^Q\big) \otimes V_{\Psi_g(\delta)}$$

is rational as a function of $t \in T^P$. It is regular on an analytically open neighborhood of T_u^P. Moreover, if $t \in T_u^P$ then $\pi(g,\xi)$ is unitary with respect to the inner products defined by (3.93).

With this in mind, let $\Gamma_{rr}(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$ be the algebra of rational sections of $\mathrm{End}(\mathcal{V}_\Xi)$ that are regular on Ξ_u. It can be described more explicitly as

$$\begin{aligned}\Gamma_{rr}\big(\Xi; \mathrm{End}(\mathcal{V}_\Xi)\big) &= \bigoplus_{(P,\delta)\in\Delta} \Gamma_{rr}\left(T^P; \mathrm{End}\big(\mathcal{H}\big(W^P\big)\otimes V_\delta\big)\right) \\ &= \bigoplus_{(P,\delta)\in\Delta} \left\{ f \in Q\big(\mathcal{O}\big(T^P\big)\big) \otimes \mathrm{End}\big(\mathcal{H}\big(W^P\big)\otimes V_\delta\big) : f \text{ is regular on } T^P_u \right\} \end{aligned} \tag{3.130}$$

Obviously this algebra contains the algebra of polynomial sections

$$\mathcal{O}\big(\Xi; \mathrm{End}(\mathcal{V}_\Xi)\big) = \bigoplus_{(P,\delta)\in\Delta} \mathcal{O}\big(T^P\big) \otimes \mathrm{End}\big(\mathcal{H}\big(W^P\big)\otimes V_\delta\big) \tag{3.131}$$

Using the standard analytic structure on Ξ, we define the algebras $C(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$ and $C^\infty(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$ of continuous (respectively smooth) sections of $\mathrm{End}(\mathcal{V}_\Xi)$ in the same way. Furthermore, if μ is a sufficiently "nice" measure on Ξ then the following formula defines a (degenerate) Hermitian inner product on $C(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$:

$$\langle f_1\,,\, f_2\rangle_\mu = \int_\Xi \mathrm{tr}\big(f_1(\xi)^* f_2(\xi)\big) d\mu(\xi) \tag{3.132}$$

We denote the corresponding Hilbert space completion by $L^2(\Xi, \mu; \mathrm{End}(\mathcal{V}_\Xi))$.

From the action of the groupoid $\mathcal{W}$ on Ξ and the above intertwiners we get an action of $\mathcal{W}$ on $\Gamma_{rr}(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$ by algebra homomorphisms :

$$(g \cdot f)(\xi) = \pi(g, g^{-1}\xi) f(g^{-1}\xi) \pi(g, g^{-1}\xi)^{-1} \tag{3.133}$$

whenever $g^{-1}\xi$ is defined. The average of f over $\mathcal{W}$ is

$$p_{\mathcal{W}}(f)(P,\delta,t) = \big|\{(Q,g) : Q \subset F_0,\, g \in \mathcal{W}_{QP}\}\big|^{-1} \sum_{Q\subset F_0} \sum_{g\in\mathcal{W}_{QP}} (g\cdot f)(P,\delta,t) \tag{3.134}$$

Notice that for $f \in \mathcal{O}(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$ and $g \in \mathcal{W}$, $g \cdot f$ and $p_{\mathcal{W}}(f)$ need not lie in $C(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$. However, if M is a $\mathcal{W}$-stable subset of Ξ on which all the intertwiners are regular, then there is a well-defined action of $\mathcal{W}$ on $C(M; \mathrm{End}(\mathcal{V}_\Xi))$. The same holds for smooth sections if M is a smooth submanifold of Ξ on top of this.

The Fourier transform is the algebra homomorphism

$$\begin{aligned} &\mathcal{F} : \mathcal{H} \to \mathcal{O}\big(\Xi; \mathrm{End}(\mathcal{V}_\Xi)\big) \\ &\mathcal{F}(h)(\xi) = \pi(\xi)(h) \end{aligned} \tag{3.135}$$

We can extend it continously to various completions of $\mathcal{H}(\mathcal{R}, q)$. After doing so, its image can be described completely with our intertwiners. For the Hilbert space completion $\mathfrak{H}(\mathcal{R}, q)$ the following was proved in [98, Theorem 4.43 and Corollary 4.45] :

Theorem 3.24 *There exists a unique "Plancherel" measure μ_{Pl} on Ξ with the properties*

1. *the support of μ_{Pl} is Ξ_u*
2. *μ_{Pl} is W_0-invariant*
3. *on every component (P, δ, T^P) μ_{Pl} is absolutely continuous with respect to the Haar measure of T_u^P*
4. *the Fourier transform extends to a bijective isometry*

$$\mathcal{F} : \mathfrak{H}(\mathcal{R}, q) \to L^2(\Xi_u, \mu_{Pl}; \mathrm{End}(\mathcal{V}_\Xi))^{\mathcal{W}}$$

   i.e. μ_{Pl} is the Plancherel measure for τ

5. *the adjoint map*

$$\mathcal{J} : L^2(\Xi_u, \mu_{Pl}; \mathrm{End}(\mathcal{V}_\Xi)) \to \mathfrak{H}(\mathcal{R}, q)$$

   satisfies $\mathcal{J}\mathcal{F} = \mathrm{id}_{\mathfrak{H}(\mathcal{R},q)}$ and $\mathcal{F}\mathcal{J} = p_{\mathcal{W}}$.

The corresponding statements for the Schwartz and C^*-completions are [39, Theorem 4.3 and Corollary 4.7] :

Theorem 3.25 *The Fourier transform induces algebra homomorphisms*

$$\begin{array}{rcr} \mathcal{H}(\mathcal{R}, q) & \to & \mathcal{O}\big(\Xi; \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}} \\ \mathcal{S}(\mathcal{R}, q) & \to & C^\infty\big(\Xi_u; \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}} \\ C_r^*(\mathcal{R}, q) & \to & C\big(\Xi_u; \mathrm{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}} \end{array}$$

*The first one is injective, the second is an isomorphism of Fréchet *-algebras and the third is an isomorphism of C^*-algebras.*

Some remarkable consequences of this theorem are

Corollary 3.26 1. *The centers of $\mathcal{S}(\mathcal{R}, q)$ and $C_r^*(\mathcal{R}, q)$ are*

$$Z\big(\mathcal{S}(\mathcal{R}, q)\big) \cong C^\infty(\Xi_u)^{\mathcal{W}}$$
$$Z\big(C_r^*(\mathcal{R}, q)\big) \cong C(\Xi_u)^{\mathcal{W}}$$

2. *Every irreducible tempered $\mathcal{H}(\mathcal{R}, q)$-representation (π, V) is a direct summand of $\pi(\xi)$, for some $\xi \in \Xi_u$. In particular we can endow V with an inner product such that π is unitary and extends to $C_r^*(\mathcal{R}, q)$.*
3. *For any $\xi \in \Xi$ and $g \in \mathcal{W}$ such that $g\xi$ is defined, the $\mathcal{H}(\mathcal{R}, q)$-representations $\pi(\xi)$ and $\pi(g\xi)$ have the same irreducible subquotients, counted with multiplicity.*

Proof. 1 comes from [39, Corollary 4.5].
2. As an $\mathcal{S}(\mathcal{R}, q)$-representation (π, V) has a single $Z(\mathcal{S}(\mathcal{R}, q))$-weight $\mathcal{W}\xi \in \Xi_u/\mathcal{W}$. Then it is also an irreducible representation of the finite dimensional C^*-algebra

$$\mathcal{S}(\mathcal{R}, q)/\{h \in \mathcal{S}(\mathcal{R}, q) : \pi(g\xi)(h) = 0 \; \forall g \in \mathcal{W}\}$$

By Theorem 3.25 every irreducible representation of this algebra is a direct summand of a parabolically induced representation $\pi(g\xi)$. Again by Theorem 3.25 this means that (π, V) extends to a (unitary) representation of $C_r^*(\mathcal{R}, q)$.
3. See [40, Proposition 6.1]. We have to show that the characters of $\pi(\xi)$ and $\pi(g\xi)$ are equal, i.e. that the function

$$\mathcal{H} \times T^P \to \mathbb{C} : (h, t) \to \text{tr}\, \pi(P, \delta, t)(h) - \text{tr}\, \pi(Q, \Psi_g(\delta), g(t))(h) \tag{3.136}$$

is identically 0. Because this is a polynomial function of t, it suffices to show that it is 0 for all $t \in T_u^P$. This follows directly from Theorem 3.25. □

Corollary 3.27 *1. An element $h \in \mathcal{S}(\mathcal{R}, q)$ is invertible in $\mathcal{S}(\mathcal{R}, q)$ if and only if it is invertible in $C_r^*(\mathcal{R}, q)$, which happens if and only if it is invertible in $B(\mathfrak{H}(\mathcal{R}, q))$.*

2. The set of invertible elements $\mathcal{S}(\mathcal{R}, q)^\times$ is open in $\mathcal{S}(\mathcal{R}, q)$, and inverting is a continuous map from this set to itself.

Proof. By [125, Proposition 4.8] and Theorem 3.25 the inclusions

$$\mathcal{S}(\mathcal{R}, q) \to C_r^*(\mathcal{R}, q) \to B(\mathfrak{H}(\mathcal{R}, q)) \tag{3.137}$$

are isospectral, which is another way to state 1. Because the set of invertibles is always open in a Banach algebra, so is

$$\mathcal{S}(\mathcal{R}, q)^\times \cong C^\infty\big(\Xi_u; \text{End}(\mathcal{V}_\Xi)\big)^{\mathcal{W}} \cap C\big(\Xi_u; \text{End}(\mathcal{V}_\Xi)\big)^\times \tag{3.138}$$

Finally, by Theorem 2.9.2 inverting is continuous on $\mathcal{S}(\mathcal{R}, q)^\times$. □

Now we will combine the Langlands classification with the Fourier transform to obtain a finer classification of irreducible $\mathcal{H}$-modules. Although the proofs of the following results are mostly in [40], we give them anyway, because we want to generalize them to Hecke algebras of reductive p-adic groups.

For $Q \subset F_0$, let $\mathcal{W}^Q$, Ξ^Q, $\pi^Q(\xi)$ etcetera denote the same as the corresponding objects without the superscript Q, but now for the affine Hecke algebra $\mathcal{H}^Q$. For $\xi = (P, W_P r, \delta, t) \in \Xi$ we put ,

$$P(\xi) = \{\alpha \in F_0 : |\alpha(t)| = 1\} \supset P \tag{3.139}$$

We will study $\pi(\xi)$ by induction in stages, first we construct $\pi^{P(\xi)}(\xi)$ and then we induce that representation to $\mathcal{H}$. The following result from [40, Proposition 6.17.1] provides the link with the Langlands classification.

Lemma 3.28 *The $\mathcal{H}^{P(\xi)}$-representation $\pi^{P(\xi)}(\xi)$ is essentially tempered and completely reducible.*

Proof. By Proposition 3.17 $\pi^{P(\xi)}(P, \delta, t|t|^{-1})$ is tempered and unitary. From the proof of Lemma 3.5 we see that

$$\pi^{P(\xi)}(\xi) \circ \psi_{|t|}^{-1} = \pi^{P(\xi)}(P, \delta, t|t|^{-1}) \tag{3.140}$$

so $\pi^{P(\xi)}(\xi)$ is completely reducible. If $r_1, \ldots, r_d$ are the $\mathcal{A}$-weights of (3.140), then $|t|r_1, \ldots, |t|r_d$ are the $\mathcal{A}$-weights of $\pi^{P(\xi)}(\xi)$. But for $x \in \mathbb{Z}P(\xi)$ we have $|r_i t(x)| = |r_i(x)|$, so $\pi^{P(\xi)}(\xi)$ is essentially tempered. $\square$

Similar to the definition of Langlands data, we need a kind of positivity condition on Ξ. Thus we say that $\xi \in \Xi^+$ if $|t| \in \overline{T_{rs}^{P,+}} = \exp \overline{\mathfrak{t}^{P,+}}$.

Proposition 3.29 *Take $\xi = (P, W_P r, \delta, t) \in \Xi^+$.*

1. *Let σ be an irreducible direct summand of $\pi^{P(\xi)}(\xi)$. Then $(P(\xi), \sigma) \in \Lambda^+$.*
2. *The functor $\mathrm{Ind}_{\mathcal{H}^{P(\xi)}}^{\mathcal{H}}$ induces an isomorphism*

$$\mathrm{End}_{\mathcal{H}}(\pi(\xi)) \cong \mathrm{End}_{\mathcal{H}^{P(\xi)}}\big(\pi^{P(\xi)}(\xi)\big)$$

3. *The irreducible quotients of $\pi(\xi)$ are precisely the modules $L(P(\xi), \sigma)$ with σ as above.*
4. *Every $L(P(\xi), \sigma)$ has an $\mathcal{A}$-weight t_σ such that there exists a root subsystem $R_\sigma \subset R_0$, of rank $|P|$, with the properties*

$$\begin{aligned} &\forall \alpha \in R_0^+ \cap R_\sigma : |t_\sigma(\alpha)| < 1 \\ &\forall \alpha \in R_0^+ \cap \big(R_\sigma^\vee\big)^\perp : |t_\sigma(\alpha)| \geq 1 \end{aligned} \tag{3.141}$$

5. *Suppose that (ρ, V) is an irreducible constituent of $\pi(\xi)$ which is not a quotient, and that t_ρ is an $\mathcal{A}$-weight of V. Then every root subsystem R_ρ with the properties (3.141) has rank $> |P|$.*

Proof. 1. By definition $r_\sigma = |t| \in T_{rs}^{P,+}$, so $(P, \sigma) \in \Lambda^+$.
2 and 3 follow from 1 and Theorem 3.7.
4. By Lemma 3.6 every $\mathcal{A}$-weight of σ is of the form $t_\sigma = w(rt)$ with r an $\mathcal{A}_P$-weight of δ and

$$w \in W_{P(\xi)} \cap W^P = \big\{ w \in W_{P(\xi)} : w(P) \subset R_0^+ \big\}$$

By [45, (2.7)] $L(P(\xi), \sigma)$ also has an $\mathcal{A}$-weight of this form. Hence we may take $R_\sigma = w(R_P)$.

5. By [45, (2.7)] every $\mathcal{A}$-weight of (ρ, V) is of the form $t_\rho = nw(rt)$, with $n \in W^Q \setminus \{e\}$ and w, r, t as in the proof of 4. Clearly, for

$$\alpha \in nw(R_P^+) \subset n\big(R_Q^+\big) \subset R_0^+$$

we have

$$\big|t_\rho(\alpha)\big| = \big|rt\big(w^{-1}n^{-1}\alpha\big)\big| < 1$$

Furthermore, since $n \neq e$, there exists a $\beta \in R_0^+$ with $n^{-1}(\beta) \in R_0^-$ and $\beta \perp n(P(\xi))^\vee$. Now $P' = nw(P) \cup \{\beta\}$ is a linearly independent set of positive roots such that $|t_\rho(\alpha)| < 1 \; \forall \alpha \in P'$. Therefore any suitable root system R_ρ must have rank at least $|P'| = |P| + 1$. □

Although it is written down differently, the proof of Proposition 3.29 is rather similar to that of the Langlands classification. In particular we use the same ideas as Langlands' geometric lemmas [80, p. 61-63].

Lemma 3.30 *Every $\xi \in \Xi$ is W_0-associate to an element of Ξ^+. If $\xi_1, \xi_2 \in \Xi^+$ are $\mathcal{W}$-associate, then $P(\xi_1) = P(\xi_2) := Q$, and $\pi^Q(\xi_1)$ and $\pi^Q(\xi_2)$ are equivalent as $\mathcal{H}^Q$-representations.*

Proof. By [61, Section 1.15] every W_0-orbit in $\mathfrak{t}$ contains a unique point in a positive chamber $\mathfrak{t}^{Q,+}$. Hence $|t_1| = |t_2|$ and $P(\xi_1) = P(\xi_2) = Q$. From Lemmas 3.5 and 3.28 we see that there is a single automorphism $\psi_{|t_1|} = \psi_{|t_2|} := \psi$ of $\mathcal{H}^Q$ such that, for $i = 1, 2$,

$$\pi^Q(\xi_i) \circ \psi^{-1} \cong \pi^Q(\xi_i') \quad \text{where} \quad \xi_i' = (P_i, \delta_i, t_i|t_i|^{-1}) \in \Xi_u^Q \tag{3.142}$$

If $g\xi_1 = \xi_2$ for some $g \in \mathcal{W}$, then also $g\xi_1' = \xi_2'$. Applying Theorem 3.25 to $\mathcal{H}^Q$, we see that $\pi^Q(\xi_1')$ and $\pi^Q(\xi_2')$ are unitarily equivalent. It follows from this and (3.142) that $\pi^Q(\xi_1)$ and $\pi^Q(\xi_2)$ are equivalent. □

Theorem 3.31 *For every irreducible $\mathcal{H}$-representation π there exists a unique association class $\mathcal{W}(P, \delta, t) \in \Xi/\mathcal{W}$ such that the following (equivalent) statements hold :*

1. *π is equivalent to an irreducible quotient of $\pi(\xi^+)$, for some $\xi^+ \in \Xi^+ \cap \mathcal{W}(P, \delta, t)$*
2. *π is equivalent to an irreducible subquotient of $\pi(P, \delta, t)$, and $|P|$ is maximal for this property*

Proof. For 1 we copy [40, Corollary 6.19]. By Theorem 3.7 there is a unique Langlands datum $(Q, \sigma) \in \Lambda^+$ such that $\pi \cong L(P, \sigma)$, and by Lemma 3.5 $\sigma \circ \psi_{r_\sigma}^{-1}$ is tempered. Now Theorem 3.25 tells us that there exists a unique association class $\mathcal{W}^Q\xi_1 = \mathcal{W}^Q(P_1, \delta_1, t_1) \in \Xi_u^Q/\mathcal{W}^Q$ such that $\pi(\xi_1)$ contains $\sigma \circ \psi_{r_\sigma}^{-1}$ as an irreducible direct summand. Put $\xi = (P_1, \delta_1, t_1 r_\sigma) \in \Xi^Q$ and, using Lemma 3.30

for $\mathcal{H}^Q$, pick $\xi^+ = (P_2, \delta_2, t_2) \in \mathcal{W}^Q\xi \cap \Xi^+$. Then σ is a direct summand of $\pi^Q(\xi^+)$, and we see from Propoistion 3.29.3 that π is an irreducible subquotient of $\pi(\xi^+)$. By Lemma 3.30 and Theorem 3.7 the class $\mathcal{W}\xi \in \Xi/\mathcal{W}$ is unique for this property.

Suppose that π is also an irreducible subquotient of $\pi(\xi_3) = \pi(P_3, \delta_3, t_3)$, where $|P_3| \geq |P|$. By Corollary 3.26.2 we may assume that $\xi_3 \in \Xi^+$. Comparing parts 4 and 5 of Proposition 3.29 we see that in fact π must be equivalent to an irreducible quotient of $\pi(\xi_3)$. But then ξ_3 is $\mathcal{W}$-associate to ξ^+ by the above. Because the class $\mathcal{W}\xi$ is unique for both properties 1 and 2, this also shows that 1 and 2 are equivalent. $\Box$.

3.4 Periodic cyclic homology

We prove comparison theorems between the periodic cyclic homology of an affine Hecke algebra, that of its Schwartz completion, and the K-theory of its C^*-completion. We reap some fruits from our previous labor in the sense that the technicalities are limited, although still substantial.

Let $\mathcal{H}(\mathcal{R}, q)$ be an affine Hecke algebra. Observe that by Theorems 2.27 and 3.25 the Chern character gives an isomorphism

$$K_*(\mathcal{S}(\mathcal{R}, q)) \otimes \mathbb{C} \xrightarrow{\sim} HP_*(\mathcal{S}(\mathcal{R}, q)) \tag{3.143}$$

However, we cannot apply Theorem 2.29 to $\mathcal{H}(\mathcal{R}, q)$, since the action of the groupoid $\mathcal{W}$ on $\mathcal{O}(\Xi; \mathrm{End}(\mathcal{V}_\Xi))$ is by rational intertwiners, which may have poles outside Ξ_u. Therefore the proof of the next theorem will involve several steps.

Theorem 3.32 *The inclusion $\mathcal{H}(\mathcal{R}, q) \to \mathcal{S}(\mathcal{R}, q)$ induces an isomorphism*

$$HP_*(\mathcal{H}(\mathcal{R}, q)) \xrightarrow{\sim} HP_*(\mathcal{S}(\mathcal{R}, q))$$

Proof. We start by constructing stratifications of the primitive ideal spectra of $\mathcal{H}(\mathcal{R}, q)$ and $\mathcal{S}(\mathcal{R}, q)$. Choose an increasing chain

$$\emptyset = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_n = \Delta$$

of $\mathcal{W}$-invariant subsets of Δ with the properties

- if $(P, \delta) \in \Delta_i$ and $|Q| > |P|$ then $\Delta_Q \subset \Delta_i$
- the elements of $\Delta_i \setminus \Delta_{i-1}$ form exactly one association class for the action of $\mathcal{W}$

To this correspond two decreasing chains of ideals

$$\begin{aligned} \mathcal{H} &= I_0 \supset I_1 \supset \cdots \supset I_n = 0 \\ \mathcal{S} &= J_0 \supset J_1 \supset \cdots \supset J_n = 0 \\ I_i &= \{h \in \mathcal{H} : \pi(P, t, \delta)(h) = 0 \text{ if } (P, \delta) \in \Delta_i, t \in T^P\} \\ J_i &= \{h \in \mathcal{S} : \pi(P, t, \delta)(h) = 0 \text{ if } (P, \delta) \in \Delta_i, t \in T_u^P\} \end{aligned} \tag{3.144}$$

For every i pick an element $(P_i, \delta_i) \in \Delta_i \setminus \Delta_{i-1}$, let $\mathcal{W}_i$ be the stabilizer of (P_i, δ_i) in $\mathcal{W}$ and write $V_i = V_{\pi(P_i,t,\delta_i)}$. By Theorem 3.25 the Fourier transform gives isomorphisms

$$J_{i-1}/J_i \cong C^\infty(T_u^{P_i}; \mathrm{End}\, V_i)^{\mathcal{W}_i} \tag{3.145}$$

On the other hand, by Theorem 3.31 the primitive ideal spectrum of (I_{i-1}/I_i) corresponds to the inverse image of $\Delta_i \setminus \Delta_j$ under the projection $\Xi \to \Delta$. Moreover the induced map $\mathrm{Prim}(I_{i-1}/I_i) \to \mathcal{W}_i \setminus T^{P_i}$ is continuous. (In fact it is the central character map for this algebra.) By Lemma 2.3 it suffices to show that each inclusion

$$I_{i-1}/I_i \to J_{i-1}/J_i \tag{3.146}$$

induces an isomorphism on periodic cyclic homology. Therefore we zoom in on J_{i-1}/J_i. By Theorem 3.23 we can extend the action of $\mathcal{W}_i$ on $C^\infty(T_u^{P_i}; \mathrm{End}\, V_i)$ to a neighborhood T' of $T_u^{P_i}$. We may take T' $\mathcal{W}_i$-equivariantly diffeomorphic to $T_u^{P_i} \times [-1,1]^{\dim T_u^{P_i}}$. Because $[-1,1]$ is compact and contractible, we can make the inner product V_i dependant of $t \in T'$, in a smooth way, such that the intertwiners $\pi(g, P_i, t, \delta_i)$ are unitary on all of T'. To avoid some technical difficlties we want to replace J_{i-1}/J_i by $C^\infty(T'; \mathrm{End}\, V_i)^{\mathcal{W}}$, but this needs some justification.

Lemma 3.33 *The inclusion $T_u^{P_i} \to T'$ and the Chern character induce isomorphisms*

$$HP_*(J_{i-1}/J_i) \xleftarrow{\sim} HP_*\left(C^\infty(T'; \mathrm{End}\, V_i)^{\mathcal{W}_i}\right) \xrightarrow{\sim} K_*\left(C(T'; \mathrm{End}\, V_i)^{\mathcal{W}_i}\right) \otimes \mathbb{C}$$

Proof. The second isomorphism follows directly from Theorems 2.27 and 2.13. For the first one, we pick a $\mathcal{W}_i$-equivariant triangulation $\Sigma \to T_u^{P_i}$ and we construct U_σ and U_σ as on page 55. Using the projection $p_u : T' \to T_u^{P_i}$ we get a closed cover of T' :

$$\{T'_\sigma : \sigma \text{ simplex of } \Sigma\}$$
$$T'_\sigma = p_u^{-1}(U_\sigma) \cong U_\sigma \times [-1,1]^{\dim T_u^{P_i}}$$

From the proof of Theorem 2.27 we see that it suffices to show that for any simplex σ we have

$$HP_*\left(C_0^\infty(U_\sigma, D_\sigma; \mathrm{End}\, V_i)^{\mathcal{W}_\sigma}\right) \cong HP_*\left(C_0^\infty(T'_\sigma, p_u^{-1}(D_\sigma); \mathrm{End}\, V_i)^{\mathcal{W}_\sigma}\right) \tag{3.147}$$

where $\mathcal{W}_\sigma$ is the stabilizer of σ in $\mathcal{W}_i$. Well, $U_\sigma \setminus D_\sigma$ is $\mathcal{W}_\sigma$-equivariantly contractible by construction, and it is an equivariant deformation retract of $T'_\sigma \setminus p_u^{-1}(D_\sigma) = p_u^{-1}(U_\sigma \setminus D_\sigma)$. So we are in the setting of Lemma 2.26 and we may use its proof. It says that exist a finite central extension G of $\mathcal{W}_\sigma$ and a linear representation

$$G \to GL(V_i) : g \to u_g$$

such that the Fréchet algebras in (3.147) are isomorphic to

$$C_0^\infty(U_\sigma, D_\sigma; \mathrm{End}\, V_i)^G \tag{3.148}$$
$$C_0^\infty(T'_\sigma, p_u^{-1}(D_\sigma); \mathrm{End}\, V_i)^G \tag{3.149}$$

The G-action on these algebras is given by

$$g(f)(t) = u_g f(g^{-1}t) u_g^{-1}$$

where we simply lifted the action of $\mathcal{W}_\sigma$ on T'_σ to G.

It is clear that the retraction $T'_\sigma \to U_\sigma$ induces a diffeotopy equivalence between (3.148) and (3.149), so it also induces the required isomorphism (3.147). □

Consider the finite collection $\mathcal{L}$ of all irreducible components of $(T^{P_i})^w$, as w runs over $\mathcal{W}_i$. These are all cosets of complex subtori of T^{P_i} and they have nonempty intersections with $T_u^{P_i}$. Extend this to a collection $\{L_j\}_j$ of cosets of subtori of T^{P_i} by including all irreducible components of intersections of any number of elements of $\mathcal{L}$. Because the action α_i of $\mathcal{W}_i$ on T^{P_i} is algebraic

$$\dim \left(\left(T^{P_i}\right)^g \cap \left(T^{P_i}\right)^w \right) < \max\{\dim \left(T^{P_i}\right)^g , \, \dim \left(T^{P_i}\right)^w\}$$

if $\alpha_i(w) \neq \alpha_i(g)$. Define $\mathcal{W}_i$-stable submanifolds

$$\begin{aligned} T_m &= \bigcup_{j:\, \dim L_j \leq m} L_j \\ T'_m &= T_m \cap T' \end{aligned}$$

and construct the following ideals

$$\begin{aligned} A_m &= \{h \in I_{i-1}/I_i : \pi(P_i, t, \delta_i)(h) = 0 \text{ if } t \in T_m\} \\ B_m &= C^\infty(T', T'_m; \mathrm{End}\, V_i)^{\mathcal{W}_i} \\ C_m &= C(T', T'_m; \mathrm{End}\, V_i)^{\mathcal{W}_i} \end{aligned} \tag{3.150}$$

Now we have $A_n = B_n = C_n = 0$ for $n \geq \dim T^{P_i}$ and

$$A_n = I_{i-1}/I_i \quad B_n = C^\infty(U; \mathrm{End}\, V_i)^{\mathcal{W}_i} \quad C_n = C(U; \mathrm{End}\, V_i)^{\mathcal{W}_i} \quad \text{for } n < 0$$

Just as in (3.146) it suffices to show that the inclusions

$$A_{m-1}/A_m \to B_{m-1}/B_m$$

induce isomorphisms on HP_*, so let us compute the periodic cyclic homologies of these quotient algebras.

Because T_m is an algebraic subvariety of T^{P_i} the spectrum of A_{m-1}/A_m consists precisely of the irreducible representations of I_{i-1}/I_i with tempered central character in $(P_i, T_m \setminus T_{m-1}, \delta_i)$. We let $r_i(t)$ be the number of $\pi \in \mathrm{Prim}(I_{i-1}/I_i)$ corresponding to (P_i, t, δ_i). From Theorem 3.7 and we see that $r_i(t|t|^s) = r_i(t)\ \forall s > -1$, and from Theorem 3.25 that $r_i(t|t|^{-1}) = r_i(t)$ if the stabilizers in $\mathcal{W}_i$ of t and $t|t|^{-1}$ are equal. Choose a minimal subset $\{L_{m,k}\}_k$ of $\mathcal{L}$ such that every m-dimensional element of $\mathcal{L}$ is conjugate under $\mathcal{W}_i$ to a $L_{m,k}$. Let $\mathcal{W}_{m,k}$ be the stabilizer of $L_{m,k}$ in $\mathcal{W}_i$ and write $r_k = r_i(t)$ for some $t \in L_{m,k} \setminus T_u^{P_i}$. By construction $\mathcal{W}_{m,k}$ acts freely on $L_{m,k} \setminus T_{m-1}$, and the spectrum of A_{m-1}/A_m is homeomorphic to

$$\begin{aligned} X_m \setminus Y_m &:= \bigsqcup_k \bigsqcup_{l=1}^{r_k} \left(L_{m,k} \setminus T_{m-1}\right)/\mathcal{W}_{m,k} \\ &= \bigsqcup_k \bigsqcup_{l=1}^{r_k} (L_{m,k}/\mathcal{W}_{m,k}) \setminus ((L_{m,k} \cap T_{m-1})/\mathcal{W}_{m,k}) \end{aligned}$$

These are separable algebraic varieties, so the morphisms of finite type algebras

$$A_{m-1}/A_m \leftarrow Z(A_{m-1}/A_m) \to \mathcal{O}_0(X_m, Y_m) \tag{3.151}$$

are spectrum preserving. Thus from Theorems 2.6 and 2.7 we get natural isomorphisms

$$HP_*(A_{m-1}/A_m) \cong HP_*\big(\mathcal{O}_0(X_m, Y_m)\big) \to \check{H}^*(X_m, Y_m; \mathbb{C}) \tag{3.152}$$

On the other hand, by [128, Théorème IX.4.3] the extension

$$0 \to C^\infty(T', T'_m; \mathrm{End}\, V_i) \to C^\infty(T'; \mathrm{End}\, V_i) \to C^\infty(T'_m; \mathrm{End}\, V_i) \to 0$$

is admissible, and since $\mathcal{W}_i$ is finite the same holds for

$$0 \to B_m \to C^\infty(T'; \mathrm{End}\, V_i)^{\mathcal{W}_i} \to C^\infty(T'_m; \mathrm{End}\, V_i)^{\mathcal{W}_i} \to 0$$

So from Theorems 2.13 and 2.27 and Proposition 2.20 we get isomorphisms

$$HP_*(B_m) \xleftarrow{\sim} K_*(B_m) \otimes \mathbb{C} \xrightarrow{\sim} K_*(C_m) \otimes \mathbb{C} \tag{3.153}$$

The spectrum of C_{m-1}/C_m is

$$\begin{aligned} X'_m \setminus Y'_m &:= (X_m \cap T'/\mathcal{W}_i) \setminus (Y_m \cap T'/\mathcal{W}_i) \\ &= \bigsqcup_k \bigsqcup_{l=1}^{r_k} (L_{m,k} \cap T'_m)/\mathcal{W}_{m,k} \setminus (L_{m,k} \cap T'_{m-1})/\mathcal{W}_{m,k} \end{aligned}$$

These are locally compact Hausdorff spaces, so the C^*-algebra homomorphisms

$$C_{m-1}/C_m \leftarrow Z(C_{m-1}/C_m) \xrightarrow{\sim} C_0(X'_m, Y'_m) \tag{3.154}$$

are spectrum preserving. By construction the stabilizer in $\mathcal{W}_i$ of $t \in T'$ is constant on the connected components of $T'_m \setminus T'_{m-1}$, so by the continuity of the intertwiners $\pi(g, P_i, t, \delta_i)$ the vector space C_{m-1}/C_m is a projective module over $C_0(X'_m, Y'_m)$. Thus by Proposition 2.21 (for $K_*(\cdot) \otimes \mathbb{Q}$) (3.154) induces isomorphisms on K-theory with rational coefficients. From this and Theorems 2.27 and 2.13 we obtain natural isomorphisms

$$\begin{aligned} HP_*(B_{m-1}/B_m) &\cong K_*(C_{m-1}/C_m) \otimes \mathbb{C} \\ &\cong K_*(Z(C_{m-1}/C_m)) \otimes \mathbb{C} \\ &\cong K_*(C_0(X'_m, Y'_m)) \otimes \mathbb{C} \\ &\cong K_*(C_0^\infty(X'_m, Y'_m)) \otimes \mathbb{C} \\ &\cong HP_*(C_0^\infty(X'_m, Y'_m)) \\ &\cong \check{H}^*(X'_m, Y'_m; \mathbb{C}) \end{aligned} \tag{3.155}$$

From (3.152) - (3.155) we construct the commutative diagram

$$\begin{array}{ccccc} HP_*(A_{m-1}/A_m) & \cong & HP_*\big(\mathcal{O}_0(X_m, Y_m)\big) & \to & \check{H}^*(X_m, Y_m; \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow \\ HP_*(B_{m-1}/B_m) & \cong & HP_*\big(C_0^\infty(X'_m, Y'_m)\big) & \to & \check{H}^*(X'_m, Y'_m; \mathbb{C}) \end{array} \tag{3.156}$$

The pair (X'_m, Y'_m) is a deformation retract of (X_m, Y_m), so all maps in this diagram are isomorphisms. Working our way back up, using excision, we find that also

$$HP_*(I_{i-1}/I_i) \to HP_*\left(C^\infty(T'; \mathrm{End}\, V_i)^{\mathcal{W}_i}\right) \to HP_*(J_{i-1}/J_i)$$

and finally

$$HP_*(\mathcal{H}(\mathcal{R}, q)) \to HP_*(\mathcal{S}(\mathcal{R}, q))$$

are isomorphisms. $\Box$

Note that Theorem 3.32 is in accordance with our earlier results for direct products of root data. If $\mathcal{R} = \mathcal{R}_1 \times \mathcal{R}_2$ then by (3.42) and Theorem 2.5 we have

$$HP_*(\mathcal{H}(\mathcal{R}, q)) \cong HP_*(\mathcal{H}(\mathcal{R}_1, q)) \otimes HP_*(\mathcal{H}(\mathcal{R}_2, q)) \tag{3.157}$$

while by (3.90), Theorem 3.25 and Corollary 2.28

$$HP_*(\mathcal{S}(\mathcal{R}, q)) \cong HP_*(\mathcal{S}(\mathcal{R}_1, q)) \otimes HP_*(\mathcal{S}(\mathcal{R}_2, q)) \tag{3.158}$$

We can pursue the path of (3.143) and Theorem 3.32 a little further. Let $\mathbf{k}$ be any (unital) subring of $\mathbb{C}$ containing $\{q(w) : w \in W\}$. As on page 63 we consider the extended Iwahori-Hecke algebra $\mathcal{H}_{\mathbf{k}}(\mathcal{R}, q)$. It makes sense to take its periodic cyclic homology in the category of $\mathbf{k}$-algebras. This is a $\mathbf{k}$-module which we denote by $HP_*(\mathcal{H}_{\mathbf{k}}(\mathcal{R}, q)|\mathbf{k})$.

Theorem 3.34 *There are natural isomorphisms*

$$HP_*(\mathcal{H}_{\mathbf{k}}(\mathcal{R}, q)|\mathbf{k}) \otimes_{\mathbf{k}} \mathbb{C} \xrightarrow{\sim} HP_*(\mathcal{H}(\mathcal{R}, q)) \xrightarrow{\sim} HP_*(\mathcal{S}(\mathcal{R}, q)) \xleftarrow{\sim}$$
$$K_*(\mathcal{S}(\mathcal{R}, q)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} K_*(C_r^*(\mathcal{R}, q)) \otimes_{\mathbb{Z}} \mathbb{C}$$

Proof. By [26, Proposition IX.5.1] we have

$$HH_n(\mathcal{H}_{\mathbf{k}}(\mathcal{R}, q)|\mathbf{k}) \otimes_{\mathbf{k}} \mathbb{C} \cong HH_n(\mathcal{H}(\mathcal{R}, q)) \tag{3.159}$$

where $HH_n(\,\cdot\,|\mathbf{k})$ means Hochschild homology in the category of $\mathbf{k}$-algebras. Now the first isomorphism follows from [81, Proposition 5.1.6]. The second isomorphism is Theorem 3.32 and the third was already noticed in (3.143). Finally, the fourth isomorphism is a consequence of Theorem 2.13. □

Apparently this is an important invariant of the labelled root datum $(\mathcal{R}, q)$. By the way, we really need complex coefficients. It does not follow from Theorem 3.34 that $HP_*(\mathcal{H}_{\mathbf{k}}(\mathcal{R}, q)|\mathbf{k})$ and $K_*(C_r^*(\mathcal{R}, q)) \otimes_{\mathbb{Z}} \mathbf{k}$ are naturally isomorphic, we merely know that they have the same (finite) rank as $\mathbf{k}$-modules. In general there is no reason why the image of a class in $K_*(C_r^*(\mathcal{R}, q))$ should land in $HP_*(\mathcal{H}_{\mathbf{k}}(\mathcal{R}, q)|\mathbf{k})$ under the composition of the above isomorphisms.

Chapter 4

Reductive p-adic groups

Iwahori and Matsumoto were the first to recognize that any finite group with a BN-pair gives rise to a finite dimensional Hecke algebra with a very nice description in terms of generators and relations. In particular this applies to a connective reductive algebraic group defined over a finite field.

On a higher level, if G is a reductive algebraic group over a non-Archimedean local field, then the Hecke algebra $\mathcal{H}(G)$ has infinite dimension. However, the valution of the field provides enough extra structure to show that $\mathcal{H}(G)$ is a direct sum of factors which tend to be Morita equivalent to affine Hecke algebras.

Thus in first section we browse through the literature on reductive groups, and we report when and how we see something that looks like an affine Hecke algebra. Meanwhile we also recall some important notions from the representation theory of totally disconnected groups (like groups over a p-adic field).

In Section 4.2 we start working towards the main new result of this chapter, namely the construction of natural isomorphisms

$$HP_*(\mathcal{H}(G)) \cong HP_*(\mathcal{S}(G), \overline{\otimes}) \cong K_*(C_r^*(G)) \tag{4.1}$$

Obviously we have to recall the definitions of the involved algebras. The reduced C^*-algebra of G is defined in a standard way, but the construction of the Schwartz algebra $\mathcal{S}(G)$, originally due to Harish-Chandra [55], is much more difficult. The greater part of Section 4.2 is used to give a proper definition of this algebra, and to characterize its representations among all G-representations.

Acknowledging the difference between $\mathcal{H}(G)$ and affine Hecke algebras, we still proceed like we did in Chapter 3. Thus for information about the primitive ideal spectra of $\mathcal{H}(G)$ and $\mathcal{S}(G)$ we turn to the Fourier transform and the Plancherel theorem for reductive p-adic groups, both of which are due to Harish-Chandra [56]. These will show that $\text{Prim}(\mathcal{H}(G))$ is a countable union of non-separated complex affine varieties. The algebras $\mathcal{S}(G)$ and $C_r^*(G)$ have the same spectrum, which turns out to be a countable union of compact non-Hausdorff spaces. The Langlands classification tells us that $\text{Prim}(\mathcal{S}(G))$ is in a sense a deformation retract of $\text{Prim}(\mathcal{H}(G))$.

Nearly all the original material of this chapter is contained in the final section. There we use all the above to lift the result 3.1 to Hecke algebras of reductive p-adic groups, which yields (4.1). We remark that we are careful with topological periodic cyclic homology, here we have to take it with respect to the completed inductive tensor product $\overline{\otimes}$..

Moreover (4.1) is related to the Baum-Connes conjecture for reductive p-adic groups by means of the diagram

$$\begin{array}{ccc} K_*^G(\beta G) & \to & K_*(C_r^*(G)) \\ \downarrow & & \downarrow \\ HP_*(\mathcal{H}(G)) & \to & HP_*(\mathcal{S}(G)) \end{array}$$

We conclude the chapter with a discussion of some subtleties of this diagram.

4.1 Hecke algebras of reductive groups

In Section 3.1 we defined Iwahori-Hecke algebras in terms of generators and relations, but this is hardly the way in which they emerged. Iwahori and Matsumoto [63, 64, 65, 89] discovered that convolution algebras associated to a reductive group and a suitable subgroup are of the type we described. We have a look at these and then we extend our view to more general convolution algebras, of smooth functions on reductive p-adic groups. We recall everything that is needed to state all the known cases in which such convolution algebras yield affine Hecke algebras or closely related structures.

The most direct way to arrive at Iwahori-Hecke algebras is through groups with a BN-pair. Recall that a group G has a BN-pair if it satisfies

1. G is generated by two subgroups B and N
2. $B \cap N$ is normal in N
3. $W := N/B \cap N$ is generated by a set $S = \{s_I : i \in I\}$ of elements of order 2
4. if $n_i \in N$ and $n_i(B \cap N) = s_i$ then $n_i B n_i \neq B$
5. for all such n_i and $n \in N$ we have $n_i B n \subset B n_i n B \cap B n B$

These axioms were first formalized by Tits, cf. [19]. Some important consequences are proven in [13, Chapitre IV.2]. For example, it turns out that (W, S) is a Coxeter system and that the group G has a Bruhat decomposition, i.e. there is a bijection between W and the double cosets of B in G, given by

$$N \ni n \to BnB \in B \setminus G/B \tag{4.2}$$

Any connected reductive group $\mathcal{G}$ over an algebraically closed field $\mathbb{K}$ has a BN-pair. To be precise, in this case $\mathcal{B}$ is a Borel subgroup of $\mathcal{G}$, $\mathcal{B} \cap \mathcal{N} = \mathcal{T}$ is a maximal torus and $\mathcal{N}$ is the normalizer of $\mathcal{T}$ in $\mathcal{G}$, see [124, Chapter 8]. With $\mathcal{G}$ and $\mathcal{T}$ one can associate a root datum

$$\mathcal{R}(\mathcal{G},\mathcal{T}) = (X, Y, R_0, R_0^\vee) \quad \text{where}$$

- X is the character lattice of $\mathcal{T}$
- Y is the cocharacter lattice of $\mathcal{T}$
- R_0 is the set of roots of $(\mathcal{G},\mathcal{T})$
- $R_0^\vee$ is the set of coroots of $(\mathcal{G},\mathcal{T})$
- $W = \mathcal{N}/\mathcal{T}$ is isomorphic to the Weyl group W_0 of R_0

This results in a bijection between isomorphism classes of connected reductive algebraic groups and root data, see [30, Exposé 24] and [38, Exposé XXV]. Under this bijection semisimple groups correspond to semisimple root data.

The most important example is of course $GL(n,\mathbb{K})$. In this group we may take for $\mathcal{B}$ the subgroup of upper triangular matrices and for $\mathcal{T}$ the subgroup of diagonal matrices. Then $\mathcal{N}$ consists of the matrices that have exactly one nonzero entry in every row and every column. In the root datum $\mathcal{R}(\mathcal{G},\mathcal{T})$ we have $X \cong Y \cong \mathbb{Z}^n$ and

$$R_0 = R_0^\vee = \{e_i - e_j : 1 \leq i,j \leq n,\ i \neq j\}$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{Z}^n$. So R_0 is the root system A_{n-1}, and the Weyl group W_0 is isomorphic to the symmetric group S_n.

Assume now that we have a group with a BN-pair such that B is almost normal in G. This means that every double coset of B is a finite union of left cosets. (Clearly B would be almost normal if it were normal in G, but this can only happen in the degenerate situation $B = G$, $N = \{e\}$.) Let $\mathbf{k}$ be a unital commutative ring and consider the $\mathbf{k}$-module $\mathcal{H}(G,B)$ of $\mathbf{k}$-valued B-biinvariant functions on G that are nonzero on only finitely many left B-cosets. Clearly this is a free module with basis $\{T_w : w \in W\}$, where T_w is the characteristic function of BwB. (This sloppy notation is justified by (4.2).) Define a measure μ on B-left invariant subsets by $\mu(H) = |B \setminus H|$. The product on $\mathcal{H}(G,B)$ is convolution with respect to μ:

$$(f_1 * f_2)(w) = \int_{B\setminus G} f_1(wx^{-1}) f_2(x) d\mu(x) \tag{4.3}$$

This notion of a Hecke algebra stems from Shimura [117, §7]. Fortunately it agrees with the definition of an Iwahori-Hecke algebra as given on page 63:

Theorem 4.1 *For $w \in W$ write*

$$q(w) = \mu(BwB) = |B \setminus BwB|$$

Then q is a label function and the relations (3.3) *and* (3.4) *hold in $\mathcal{H}(G,B)$.*

Proof. This result is due to Iwahori [63, Theorem 3.2] and Matsumoto [89, Théorème 4]. A full proof can be found in [46, Theorem 8.4.6]. □

In the above situation of a reductive group $\mathcal{G}$ over an algebraically closed field $\mathbb{K}$, $\mathcal{B}$ cannot be almost normal, because it has lower dimension than $\mathcal{G}$ and $\mathbb{K}$ is infinite. However, suppose that the characteristic p of $\mathbb{K}$ is nonzero and that $\mathcal{G}$ is defined over a finite field $\mathbb{F}_q$. The group $\mathcal{G}(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points still has a BN-pair, where $\mathcal{B}(\mathbb{F}_q)$ is a Borel subgroup. The associated Hecke algebras were studied by Iwahori [63] (for Chevalley groups) and by Howlett and Lehrer, see [27]. In these cases W is a subgroup of the finite Weyl group W_0 of $\mathcal{R}(\mathcal{G}, \mathcal{T})$, and the numbers $q(s)$ are certain powers of p.

Now we turn to p-adic groups. Recall that a non-Archimedean local field is a field $\mathbb{F}$ with a discrete valuation

$$v : \mathbb{F} \to \mathbb{Z} \cup \{\infty\} \tag{4.4}$$

such that $\mathbb{F}$ is complete with respect to the induced norm $\|x\|_{\mathbb{F}} = q^{-v(x)}$. Here q is the cardinality of the residue field $\mathcal{O}/\mathfrak{P}$,

$$\mathcal{O} = \{x \in \mathbb{F} : v(x) \geq 0\} \tag{4.5}$$

being the ring of integers of $\mathbb{F}$ and

$$\mathfrak{P} = \{x \in \mathbb{F} : v(x) > 0\} \tag{4.6}$$

its unique maximal ideal. This implies that $\mathbb{F}$ is a totally disconnected, nondiscrete, locally compact Hausdorff space. If its characteristic is zero then $\mathbb{F}$ is isomorphic to a finite algebraic extension of the field of p-adic numbers $\mathbb{Q}_p$. On the other hand, if $\mathrm{char}(\mathbb{F}) > 0$ then $\mathcal{O} \cong \mathbb{F}_q[[t]]$, the ring of formal power series over the finite field $\mathbb{F}_q$. With a slight abuse of terminology, non-Archimedean local fields are also known as p-adic fields.

So let $\mathbb{F}$ be a non-Archimedean local field and $\mathcal{G}$ a connected reductive algebraic group that is defined over $\mathbb{F}$. Consider the group $G = \mathcal{G}(\mathbb{F})$ of $\mathbb{F}$-rational points of $\mathcal{G}$. Affine Hecke algebras play an important role in the representation theory of such groups, as we will try to explain. We are mainly interested in smooth representations, i.e. representations of G on a complex vector space V such that for every $v \in V$ the group $\{g \in G : gv = v\}$ is open. Recall that, because $\mathbb{F}$ is non-Archimedean, the identity element e of G has a countable neighborhood basis consisting of compact open subgroups. Hence smooth representations can also be characterized by the condition

$$V = \bigcup_K V^K$$

where K runs over all compact open subgroups of G. We denote the category of smooth G-representations by $\mathrm{Rep}(G)$, and the set of equivalence classes of irreducible smooth G-representations by $\mathrm{Irr}(G)$. We call a map from G to a Hausdorff space smooth if it is uniformly locally constant, i.e. if it is bi-invariant for some compact open subgroup of G.

Furthermore, because G is reductive, it has an affine building βG, also known as the Bruhat-Tits building of G. We quickly recall the construction and terminology of this polysimplicial complex, referring to [19, 127] for more detailed information. Let $A_0 = \mathcal{A}_0(\mathbb{F})$ be a maximal $\mathbb{F}$-split torus of $\mathcal{G}$, $X^*(A_0) = X^*(\mathcal{A}_0)$ its character lattice, and put $\mathfrak{a}_0^* = X^*(A_0) \otimes_{\mathbb{Z}} \mathbb{R}$. Then βG is $G \times \mathfrak{a}_0^*$ modulo a certain equivalence relation.

The affine building is a universal space for proper G-actions. Such a universal space always exists, based on general categorical considerations, but it is unique only up to homotopy. On the other hand, the proof that βG really has the required properties is very tricky, and ultimately relies on the existence of "valuated root data" [20].

The images of $\mathfrak{a}_0^*$ under G are the appartments of βG, and a polysimplex of maximal dimension in βG is called a chamber. The stabilizer I of such a chamber (or equivalently of an interior point of a chamber) is an Iwahori subgroup of G. More generally the stabilizer of an arbitrary point of βG is known as a parahoric subgroup. If $x_0 \in \beta G$ is a "special" point (in particular it must lie in a polysimplex of minimal dimension) then its stabilizer K_0 is a "good" maximal compact subgroup of G in the sense of [118, §0.6]. This implies that

$$G = PK_0 = K_0 P \tag{4.7}$$

for any parabolic subgroup P of G containing A_0.

Normalize the Haar measure μ on G by $\mu(K_0) = 1$. Any compact open $K < G$ is almost normal, so we can consider the convolution algebra $\mathcal{H}(G,K)$ of compactly supported $\mathbb{C}$-valued K-biinvariant functions on G. For example, if $G = GL_2(\mathbb{Q}_p)$ and $K = GL_2(\mathbb{Z}_p)$, then $\mathcal{H}(G,K)$ is classical algebra of Hecke operators, hence the "$\mathcal{H}$" for "Hecke" algebra.

If $K' \subset K$ is another compact subgroup then there is a natural inclusion $\mathcal{H}(G,K) \to \mathcal{H}(G,K')$. The inductive limit of this system of inclusions (i.e. the union), over all compact open subgroups, is called the Hecke algebra $\mathcal{H}(G)$ of G. It consists of all compactly supported smooth functions on G. For every K we define the idempotent $e_K \in \mathcal{H}(G)$ by

$$e_K(g) = \begin{cases} \mu(K)^{-1} & \text{if} \quad g \in K \\ 0 & \text{if} \quad g \notin K \end{cases} \tag{4.8}$$

This gives the useful identification

$$e_K \mathcal{H}(G) e_K = \mathcal{H}(G)^{K \times K} = \mathcal{H}(G,K) \tag{4.9}$$

By construction a smooth G-representation is the same thing as a nondegenerate representation of $\mathcal{H}(G)$. There are natural maps

$$\begin{array}{llllll} \mathrm{Rep}(G) & \to & \mathrm{Rep}\big(\mathcal{H}(G,K)\big) & : & V & \to & V^K = \pi(e_K)V \\ \mathrm{Rep}\big(\mathcal{H}(G,K)\big) & \to & \mathrm{Rep}(G) & : & W & \to & \mathrm{Ind}_{\mathcal{H}(G,K)}^{\mathcal{H}(G)} W \end{array} \tag{4.10}$$

Bernstein [9, Corollaire 3.9] showed that there exist arbitrarily small K for which these maps define equivalences between the category of nondegenerate $\mathcal{H}(G,K)$-representations and the category of those smooth G-representations that are generated by their K-fixed vectors. Thus $\mathcal{H}(G,K)$ covers a clear part of the representation theory of G.

It is quite possible that $\mathcal{H}(G,K)$ is an extended Iwahori-Hecke algebra. For example, suppose that $\mathcal{G}$ is split over $\mathbb{F}$, let $A_0 = \mathcal{A}_0(\mathbb{F})$ be a maximal split torus and $B = \mathcal{B}(\mathbb{F})$ a Borel subgroup containing $\mathcal{T}$. Assume that $\mathcal{B}$ is defined over $\mathcal{O}$, so that $\mathcal{B}(\mathcal{O}/\mathfrak{P})$ is a Borel subgroup of $\mathcal{G}(\mathcal{O}/\mathfrak{P})$. The inverse image I of $\mathcal{B}(\mathcal{O}/\mathfrak{P})$ under the quotient map $\mathcal{G}(\mathcal{O}) \to \mathcal{G}(\mathcal{O}/\mathfrak{P})$ is an Iwahori subgroup of G. It is known [65] that $\mathcal{H}(G,I)$ is an affine Hecke algebra. The Weyl group of the associated root datum is $W = N_G(A_0)/\mathcal{A}_0(\mathcal{O})$ and it decomposes as $W = W_0 \ltimes X^*(A_0)$ where $W_0 = N_{\mathcal{G}(\mathcal{O})}(A_0)/\mathcal{A}_0(\mathcal{O})$. Finally, the value of the label function on any simple reflection is $q = |\mathcal{O}/\mathfrak{P}|$.

Or suppose that $\mathcal{G}$ is simply connected, but not necessarily split over $\mathbb{F}$. Let N be the stabilizer of an appartment in the affine building of G, and I the stabilizer of a chamber of this appartment. According to [20, Proposition 5.2.10] (I,N) is a BN-pair in G, so $\mathcal{H}(G,I)$ is an Iwahori-Hecke algebra. The Coxeter group $W = N/I \cap N$ is an affine Weyl group coming from a root datum that is contained in $\mathcal{R}(\mathcal{G},\mathcal{T})$, for a suitable torus $\mathcal{T}$.

An important decomposition of the category $\mathrm{Rep}(G)$ was discovered by Bernstein [9]. To describe it we introduce several classes of smooth representations.

We call $(\pi, V) \in \mathrm{Rep}(G)$

- admissible if V^K has finite dimension for every compact open subgroup K
- supercuspidal if it is admissible and all matrix coefficients of π have compact support modulo the center of G

By [9, Corollaire 3.4] for every K the algebra $\mathcal{H}(G,K)$ is of finite type, so all its irreducible representations have finite dimension. In combination with [9, Corollaire 3.9] this shows that every irreducible smooth representation is automatically admissible. Supercuspidal representations are also known as absolutely cuspidal (or just cuspidal) representations, but there seems to be no agreement in the terminology here.

There is a natural notion of the contragredient of a smooth representation. Let

$$\breve{V}^K = \{ f \in V^* : f \circ \pi(e_K) = f \} \tag{4.11}$$

be the dual space of V^K and define

$$\breve{V} = \bigcup_K \breve{V}^K \tag{4.12}$$

Then $(\breve{\pi}, \breve{V})$ is the contragredient representation of (π, V). By construction it is smooth, and it is admissible whenever V is.

Suppose that P is a parabolic subgroup of G and that M is a Levi subgroup of P. Although G and M are unimodular, the modular function δ_P of P is in general not constant. Let σ be an irreducible supercuspidal representation of M. Under these conditions we call (M, σ) a cuspidal pair. From this we construct a parabolically induced G-representation

$$I_P^G(\sigma) = \mathrm{Ind}_P^G\big(\delta_P^{1/2} \otimes \sigma\big) \tag{4.13}$$

This means that we first inflate σ to P, and then we apply the normalized induction functor, i.e. we twist it by $\delta_P^{1/2}$ and take the smooth induction to G. This twist is useful to preserve unitarity, cf. [28, Theorem 3.2].

For every $(\pi, V) \in \mathrm{Irr}(G)$ there is a cuspidal pair (M, σ), uniquely determined up to G-conjugacy, such that V is a subquotient of $I_P^G(\sigma)$. If P' is another parabolic subgroup of G containing M then $I_{P'}^G(\sigma)$ and $I_P^G(\sigma)$ have the same irreducible subquotients, but they need not be isomorphic.

To define a suitable equivalence relation on the set of cuspidal pairs, we now recall a particular kind of characters. Let H be any algebraic group over $\mathbb{F}$, and consider the subgroup

$${}^0H = \{h \in H : v(\gamma(h)) = 0 \ \forall \gamma \in X^*(H)\} \tag{4.14}$$

This is an open normal subgroup of G which contains every compact subgroup, and $H/{}^0H$ is a free abelian group. An unramified character of H is a homomorphism $\chi : H \to \mathbb{C}^\times$ whose kernel contains ${}^0H$. The group of these forms a complex torus $X_{nr}(H)$ and the map $X^*(H) \otimes_{\mathbb{Z}} \mathbb{C}^\times \to X_{nr}(H)$ defined by

$$\gamma \otimes z \ \longrightarrow \ \Big(h \to z^{v(\gamma(h))}\Big)$$

is an isomorphism. We will denote the compact torus of unitary unramified characters by

$$X_{unr}(H) = \mathrm{Hom}\big(H/{}^0H, S^1\big) \tag{4.15}$$

We say that two cuspidal pairs (M, σ) and (M', σ') are inertially equivalent if there exist $\chi \in X_{nr}(M')$ and $g \in G$ such that $M' = g^{-1}Mg$ and $\sigma' \otimes \chi \cong \sigma^g$.

With an inertial equivalence class $\mathfrak{s} = [M, \sigma]_G$ we associate a subcategory of $\mathrm{Rep}(G)^{\mathfrak{s}}$ of $\mathrm{Rep}(G)$. By definition its objects are those smooth representations π with the following property: for every irreducible subquotient ρ of π there is a $(M, \sigma) \in \mathfrak{s}$ such that ρ is a subrepresentation of $I_P^G(\sigma)$.

These $\mathrm{Rep}(G)^{\mathfrak{s}}$ give rise to the Bernstein decomposition [9, Proposition 2.10]:

$$\mathrm{Rep}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathrm{Rep}(G)^{\mathfrak{s}} \tag{4.16}$$

The set $\mathfrak{B}(G)$ of Bernstein components is countably infinite. We have a corresponding decomposition of the Hecke algebra of G into two-sided ideals:

$$\mathcal{H}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} \mathcal{H}(G)^{\mathfrak{s}} \tag{4.17}$$

with $\text{Rep}\big(\mathcal{H}(G)^{\mathfrak{s}}\big) = \text{Rep}(G)^{\mathfrak{s}}$. By [24, Proposition 3.3] there exists an idempotent $e_{\mathfrak{s}} \in \mathcal{H}(G)$ such that

- $\mathcal{H}(G)^{\mathfrak{s}} = \mathcal{H}(G) e_{\mathfrak{s}} \mathcal{H}(G)$
- $\text{Rep}(G)^{\mathfrak{s}}$ is equivalent to $\text{Rep}\big(e_{\mathfrak{s}} \mathcal{H}(G) e_{\mathfrak{s}}\big)$

Under these conditions $e_{\mathfrak{s}} \mathcal{H}(G) e_{\mathfrak{s}}$ is a finite type algebra, whose center was already described by Bernstein. The set D_σ of all cuspidal pairs of the form $(M, \sigma \otimes \chi)$ is in bijection with $X_{nr}(M)$, so it has the structure of a complex torus. Put

$$N(M,\sigma) = \big\{ g \in G : g M g^{-1} = M \text{ and } [M, \sigma^g]_M = [M,\sigma]_M \big\} \tag{4.18}$$

Then $W_\sigma = N(M,\sigma)/M$ is a finite group acting on D_σ, so D_σ / W_σ is an irreducible algebraic variety. [9, Théorème 2.13] tells us that

$$Z\big(e_{\mathfrak{s}} \mathcal{H}(G) e_{\mathfrak{s}}\big) \cong \mathcal{O}(D_\sigma / W_\sigma) = \mathcal{O}(D_\sigma)^{W_\sigma} \tag{4.19}$$

For any compact open $K < G$ write

$$\begin{aligned} \mathfrak{B}(G,K) &= \big\{ \mathfrak{s} \in \mathfrak{B}(G) : \mathcal{H}(G,K)^{\mathfrak{s}} \neq 0 \big\} \\ \mathcal{H}(G,K)^{\mathfrak{s}} &= \mathcal{H}(G)^{\mathfrak{s}} \cap \mathcal{H}(G,K) \end{aligned}$$

Proposition 4.2 *1. $\mathfrak{B}(G,K)$ is finite for any compact open $K < G$.*

2. If K is a normal subgroup of a good maximal compact subgroup K_0 then $\mathcal{H}(G,K)$ is Morita equivalent to $\oplus_{\mathfrak{s} \in \mathfrak{B}(G,K)} \mathcal{H}(G)^{\mathfrak{s}}$ and

$$Z\big(\mathcal{H}(G,K)\big) \cong \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G,K)} Z\big(e_{\mathfrak{s}} \mathcal{H}(G) e_{\mathfrak{s}}\big)$$

3. For any $\mathfrak{s} \in \mathfrak{B}(G)$ there exists a compact open $K_{\mathfrak{s}} < G$ such that for every compact open $K \subset K_{\mathfrak{s}}$ the algebras

$$\mathcal{H}(G,K_{\mathfrak{s}})^{\mathfrak{s}} \;,\; \mathcal{H}(G,K)^{\mathfrak{s}} \text{ and } \mathcal{H}(G)^{\mathfrak{s}}$$

are Morita equivalent.

Proof. All these results are due to Bernstein. 2 is a direct consequence of [9, Corollaire 3.9] and (4.19). 1 and 3 follow from this in combination with [9, (3.7)], as was remarked in [7, p. 143]. □

We may also consider more general algebras associated to (G,K). Let (ρ, V) be an irreducible smooth representation of K, and $(\check{\rho}, \check{V})$ its contragredient. Notice that ρ is smooth and has finite dimension. Define

$$\mathcal{H}(G,K,\rho) = \Big\{ f : G \to \text{End}_{\mathbb{C}}(\check{V}) : f(k_1 g k_2) = \check{\rho}(k_1) f(g) \check{\rho}(k_2) \; \forall k_1, k_2 \in K, g \in G \Big\} \tag{4.20}$$

This is a unital algebra under the convolution product, its elements being smooth functions on G. Consider the idempotent $e_\rho \in \mathcal{H}(G)$ defined by

$$e_\rho(g) = \begin{cases} \mu(K)^{-1} \dim(V) \operatorname{tr}\big(\rho(g^{-1})\big) & \text{if} \quad g \in K \\ 0 & \text{if} \quad g \notin K \end{cases} \tag{4.21}$$

By [23, Proposition 4.2.4] there is a natural isomorphism

$$\mathcal{H}(G, K, \rho) \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(V) \cong e_\rho \mathcal{H}(G) e_\rho \tag{4.22}$$

Let $\operatorname{Rep}_\rho(G)$ be the subcategory of $\operatorname{Rep}(G)$ consisting of all representations (π, U) for which $\mathcal{H}(G) e_\rho U = U$. According to [23, Proposition 4.2.3] there are natural bijections between the sets of irreducible objects of

- $\operatorname{Rep}_\rho(G)$
- $\operatorname{Rep}\big(e_\rho \mathcal{H}(G) e_\rho\big)$
- $\operatorname{Rep}\big(\mathcal{H}(G, K, \rho)\big)$

If moreover $\operatorname{Rep}_\rho(G)$ is closed for taking subquotients (of G-representations) then there exists a finite subset $\mathfrak{S} \subset \mathfrak{B}(G)$ such that

$$\operatorname{Rep}_\rho(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{S}} \operatorname{Rep}(G)^{\mathfrak{s}} \tag{4.23}$$

In the terminology of Bushnell and Kutzko (K, ρ) is an $\mathfrak{S}$-type [24, (3.12)]. Of special interest is the case when $\mathfrak{S}$ consists of a single element $\mathfrak{s}$, for then $\mathcal{H}(G, K, \rho)$ is Morita equivalent to $\mathcal{H}(G)^{\mathfrak{s}}$. It is known that under this and certain extra conditions $\mathcal{H}(G, K, \rho)$ is isomorphic to an affine Hecke algebra [108, Theorem 6.3], sometimes with unequal labels [87, Section 1], or to the twisted crossed product of such a thing with a (twisted) group algebra [94, Theorem 7.12].

Using this approach it has been shown that $\mathcal{H}(G)^{\mathfrak{s}}$ is Morita equivalent to an affine Hecke algebra for every $\mathfrak{s} \in \mathfrak{B}(GL(n, \mathbb{F}))$ [23, 25], and to a "twisted" affine Hecke algebra for every $\mathfrak{s} \in \mathfrak{B}(SL(n, \mathbb{F}))$ [48, Theorem 11.1].

4.2 Harish-Chandra's Schwartz algebra

In this section G will be a connected reductive algebraic group defined over a non-Archimedean local field $\mathbb{F}$. The reduced C^*-algebra of G is defined in a standard way, but we need to go to some lengths to construct Harish-Chandra's Schwartz algebra. Once this is done we characterize its representations among admissibe G-representations and formulate the Langlands classification for reductive p-adic groups.

Define the adjoint and the trace of $f \in \mathcal{H}(G)$ by

$$f^*(g) = \overline{f(g^{-1})} \qquad \tau(f) = f(e)$$

This gives rise to a bitrace

$$(f, f') = \tau(f^* f')$$

making $\mathcal{H}(G)$ into a Hilbert algebra. The Hilbert space completion of $\mathcal{H}(G)$ is the space $L^2(G)$ of all square-integrable functions on G. It carries two natural G-actions, left and right translation:

$$\big(\lambda(g)f\big)(h) = f(g^{-1}h)$$
$$\big(\rho(g)f\big)(h) = f(hg)$$

Now $\lambda(g)$ and $\rho(g)$ are bounded operators of the same norm. They extend naturally to representations of $\mathcal{H}(G)$, so we get an injection

$$\lambda : \mathcal{H}(G) \to B(L^2(G))$$

The reduced C^*-algebra $C_r^*(G)$ is the closure of $\lambda(\mathcal{H}(G))$ in $B(L^2(G))$. It is a separable nonunital C^*-algebra whose representations correspond to the unitary G-representations that are weakly contained in the left regular representation $(\lambda, L^2(G))$ of G.

Usually the reduced C^*-algebra of a locally compact group H is defined as the completion of $C_c(H)$ or $L^1(H)$ in $B(L^2(H))$. However if H is totally disconnected we may just as well start with smooth functions only.

For a compact open $K < G$ we let $C_r^*(G, K)$ be the completion of $\mathcal{H}(G, K)$ in $B(L^2(G))$. It is a unital type I C^*-algebra and it equals

$$e_K C_r^*(G) e_K = C_r^*(G)^{K \times K} = C_r^*(G, K) \tag{4.24}$$

Let us mention some general facts about the structure of $C_r^*(G)$. They can be read off from Theorem 4.10, but it seems appropriate to formulate them here already. This algebra can be recovered as the inductive limit of the above subalgebras, over all compact open subgroups, partially ordered by inclusion:

$$C_r^*(G) = \varinjlim C_r^*(G, K) \tag{4.25}$$

Moreover it has a Bernstein decomposition, analogous to (4.17), with a direct sum in the C^*-algebra sense:

$$C_r^*(G) = \varinjlim_{\mathfrak{S}} \bigoplus_{\mathfrak{s} \in \mathfrak{S}} C_r^*(G)^{\mathfrak{s}} \tag{4.26}$$

where $\mathfrak{S}$ runs over all finite subsets of $\mathfrak{B}(G)$. Here $C_r^*(G)^{\mathfrak{s}}$ is the two-sided ideal of $C_r^*(G)$ generated by $\mathcal{H}(G)^{\mathfrak{s}}$. Every subalgebra $C_r^*(G, K)$ lives in only finitely many Bernstein components.

The construction of the Schwartz algebra of G is more complicated, we need to introduce a lot of things to achieve it.

A p-pair is a pair (P, A) consisting of a parabolic subgroup P of G, and the identity component A of the maximal split torus in the center of some Levi subgroup M of P. Then

$$M = Z_G(A) = A \times {}^0M \tag{4.27}$$

$$P = MN = A^0MN = Z_G(A)N \tag{4.28}$$

where N is the unipotent radical of P. For example (G, A_G) is a p-pair, where A_G is the maximal split torus of $Z(G)$.

There is a unique p-pair $(\bar{P}, A)$ such that $\bar{P} \cap P = M$. The parabolic subgroup $\bar{P}$ is called the opposite of P. Clearly $\bar{P} = M\bar{N}$ where $\bar{N} \cap N = \{1\}$.

Let (Q, B) be another parabolic pair. Write $W(A|G|B)$ for the set of all homomorphisms $B \to A$ induced by inner automorphisms of G. If $B = A$ then this is a group :

$$W(G|A) := W(A|G|A) = N_G(A)/Z_G(A) = N_G(A)/M \tag{4.29}$$

We say that (P, A) dominates (Q, B), written $(P, A) \geq (Q, B)$, if $P \supset Q$ and $A \subset B$.

Recall that we have chosen a maximal split torus A_0 of G, and let P_0 be a minimal parabolic subgroup containing it. We call a p-pair (P, A) and its Levi factor M semi-standard if $A \subset A_0$, or equivalently $A_0 \subset M$. If moreover $(P, A) \geq (P_0, A_0)$, then we say that (P, A) is standard. Every p-pair is conjugate to a standard p-pair.

Let $X^*(A)$ be the character lattice of A and put

$$\begin{aligned} \mathfrak{a} &= \mathrm{Hom}_{\mathbb{Z}}(X^*(A), \mathbb{R}) \\ \mathfrak{a}^* &= X^*(A) \otimes_{\mathbb{Z}} \mathbb{R} \end{aligned} \tag{4.30}$$

There is a natural homomorphism $H_M : M \to \mathfrak{a}$, defined by the equivalent conditions

$$\begin{aligned} \langle \chi \, , \, H_M(m) \rangle &= -v(\chi(m)) \\ q^{\langle \chi \, , \, H_M(m) \rangle} &= \|\chi(m)\|_{\mathbb{F}} \end{aligned} \tag{4.31}$$

where $\chi \in X^*(A)$ and q is the module of $\mathbb{F}$. Conversely, if $\nu \in \mathfrak{a}^*$ then we define an unramified character χ_ν of M by

$$\chi_\nu(m) = q^{\langle \nu \, , \, H_M(m) \rangle} \tag{4.32}$$

For a parabolic subgroup Q with $P \subset Q \subset G$, let $\Sigma(Q, A) \subset \mathfrak{a}^*$ be the set of roots of Q with respect to A. By this we mean the set of $\alpha \in X^*(A) \setminus \{1\}$ such that $\mathfrak{q}_\alpha$ is nonzero, where $\mathfrak{q}$ is the Lie algebra of Q and

$$\mathfrak{q}_\alpha := \{x \in \mathfrak{q} : \mathrm{Ad}(a)x = \alpha(a)x \quad \forall a \in A\}$$

Then $\Sigma(G, A)$ is a root system and $\Sigma(P, A)$ is a positive system of roots. Let $\Delta(P, A)$ be the corresponding set of simple roots. The Weyl group of $\Sigma(G, A)$ is naturally isomorphic to $W(G|A)$. Notice that $\Sigma(\bar{P}, A) = -\Sigma(P, A)$.

The minimal p-pair (P_0, A_0) gives us a root system

$$\Sigma_0 = \Sigma(G, A_0) \subset \mathfrak{a}_0^*$$

with simple roots $\Delta_0 = \Delta(P_0, A_0)$ and Weyl group $W_0 = W(G|A_0)$. Fix a W_0-invariant inner product on $\mathfrak{a}_0^*$, so that we may identify this vector space with its dual $\mathfrak{a}_0$.

If (P, A) is standard then $\Delta(P, A)$ is the set of nonzero projections of Δ_0 on $\mathfrak{a}^*$, and $W(M|A_0)$ is the parabolic subgroup of W_0 generated by

$$\{s_\alpha : \alpha \in \Delta_0, \, \alpha(A) = 1\}$$

Let us also introduce the associated sets of positive elements in $\mathfrak{a}^*$ and A :

$$\begin{aligned}
\mathfrak{a}^{*,+} &= \{\nu \in \mathfrak{a}^* : \langle \nu \, , \, \alpha \rangle > 0 \; \forall \alpha \in \Delta(P, A)\} \\
\bar{\mathfrak{a}}^{*,+} &= \{\nu \in \mathfrak{a}^* : \langle \nu \, , \, \alpha \rangle \geq 0 \; \forall \alpha \in \Delta(P, A)\} \\
A^+ &= \{a \in A : \|\alpha(a)\|_{\mathbb{F}} > 1 \; \forall \alpha \in \Delta(P, A)\} \\
\bar{A}^+ &= \{a \in A : \|\alpha(a)\|_{\mathbb{F}} \geq 1 \; \forall \alpha \in \Delta(P, A)\}
\end{aligned}$$

In order to say when a function on G is rapidly decreasing, we need a length function on this group. For $x \in GL(m, \mathbb{F})$ let x_{ij} and x^{ij} be the entries of x and x^{-1}, and define

$$\mathcal{N}(x) = \max\{-v(x_{ij}), -v(x^{ij}) : 1 \leq i, j \leq m\} \tag{4.33}$$

Notice that for all $x, y \in GL(m, \mathbb{F})$

$$0 \leq \mathcal{N}(xy) \leq \mathcal{N}(x) + \mathcal{N}(y) \tag{4.34}$$

Pick an injective homomorphism $\tau : G \to GL(m, \mathbb{F})$ and put

$$\sigma = \mathcal{N} \circ \gamma : G \to \mathbb{Z}_{\geq 0} \tag{4.35}$$

Then σ is a continuous length function on G. Let δ_{P_0} be the modular function of P_0. Using the decomposition (4.7) we extend this to a function δ_0 on G satisfying

$$\delta_0(pk) = \delta_{P_0}(p) \qquad p \in P_0, k \in K_0 \tag{4.36}$$

Harish-Chandra's spherical Ξ-function is

$$\Xi(g) = \int_{K_0} \delta_0(kg) d\mu(k) \tag{4.37}$$

Important properties of this function can be found in [132, Paragraphe II] and [118, §4.2]. For $n \in \mathbb{N}$ consider the following norm on $\mathcal{H}(G)$:

$$\nu_n(f) = \sup_{g \in G} |f(g)| \Xi(g)^{-1} (\sigma(g) + 1)^n \tag{4.38}$$

We say that $f \in C(G)$ decreases rapidly if $\nu_n(f) < \infty \quad \forall \in \mathbb{N}$. Clearly the ν_n depend on the choice of $\tau : G \to GL(m, \mathbb{F})$, but the topology defined by the family $\{\nu_n\}_{n=1}^\infty$ does not. For any compact open $K < G$ let $S(G, K)$ be the completion of $\mathcal{H}(G, K)$ for this family of norms. According to Vignéras [130, Theorem 29] this is a unital, nuclear Fréchet *-algebra, and a dense subalgebra of $C_r^*(G, K)$. Moreover an element of $\mathcal{S}(G, K)$ is invertible if and only if it is invertible in $C_r^*(G, K)$, so $\mathcal{S}(G, K)$ is closed under the holomorphic functional calculus of $C_r^*(G, K)$.

For $K' \subset K$ there is still an inclusion $S(G, K) \to S(G, K')$, so we can take the inductive limit over all compact open subgroups of G. This yields Harish-Chandra's Schwartz algebra:

$$\mathcal{S}(G) = \varinjlim \mathcal{S}(G, K) \tag{4.39}$$

By definition it consists of all rapidly decreasing smooth functions on G. The obvious analogue of (4.9) is

$$e_K \mathcal{S}(G) e_K = \mathcal{S}(G)^{K \times K} = \mathcal{S}(G, K) \tag{4.40}$$

Compared to the above C^*-algebras, $\mathcal{S}(G)$ inherits fewer topological properties from its subalgebras. Namely, it is a complete Hausdorff locally convex algebra, but it is not metrizable, and its multiplication is only separately continuous [132, §III.6].

It does have a Bernstein decomposition

$$\mathcal{S}(G) = \bigoplus_{\mathfrak{s} \in \mathfrak{B}(G)} S(G)^{\mathfrak{s}} \tag{4.41}$$

where $S(G)^{\mathfrak{s}}$ is the completion of $\mathcal{H}(G)^{\mathfrak{s}}$, a two-sided ideal in $\mathcal{S}(G)$. This follows from Theorem 4.9, but of course it can be proved without using the full strength of that result.

To characterize those G-representations that extend to $\mathcal{S}(G)$ we need to know more about smooth representations.

Let (π, V) be a smooth G-representation, and P a parabolic subgroup with unipotent radical N and a Levi factor M. The Jacquet module associated to these data is

$$\begin{aligned} V_P &= V / V(N) \\ V(N) &= \text{span}\{\pi(n)v - v : n \in N, v \in V\} \end{aligned} \tag{4.42}$$

We make it into an M-representation (π_P, V_P) by

$$\pi_P(m) j_P(v) = \delta_P^{-1/2}(m) j_P(\pi(m)v) \tag{4.43}$$

where $j_P : V \to V/V(N)$ is the natural projection. By Frobenius reciprocity we get, for any smooth M-representation σ :

$$\text{Hom}_G(\pi, I_P^G \sigma) \cong \text{Hom}_M(\pi_P, \sigma) \tag{4.44}$$

For $\chi \in \mathrm{Hom}(A_G, \mathbb{C}^\times)$ define the generalized weight space

$$V_\chi = \{v \in V : \exists n \in \mathbb{N} : (\pi(a) - \chi(a))^n v = 0 \ \forall a \in A_G\} \tag{4.45}$$

If $V_\chi \neq 0$ then we call χ an exponent of (π, V). If

$$\pi(a)v = \chi(a)v \ \forall v \in V, a \in A_G$$

then we say that V admits the central character χ. Similarly for a p-pair (P, A) and a smooth M-representation we have exponents in $\mathrm{Hom}(A, \mathbb{C}^\times)$. The set of exponents of the Jacquet module $V_{\bar{P}}$ is

$$\mathcal{X}_\pi(P, A) = \{\chi \in \mathrm{Hom}(A, \mathbb{C}^\times) : V_{\bar{P},\chi} \neq 0\} \tag{4.46}$$

Notice the curious shift in the notation, from $\bar{P}$ to P. This is designed to make a nicer formulation of the Langlands classification possible.

Let us characterize square-integrable, discrete series and tempered G-representations. Our first description is due to Casselman [29, Theorem 4.4.6].

Proposition 4.3 *Let π be an admissible G-representation which admits a unitary central character. The following are equivalent :*

- *Every matrix coefficient of π is square-integrable on G/A_G*
- *If (P, A) is a semi-standard p-pair, $\chi \in \mathcal{X}_\pi(P, A)$ and $a \in \bar{A}^+$ is such that there is an $\alpha \in \Delta_0$ with $|\alpha(a)| \neq 1$, then $|\chi(a)| < 1$*
- *For every semi-standard p-pair (P, A) and every $\chi \in \mathcal{X}_\pi(P, A)$ we can write*

$$\log|\chi| = \sum_{\alpha \in \Delta(P,A)} \chi_\alpha \alpha \quad \text{with } \chi_\alpha < 0$$

We say that π is square-integrable if it satisfies these conditions.

Every square-integrable representation is unitary and completely reducible, see [118, Corollary 1.11.8] or [132, Lemme III.1.3]. A more restrictive notion is that of a discrete series representation.

Proposition 4.4 *Let (π, V) be an irreducible admissible G-representation. The following are equivalent :*

- *(π, V) is a subrepresentation of $(\lambda, L^2(G))$*
- *G is semi-simple and π is square-integrable*

If (π, V) satisfies these conditions then it is called a discrete series representation.

By [41, Proposition 18.4.2] such a representation does indeed give an isolated point in $\mathrm{Prim}(C_r^*(G))$.

A (smooth) function f on G is tempered if there exist $C, N \in (0, \infty)$ such that

$$|f(g)| \leq C\Xi(g)(1+\sigma(g))^N \quad \forall g \in G \tag{4.47}$$

Proposition 4.5 *Let π be an admissible G-representation. The following are equivalent :*

- *π extends continuously to $\mathcal{S}(G)$*
- *Every matrix coefficient of π is a tempered function*
- *If (P, A) is a semi-standard p-pair, $\chi \in \mathcal{X}_\pi(P, A)$ and $a \in A^+$, then $|\chi(a)| \leq 1$*
- *For every semi-standard p-pair (P, A) and every $\chi \in \mathcal{X}_\pi(P, A)$ we can write*

$$\log|\chi| = \sum_{\alpha \in \Delta(P,A)} \chi_\alpha \alpha \quad \text{with } \chi_\alpha \leq 0$$

The representation π is said to be tempered if and only if these conditions hold.

Proof. Almost everything follows from [132, Proposition III.2.2 and §III.7]. The only thing left is to show that all matrix coefficients of an admissible $\mathcal{S}(G)$-representation are tempered. This follows from the admissibility, combined with the observation that the collection of tempered K-biinvariant functions on G is the linear dual of $\mathcal{S}(G, K)$. $\square$

The properties temperedness and pre-unitarity are preserved under normalized induction:

Proposition 4.6 *Let (P, A) be a semi-standard p-pair and (π, V) an admissible M-representation. Then*

1. *$I_P^G(\pi)$ is tempered if and only if π is tempered*
2. *$I_P^G(\pi)$ is pre-unitary if and only if π is pre-unitary*

Proof. 1. comes from [132, Lemme III.2.3].
2. It is clear that $I_P^G(\pi)$ cannot be pre-unitary if π is not. It remains to produce a G-invariant inner product on $I_P^G(V)$, given an M-invariant inner product on V. This is achieved by setting

$$\langle f\,,\, f'\rangle = \int_{K_0} \langle f(k)\,,\, f'(k)\rangle d\mu(k) \tag{4.48}$$

Notice that $I_P^G(V)$ is usually not complete with respect to this inner product, even is V is. $\square$

Like in Section 3.2, let Λ be the set of triples (P, σ, ν), where (P, A) is a standard p-pair, σ is an irreducible tempered representation of $M = Z_G(A)$ and $\nu \in \mathfrak{a}^*$. With such a triple we associate the admissible G-representation

$$I(P, \sigma, \nu) = I_P^G(\sigma \otimes \chi_\nu) = \mathrm{Ind}_P^G\big(\sigma \otimes \chi_\nu \otimes \delta_P^{1/2}\big) \tag{4.49}$$

The set of Langlands data is

$$\Lambda^+ = \big\{(P, \sigma, \nu) \in \Lambda : \nu \in \mathfrak{a}^{*,+}\big\} \tag{4.50}$$

A somewhat extended version of the Langlands classification for reductive p-adic groups reads :

Theorem 4.7 *Let (P, σ, ν), $(P', \sigma', \nu') \in \Lambda^+$.*

1. *The G-representation $I(P, \sigma, \nu)$ is indecomposable and has a unique irreducible quotient, which we call $J(P, \sigma, \nu)$.*
2. *For every $\pi \in \mathrm{Irr}(G)$ there is precisely one Langlands datum (P, σ, ν) such that π is equivalent to $J(P, \sigma, \nu)$.*
3. *If $J(P, \sigma, \nu)$ is equivalent to a subquotient of $I(P', \sigma', \nu')$, then $\nu' - \nu \in \bar{\mathfrak{a}}_0^{*,+}$ and $P' \subset P$. If $P' = P$ then also $\sigma' = \sigma$ and $\nu' = \nu$.*

Proof. For 1 and 2 see [119] or [11, §XI.2]. As concerns 3, by [11, Lemma XI.2.13] we have $\nu' - \nu \in \bar{\mathfrak{a}}_0^{*,+}$. Now it follows from the definition of Λ^+ that $P' \subset P$. Suppose that $(P, \sigma, \nu) \neq (P', \sigma', \nu')$ while $P = P'$. Then, again by [11, Lemma XI.2.13], $\nu \neq \nu'$. But by Frobenius reciprocity σ is equivalent to a subquotient of $I_P^G(\sigma' \otimes \chi_{\nu'})(N)$. Hence $\nu = \nu' \circ w$ for some $w \in W(G|A) \setminus \{1\}$. Since $\nu' \in \mathfrak{a}'^{*,+}$ there is an $\alpha \in \Delta_0$ with $\langle \nu' , \alpha \rangle > 0$ but $\langle \nu , \alpha \rangle < 0$. This contradicts the positivity of ν with respect to $P' = P$. □

4.3 The Plancherel theorem

The Plancherel formula for G is an explicit decomposition of the trace τ in terms of the traces of irreducible G-representations. Closely related is the Plancherel theorem, which describes the image of $\mathcal{S}(G)$ under the Fourier transform. The crucial theorems in this section are due to Harish-Chandra [56], but unfortunately he never published the proofs. Based upon Harish-Chandra's notes, Waldspurger [132] provided full proofs of these results, which we will describe in as much detail as we need. We try to set up a complete analogy with Section 3.3. In particular we refine the Langlands classification using parabolic induction in stages.

We start with a semi-standard p-pair (P, A) and an irreducible square-integrable M-representation (ω, E). Let $(\check{\omega}, \check{E})$ be its contragredient, and construct the admissible $G \times G$-representation

$$L(\omega, P) = I_{P\times P}^{G\times G}(E \otimes \check{E}) = I_P^G(E) \otimes I_P^G(\check{E}) \tag{4.51}$$

Using (4.48) we make $I_P^G(E)^K$ into a finite dimensional Hilbert space, for every compact open $K < G$. This allows us to identify $I_P^G(E)^{\vee}$ with $I_P^G(\check{E})$ as representations, and with $I_P^G(E)$ as vector spaces. Thus we can turn $L(\omega, P)$ into a nonunital *-algebra with

$$(f_1 \otimes f_2)(f_3 \otimes f_4) = \langle f_2\,,\, f_3\rangle f_1 \otimes f_4 \tag{4.52}$$
$$(f_1 \otimes f_2)^* = f_2 \otimes f_1 \tag{4.53}$$

Notice that for every $\chi \in X_{unr}(M)$ the representation $\chi \otimes \omega$ is still square-integrable, and that $L(\chi \otimes \omega, P)$ can be identified with $L(\omega, P)$. Let K_ω be the set of $k \in X_{nr}(M)$ such that $k \otimes \omega$ is equivalent to ω. This is a finite subgroup of $X_{unr}(M)$. For every $k \in K_\omega$ we pick a unitary intertwiner

$$\tilde{\omega}_k : (k \otimes \omega, E) \to (\omega, E) \tag{4.54}$$

This induces an automorphism of $L(\omega, P)$ by

$$I(k, \omega) = I_{P\times P}^{G\times G}(\tilde{\omega}_k \otimes \tilde{\omega}_k^{-t}) = I_P^G(\tilde{\omega}_k) \otimes I_P^G(\tilde{\omega}_k^{-t}) \tag{4.55}$$

where $\tilde{\omega}_k^{-t}$ is the inverse transpose of $\tilde{\omega}_k$. Then $I(k, \omega) \in \mathrm{Aut}_{G\times G}\big(L(\omega, P)\big)$ is independent of the choice of $\tilde{\omega}_k$, and in general nontrivial, cf. [132, §VI.1].

It is more difficult to define intertwiners corresponding to elements of the various Weyl groups. First we notice that for any p-pair (P', A) with the same Levi factor M, ω can also be lifted to a representation of P' that is trivial on N'. Let (Q, A^g), with $g \in G$, be yet another semi-standard p-pair, and put

$$n = [g] \in W(A^g|G|A)$$

The equivalence class of the M^g representation $(\omega^{g^{-1}}, E)$ depends only on n, and is therefore denoted by $n\omega$.

In [132, Paragraphe V] certain normalized intertwiners ${}^o c_{Q|P}(n, \omega)$ are constructed. Preferring the simpler notation $I(n, \omega)$, we recall their properties.

Theorem 4.8 *Let (P, A), (P', A') and (Q, B) be semi-standard p-pairs, and $n \in W(B|G|A)$. There exists an intertwiner*

$$I(n, \chi \otimes \omega) \in \mathrm{Hom}_{G\times G}(L(\omega, P), L(n\omega, Q))$$

with the following properties :

- $\chi \to I(n, \chi \otimes \omega)$ *is a rational function on* $X_{nr}(M)$

- *$I(n, \chi \otimes \omega)$ is unitary and regular for $\chi \in X_{unr}(M)$.*

- *If $n' \in W(A'|G|B)$ then*

$$I(n', n(\chi \otimes \omega))I(n, \chi \otimes \omega) = I(n'n, \chi \otimes \omega)$$

Let $\Gamma_{rr}\big(X_{nr}(M); L(\omega, P)\big)$ be the algebra of rational sections that are rational on $X_{unr}(M)$. We define an action of K_ω on this algebra by

$$kf(\chi) = I(k, \omega)f(k^{-1}\chi) \tag{4.56}$$

Similarly, for n as in Theorem 4.8 we define an algebra homomorphism

$$\begin{aligned} &n : \Gamma_{rr}\big(X_{nr}(M)); L(\omega, P)\big) \to \Gamma_{rr}\big(X_{nr}(Z_G(B)); L(n\omega, Q)\big) \\ &nf(\chi) = I(n, \omega)f(\chi \circ n) \end{aligned} \tag{4.57}$$

To define the Fourier transform we construct a scheme containing all tempered G-representations. For every Levi subgroup M of G choose a set Δ_M of irreducible square-integrable M-representations, with the property that for every square-integrable $\pi \in \mathrm{Irr}(M)$ there exists exactly one $\omega \in \Delta_M$ such that π is equivalent to $\chi \otimes \omega$, for some $\chi \in X_{nr}(M)$.

Let Ξ be the scheme consisting of all quadruples (P, A, ω, χ), with (P, A) a semi-standard p-pair, $\omega \in \Delta_M$ and $\chi \in X_{nr}(M)$. This is a countable disjoint union of complex algebraic tori. Let Ξ_u be the smooth submanifold obtained by the restriction $\chi \in X_{unr}(M)$. Notice that Ξ is naturally a finite cover of the set Θ defined in [132, p. 305]. For $\xi = (P, A, \omega, \chi) \in \Xi$ we put $\pi(\xi) = I_P^G(\chi \otimes \omega)$. Let $\mathcal{L}_\Xi$ be the vector bundle over Ξ which is trivial on every component and whose fiber at ξ is $L(\omega, P)$. We say that a section of this bundle is algebraic or rational if it is supported on only finitely many components, and has the required property on every component. Now we define the Fourier transform

$$\begin{aligned} &\mathcal{F} : \mathcal{H}(G) \to \mathcal{O}(\Xi; \mathcal{L}_\Xi) \\ &\mathcal{F}(f)(P, A, \omega, \chi) = I(P, A, \omega, \chi)(f) \in L(\omega, P) \end{aligned} \tag{4.58}$$

This is not the same as $\hat{f}(\chi \otimes \omega, P)$, as in [132, §VII.1]! We adjusted the latter to make $\mathcal{F}$ multiplicative. Fortunately the difference is not too big, so most results remain valid.

We construct a locally finite groupoid $\mathcal{W}$ as follows. The objects of $\mathcal{W}$ are triples (P, A, ω) with (P, A) a semi-standard p-pair and $\omega \in \Delta_M$. The morphisms from (Q, B, η) to (P, A, ω) are pairs (k, n) with the following properties

- $k \in K_\omega$

- $n \in W(A|G|B)$ and $nB = A$

- $n\eta$ is equivalent to $\chi \otimes \omega$, for some $\chi \in X_{nr}(M)$

The multiplication in $\mathcal{W}$, if possible, is given by

$$(k,n)(k',n') = (k(k' \circ n), nn') \tag{4.59}$$

Let $\Gamma_{rr}(\Xi;\mathcal{L}_\Xi)$ be the algebra of rational sections of L_Ξ that are regular on Ξ_u :

$$\begin{aligned}\Gamma_{rr}(\Xi;\mathcal{L}_\Xi) &= \bigoplus_{(P,A,\omega)} \Gamma_{rr}\big(X_{nr}(M); L(\omega,P)\big) \\ &= \bigoplus_{(P,A,\omega)} \big\{ f \in Q(\mathcal{O}(X_{nr}(M))) \otimes L(\omega,P) : f \text{ is regular on } X_{unr}(M) \big\}\end{aligned} \tag{4.60}$$

From (4.56) and (4.57) we get an action of the groupoid $\mathcal{W}$ on this algebra. By construction the image of $\mathcal{H}(G)$ under the Fourier transform consists of $\mathcal{W}$-invariant sections.

Because (ω, E) is admissible

$$C^\infty(X_{unr}(M)) \otimes L(\omega,P)^{K\times K} \cong C^\infty(X_{unr}(M)) \otimes I_P^G(E)^K \otimes I_P^G(\check{E})^K \tag{4.61}$$

is in a natural way a Fréchet space, for every compact open $K < G$. Endow $C^\infty(X_{unr}(M)) \otimes L(\omega,P)$ with the inductive limit topology. This also gives a topology on $C_c^\infty(\Xi_u;\mathcal{L}_\Xi)$, as the inductive limit of finite direct sums of such algebras. Notice that these are all *-algebras by (4.63). Clearly the action of $\mathcal{W}$ extends continuously to $C_c^\infty(\Xi_u;\mathcal{L}_\Xi)$.

Now the Plancherel theorem for reductive p-adic groups [132, p. 320] tells us that

Theorem 4.9 *The Fourier transform*

$$\mathcal{F} : \mathcal{S}(G) \to C_c^\infty(\Xi_u;\mathcal{L}_\Xi)^{\mathcal{W}}$$

*is an isomorphism of topological *-algebras.*

This guides us to the Fourier transform of $C_r^*(G)$. For (ω, E) as on page 119, let $\mathcal{K}(\omega,P)$ be the algebra of compact operators on the Hilbert space completion of $I_P^G(E)$. Notice that

$$\mathcal{K}(\omega,P) = \varinjlim L(\omega,P)^{K\times K} \tag{4.62}$$

in the C^*-algebra sense, and that the intertwiner $I(n,\omega)$ extends to $\mathcal{K}(\omega,P)$ because it is unitary. Let $\mathcal{K}_\Xi$ be the vector bundle over Ξ whose fiber at (P,A,ω,χ) is $\mathcal{K}(\omega,P)$, and $C_0(\Xi_u;\mathcal{K}_\Xi)$ the C^*-completion of

$$\bigoplus_{(P,A,\omega)} C(X_{unr}(M); \mathcal{K}(\omega,P))$$

Plymen [104, Theorem 2.5] proved that

Theorem 4.10 *The Fourier transform extends to an isomorphism of C^*-algebras*

$$C_r^*(G) \xrightarrow{\sim} C_0(\Xi_u;\mathcal{K}_\Xi)^{\mathcal{W}}$$

The subalgebras $\mathcal{S}(G,K)$ are more manageable than $\mathcal{S}(G)$, so it pays off to describe their images under $\mathcal{F}$.

Theorem 4.11 *Let K be a compact open subgroup of G. There exists a finite set of triples (P_i, A_i, ω_i), $i = 1, \ldots, n_K$, such that the Fourier transform induces algebra homomorphisms*

$$\begin{array}{rcl} \mathcal{H}(G,K) & \to & \bigoplus_{i=1}^{n_K} \big(\mathcal{O}(X_{nr}(M_i)) \otimes L(\omega_i, P_i)^{K\times K}\big)^{\mathcal{W}_i} \\ \mathcal{S}(G,K) & \to & \bigoplus_{i=1}^{n_K} \big(C^\infty(X_{unr}(M_i)) \otimes L(\omega_i, P_i)^{K\times K}\big)^{\mathcal{W}_i} \\ C_r^*(G,K) & \to & \bigoplus_{i=1}^{n_K} \big(C(X_{unr}(M_i)) \otimes L(\omega_i, P_i)^{K\times K}\big)^{\mathcal{W}_i} \end{array}$$

*where $\mathcal{W}_i$ is the isotropy group of (P_i, A_i, ω_i) in $\mathcal{W}$. The first map is injective, the second is an isomorphism of Fréchet *-algebras and the third is an isomorphism of C^*-algebras. For every $w \in \mathcal{W}_i$ there is a rational, unitary element*

$$u_w \in C^\infty(X_{unr}(M_i)) \otimes L(\omega_i, P_i)^{K\times K}$$

such that for every $f \in C(X_{unr}(M_i)) \otimes L(\omega_i, P_i)^{K\times K}$

$$w f(\chi) = u_w(\chi) f(w^{-1}\chi) u_w^{-1}(\chi) \tag{4.63}$$

Proof. By [132, Théorème VIII.1.2] there is only a finite number of association classes among the objects of $\mathcal{W}$ on which the idempotent e_K does not act as zero. Pick one representant (P_i, A_i, ω_i) in every such association class. From (4.58) and Theorems 4.9 and 4.10 we immediately get the required description of the Fourier transforms of $\mathcal{H}(G,K)$, $\mathcal{S}(G,K)$ and $C_r^*(G,K)$.

Every automorphism of $L(\omega_i, P_i)^{K\times K} \cong \mathrm{End}\big(I_P^G(E)^K\big)$ is inner, so the formula (4.63) holds for some u_w. Using Theorem 4.8 we can arrange that u_w is rational on $X_{nr}(M_i)$ and unitary on $X_{unr}(M_i)$. □

Purely representation-theoretic consequences of the above isomorphisms are:

Corollary 4.12 *1. Every irreducible tempered G-representation is a direct summand of $I(\xi)$, for some $\xi \in \Xi_u$.*

2. For any $w \in \mathcal{W}$ and $\xi \in \Xi$ such that $w\xi$ is defined, the G-representations $I(\xi)$ and $I(w\xi)$ have the same irreducible subquotients, counted with multiplicity.

Proof. 1. Let K be a compact open subgroup of G such that $V^K \neq 0$, and let V' be an irreducible submodule of V^K, considered as a $\mathcal{S}(G,K)$-representation. With Theorem 4.11 and the same argument as in the proof of Corollary 3.26.2 we deduce that V' is a direct summand of $I(P_i, A_i, \omega_i, \chi)^K$ for some $\chi \in X_{unr}(M)$. Because V' generates V as a G-module, V is a constituent of $I(P_i, A_i, \omega_i, \chi)$. By Proposition 4.6.2 the latter is completely reducible, so V is in fact equivalent to a direct summand.of $I(P_i, A_i, \omega_i, \chi)$.

2. By [29, Corollary 2.3.3] we have to show that the characters of $I(\xi)$ and $I(w\xi)$ are the same, i.e. that the function

$$\mathcal{H}(G) \times X_{nr}(M) \to \mathbb{C} : (f, \chi) \to \operatorname{tr} I(P, A, \omega, \chi)(f) - \operatorname{tr} I(wP, wA, w\omega, \chi \circ w^{-1})(f) \tag{4.64}$$

is identically 0. Because this is a polynomial function of χ, it suffices to show that it is 0 on $\mathcal{H}(G) \times X_{unr}(M)$. This is an immediate consequence of Theorem 4.9. □

For $\xi = (P, A, \omega, \chi) \in \Xi$ we define

$$\begin{aligned} A(\xi) &= \{a \in A : \alpha(a) = 1 \text{ if } \langle \log |\chi| \,, \alpha \rangle = 0 \ \forall \alpha \in \Sigma(P, A)\} \\ M(\xi) &= Z_G(A(\xi)) \\ P(\xi) &= PM(\xi) \\ \omega(\xi) &= I_{M(\xi) \cap P}^{M(\xi)} \big(\omega \otimes \chi |\chi|^{-1} \big) \\ \nu(\xi) &= \log |\chi| \end{aligned} \tag{4.65}$$

By Proposition 4.6 $\omega(\xi)$ is a pre-unitary tempered $M(\xi)$-representation. Like in [78, §XI.9] these objects are designed to divide parabolic induction into stages:

$$\begin{aligned} I_{P(\xi)}^G \big(|\chi| \otimes \omega(\xi) \big) &\cong \operatorname{Ind}_{P(\xi)}^G \big(\delta_{P(\xi)}^{1/2} \otimes |\chi| \otimes \omega(\xi) \big) \\ &\cong \operatorname{Ind}_{P(\xi)}^G \big(\delta_{P(\xi)}^{1/2} \otimes |\chi| \otimes \operatorname{Ind}_{M(\xi) \cap P}^{M(\xi)} \big(\delta_{P \cap M(\xi)}^{1/2} \otimes \chi |\chi|^{-1} \otimes \omega \big) \big) \\ &\cong \operatorname{Ind}_{PM(\xi)}^G \big(\delta_{PM(\xi)}^{1/2} \otimes \operatorname{Ind}_{M(\xi) \cap P}^{M(\xi)} \big(\delta_{P \cap M(\xi)}^{1/2} \otimes \chi \otimes \omega \big) \big) \\ &\cong \operatorname{Ind}_{PM(\xi)}^G \big(\operatorname{Ind}_{M(\xi) \cap P}^{M(\xi)} \big(\delta_{PM(\xi)}^{1/2} \otimes \delta_{P \cap M(\xi)}^{1/2} \otimes \chi \otimes \omega \big) \big) \\ &\cong \operatorname{Ind}_P^G \big(\delta_P^{1/2} \otimes \chi \otimes \omega \big) \quad = \quad I(\xi) \end{aligned} \tag{4.66}$$

We say that $(P, A, \omega, \chi) \in \Xi^+$ if (P, A) is standard and $\log |\chi| \in \bar{a}^{*,+}$. This choice of a "positive cone" in Ξ is justified by the next result.

Lemma 4.13 *Every $\xi \in \Xi$ is $\mathcal{W}$-associate to an element of Ξ^+. If $\xi_1, \xi_2 \in \Xi^+$ are $\mathcal{W}$-associate, then the objects $A(\xi_i)$, $M(\xi_i)$, $P(\xi_i)$ and $\nu(\xi_i)$ are the same for $i = 1$ and $i = 2$, while $\omega(\xi_1)$ and $\omega(\xi_2)$ are equivalent $M(\xi_i)$-representations.*

Proof. Every p-pair is conjugate to a standard p-pair, and by [61, Section 1.15] every W_0-orbit in $\mathfrak{a}_0^*$ contains a unique point in postive chamber $\mathfrak{a}^{*,+}$. This proves the first claim, and it also shows that

$$\log |\chi_1| = \log |\chi_2| \in \mathfrak{a}_0^* \tag{4.67}$$

Hence the ν's, A's and M's are the same for $i = 1$ and $i = 2$. Because

$$\Delta(P_i, A_i) = \{\alpha\big|_{\mathfrak{a}^*} : \alpha \in \Delta_0, \ \langle \log |\chi_i| \,, \alpha \rangle > 0\} \tag{4.68}$$

we must also have $P(\xi_1) = P(\xi_2)$. If now $w \in \mathcal{W}$ is such that $w\omega_1 \cong \omega_2$, then by Theorem 4.8, applied to $M(\xi_i)$, there is a unitary intertwiner between $\omega(\xi_1)$ and $\omega(\xi_2)$. $\square$

Some immediate consequences of the above definitions and Theorem 4.7 are:

Proposition 4.14 *Take $\xi = (P, A, \omega, \chi) \in \Xi^+$.*

1. *Let τ be an irreducible direct summand of $\omega(\xi)$. Then $(P(\xi), \tau, \nu(\xi)) \in \Lambda^+$.*
2. *The functor $I_{P(\xi)}^G$ induces an isomorphism*

$$\mathrm{End}_G(I(\xi)) \cong \mathrm{End}_{M(\xi)}(\omega(\xi))$$

3. *The irreducible quotients of $I(\xi)$ are precisely the modules $J(P(\xi), \tau, \nu(\xi))$ with τ as above.*

Theorem 4.15 *For every $\pi \in \mathrm{Irr}(G)$ there exists a unique association class $\mathcal{W}(P, A, \omega, \chi) \in \Xi/\mathcal{W}$ such that the following equivalent statements hold :*

1. *π is equivalent to an irreducible quotient of $I(\xi^+)$, for some $\xi^+ \in \mathcal{W}(P, A, \omega, \chi) \cap \Xi^+$.*
2. *π is equivalent to an irreducible subquotient of $I(P, A, \omega, \chi)$, and P is maximal for this property.*

Proof. 1. Let (Q, σ, ν) be the Langlands datum associated to π. Write $\mathcal{W}^M, \Xi^M$ etcetera for $\mathcal{W}, \Xi$, but now corresponding to M instead of G. By Theorem 4.9 there exists a unique association class

$$\mathcal{W}^M \xi = \mathcal{W}^M(P, A, \omega, \chi) \in \Xi_u^M / \mathcal{W}^M$$

such that σ is a direct summand of $I^M(\xi) = I_P^M(\omega \otimes \chi)$. Pick $\xi^+ \in \mathcal{W}^M \xi \cap \Xi^+$. By Proposition 4.14.3 π is equivalent to an irreducible quotient of $I(\xi^+)$, and by Lemma 4.13 and Theorem 4.9 the class $\mathcal{W}\xi^+ = \mathcal{W}\xi \in \Xi/\mathcal{W}$ is unique for this property.

2. Suppose that $\xi' = (P', A', \omega', \chi') \in \Xi^+$ and that π is equivalent to a subquotient of $I(\xi')$ which is not a quotient. By Theorem 4.7.3 we have $\nu(\xi') - \nu(\xi) \in \mathfrak{a}(\xi')^{*,+}$ and $A(\xi) \subsetneq A(\xi')$. For $\alpha \in \Delta_0$ we have

$$\langle \nu(\xi') \, , \, \alpha \rangle = 0 \;\Rightarrow\; \langle \nu(\xi) \, , \, \alpha \rangle = 0$$

so by (4.65) $A \subsetneq A'$ and $P \supsetneq P'$. Therefore the conditions 1 and 2 are equivalent. $\square$

4.4 Noncommutative geometry

Now that we know quite something about the representation theory of reductive p-adic groups, we can turn to the study of their noncommutative geometry with more confidence. More specifically, we compare the periodic cyclic homologies of $\mathcal{H}(G)$ and $\mathcal{S}(G)$, and the K-theory of $C_r^*(G)$. Although these results are new and technical, this section remains very short, because we already did most of the hard work. We will discuss these comparison theorems in relation with the Baum-Connes conjecture.

First we will prove an analogue of (3.143) for the Hecke algebra of a reductive p-adic group. Since $\mathcal{S}(G)$ is defined as an inductive limit of Fréchet algebras, we take its periodic cyclic homology with respect to the completed inductive tensor product $\overline{\otimes}$.

Theorem 4.16 *Let $\mathfrak{s} \in \mathfrak{B}(G)$ be a Bernstein component and $K_\mathfrak{s}$ a compact open subgroup of G as in Proposition 4.2.3. The Chern character for $\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}$ induces an isomorphism*

$$K_*\big(C_r^*(G)^\mathfrak{s}\big) \otimes \mathbb{C} \xrightarrow{\sim} HP_*\big(\mathcal{S}(G)^\mathfrak{s}, \overline{\otimes}\big)$$

The direct sum of these maps, over all $\mathfrak{s} \in \mathfrak{B}(G)$, is a natural isomorphism

$$K_*(C_r^*(G)) \otimes \mathbb{C} \xrightarrow{\sim} HP_*(\mathcal{S}(G), \overline{\otimes})$$

Proof. Since $\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}$ is a direct summand of $\mathcal{S}(G, K_\mathfrak{s})$, by Theorem 4.11 there are finitely many components (P_i, A_i, ω_i) of Ξ such that

$$\begin{array}{lll} \mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s} & \cong & \bigoplus_i \big(C^\infty(X_{unr}(M_i)) \otimes L(\omega_i, P_i)^{K_\mathfrak{s} \times K_\mathfrak{s}}\big)^{\mathcal{W}_i} \\ C_r^*(G, K_\mathfrak{s})^\mathfrak{s} & \cong & \bigoplus_i \big(C(X_{unr}(M_i)) \otimes L(\omega_i, P_i)^{K_\mathfrak{s} \times K_\mathfrak{s}}\big)^{\mathcal{W}_i} \end{array} \tag{4.69}$$

Note that the single Bernstein component $\mathfrak{s}$ generally contains more than component of Ξ. According to Theorem 2.13 the inclusion induces an isomorphism

$$K_*\big(\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s}\big) \xrightarrow{\sim} K_*\big(C_r^*(G, K_\mathfrak{s})^\mathfrak{s}\big) \tag{4.70}$$

and by Theorem 2.27 the Chern character induces an isomorphism

$$K_*\big(\mathcal{S}(G, K)^\mathfrak{s}\big) \otimes \mathbb{C} \xrightarrow{\sim} HP_*\big(\mathcal{S}(G, K)^\mathfrak{s}\big) \tag{4.71}$$

For any compact open $K \subset K_\mathfrak{s}$ the algebra $L(\omega_i, P_i)^{K \times K}$ is finite dimensional and simple, so the inclusion

$$L(\omega_i, P_i)^{K^\mathfrak{s} \times K^\mathfrak{s}} \to L(\omega_i, P_i)^{K \times K}$$

is of the type $M_n(\mathbb{C}) \subset M_m(\mathbb{C})$. Therefore

$$\mathcal{S}(G, K_\mathfrak{s})^\mathfrak{s} \to \mathcal{S}(G, K)^\mathfrak{s} \tag{4.72}$$

induces an isomorphism on HH_*, HC_*, HP_* and K_*. From this, (4.26) and the properties of the topological K-functor (cf. page 39) we see that

$$
\begin{aligned}
K_*(C_r^*(G)) &\cong K_* \Big(\varinjlim_{\mathfrak{S}} \bigoplus_{\mathfrak{s}\in\mathfrak{S}} C_r^*(G)^{\mathfrak{s}} \Big) \\
&\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} K_* \left(C_r^*(G)^{\mathfrak{s}} \right) \\
&\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \varinjlim K_* \left(C_r^*(G,K)^{\mathfrak{s}} \right) \\
&\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} K_* \big(C_r^*(G,K_{\mathfrak{s}})^{\mathfrak{s}} \big) \\
&\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} K_* \big(\mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}} \big)
\end{aligned} \tag{4.73}
$$

Since the algebras $\mathcal{S}(G,K)^{\mathfrak{s}}$ are all nuclear Fréchet, and have the same Hochschild homology for fixed $\mathfrak{s} \in \mathfrak{B}(G)$ and $K \subset K_{\mathfrak{s}}$, we may use the continuity and additivity of $HP_*(\cdot , \overline{\otimes})$, as described on page 32.

$$
\begin{aligned}
HP_*(\mathcal{S}(G),\overline{\otimes}) &\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} HP_*(\mathcal{S}(G)^{\mathfrak{s}},\overline{\otimes}) \\
&\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \varinjlim HP_*(\mathcal{S}(G,K)^{\mathfrak{s}},\overline{\otimes}) \\
&\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} HP_*(\mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}},\overline{\otimes}) \\
&= \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} HP_*(\mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}},\widehat{\otimes})
\end{aligned} \tag{4.74}
$$

Now the theorem follows from the combination of (4.71), (4.73) and (4.74). □

The analogue of Theorem 3.32 was suggested in [7, Conjecture 8.9] and in [3, Conjecture 1] :

Theorem 4.17 *The inclusions $\mathcal{H}(G)^{\mathfrak{s}} \to \mathcal{S}(G)^{\mathfrak{s}}$ induce isomorphisms*

$$
\begin{aligned}
HP_*(\mathcal{H}(G)^{\mathfrak{s}}) &\xrightarrow{\sim} HP_*(\mathcal{S}(G)^{\mathfrak{s}},\overline{\otimes}) \\
HP_*(\mathcal{H}(G)) &\xrightarrow{\sim} HP_*(\mathcal{S}(G),\overline{\otimes})
\end{aligned}
$$

Proof. Just as in (4.74) we have

$$\begin{aligned} HP_*(\mathcal{H}(G)) &\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} HP_*(\mathcal{H}(G)^{\mathfrak{s}}) \\ &\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} \varinjlim HP_*(\mathcal{H}(G,K)^{\mathfrak{s}}) \\ &\cong \bigoplus_{\mathfrak{s}\in\mathfrak{B}(G)} HP_*(\mathcal{H}(G,K_{\mathfrak{s}})^{\mathfrak{s}}) \end{aligned} \tag{4.75}$$

Therefore we only have to show that every inclusion

$$\mathcal{H}(G,K_{\mathfrak{s}})^{\mathfrak{s}} \to \mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}}$$

induces an isomorphism on periodic cyclic homology. Number the direct summands in (4.69), such that

$$\mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}} \cong \bigoplus_{j=1}^{n_{\mathfrak{s}}} \big(C^\infty(X_{unr}(M_j)) \otimes L(\omega_j,P_j)^{K_{\mathfrak{s}}\times K_{\mathfrak{s}}} \big)^{\mathcal{W}_j} \tag{4.76}$$

and $\dim M_i \leq \dim M_j$ if $i \leq j$. Now we construct two chains of ideals

$$\begin{aligned} &\mathcal{H}(G,K_{\mathfrak{s}})^{\mathfrak{s}} = I_0 \supset I_1 \supset \cdots \supset I_{n_{\mathfrak{s}}} = 0 \\ &\mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}} = J_0 \supset J_1 \supset \cdots \supset J_{n_{\mathfrak{s}}} = 0 \\ &I_j = \{ h \in \mathcal{H}(G,K_{\mathfrak{s}})^{\mathfrak{s}} : I(P_j,A_j,\omega_j,\chi)(h) = 0 \text{ if } \chi \in X_{nr}(M_j) \text{ and } j \leq i \} \\ &J_j = \{ h \in \mathcal{S}(G,K_{\mathfrak{s}})^{\mathfrak{s}} : I(P_j,A_j,\omega_j,\chi)(h) = 0 \text{ if } \chi \in X_{unr}(M_j) \text{ and } j \leq i \} \end{aligned}$$

Writing $V_i = I_{P_i}^G(E_i)^{K_{\mathfrak{s}}}$, we clearly have

$$J_{i-1}/J_i \cong \big(C^\infty(X_{unr}(M_i)) \otimes \operatorname{End} V_i \big)^{\mathcal{W}_i}$$

From now on we can follow the proof of Theorem 3.32. We have to substitute Theorems 4.7, 4.8, 4.11 and 4.15 for, respectively, Theorems 3.7, 3.23, 3.25 and 3.31. $\Box$.

This theorem should be compared with the work of Meyer [91].

It is also interesting to compare Theorems 4.16 and 4.17 with other homological invariants of reductive p-adic groups. One such is the chamber homology of the Bruhat-Tits building βG, equivariant with respect to G. This is a sequence of complex vector spaces $H_n^G(\beta G)$, $n = 0,1,2,\ldots$, first defined in [5, §6]. It was proved simultaneously in [59] and [111] that there are natural isomorphisms

$$HP_i(\mathcal{H}(G)) \cong \bigoplus_{n\in\mathbb{Z}} H_{i+2n}^G(\beta G) \tag{4.77}$$

Closely related is the G-equivariant K-homology of βG, as defined in [5, §3]. According to Voigt [131, Theorem 6.8] there exists a natural equivariant Chern character

$$ch_*^G : K_*^G(\beta G) \to H_*^G(\beta G) \tag{4.78}$$

which becomes an isomorphism after tensoring with $\mathbb{C}$. Notice that Voigt uses an alternative but equivalent definition of $H_*^G(\beta G)$. The Baum-Connes conjecture for reductive p-adic groups asserts that the so-called assembly map

$$\mu : K_j^G(\beta G) \to K_j(C_r^*(G)) \tag{4.79}$$

is an isomorphism. This was proved by Lafforgue [79], as a part of a much more general result. Putting all these things together we more or less arrive at [7, Proposition 9.4] :

Theorem 4.18 *In following diagram the horizontal maps are natural isomorphisms, and the vertical maps become natural isomorphisms after tensoring with* $\mathbb{C}$.

$$\begin{array}{ccccc} K_*^G(\beta G) & & \longrightarrow & & K_*(C_r^*(G)) \\ \downarrow & & & & \downarrow \\ H_*^G(\beta G) & \cong & HP_*(\mathcal{H}(G)) & \to & HP_*(\mathcal{S}(G), \overline{\otimes}) \end{array}$$

Because of the naturality, the diagram is probably commutative, but the author does not know how to prove this. Unfortunately the definitions are so complicated that it already is difficult to find any element of $K_*^G(\beta G)$ for which the diagram can be seen to commute by direct computation.

For the groups $GL_n(\mathbb{F})$ and $SL_n(\mathbb{F})$ partial results in the direction of Theorem 4.18 were proved by Baum, Higson and Plymen in [6]. In fact, in [6] the Baum-Connes conjecture for these groups is proved precisely with the above diagram. However, the argument uses the commutativity of the diagram in an essential way, and unfortunately the authors do not provide any support of the (implicit) claim that it does commute.

Chapter 5

Parameter deformations in affine Hecke algebras

So far we have always written affine Hecke algebras as deformations of a group algebra, but we have not really done anything with this. Ideally speaking, several properties of an affine Hecke algebra $\mathcal{H}(\mathcal{R}, q)$ should be independent of the parameters $q(s)$. This intuitive idea comes from finite dimensional algebras, where it is very clear from Tits' deformation theorem. Roughly speaking, it tells us that if two semisimple algebras can be continuously deformed into each other, then they are isomorphic. This is so because there are only countably many isomorphism classes of such algebras, and they lie discrete in some sense. For infinite dimensional algebras nothing similar holds, so there we have to find more subtle invariants and arguments.

This has been done for affine Hecke algebras with equal labels. Kazhdan and Lusztig [76] gave a complete geometric parametrization of the irreducible representations of such algebras. This parametrization is independent of $q \in \mathbb{C}^\times$, except in few tricky cases where q is a proper root of unity. Baum and Nistor [8] showed that this leads to an isomorphism

$$HP_*\big(\mathcal{H}(\mathcal{R}, q)\big) \cong HP_*\big(\mathbb{C}[W]\big) \tag{5.1}$$

Acknowledging that these results cannot readily be carried over to the unequal label case, we follow another path, more analytic in nature. By careful estimates in the Schwartz algebra $\mathcal{S}(\mathcal{R}, q)$ we show that the following things all depend continuously on q: the operator norm, multiplication, inverting and, the holomorphic functional calculus.

Equipped with these tools and the knowledge from Chapter 3 we attack a special kind of parameter deformation, scaling the label function. Such deformations were studied first by Opdam [98]. On the level of central characters this comes down to scaling the absolute value by a real factor ϵ. For $\epsilon > 0$ we construct isomorphisms of pre-C^*-algebras

$$\phi_\epsilon : \mathcal{S}(\mathcal{R}, q^\epsilon) \xrightarrow{\sim} \mathcal{S}(\mathcal{R}, q) \tag{5.2}$$

They depend continuously on ϵ, but the limit

$$\phi_0 : \mathcal{S}(W) \to \mathcal{S}(\mathcal{R}, q) \tag{5.3}$$

is no longer surjective. Nevertheless this map seems to behave well. In view of (5.1) one is naturally lead to conjecture that

$$HP_*(\phi_0) : HP_*\big(\mathcal{S}(W)\big) \to HP_*\big(\mathcal{S}(\mathcal{R}, q)\big) \tag{5.4}$$

is an isomorphism for any positive label function q. We provide various equivalent reformulations of this statement.

Conjecture (5.4) can be derived from the stronger conjecture, originally due to Baum, Connes and Higson [5], that

$$K_*\big(C_r^*(\mathcal{R}, q)\big) \cong K_*\big(C_r^*(W)\big) \tag{5.5}$$

We show that (5.5) is equivalent to the existence of a natural bijection between the Grothendieck groups of irreducible representations of $C_r^*(W)$ and of $C_r^*(\mathcal{R}, q)$. At the end of the chapter we give some clues in support of these conjectures.

5.1 The finite dimensional and equal label cases

We recall what is already known about deformations of Iwahori-Hecke algebras obtained by varying the label function q. For Hecke algebras of finite type this is very clear: as long as they are semisimple they are rigid under deformations. But this is specific for the finite case, as it relies on the classification of finite dimensional semisimple algebras.

For any extended Iwahori-Hecke algebra $\mathcal{H}(\mathcal{R}, q)$ with equal labels a complete parametrization of irreducible representations is available. This is a refinement of the Langlands classification, and it is essentially independent of q. The link between different q's is made via Lusztig's asymptotic Hecke algebra J, which allows a weakly spectrum preserving morphism $\mathcal{H}(\mathcal{R}, q) \to J$. From this we will see that the periodic cyclic homology of $\mathcal{H}(\mathcal{R}, q)$ is independent of q, as long as it is not a proper root of unity.

Recall that an algebra A is semisimple if its Jacobson radical is 0, which means that for every nonzero $a \in A$ there is an irreducible A-representation π such that $\pi(a) \neq 0$. For example, by [41, Théorème 2.7.3] every C^*-algebra is semisimple. The structure of finite dimensional semisimple algebras is described in a famous theorem of Wedderburn [135] :

Theorem 5.1 *Let A be a finite dimensional semisimple algebra over a field $\mathbb{F}$. There exist natural numbers n_i and division algebras D_i over $\mathbb{F}$ such that*

$$A \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i)$$

If $\mathbb{F}$ is algebraically closed then $D_i = \mathbb{F}\ \forall i$.

Let G be any finite group. By Maschke's theorem the group algebra $\mathbb{C}[G]$ is semisimple. Let $\{T_g : g \in G\}$ be its canonical basis, and $\mathbf{k} = \mathbb{C}[x_1, \ldots, x_r]$ a polynomial ring over $\mathbb{C}$. Let A be a $\mathbf{k}$-algebra whose underlying $\mathbf{k}$-module is $\mathbf{k}[G]$ and whose multiplication is defined by

$$T_g \cdot T_h = \sum_{w \in G} a_{g,h,w} T_w \tag{5.6}$$

for certain $a_{g,h,w} \in \mathbf{k}$. For any point $q \in \mathbb{C}^r$ we can endow the vector space $\mathbb{C}[G]$ with the structure of an associative algebra by

$$T_g \cdot_q T_h = \sum_{w \in G} a_{g,h,w}(q) T_w \tag{5.7}$$

We denote the resulting algebra by $\mathcal{H}(G, q)$. It is isomorphic to the tensor product $A \otimes_{\mathbf{k}} \mathbb{C}$ where $\mathbb{C}$ has the $\mathbf{k}$-module structure obtained from evaluating at q. Assume moreover that there exists a $q^0 \in \mathbb{C}^r$ such that

$$\mathcal{H}(G, q^0) = \mathbb{C}[G]$$

We express the rigidity of finite dimensional semisimple algebras by the following special case of Tits's deformation theorem [27, p. 357 - 359]:

Theorem 5.2 *There exists a polynomial $P \in \mathbf{k}$ such that the following are equivalent :*

- $P(q) \neq 0$
- $\mathcal{H}(G, q)$ *is semisimple*
- $\mathcal{H}(G, q) \cong \mathbb{C}[G]$

Now let (W, S) be a finite Coxeter system, q a label function om W and $\mathcal{H}(W, q)$ the associated Iwahori-Hecke algebra, as in Section 3.1. This is consistent with the above notation. We want to know under which conditions this algebra is semisimple. Clearly this is the case if $q(w) > 0\ \forall w \in W$, for then $\mathcal{H}(W, q)$ is a C^*-algebra by (3.80).

But the polynomials $P(q)$ of Theorem 5.2 have also been determined explicitly. If we are in the equal label case $q(s) = q\ \forall s \in S$ then we may take

$$P(q) = q \sum_{w \in W} q^{\ell(w)} \tag{5.8}$$

except that we must omit the factor q if W is of type $(A_1)^n$, see [54]. More generally, Gyoja [53, p. 569] showed that if (W,S) is irreducible and S consists of two conjugacy classes, then in most cases we may take

$$\begin{array}{rcl} P(q_1,q_2) & = & q_1^{|W|} q_2 W(q_1,q_2) W(q_1^{-1},q_2) \\ W(q_1,q_2) & = & \sum_{w \in W} q(w) \end{array} \tag{5.9}$$

So generically there is an isomorphism

$$\mathcal{H}(W,q) \cong \mathbb{C}[W] \tag{5.10}$$

We will see later how it can be constructed explicitly. From our somewhat simple point of view this is all there is to say about parameter deformations of finite dimensional Hecke algebras. If they are semisimple then they are isomorphic, and if not, then they have nilpotent ideals and look very different from $\mathbb{C}[W]$.

Let $\mathcal{R} = (X,Y,R_0,R_0^\vee,F_0)$ be a root datum, $q \in \mathbb{C}^\times$, and consider the affine Hecke algebra with equal labels $\mathcal{H}(\mathcal{R},q)$. The irreducible representations of this algebra have been classified completely by Kazhdan and Lusztig [76]. For this very deep result they showed among others that $\mathcal{H}(\mathcal{R},q)$ is isomorphic to the equivariant algebraic K-theory of a certain variety.

Let G be the unique complex reductive algebraic group with root datum $\mathcal{R}^\vee = (Y,X,R_0^\vee,R_0)$, and $\mathfrak{g}$ its Lie algebra. For reasons of a much more general nature G is called the Langlands dual group. Then $T = \mathrm{Hom}_{\mathbb{Z}}(X,\mathbb{C}^\times)$ can be identified with a maximal torus of G. Since every semisimple element of G is conjugate to an element of T, and since $N_G(T)/Z_G(T) \cong W_0$, we can parametrize the central character of an (irreducible) $\mathcal{H}(\mathcal{R},q)$-module by a unique conjugacy class of semisimple elements in G. So, let $s \in G$ be semisimple and write

$$\mathfrak{n}(s,q) = \{N \in \mathfrak{g} : N \text{ nilpotent}\,, \mathrm{Ad}(s)N = qN\} \tag{5.11}$$

The G-conjugacy classes of pairs (s,N) with $N \in \mathfrak{n}(s,q)$ are called Deligne-Langlands parameters. They give an almost complete description of $\mathrm{Prim}(\mathcal{H}(\mathcal{R},q))$. For instance it works perfectly if $\mathcal{R}$ is of type GL_n, see [137]. In a sense this is equivalent to the Langlands classification in Theorem 4.7. To find really all irreducible representations we must add one extra ingredient. Let

$$Z(s,N) = \{g \in G : gs = sg,\ \mathrm{Ad}(g)N = N\} \tag{5.12}$$

be the simultaneous centralizer of s and N, and $Z^0(s,N)$ its identity component. Assume that $q \in \mathbb{C}^\times$ is not a proper root of unity, i.e. either $q=1$ or q is not a root of unity. In these cases there is a bijection between $\mathrm{Prim}(\mathcal{H}(\mathcal{R},q))$ and G-conjugacy classes of triples (s,N,ρ), where $s \in G$ is semisimple, $N \in \mathfrak{n}(s,q)$ and ρ is a "geometric" irreducible representation of the finite group $Z(s,N)/Z^0(s,N)$. This was proved for $q=1$ in [72, Theorem 4.1] and for q not a root of unity and

X equal to the weight lattice of $R_0^\vee$ in [76, Theorem 7.12]. Later it was shown in [105, Theorem 2] that this condition on X is not necessary.

Another construction which is particular for the equal label case is Lusztig's asymptotic Hecke algebra [84, 85]. This is a finite type algebra J with a basis $\{t_w : w \in W\}$ over $\mathbb{C}$. It decomposes as a finite direct sum of two-sided ideals

$$\begin{array}{lll} J & = & \bigoplus_{i=0}^{|R_0^+|} J^i \\ J^i & = & \text{span}\{t_w : a(w) = i\} \end{array} \tag{5.13}$$

where a is Lusztig's a-function. For every $q \in \mathbb{C}^\times$ there is an injective morphism of finite type algebras

$$\phi_q : \mathcal{H}(\mathcal{R}, q) \to J \tag{5.14}$$

If q is not a proper root of unity, then ϕ_q induces a bijection on irreducible representations. Namely, for any irreducible J-representation π the $\mathcal{H}(\mathcal{R}, q)$-representation $\pi \circ \phi_q$ has a unique irreducible constituent of minimal "a-weight". This implies that the morphisms of finite type algebras

$$\phi_q^{-1}\Big(\bigoplus_{i \geq k} J^i\Big) \Big/ \phi_q^{-1}\Big(\bigoplus_{i > k} J^i\Big) \longrightarrow J^k \tag{5.15}$$

are spectrum preserving. Lusztig [85, Corollary 3.6] proved this in the case $W = W_{\text{aff}}$, but using the aforementioned result of Reeder [105] his proof can be extended to general root data. Combining this with Theorem 2.7 and Lemma 2.3 we arrive at an extended version of [8, Theorem 11] :

Theorem 5.3 *Assume that q is not a proper root of unity. Then*

$$HP_*(\phi_q) : HP_*(\mathcal{H}(\mathcal{R}, q)) \to HP_*(J)$$

is an isomorphism. Consequently

$$HP_*(\mathcal{H}(\mathcal{R}, q)) \cong HP_*(\mathbb{C}[W])$$

It is expected that an asymptotic Hecke algebra can also be constructed for finite or affine Coxeter systems with unequal labels [88, Chapter 18]. Assuming certain conjectures [88, Chapter 15] one can construct algebra homomorphisms $\phi_q : \mathcal{H}(W, q) \to J$ for any label function with the following property: there exist $v \in \mathbb{C}^\times$ and $n_s \in \mathbb{N}$ such that $q(s) = v^{n_s}\ \forall s \in S$.

For finite W the map ϕ_q is an isomorphism if and only if $\mathcal{H}(W, q)$ is semisimple [88, (20.1.e)]. In this way one can find explicit formulas for the isomorphisms from Theorem 5.2.

For affine W ϕ_q has a nilpotent kernel [88, Proposition 18.12] and in general it is not surjective. It is unknown whether ϕ_q is spectrum preserving in any sense. The problem is that in general there is no definite classification of all irreducible representations of an affine Hecke algebra. Apparently the linkwith the Langlands dual group is much weaker for unequal parameters.

Nevertheless in some cases the Deligne-Langlands philosophy outlined above can be generalized. Namely, along these lines a classification of the irreducible representations of $\mathcal{H}(\mathcal{R}, q)$ has been obtained in [74] for $\mathcal{R}$ of type B_n/C_n, for almost all label functions q.

5.2 Estimating norms

In this section we lay the analytic foundations for all our coming results on parameter deformations. In Section 3.2 we defined various norms on an affine Hecke algebra $\mathcal{H}(\mathcal{R}, q)$: the norm $\| \cdot \|_\tau$ associated with the trace τ, the operator norm $\| \cdot \|_o$ and the Schwartz norms p_n. We will show that the operator norm, the multiplication and the inverse of an element depend continuously on q. From this we deduce the holomorphic functional calculus on affine Hecke algebras is continuous in a very general sense.

We also reconstruct the Schwartz algebra $\mathcal{S}(\mathcal{R}, q)$ in a different way. With this construction we can show in straightforward fashion that $\mathcal{S}(\mathcal{R}, q)$ is holomorphically closed in $C_r^*(\mathcal{R}, q)$.

With respect to the bases $\{N_w : w \in W\}$ the norms $\|\cdot\|_\tau$ and p_n are independent of q. Therefore we can identify all the Hilbert spaces $\mathfrak{H}(\mathcal{R}, q)$ and all the Schwartz spaces $\mathcal{S}(\mathcal{R}, q)$ by means of this basis. When we want to consider them in this way, only as topological vector spaces and without a specified label function, we write $\mathfrak{H}(\mathcal{R})$ and $\mathcal{S}(\mathcal{R})$. To indicate that $x \in \mathcal{S}(\mathcal{R})$ should be considered as an element of $\mathcal{S}(\mathcal{R}, q)$ we sometimes denote it by (x, q). Furthermore, to distinguish the various products we add a subscript, so $\cdot_q$ is the multiplication in $C_r^*(\mathcal{R}, q)$.

Let $L_\mathcal{R}$ be the space of label functions on $\mathcal{R}$ satisfying the positivity Condition 3.8. Recall that for a simple reflection $s_i \in S_{\text{aff}}$ we put

$$q_i = q(s_i) \quad \text{and} \quad \eta_i = q_i^{1/2} - q_i^{-1/2} \tag{5.16}$$

These numbers determine q uniquely, and their domain is only limited by the conditions $q_i > 0$ and $q_i = q_j$ whenever s_i and s_j are conjugate in W. Hence the parameterspace $L_\mathcal{R}$ is homeomorphic to $\mathbb{R}^n$ for a certain n. We will use the standard topology on $L_\mathcal{R}$, induced by the metric

$$\rho(q, q') = \max_{s_i \in S_{\text{aff}}} |\eta_i - \eta_i'| \tag{5.17}$$

We already know that the group W with the length function $\mathcal{N}$ is of polynomial growth, but we need a more explicit estimate on number of elements of a fixed length.

Lemma 5.4 *There exists a real number $C_\mathcal{N}$ such that $\forall n \in \mathbb{N}$*

$$\#\big\{w \in W : \mathcal{N}(w) = n\big\} < C_\mathcal{N}\, (n+1)^{\text{rk}(X)-1}$$

Proof. Let r denote the rank of X, and pick a linear bijection $L : X \otimes \mathbb{R} \to \mathbb{R}^r$ such that

- $L(X) \subset \mathbb{Z}^r$
- $\forall x \in X^+ : \mathcal{N}(x) = \|L(x)\|_1$

This is possible because, on $X^+, \mathcal{N}$ is additive and takes values in $\mathbb{N}$. By (3.25) we can write any $w \in W$ as

$$w = uxv \quad \text{with} \quad u, v \in W_0\,, x \in X^+$$

If $\mathcal{N}(w) = n$, then clearly

$$n - |R_0| \leq n - \mathcal{N}(u) - \mathcal{N}(v) \leq \mathcal{N}(x) \leq n + \mathcal{N}(u) + \mathcal{N}(v) \leq n + |R_0| \tag{5.18}$$

Therefore we can estimate

$$\begin{aligned} |W_0|^{-2}\#\big\{w \in W : \mathcal{N}(w) = n\big\} &\leq \#\big\{x \in X^+ : n - |R_0| \leq \mathcal{N}(x) \leq n + |R_0|\big\} \\ &= \#\big\{y \in L(X^+) : n - |R_0| \leq \|y\|_1 \leq n + |R_0|\big\} \\ &\leq \#\big\{y \in \mathbb{Z}^r : n - |R_0| \leq \|y\|_1 \leq n + |R_0|\big\} \end{aligned}$$

For the sake of calculation we assume now that $n > |R_0|$. This is allowed because there are only finitely many $w \in W$ of smaller length. We continue our estimate:

$$\begin{aligned} &\leq (n + |R_0| + 1)^r - (n - |R_0| - 1)^r \\ &= \sum_{i=0}^{r} \binom{r}{i} n^{r-i}\big(|R_0| + 1\big)^i\big(1 - (-1)^i\big) \\ &< n^{r-1} \sum_{i \leq r,\, i \text{ odd}} 2\binom{r}{i}\big(|R_0| + 1\big)^i \end{aligned}$$

So our candidate for $C_{\mathcal{N}}$ is

$$|W_0|^2 \sum_{i \leq r,\, i \text{ odd}} 2\binom{r}{i}\big(|R_0| + 1\big)^i$$

We only have to check whether it works also for $\mathcal{N}(w) \leq |R_0|$ and, if not, increase it accordingly. $\square$

Put $b = \mathrm{rk}(X) + 1$. By Lemma 5.4 the following sum converges to a limit C_b:

$$C_b := \sum_{w \in W} (\mathcal{N}(w) + 1)^{-b} < \sum_{n=0}^{\infty} C_{\mathcal{N}}(n + 1)^{\mathrm{rk}(X)-1}(n + 1)^{-\mathrm{rk}(X)-1} < \infty \tag{5.19}$$

This implies that for any $x = \sum_u x_u N_u \in \mathcal{S}(\mathcal{R})$ and $n \in \mathbb{N}$

$$\sum_u |x_u|(\mathcal{N}(u) + 1)^n \leq \sum_u \sup_v \big\{|x_v|(\mathcal{N}(v) + 1)^{n+b}\big\}(\mathcal{N}(u) + 1)^{-b} = C_b p_{n+b}(x) \tag{5.20}$$

$$\|x\|_\tau \leq \sum_u |x_u| \leq C_b p_b(x) \tag{5.21}$$

Since $\ell(w) \leq \mathcal{N}(w)\ \forall w \in W$, these inequalities a fortiori remain valid if we replace $\mathcal{N}$ by ℓ.

Let $u, v, w \in W$ and let $u = \omega s_1 \cdots s_{\ell(u)}$ be a reduced expression, where $\ell(\omega) = 0$ and $s_i \in S_{\text{aff}}$. (The s_i need not all be different.) For $I \subset \{1, 2, \dots, \ell(u)\}$ we put $\eta_I = \prod_{i \in I} \eta_i$ and

$$u_I = \omega \tilde{s}_1 \cdots \tilde{s}_{\ell(u)} \qquad \text{where} \qquad \tilde{s}_i = \begin{cases} s_i & \text{if} \quad i \notin I \\ e & \text{if} \quad i \in I \end{cases}$$

Theorem 5.5

$$N_u \cdot_q N_v = \sum_{I \subset \{1,2,\dots,\ell(u)\}} \eta_I D_v^u(I) N_{u_I v} \qquad \text{where}$$

- $D_v^u(I)$ *is either 0 or 1*
- $D_u^v(\emptyset) = 1$ *and* $D_v^u(I) = 0$ *if* $|I| > |R_0^+|$
- $\sum_{I \subset \{1,2,\dots,\ell(u)\}} D_v^u(I) < 3(\ell(u)+1)^{|R_0^+|}$

Proof. It follows from the multiplicaton rules (3.37) that

$$N_{s_i} \cdot_q N_v = N_{s_i v} + D_v^{s_i}(i) \eta_i N_v \quad \text{where} \quad D_v^{s_i}(i) = \begin{cases} 0 & \text{if} \quad \ell(s_i v) > \ell(v) \\ 1 & \text{if} \quad \ell(s_i v) < \ell(v) \end{cases} \tag{5.22}$$

The expression for $N_u \cdot_q N_v$, with $D_u^v(I)$ being 0 or 1 and $D_u^v(\emptyset) = 1$, follows from this, with induction to $\ell(u)$. By [83, Theorem 7.2] for fixed $w \in W$ the sum

$$\sum_{I : u_I = w} \eta_I D_v^u(I)$$

is a polynomial of degree at most $|R_0^+|$ in the η_i. Therefore $D_v^u(I) = 0$ whenever $|I| > |R_0^+|$. Consequently

$$\begin{aligned} \sum_{I \subset \{1,2,\dots,\ell(u)\}} D_v^u(I) \ &\leq\ \#\big\{ I \subset \{1,2,\dots,\ell(u)\} : |I| \leq |R_0^+| \big\} \ \leq\ \sum_{j=0}^{|R_0^+|} \binom{\ell(u)}{j} \\ &\leq\ \frac{\ell(u)!}{(\ell(u) - |R_0^+|)!} \sum_{j=0}^{|R_0^+|} \frac{1}{j!} \ <\ 3(\ell(u)+1)^{|R_0^+|} \end{aligned} \tag{5.23}$$

where we should interpret $(\ell(u) - |R_0^+|)!$ as 1 if $|R_0^+| \geq \ell(u)$. □

Let $\eta > 0$ and put $C_\eta = 3 C_b \max\big\{1, \eta^{|R_o^+|}\big\}$.

Proposition 5.6 *For all $q, q' \in B_\rho(q^0, \eta)$, $x \in \mathcal{S}(\mathcal{R})$ the following estimates hold.*

$$\begin{array}{rcl} \|\lambda(x,q)\|_{B(\mathfrak{H}(\mathcal{R}))} = \|(x,q)\|_o & \leq & C_\eta p_{b+|R_0^+|}(x) \\ \|\lambda(x,q) - \lambda(x,q')\|_{B(\mathfrak{H}(\mathcal{R}))} & \leq & \rho(q,q')C_\eta p_{b+|R_0^+|}(x) \end{array}$$

In particular $\mathcal{S}(\mathcal{R},q) \subset C_r^*(\mathcal{R},q)$ and every finite dimensional $C_r^*(\mathcal{R},q)$-representation is tempered.

Proof. Let $y = \sum_v y_v N_v \in \mathcal{S}(\mathcal{R})$. By (5.20) and Theorem 5.5 we have

$$\begin{aligned} \|x \cdot_q y\|_\tau &= \left\| \sum_{u,v} x_u y_v N_u \cdot_q N_v \right\|_\tau \\ &= \left\| \sum_{u,v} x_u y_v \sum_I \eta_I D_v^u(I) N_{u_I v} \right\|_\tau \\ &\leq \sum_u |x_u| \sum_{I:|I|\leq|R_0^+|} |\eta_I| \left\| \sum_v |y_v| N_{u_I v} \right\|_\tau \\ &\leq \sum_u |x_u| (\ell(u)+1)^{|R_0^+|} 3 \max\left\{1, \eta^{|R_0^+|}\right\} \|y\|_\tau \\ &\leq C_\eta p_{b+|R_o^+|}(x) \|y\|_\tau \end{aligned} \tag{5.24}$$

Since $\mathcal{S}(\mathcal{R})$ is dense in $\mathfrak{H}(\mathcal{R})$, this gives the estimate, by the very definition of the operator norm on $B(\mathfrak{H}(\mathcal{R}))$. In particular we get a continuous embedding $\mathcal{S}(\mathcal{R},q) \to C_r^*(\mathcal{R},q)$, so every finite dimensional representation of the latter algebra is tempered by Lemma 3.14.

$$\begin{aligned} \|x \cdot_q y - x \cdot_{q'} y\|_\tau &= \left\| \sum_{u,v} x_u y_v (N_u \cdot_q N_v - N_u \cdot_{q'} N_v) \right\| \tau \\ &= \left\| \sum_{u,v} x_u y_v \sum_I (\eta_I - \eta'_I) D_v^u(I) N_{u_I v} \right\|_\tau \\ &\leq \left\| \sum_{u,v} x_u y_v \sum_I \rho(q.q') |I| \eta^{|I|-1} D_v^u(I) N_{u_I v} \right\|_\tau \\ &\leq \rho(q,q') \sum_u |x_u| \sum_{I:|I|\leq|R_0^+|} |I| \eta^{|I|-1} \left\| \sum_v |y_v| N_{u_I v} \right\|_\tau \\ &\leq \rho(q,q') \sum_u |x_u| (\ell(u)+1)^{|R_0^+|} 3 \max\left\{1, \eta^{|R_0^+|}\right\} \|y\|_\tau \\ &\leq \rho(q,q') C_\eta p_{b+|R_o^+|}(x) \|y\|_\tau \end{aligned} \tag{5.25}$$

Between lines 4 and 5 we used as small calculation like (5.23) :

$$\begin{aligned} \sum_{I:|I|\leq|R_0^+|} |I|\eta^{|I|-1} &\leq \sum_{j=0}^{|R_0^+|} \binom{\ell(u)}{j} j\eta^{j-1} \\ &\leq \frac{\ell(u)!}{(\ell(u)-|R_0^+|)!} \sum_{j=0}^{|R_o^+|} \frac{j}{j!} \max\left\{1,\eta^{|R_0^+|-1}\right\} \\ &\leq (\ell(u)+1)^{|R_0^+|}\, 3 \max\left\{1,\eta^{|R_0^+|}\right\} \end{aligned} \tag{5.26}$$

Note that we did not really use that y lies in the subspace $\mathcal{S}(\mathcal{R})$ of $\mathfrak{H}(\mathcal{R})$, it only helps to ensure that all intermediate expressions are well-defined. □

Now we can also estimate the behaviour of the Schwartz norms p_n under multiplication. Put $b' = 2b + |R_0^+| = 2\,\mathrm{rk}(X) + |R_0^+| + 2$.

Proposition 5.7 *Let $n \in \mathbb{N}$, $q, q' \in B_\rho(q^0,\eta)$ and $x_i = \sum_{u\in W} x_{iu} N_u \in \mathcal{S}(\mathcal{R}, q)$. Then*

$$\begin{array}{lcr} p_n(x_1 \cdot_q \cdots \cdot_q x_m) & \leq & \prod_{i=1}^m C_\eta C_b p_{n+b'}(x_i) \\ p_n(x_1 \cdot_q \cdots \cdot_q x_m - x_1 \cdot_{q'} \cdots \cdot_{q'} x_m) & \leq & \rho(q,q') \prod_{i=1}^m C_\eta C_b p_{n+b'}(x_i) \end{array}$$

Proof. This can be deduced with a piece of careful bookkeeping:

$$\begin{array}{lr} p_n(x_1 \cdot_q \cdots \cdot_q x_m) & \leq \\ p_n(\sum_{u_i\in W} x_{1u_1}\cdots x_{mu_m} N_{u_1} \cdot_q \cdots \cdot_q N_{u_m}) & \leq \\ \sum_{u_i\in W} |x_{1u_1}\cdots x_{mu_m}| (\mathcal{N}(u_1)+\cdots+\mathcal{N}(u_m)+1)^n \prod_{i=1}^m \|(N_{u_i},q)\|_o & \leq \\ \sum_{u_i\in W} |x_{1u_1}\cdots x_{mu_m}| \prod_{i=1}^m C_\eta (\mathcal{N}(u_i)+1)^{n+b+|R_0^+|} & = \\ \prod_{i=1}^m C_\eta \sum_{u\in W} |x_{iu}| (\mathcal{N}(u)+1)^{n+b+|R_0^+|} & \leq \\ \prod_{i=1}^m C_\eta C_b p_{n+b'}(x_i) & \end{array}$$

$$\begin{array}{lr} p_n(N_{u_1} \cdot_q \cdots \cdot_q N_{u_m} - N_{u_1} \cdot_{q'} \cdots \cdot_{q'} N_{u_m}) & \leq \\ \sum_{j=1}^{m-1} p_n(N_{u_1} \cdot_q \cdots \cdot_q N_{u_j} \cdot_q N_{u_{j+1}} \cdot_{q'} \cdots \cdot_{q'} N_{u_m} - & \\ \qquad N_{u_1} \cdot_q \cdots \cdot_q N_{u_j} \cdot_{q'} N_{u_{j+1}} \cdot_{q'} \cdots \cdot_{q'} N_{u_m}) & \leq \\ \sum_{j=1}^{m-1} \rho(q,q') \prod_{i=1}^m C_\eta (\mathcal{N}(u_i)+1)^{n+b+|R_0^+|} & \leq \\ \rho(q,q') \prod_{i=1}^m C_\eta (\mathcal{N}(u_i)+1)^{n+b+|R_0^+|} & \end{array}$$

$$\begin{array}{lr} p_n(x_1 \cdot_q \cdots \cdot_q x_m - x_1 \cdot_{q'} \cdots \cdot_{q'} x_m) & \leq \\ \sum_{u_i\in W} |x_{1u_1}\cdots x_{mu_m}| p_n(N_{u_1} \cdot_q \cdots \cdot_q N_{u_m} - N_{u_1} \cdot_{q'} \cdots \cdot_{q'} N_{u_m}) & \leq \\ \sum_{u_i\in W} |x_{1u_1}\cdots x_{mu_m}| \rho(q,q') \prod_{i=1}^m C_\eta (\mathcal{N}(u_i)+1)^{n+b+|R_0^+|} & = \\ \rho(q,q') \prod_{i=1}^m C_\eta \sum_{u\in W} |x_{iu}| (\mathcal{N}(u)+1)^{n+b+|R_0^+|} & \leq \\ \rho(q,q') \prod_{i=1}^m C_\eta p_{n+b'}(x_i) & \end{array}$$

In these calculations we used (5.20) and Proposition 5.6 several times. □

Knowing how to handle multiple products in $\mathcal{S}(\mathcal{R}, q)$, we can even make some rough estimates for the holomorphic functional calculus.

Corollary 5.8 *Let $f : z \to \sum_{m=0}^{\infty} a_m z^m$ be a holomorphic function on a neighborhood of $0 \in \mathbb{C}$ and define another holomorphic function $\tilde{f}$ (with the same radius of convergence) by $\tilde{f}(z) = \sum_{m=0}^{\infty} |a_m| z^m$. For any $n \in \mathbb{N}$, $x \in \mathcal{S}(\mathcal{R}, q)$ and $q, q' \in B_\rho(q^0, \eta)$ such that $f(x, q)$ and $f(x, q')$ are defined we have*

$$\begin{array}{rcl} p_n(f(x,q)) & \leq & \tilde{f}\big(C_\eta C_b p_{n+b'}(x)\big) \\ p_n(f(x,q) - f(x,q')) & \leq & \rho(q,q')\tilde{f}\big(C_\eta C_b p_{n+b'}(x)\big) \end{array}$$

Proof. By Proposition 5.7 we have

$$\begin{aligned} p_n(f(x,q)) &= p_n\left(\sum_{m=0}^{\infty} a_m (x,q)^m\right) \\ &\leq \sum_{m=0}^{\infty} |a_m| p_n((x,q)^m) \\ &\leq \sum_{m=0}^{\infty} |a_m| \big(C_\eta C_b p_{n+b'}(x)\big)^m \\ &= \tilde{f}\big(C_\eta C_b p_{n+b'}(x)\big) \end{aligned}$$

$$\begin{aligned} p_n(f(x,q) - f(x,q')) &= p_n\left(\sum_{m=0}^{\infty} a_m ((x,q)^m - (x,q')^m)\right) \\ &\leq \sum_{m=0}^{\infty} |a_m| p_n((x,q)^m - (x,q')^m) \\ &\leq \sum_{m=0}^{\infty} |a_m| \rho(q,q') \big(C_\eta C_b p_{n+b'}(x)\big)^m \\ &= \rho(q,q')\tilde{f}\big(C_\eta C_b p_{n+b'}(x)\big) \end{aligned}$$

The right hand sides of these inequalities might be infinite, but that is no problem. □

With this result we can show that inverting is continuous as a function of x and q.

Proposition 5.9 *The set of invertible elements $\bigcup_{q \in L_\mathcal{R}} \mathcal{S}(\mathcal{R}, q)^\times$ is open in $\mathcal{S}(\mathcal{R}) \times L_\mathcal{R}$, and inverting is a continuous map from this set to itself.*

Proof. First we recall that if $\|(z,q)\|_o < 1$, then z is invertible in $C_r^*(\mathcal{R},q)$, with inverse
$\sum_{n=0}^\infty (1-z)^n$. Take $q,q' \in B_\rho(q^0,\eta)$, $y \in \mathcal{S}(\mathcal{R})$, $x \in \mathcal{S}(\mathcal{R},q)^\times$ and write $a = (x,q)^{-1}$. If the sum converges, then

$$a \cdot_{q'} \sum_{m=1}^\infty (1-(x+y)\cdot_{q'} a, q')^m = a \cdot_{q'} ((x+y)\cdot_{q'} a, q')^{-1} - a \cdot_{q'} 1 = (x+y,q')^{-1} - a \quad (5.27)$$

By Proposition 5.7

$$\begin{aligned} p_n((x+y)\cdot_{q'} a - 1) &\leq p_n(x \cdot_{q'} a - x \cdot a) + p_n(y \cdot_{q'} a) \\ &\leq \rho(q,q') C_\eta^2 C_b^2 p_{n+b'}(x) p_{n+b'}(a) + C_\eta^2 C_b^2 p_{n+b'}(y) p_{n+b'}(a) \quad (5.28) \end{aligned}$$

Let U be the open neighborhood of (x,q) consisting of those $(x+y,q') \in \mathcal{S}(\mathcal{R}) \times B_\rho(q^0,\eta)$ for which

$$\begin{aligned} \rho(q,q') C_\eta^3 C_b^2 p_{3b+|R_0|}(x) p_{3b+|R_0|}(a) &< 1/2 \\ C_\eta^3 C_b^2 p_{3b+|R_0|}(y) p_{3b+|R_0|}(a) &< 1/2 \end{aligned}$$

By (5.28) and Proposition 5.6 we have

$$\|((x+y)\cdot_{q'} a - 1, q')\|_o < 1 \qquad \forall (x+y,q') \in U$$

so every element of U is invertible. To show the continuity of inverting we consider the function

$$f(z) = \sum_{m=1}^\infty z^m = z/(1-z)$$

By (5.27) and Corollary 5.8 we have

$$\begin{aligned} p_n((x+y,q')^{-1} - a) &\leq C_b^2 C_\eta^2 p_{n+b'}(a) p_{n+b'}\big(f(1-(x+y)\cdot_{q'} a, q')\big) \\ &\leq C_b^2 C_\eta^2 p_{n+b'}(a) f\big(C_b C_\eta p_{n+2b'}(1-(x+y)\cdot_{q'} a)\big) \end{aligned}$$

Since $f(0) = 0$ we deduce from (5.28) that this expression is small whenever $\rho(q,q')$ and y are small. □

With this result we can see that the holomorphic functional calculus is continuous in the most general sense.

Corollary 5.10 *Let $V \subset \mathcal{S}(\mathcal{R})$, $Q \subset L_\mathcal{R}$ and $U \subset \mathbb{C}$ be open subsets such that the spectrum of every $(x,q) \in V \times Q$ is contained in U. Then the map*

$$C^{an}(U) \times V \times Q \to \mathcal{S}(\mathcal{R}) : (f,x,q) \to f(x,q)$$

is continuous.

Proof. Recall from Theorem 2.9.4 that

$$f(x,q) = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - x,q)^{-1} d\lambda$$

for a suitable contour $\Gamma \subset U$ around $\mathrm{sp}(x,q)$. By Corollary 3.27.2 $\mathrm{sp}(x,q)$ is compact, and by Proposition 5.9 it depends continuously on x and q. Therefore we can find a contour which is suitable for every (x',q') in a small neighborhood of (x,q). Now apply Proposition 5.9 and Theorem 2.9.3. □

Let $\mathcal{H}(\mathcal{R})^*$ be the algebraic dual of $\mathcal{H}(\mathcal{R})$, which we identify, using the bitrace τ, with the space of all formal infinite sums $\sum_{w\in W} x_w N_w$. The length function $\mathcal{N}$ may also be considered as an endomorphism of $\mathcal{H}(\mathcal{R})^*$:

$$\lambda(\mathcal{N}) : \sum_{w\in W} x_w N_w \to \sum_{w\in W} \mathcal{N}(w) x_w N_w \tag{5.29}$$

This is an unbounded operator on $\mathfrak{H}(\mathcal{R})$, but it does restrict to a continuous endomorphism of $\mathcal{S}(\mathcal{R})$. For $T \in B(\mathfrak{H}(\mathcal{R}))$ put $D(T)= [\lambda(\mathcal{N}),T]$. Inspired by the work of Vignéras [130, Section 7] we study the space

$$V_{\mathcal{N}}^\infty(\mathcal{R},q) = \{x \in \mathcal{H}(\mathcal{R})^* : D^n(\lambda(x)) \in B(\mathfrak{H}(\mathcal{R}))\ \forall n \in \mathbb{Z}_{\geq 0}\} \tag{5.30}$$

We use the topology defined by the collection of seminorms

$$\left\{ \|D^n(\lambda(\,\cdot\,))\|_{B(\mathfrak{H}(\mathcal{R}))},\ n \in \mathbb{Z}_{\geq 0} \right\} \tag{5.31}$$

In fact we already know this space:

Lemma 5.11

$$V_{\mathcal{N}}^\infty(\mathcal{R},q) = \mathcal{S}(\mathcal{R},q)$$

Proof. From the proof of Proposition 5.6 we see that for any $y = \sum_{v\in W} y_v N_v \in \mathfrak{H}(\mathcal{R})$, $n \in \mathbb{Z}_{\geq 0}$, $u \in W$

$$\begin{aligned}
\|D^n(\lambda(N_u)y\|_\tau &= \left\| \sum_v y_v \sum_{i=0}^n (-1)^i \binom{n}{i} \lambda(\mathcal{N})^{n-i}\lambda(N_u)\lambda(\mathcal{N})^i N_v \right\|_\tau \\
&= \left\| \sum_v y_v \sum_{i=0}^n (-1)^i \binom{n}{i} \sum_I \eta_I D_v^u(I)\mathcal{N}(u_I v)^{n-i}\mathcal{N}(v)^i N_{u_I v} \right\|_\tau \\
&= \left\| \sum_v y_v \sum_I \eta_I D_v^u(I)(\mathcal{N}(u_I v) - \mathcal{N}(v))^n N_{u_I v} \right\|_\tau \\
&\leq \mathcal{N}(u)^n \left\| \sum_v |y_v| \sum_I |\eta_I| D_v^u(I) N_{u_I v} \right\|_\tau \\
&\leq \mathcal{N}(u)^n 3(\mathcal{N}(u)+1)^{|R_o^+|} \max\left\{1, \eta^{|R_0^+|}\right\} \left\| \sum_v |y_v| N_{u_I v} \right\|_\tau \\
&= \mathcal{N}(u)^n 3(\mathcal{N}(u)+1)^{|R_o^+|} \max\left\{1, \eta^{|R_0^+|}\right\} \|y\|_\tau
\end{aligned}$$

where $\eta = \rho(q, q^0)$. Hence, for $x = \sum_u x_u N_u \in \mathcal{H}(\mathcal{R})^*$

$$\begin{aligned} \|D^n(\lambda(x))\|_o &= \left\| \sum_u x_u D^n(\lambda(N_u)) \right\|_o \\ &\leq \sum_u |x_u| 3(\mathcal{N}(u)+1)^{n+|R_o^+|} \max\left\{1, \eta^{|R_0^+|}\right\} \\ &\leq C_\eta p_{n+b+|R_0^+|}(x) \end{aligned}$$

On the other hand, since $\Omega' = \{\omega \in W : \mathcal{N}(\omega) = 0\}$ is finite,

$$\begin{aligned} p_n(x)^2 &\leq \sum_{u \in W} (\mathcal{N}(u)+1)^{2n} |x_u|^2 \\ &\leq \sum_{\omega \in \Omega'} |x_\omega|^2 + 4^n \sum_{u \in W} \mathcal{N}(u)^{2n} |x_u|^2 \\ &\leq |\Omega'| \, \|x\|_\tau^2 + 4^n \, \|\lambda(N)^n x\|_\tau^2 \\ &= |\Omega'| \, \|\lambda(x) N_e\|_\tau^2 + 4^n \, \|D^n(\lambda(x)) N_e\|_\tau^2 \\ &\leq |\Omega'| \, \|\lambda(x)\|_o^2 + 4^n \, \|D^n(\lambda(x))\|_o^2 \\ &\leq \left(|\Omega'|^{1/2} \, \|\lambda(x)\|_o + 2^n \, \|D^n(\lambda(x))\|_o \right)^2 \end{aligned}$$

Therefore the collections of seminorms $\{p_n : n \in \mathbb{Z}_{\geq 0}\}$ and (5.31) are equivalent. □

So we found a different way to construct the Schwartz algebra of an affine Hecke algebra. An advantage of this construction is that it allows us to prove Corollary 3.27 in a more elementary way, relying only on the density of $V_{\mathcal{N}}^\infty(\mathcal{R}, q)$ in $C_r^*(\mathcal{R}, q)$ and not on any representation theory.

Theorem 5.12 *1. $V_{\mathcal{N}}^\infty(\mathcal{R}, q)$ is a complete locally convex algebra with jointly continuous multiplication.*

2. $V_{\mathcal{N}}^\infty(\mathcal{R}, q)^\times$ is open in $V_{\mathcal{N}}^\infty(\mathcal{R}, q)$, and inverting is a continuous map from this set to itself.

3. An element of $V_{\mathcal{N}}^\infty(\mathcal{R}, q)$ is invertible if and only if it is invertible in $C_r^*(\mathcal{R}, q)$.

Proof. 1. By Lemma 5.11 $V_{\mathcal{N}}^\infty(\mathcal{R}, q)$ is a Fréchet space. Since D is a derivation, it is also a topological algebra with jointly continuous multiplication.
2. See [130, Lemma 16]. Suppose that $x \in V_{\mathcal{N}}^\infty(\mathcal{R}, q)$ and $\|x\|_o < 1$. Then $1 - x \in C_r^*(\mathcal{R}, q)^\times$ and $(1-x) = \sum_{n=0}^\infty x^n \in C_r^*(\mathcal{R}, q)$. We have to show that this sum converges in $V_{\mathcal{N}}^\infty(\mathcal{R}, q)$. For $n, r \in \mathbb{N}$

$$D^r(\lambda(x)) = \sum_{r_1 + \cdots + r_n = r} \frac{r! D^{r_1}(\lambda(x)) \cdots D^{r_n}(\lambda(x))}{r_1! \cdots r_n!}$$

Every product $D^{r_1}(\lambda(x))\cdots D^{r_n}(\lambda(x))$ contains at least $n-r$ factors $\lambda(x)$ and at most r factors of the form $D^i(\lambda(x))$ with $i>0$. Therefore

$$\|D^r(\lambda(x))\|_{B(\mathfrak{H}(\mathcal{R}))} \le n^r M^r \|x\|_o^n$$
$$M = \max\left\{\left\|D^i(\lambda(x))\right\|_{B(\mathfrak{H}(\mathcal{R}))} \|x\|_o^{-1} : 0 < i \le r\right\}$$

This gives

$$\left\|D^r(\lambda(1-x)^{-1})\right\|_{B(\mathfrak{H}(\mathcal{R}))} \le \sum_{n=0}^{\infty} n^r M^r \|x\|_o^n \le r! M^r (1-\|x\|_o)^{-r-1}$$

from which we conclude that indeed $(1-x)^{-1} \in V^\infty_{\mathcal{N}}(\mathcal{R},q)$ and that inverting in $V^\infty_{\mathcal{N}}(\mathcal{R},q)$ is continuous around 1. In general, if $y \in V^\infty_{\mathcal{N}}(\mathcal{R},q)^\times$ then we can use the "translation" $\lambda(y^{-1})$ to show that $V^\infty_{\mathcal{N}}(\mathcal{R},q)^\times$ contains an open neighborhood of y and that inverting is continuous on this set.
3. Suppose that $z \in V^\infty_{\mathcal{N}}(\mathcal{R},q) \cap C^*_r(\mathcal{R},q)^\times$. By Lemma 5.11 $V^\infty_{\mathcal{N}}(\mathcal{R},q)$ is dense in $C^*_r(\mathcal{R},q)$, so we can find $y \in z^{-1}B \cap Bz^{-1}$, where

$$B = \{x \in C^*_r(\mathcal{R},q) : \|1-x\|_o < 1\}$$

By the above

$$yz, zy \in B \cap V^\infty_{\mathcal{N}}(\mathcal{R},q) \subset V^\infty_{\mathcal{N}}(\mathcal{R},q)^\times$$

so z is also invertible in $V^\infty_{\mathcal{N}}(\mathcal{R},q)$. □

Note that it does not follow from these considerations that $V^\infty_{\mathcal{N}}(\mathcal{R},q)$ is a m-algebra. To prove that we still have to use Theorem 3.25.

Consider the bundle of Banach spaces $\bigsqcup_{q\in L_\mathcal{R}} C^*_r(\mathcal{R},q)$ over $L_\mathcal{R}$. For any fixed $x \in \mathcal{S}(\mathcal{R})$ the constant function $q \to x$ is a section this bundle, and by Proposition 5.6 the function $q \to \|(x,q)\|_o$ is continuous on $L_\mathcal{R}$. So by [41, Proposition 10.2.3] there is a unique collection Γ of sections of $\bigsqcup_{q\in L_\mathcal{R}} C^*_r(\mathcal{R},q)$ containing all these constant sections, which makes this into a field of C^*-algebras, in the sense of Dixmier [41, Section 10.3]. By construction Γ contains all continuous maps $L_\mathcal{R} \to \mathcal{S}(\mathcal{R})$. In particular for any compact set $Q \subset \mathcal{L}_\mathcal{R}$ we can construct the unital C^*-algebra $C^*_r(\mathcal{R},Q) = \Gamma\big|_Q$, which contains $C(Q;\mathcal{S}(\mathcal{R}))$ as a dense subalgebra.

For a related construction, assume that Q is a smooth submanifold of $L_\mathcal{R}$, not necessarily compact. Although the author is not aware of any precise definition, he believes it makes sense to call $\bigsqcup_{q\in L_\mathcal{R}} \mathcal{S}(\mathcal{R},q)$ a field of Fréchet algebras. By Proposition 5.7 the set of smooth sections

$$\mathcal{S}(\mathcal{R},Q) = C^\infty(Q;\mathcal{S}(\mathcal{R})) \tag{5.32}$$

is a Fréchet space and a topological algebra with jointly continuous multiplication. However, the author does not know whether $\mathcal{S}(\mathcal{R},Q)$ is a m-algebra in general. For every $q \in Q$ we can obtain submultiplicative seminorms on $\mathcal{S}(\mathcal{R},q)$ from the

Fourier transform, but these are not easily expressible in terms of the elements N_w. Therefore it is not clear whether we can choose them in some sense continuous in q and "glue" such seminorms to a family of seminorms that defines the topology of $\mathcal{S}(\mathcal{R}, Q)$.

Seminorms that we can construct are related to the principal series representations, which exist for every $q \in L_{\mathcal{R}}$. From Theorem 3.3.1 we get a linear bijection

$$\mathcal{H}(\mathcal{R}, q) \to \mathcal{H}(W_0, q) \otimes \mathbb{C}[X]$$

This can be extended to a continuous map

$$\phi_{\theta,q} : \mathcal{S}(\mathcal{R}, q) \to \mathcal{H}(W_0, q) \otimes \mathcal{S}(X) \tag{5.33}$$

which is essentially the direct integral of all unitary principal series representations. In general $\phi_{\theta,q}$ is neither injective nor surjective. For $n \in \mathbb{N}$ we define the following norm on $\mathcal{H}(W_0, q) \otimes \mathcal{S}(X)$:

$$\sigma_n \left(\sum_{w \in W_0, x \in X} y_{w,x} N_w \theta_x \right) = \sum_{w \in W_0, x \in X} (\mathcal{N}(x) + 1)^n \tag{5.34}$$

We compose it with $\phi_{\theta,q}$ to get the seminorm $\sigma_{n,q} = \sigma_n \circ \phi_{\theta,q}$ on $\mathcal{S}(\mathcal{R}, q)$. These seminorms are continuous in q:

Lemma 5.13 *Let $n \in \mathbb{N}$, $\eta > 0$ and $q, q' \in B_\rho(q^0, \eta)$. There exists a real number $C_{n,\eta}$ such that for all $z \in \mathcal{S}(\mathcal{R})$*

$$\begin{array}{rcl} \sigma_{n,q}(z) & \leq & C_{n,\eta} p_{n+b+|R_0^+|}(z) \\ |\sigma_{n,q}(z) - \sigma_{n,q'}(z)| & \leq & \rho(q, q') C_{n,\eta} p_{n+b+|R_0^+|}(z) \end{array}$$

Proof. Let $x^+ \in X^+$ and put

$$P = \{\alpha \in F_0 : \alpha^\vee(x^+) = 0\}$$

By [61, Proposition 1.15] we have

$$W_P = \{w \in W_0 : wx^+ = x^+\}$$

and hence

$$W_0 x^+ W_0 = W^P x^+ W_0 = W_0 x^+ \left(W^P\right)^{-1}$$

From (3.41) we see that the N_u with $u \in W_P$ commute with $\theta_{x^+} \in \mathcal{H}(\mathcal{R}, q)$. Pick any $w \in W_0 x^+ W_0$. From Theorem 5.5 we see that there is a unique way to write

$$N_w = \sum_{u \in W_0, v \in W^P} c^w_{u,v} N_u \theta_{x^+} N_{v^{-1}} \tag{5.35}$$

where every $c^w_{u,v}$ is a polynomial in the η_i of degree at most $|R_0|$. From the length formula in Proposition 3.1 we see that the $c^w_{u,v}$ depend only on P, in the following sense. If $w = w_1 x^+ w_2$ $(w_1, w_2 \in W_0)$ and $w' = w_1 x' w_2$ where $x' \in X^+$ and $\{\alpha \in F_0 : \alpha^\vee(x') = 0\} = P$, then $c^{w'}_{u,v} = c^w_{u,v}$. In particular there are only finitely many different $c^w_{u,v}$, less than $|W_0|^2 2^{|F_0|}$.

Assume for simplicity that $\eta \geq 1$, so

$$\left\| (N_v, q) \right\|_o \leq (2\eta)^{|R_0^+|} \quad \forall v \in W_0$$

Let K be an upper bound for the absolute values of all $c^w_{u,v}$, also under the condition $q \in B_\rho(q^0, \eta)$. From a repeated application of (3.41) we see that $\theta_{x^+} N_{v^{-1}}$ equals a sum of at most $(2\mathcal{N}(x^+) + 2)^{|R_0^+|}$ terms of the form $\eta_I N_{v'} \theta_x$, where $\mathcal{N}(x) \leq \mathcal{N}(x^+)$, $v' \in W_0$ and I is a multi-index with $|I| \leq |R_0^+|$. This leads to the estimate

$$\begin{aligned}
\sigma_{n,q}(N_w) &= \sigma_n \Big(\sum_{u,v} c^w_{u,v} N_u \sum_{I,v',x} \eta_I N_{v'} \theta_x \Big) \\
&\leq \sum_v |W_0|^{1/2} \Big\| \sum_u c^w_{u,v} N_u \Big\|_\tau \sum_{I,v',x} \left\| (N_{v'}, q) \right\|_o |\eta_I| (\mathcal{N}(x) + 1)^n \\
&\leq |W_0| K (2\eta)^{|R_0^+|} \eta^{|R_o^+|} (2\mathcal{N}(x^+) + 2)^{|R_0^+|} (\mathcal{N}(x^+) + 1)^n \\
&= |W_0| K (2\eta)^{|R_0|} (\mathcal{N}(x^+) + 1)^{n + |R_0^+|} \\
&\leq |W_0| K (2\eta)^{|R_0|} (|R_0^+| + 1)^{n + |R_0^+|} (\mathcal{N}(w) + 1)^{n + |R_0^+|}
\end{aligned}$$

For $z = \sum_{w \in W} z_w N_w \in \mathcal{S}(\mathcal{R})$ we obtain

$$\begin{aligned}
\sigma_{n,q}(z) &\leq \sum_{w \in W} |z_w| \sigma_{n,q}(N_w) \\
&\leq |W_0| K (2\eta)^{|R_0|} (|R_0^+| + 1)^{n + |R_0^+|} \sum_{w \in W} |z_w| (\mathcal{N}(w) + 1)^{n + |R_0^+|} \\
&\leq |W_0| K (2\eta)^{|R_0|} (|R_0^+| + 1)^{n + |R_0^+|} C_b p_{n + b + |R_0^+|}(z)
\end{aligned}$$

Plugging the description of $\theta_{x^+} N_{v^{-1}}$ into (5.35) we see that

$$N_w = \sum_{u \in W_0, x \in X} y^w_{u,x} N_u \theta_x \tag{5.36}$$

where the $y^w_{u,x}$ are polynomials in the η_i of degree at most $2|R_0|$. Therefore we can write

$$\phi_{\theta,q}(z) = \sum_{I : |I| \leq 2|R_0|} \sum_{u \in W_0, x \in X} \eta_I z_{I,u,x} N_u \theta_x := \sum_I \eta_I z_I \tag{5.37}$$

Since the finite collection $\{\eta_I : |I| \leq 2|R_0|\}$ is linearly independent, considered as functions of the η_i, we can find constants K_n such that

$$\sigma_n(z_I) \leq K_n p_{n + b + |R_0^+|}(z) \quad \forall I$$

Hence

$$
\begin{aligned}
|\sigma_{n,q}(z) - \sigma_{n,q'}(z)| &\leq \sum_{I : |I| \leq 2|R_0|} |\eta_I - \eta'_I| \sigma_n(z_I) \\
&\leq \rho(q,q') 2 |R_0| \eta^{2|R_0|} K_n p_{n+b+|R_0^+|}(z)
\end{aligned}
$$

To finish the proof we take for $C_{n,\eta}$ the maximum of
$|W_0| K(2\eta)^{|R_0|} (|R_0^+| + 1)^{n+|R_0^+|} C_b$ and $2|R_0|\eta^{2|R_0|} K_n$. □

We conclude this section with an important remark. All the estimates obtained here can be generalized to $M_k(\mathcal{S}(\mathcal{R},q))$ for any $k \in \mathbb{N}$. Of course we first have to (re)define

$$
p_n^{(k)}\big((z_{i,j})_{i,j=1}^k \big) = \max_{1 \leq i,j \leq k} p_n(z_{i,j}) \tag{5.38}
$$

but then all our results can be extended by standard techniques. In the coming sections we assume that this has been done, and we attach a superscript (k) to the modified constants.

5.3 Scaling the labels

Fix a root datum $\mathcal{R}$ and a label function q. Instead of considering general deformations of q, we concentrate on the scaled label functions q^ϵ for $\epsilon \in [-1, 1]$. Opdam [98, Section 5] was the first to realize that a great deal of the representation theory of $\mathcal{H}(\mathcal{R}, q^\epsilon)$ can be "scaled" accordingly. We will construct isomorphisms $\mathcal{S}(\mathcal{R}, q^\epsilon) \to \mathcal{S}(R, q)$ for $\epsilon > 0$ and an injection $\mathcal{S}(W) = S(\mathcal{R}, q^0) \to \mathcal{S}(\mathcal{R}, q)$, all depending continuously on ϵ. This requires a lot of long calculations, which rely on the technical parts of Chapter 3 and Section 5.2.

Recall from [98, Section 6.3] that we always have a canonical isomorphism

$$
\mathcal{H}(\mathcal{R}, q) \xrightarrow{\sim} \mathcal{H}(\mathcal{R}, q^{-1}) : N_w \to (-1)^{\ell(w)} N_w \tag{5.39}
$$

This map preserves * and τ, and it extends to the Schwartz and C^*-completions. However, our scaling maps will not lead to this map for $\epsilon = -1$.

To facilitate the study of $\mathcal{S}(\mathcal{R}, q)$ we want to regard it not only as an algebra of invariant sections (via the Fourier transform), but also as the image of a projector in a larger algebra. Let $\mathcal{P}'(F_0)$ be a complete set of representatives for the action of $\mathcal{W}$ on the power set of F_0. By Theorem 3.25 there are canonical direct sum decompositions

$$
\begin{array}{lllll}
\mathcal{S}(\mathcal{R}, q) & = & \bigoplus_{P \in \mathcal{P}'(F_0)} \mathcal{S}(\mathcal{R}, q)_P & = & \bigoplus_{P \in \mathcal{P}'(F_0)} \mathcal{S}(\mathcal{R}, q)\, e_P \\
C_r^*(\mathcal{R}, q) & = & \bigoplus_{P \in \mathcal{P}'(F_0)} C_r^*(\mathcal{R}, q)_P & = & \bigoplus_{P \in \mathcal{P}'(F_0)} C_r^*(\mathcal{R}, q)\, e_P
\end{array} \tag{5.40}
$$

where the e_P are central idempotents in $\mathcal{S}(\mathcal{R},q)$. But this can be refined. Let Δ' be a collection of representatives for the action of $\mathcal{W}$ on Δ. For $(P,\delta)\in\Delta$, we let $\mathcal{W}_\delta$ be the isotropy group of (P,δ) in $\mathcal{W}$. The Fourier transform gives an isomorphism

$$\mathcal{F}' : \mathcal{S}(\mathcal{R},q) \xrightarrow{\sim} \bigoplus_{(P,\delta)\in\Delta'} C^\infty\left(T_u^P; \mathrm{End}\big(\mathcal{H}(W^P)\otimes V_\delta\big)\right)^{\mathcal{W}_\delta} := \bigoplus_{(P,\delta)\in\Delta'} A_\delta^{\mathcal{W}_\delta} \quad (5.41)$$

We must be careful when taking invariants, since

$$\mathcal{W}_\delta \to A_\delta^\times : g \to \pi(g,P,\delta,\cdot) \quad (5.42)$$

is not necessarily a group homomorphism. In fact, by (3.129) it is a projective representation. By Schur's theorem [113] there exists a finite central extension

$$\{e\} \to N_\delta \to \Gamma_\delta \to \mathcal{W}_\delta \to \{e\} \quad (5.43)$$

such that every projective representation of $\mathcal{W}_\delta$ lifts to a unique linear representation of Γ_δ. This lift does not depend on the $\tilde\delta_g$ that we chose in (3.112) to construct $\pi(g,\xi)$ for $\xi=(P,\delta,t)$. In fact, the problems with (5.42) arise only from the ambiguity in the definition of $\tilde\delta_g$. Lifting things to a linear representation of Γ_δ is therefore equivalent to picking, for every lift $\gamma\in\Gamma_\delta$ of $g\in\mathcal{W}_\delta$, a multiple $\tilde\delta_\gamma$ of $\tilde\delta_g$ such that

$$\Gamma_\delta \to A_\delta^\times : \gamma \to \pi(\gamma,P,\delta,\cdot)$$

becomes multiplicative. Writing

$$u_\gamma(\xi) = \pi(\gamma,\gamma^{-1}\xi)$$

(3.129) becomes the cocycle relation

$$u_{\gamma\gamma'} = u_\gamma u_{\gamma'}^\gamma \quad (5.44)$$

Notice the similarity with the proof of Lemma 2.26. Consider the crossed product

$$A_\delta \rtimes \Gamma_\delta \cong \mathrm{End}\big(\mathcal{H}(W^P)\otimes V_\delta\big)\otimes C^\infty\big(T_u^P\big)\rtimes\Gamma_\delta \quad (5.45)$$

with respect to the action of Γ_δ on T_u^P. Lemma A.1 gives some information about this algebra. We still need to determine $A_\delta^{\mathcal{W}_\delta} = A_\delta^{\Gamma_\delta}$, but using the multiplication in $A_\delta\rtimes\Gamma_\delta$ we can write the group action as

$$\gamma(a) = u_\gamma\gamma a\gamma^{-1}u_\gamma^{-1} \quad (5.46)$$

Moreover by (5.44)

$$\Gamma_\delta \to \big(A_\delta\rtimes\Gamma_\delta\big)^\times : \gamma \to u_\gamma\gamma$$

is a unitary representation, so

$$p_\delta(u) := |\Gamma_\delta|^{-1}\sum_{\gamma\in\Gamma_\delta} u_\gamma\gamma \in A_\delta\rtimes\Gamma_\delta \quad (5.47)$$

is a projection. By Lemma A.2 the map

$$A_\delta^{\Gamma_\delta} \to p_\delta(u)\big(A_\delta \rtimes \Gamma_\delta\big)p_\delta(u) : a \to p_\delta(u) a p_\delta(u) \tag{5.48}$$

is an isomorphism of pre-C^*-algebras.

Consider for $\epsilon \in [-1,1]$ the affine Hecke algebras $\mathcal{H}_\epsilon = \mathcal{H}(\mathcal{R}, q^\epsilon)$ with label functions $q^\epsilon(w) = q(w)^\epsilon$. Let $c_{\alpha,\epsilon}$ and $\imath^o_{w,\epsilon}$ be the c-functions and normalized intertwiners for these algebras. From the formula (3.101) we see that the residual cosets are also related by scaling in suitable directions. Their tempered forms all approach T_u when $\epsilon \to 0$. iowe@$\imath^o_{w,\epsilon}$

Take $r \in T$ and write $r = r_u \exp(r_s)$ with $r_u \in T_u$ and $r_s \in \mathfrak{t}_{rs}$. Let $B \subset \mathrm{Lie}(T)$ be a ball satisfying Conditions 3.21. Then ϵB satisfies these conditions with respect to $r_u \exp(\epsilon r_s)$ and q^ϵ, except that it is not open for $\epsilon = 0$. (Here we use $|\epsilon| \leq 1$.) We put

$$U_\epsilon = W_0\big(r_u \exp(\epsilon(r_s + B))\big)$$

and we define a W_0-equivariant scaling map

$$\begin{aligned} &\sigma_\epsilon : U \to U_\epsilon \\ &\sigma_\epsilon\big(w(r\exp(b))\big) = w\big(r_u \exp(\epsilon(r_s + b))\big) \end{aligned} \tag{5.49}$$

Assume now that $0 \neq \epsilon \in [-1,1]$. As was noted in [98, Lemma 5.1], σ_ϵ is an analytic diffeomorphism. We can combine it with (3.117) and Theorem 3.22 to construct algebra isomorphisms

$$\begin{aligned} &\rho_\epsilon : \mathcal{H}_\epsilon^{me}(U_\epsilon) \xrightarrow{\sim} \mathcal{H}^{me}(U) \\ &\rho_\epsilon\left(\sum_{w\in W_0} a_w \imath^o_{w,\epsilon}\right) = \sum_{w\in W_0} (a_w \circ \sigma_\epsilon)\imath^o_w \qquad a_w \in C^{me}(U_\epsilon) \end{aligned} \tag{5.50}$$

We intend to show that these maps depend analytically on ϵ and have a well-defined limit as $\epsilon \to 0$. Notice that $\sigma_0 = \lim_{\epsilon\to 0}\sigma_\epsilon$ is a locally constant map with range $W_0 r_u$.

Lemma 5.14 *For $\epsilon \neq 0$ and $\alpha \in R_1$ write $d_{\alpha,\epsilon} = (c_{\alpha,\epsilon} \circ \sigma_\epsilon)c_\alpha^{-1}$. This defines a bounded invertible analytic function of u and ϵ which extends to a function on $\overline{U} \times [-1,1]$ with the same properties.*

Proof. This is an extended version of [98, Lemma 5.2]. Let us write

$$\begin{aligned} d_{\alpha,\epsilon}(u) \quad = \quad \frac{f_1 f_2 f_3 f_4}{g_1 g_2 g_3 g_4}(u) \quad = \quad & \frac{1+\theta_{-\alpha/2}(u)}{1+\theta_{-\alpha/2}(\sigma_\epsilon(u))} \times \\ \frac{1+q_{\alpha^\vee}^{-\epsilon/2}\theta_{-\alpha/2}(\sigma_\epsilon(u))}{1+q_{\alpha^\vee}^{-1/2}\theta_{-\alpha/2}(u)} & \frac{1-\theta_{-\alpha/2}(u)}{1-\theta_{-\alpha/2}(\sigma_\epsilon(u))} \frac{1-q_{\alpha^\vee}^{-\epsilon/2}q_{2\alpha^\vee}^{-\epsilon}\theta_{-\alpha/2}(\sigma_\epsilon(u))}{1-q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}\theta_{-\alpha/2}(u)} \end{aligned}$$

We see that $d_{\alpha,\epsilon}(u)$ extends to an invertible analytic function on $\overline{U} \times [-1,1]$ if none of the quotients f_n/g_n has a zero or a pole on this domain. By Condition

3.21.2 there is a unique $b \in w(r_s + \overline{B})/2$ such that $u = w(r_u)\exp(2b)$. This forms a coordinate system on $w(r\exp(\overline{B}))$, and $\sigma_\epsilon(u) = w(r_u)\exp(2b\epsilon)$. By Condition 3.21.4 if either $f_n(u) = 0$ or $g_n(u) = 0$ for some $u \in w(r\exp(\overline{B})) \subset \overline{U}$, then $f_n(w(r)) = g_n(w(r)) = 0$. One can easily check that in this situation

$$\frac{f_n(u)}{g_n(u)} = \left(\frac{1-e^{-\alpha(b)\epsilon}}{1-e^{-\alpha(b)}}\right)^{(-1)^n}$$

Again by Condition 3.21.2 the only critical points of this function are those for which $\alpha(b) = 0$. If $\epsilon \neq 0$ then both the numerator and the denominator have a zero of order 1 at such points, so the singularity is removable. For the case $\epsilon = 0$ we need to have a closer look. In our new coordinate system we can write

$$\begin{aligned} c_{\alpha,\epsilon}(\sigma_\epsilon(u)) &= \frac{f_2(u)f_4(u)}{g_1(u)g_3(u)} \\ &= \frac{r_u(w^{-1}\alpha/2) + \left(q_{\alpha^\vee}^{-1/2}e^{-\alpha(b)}\right)^\epsilon}{r_u(w^{-1}\alpha/2) + e^{-\alpha(b)\epsilon}} \frac{r_u(w^{-1}\alpha/2) - \left(q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}e^{-\alpha(b)}\right)^\epsilon}{r_u(w^{-1}\alpha/2) - e^{-\alpha(b)\epsilon}} \end{aligned}$$

Standard calculations using L'Hospital's rule show that

$$\lim_{\epsilon\to 0} c_{\alpha,\epsilon}(\sigma_\epsilon(u)) = \begin{cases} 1 & \text{if} \quad r_u(w^{-1}\alpha) \neq 1 \\ \dfrac{\alpha(b) + \log(q_{\alpha^\vee})/2}{\alpha(b)} & \text{if} \quad r_u(w^{-1}\alpha/2) = -1 \\ \dfrac{\alpha(b) + \log(q_{2\alpha^\vee}) + \log(q_{\alpha^\vee})/2}{\alpha(b)} & \text{if} \quad r_u(w^{-1}\alpha/2) = 1 \end{cases}$$

Thus at least $d_{\alpha,0} = \lim_{\epsilon\to 0} d_{\alpha,\epsilon}$ exists as a meromorphic function on $\overline{U}$. For $r_u(w^{-1}\alpha) \neq 1$,
$d_{\alpha,0} = c_\alpha^{-1}$ is invertible by Condition 3.21.4. For $r_u(w^{-1}\alpha/2) = -1$ we have

$$d_{\alpha,0}(u) = \frac{1-e^{-\alpha(b)}}{\alpha(b)} \frac{\alpha(b) + \log(q_{\alpha^\vee})/2}{1 - q_{\alpha^\vee}^{-1/2}e^{-\alpha(b)}} \frac{1+e^{-\alpha(b)}}{1 + q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}e^{-\alpha(b)}}$$

while for $r_u(w^{-1}\alpha/2) = 1$

$$d_{\alpha,0}(u) = \frac{1-e^{-\alpha(b)}}{\alpha(b)} \frac{1+e^{-\alpha(b)}}{1 + q_{\alpha^\vee}^{-1/2}e^{-\alpha(b)}} \frac{\alpha(b) + \log(q_{2\alpha^\vee}) + \log(q_{\alpha^\vee})/2}{1 - q_{\alpha^\vee}^{-1/2}q_{2\alpha^\vee}^{-1}e^{-\alpha(b)}}$$

These expression define invertible functions by Condition 3.21.2. We conclude that indeed $d_{\alpha,\epsilon}(u)$ and $d_{\alpha,\epsilon}^{-1}(u)$ are analytic functions on $\overline{U} \times [-1,1]$. Since this domain is compact, they are bounded. □

Note that $d_{\alpha,1} = 1$ and that

$$d_{\alpha,-1}(u) = \frac{r_u(w^{-1}\alpha) - e^{-2\alpha(b)}}{r_u(w^{-1}\alpha) - e^{2\alpha(b)}} \frac{r_u(w^{-1}\alpha/2) + q_{\alpha^\vee}^{1/2} e^{\alpha(b)}}{r_u(w^{-1}\alpha/2) + q_{\alpha^\vee}^{-1/2} e^{-\alpha(b)}} \times \frac{r_u(w^{-1}\alpha/2) - q_{\alpha^\vee}^{1/2} q_{2\alpha^\vee} e^{\alpha(b)}}{r_u(w^{-1}\alpha/2) - q_{\alpha^\vee}^{-1/2} q_{2\alpha^\vee}^{-1} e^{-\alpha(b)}} \tag{5.51}$$

If either $r_u(w^{-1}\alpha) = \theta_{-\alpha}(w(r_u)) = 1$ or $q_{\alpha^\vee} = q_{2\alpha^\vee} = 1$ this simplifies considerably to

$$d_{\alpha,-1}(u) = q_{\alpha^\vee} q_{2\alpha^\vee} \tag{5.52}$$

We can use Lemma 5.14 to show that the maps ρ_ϵ preserve analyticity:

Lemma 5.15 *The isomorphisms* (5.50) *restrict to isomorphisms*

$$\rho_\epsilon : \mathcal{H}_\epsilon^{an}(U_\epsilon) \xrightarrow{\sim} \mathcal{H}^{an}(U)$$

These maps have a well-defined limit homomorphism

$$\rho_0 = \lim_{\epsilon \to 0} \rho_\epsilon : \mathbb{C}[W] \to \mathcal{H}^{an}(U)$$

such that for every $w \in W$ the function

$$[-1,1] \to \mathcal{H}^{an}(U) : \epsilon \to \rho_\epsilon(N_w)$$

is analytic.

Proof. The first statement is [98, Theorem 5.3], but for the remainder we need to prove this anyway. It is clear that ρ_ϵ restricts to an isomorphism between $C^{an}(U_\epsilon)$ and $C^{an}(U)$. For a simple reflection $s \in S_0$ corresponding to $\alpha \in F_1$ we have

$$\begin{aligned} N_s + q(s)^{-\epsilon/2} &= q^{-\epsilon/2} c_{\alpha,\epsilon}(\imath_{s,\epsilon}^o + 1) \\ \rho_\epsilon(N_s) &= q^{\epsilon/2}(c_{\alpha,\epsilon} \circ \sigma_\epsilon)(\imath_s^o + 1) - q(s)^{-\epsilon/2} \\ &= q(s)^{(\epsilon-1)/2}(c_{\alpha,\epsilon} \circ \sigma_\epsilon) c_\alpha^{-1}\big(N_s + q(s)^{-1/2}\big) - q(s)^{-\epsilon/2} \\ &= q(s)^{(\epsilon-1)/2} d_{\alpha,\epsilon}\big(N_s + q(s)^{-1/2}\big) - q(s)^{-\epsilon/2} \end{aligned} \tag{5.53}$$

By Lemma 5.14 such elements are analytic in $\epsilon \in [-1,1]$ and $u \in U$, so in particular they belong to $\mathcal{H}^{an}(U)$. Moreover, since every $d_{\alpha,\epsilon}$ is invertible, the set $\{\rho_\epsilon(N_w) : w \in W_0\}$ is a basis for $\mathcal{H}^{an}(U)$ as a $C^{an}(U)$-module. Therefore ρ_ϵ restricts to an isomorphism between $\mathcal{H}_\epsilon^{an}(U_\epsilon)$ and $\mathcal{H}^{an}(U)$ for $\epsilon \neq 0$.

For any $x \in X$, $\rho_\epsilon(\theta_x) = \theta_x \circ \sigma_\epsilon$ depends analytically on ϵ, as a function on U. Combined with (5.35) this shows that $\rho_\epsilon(N_w)$ is analytic in $\epsilon \in [-1,1]$ for any $w \in W$. Thus ρ_0 exists as a linear map. But, being a limit of algebra homomorphisms, it must also be multiplicative. □

Note that ρ_0 is never injective or surjective because σ_ϵ is not. Moreover ρ_{-1} is not the isomorphism (5.39). In general (5.51) is not even rational, so $\rho_{-1}(N_s)$ cannot lie in $\mathcal{H}$. In the simple case $r_u(w^{-1}\alpha) = 1$ we have $\rho_{-1}(\theta_\alpha) = \theta_{-\alpha}$ and, by (5.52) and (5.53)

$$\rho_{-1}(N_s) = N_s + q(s)^{-1/2} - q(s)^{1/2} = N_s^{-1}$$

Usually the maps ρ_ϵ do not preserve the *, but this can be fixed. For $\epsilon \in [-1, 1]$ consider the element

$$M_\epsilon = \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \prod_{\alpha \in R_1^+} d_{\alpha,\epsilon} \in \mathcal{H}^{an}(U)$$

We will use M_ϵ to extend [98, Corollary 5.7]. However, this result contained a small mistake: the construction of the element A_ϵ in [98] was unfortunately influenced by an inessential oversight. To correct this we replace it by M_ϵ.

Theorem 5.16 *For all $\epsilon \in [-1, 1]$ M_ϵ is invertible, positive and bounded. It has a positive square root in $\mathcal{H}^{an}(U)$ and the map $\epsilon \to M_\epsilon^{1/2}$ is analytic.*

$$\tilde{\rho}_\epsilon : \mathcal{H}_\epsilon^{an}(U_\epsilon) \to \mathcal{H}^{an}(U)$$
$$\tilde{\rho}_\epsilon(h) = M_\epsilon^{1/2} \rho_\epsilon(h) M_\epsilon^{-1/2}$$

*is a homomorphism of topological *-algebras, and an isomorphism if $\epsilon \neq 0$. For any $w \in W$ the function*

$$[-1, 1] \to \mathcal{H}^{an}(U) : \epsilon \to \tilde{\rho}_\epsilon(N_w)$$

is analytic.

Proof. By Lemmas 5.14 and 5.15 the M_ϵ are invertible, bounded and analytic in ϵ. Consider, for $\epsilon \neq 0$, the automorphism μ_ϵ of $\mathcal{H}^{me}(U)$ given by

$$\mu_\epsilon(h) = \rho_\epsilon(\rho_\epsilon^{-1}(h)^*)^*$$

On one hand, for $f \in C^{me}(U)$ we have by definition (3.118) and the W_0-equivariance of σ_ϵ

$$\begin{aligned} \mu_\epsilon(f) &= \rho_\epsilon((f \circ \sigma_\epsilon)^*)^* \\ &= \rho_\epsilon\big(N_{w_0}(f^{-w_0} \circ \sigma_\epsilon) N_{w_0,\epsilon}^{-1}\big)^* \\ &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* \big(f^{-w_0}\big)^* \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} f N_{w_0}^{-1} \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \prod_{\alpha \in R_1^+} d_{\alpha,\epsilon} f \prod_{\alpha \in R_1^+} d_{\alpha,\epsilon}^{-1} N_{w_0}^{-1} \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= M_\epsilon f M_\epsilon^{-1} \end{aligned} \tag{5.54}$$

On the other hand, suppose that the simple reflections s and $s' = w_0 s w_0 \in S_0$ correspond to α and $\alpha' = -w_0\alpha \in F_1$. Using (3.123) and (3.121) we find

$$\begin{aligned} M_\epsilon \imath_s^o M_\epsilon^{-1} &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \prod_{\alpha\in R_1^+} d_{\alpha,\epsilon} \imath_s^o \prod_{\alpha\in R_1^+} d_{\alpha,\epsilon}^{-1} N_{w_0}^{-1} \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \imath_s^o d_{\alpha,\epsilon}^{-1} d_{-\alpha,\epsilon} N_{w_0}^{-1} \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \imath_s^o \frac{c_{-\alpha}}{c_\alpha} N_{w_0}^{-1} N_{w_0} \frac{c_{\alpha,\epsilon}\circ\sigma_\epsilon}{c_{-\alpha,\epsilon}\circ\sigma_\epsilon} N_{w_0}^{-1} \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* (\imath_{s'}^0)^* \left(\frac{c_{\alpha',\epsilon}\circ\sigma_\epsilon}{c_{-\alpha',\epsilon}\circ\sigma_\epsilon}\right)^* \rho_\epsilon(N_{w_0,\epsilon})^* \\ &= \left(\rho_\epsilon(N_{w_0,\epsilon}) \frac{c_{\alpha',\epsilon}\circ\sigma_\epsilon}{c_{-\alpha',\epsilon}\circ\sigma_\epsilon} \imath_{s'}^0 \rho_\epsilon(N_{w_0,\epsilon}^{-1})\right)^* \\ &= \rho_\epsilon\left(N_{w_0} \frac{c_{\alpha'}}{c_{-\alpha'}} \imath_{s',\epsilon}^o N_{w_0,\epsilon}^{-1}\right)^* \\ &= \rho_\epsilon\left((\imath_{s,\epsilon}^o)^*\right)^* = \mu_\epsilon(\imath_s^o) \end{aligned} \tag{5.55}$$

Since $C^{me}(U)$ and the $\imath_s^0$ generate $\mathcal{H}^{me}(U)$, we conclude that

$$\mu_\epsilon(h) = M_\epsilon h M_\epsilon^{-1} \qquad \forall h \in \mathcal{H}^{me}(U)$$

Now we can see that

$$\begin{aligned} \rho_\epsilon\big(N_{w_0,\epsilon}^{-1}\big)^* &= \rho_\epsilon\big((N_{w_0,\epsilon}^*)^{-1}\big)^* = \rho_\epsilon\big((N_{w_0,\epsilon}^{-1})^*\big)^* \\ &= \mu_\epsilon\big(\rho_\epsilon(N_{w_0,\epsilon}^{-1})\big) = M_\epsilon \rho_\epsilon(N_{w_0,\epsilon}^{-1}) M_\epsilon^{-1} \\ N_e &= M_\epsilon^{-1} \rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \prod_{\alpha\in R_1^+} d_{\alpha,\epsilon} = \rho_\epsilon(N_{w_0,\epsilon}^{-1}) M_\epsilon^{-1} N_{w_0} \textstyle\prod_{\alpha\in R_1^+} d_{\alpha,\epsilon} \\ M_\epsilon &= N_{w_0} \prod_{\alpha\in R_1^+} d_{\alpha,\epsilon} \rho_\epsilon(N_{w_0,\epsilon}^{-1}) = \big(\rho_\epsilon(N_{w_0,\epsilon}^{-1})^* \big(N_{w_0} \prod_{\alpha\in R_1^+} d_{\alpha,\epsilon}\big)^*\big)^* \\ &= \big(\rho_\epsilon(N_{w_0,\epsilon}^{-1})^* N_{w_0} \prod_{\alpha\in R_1^+} d_{\alpha,\epsilon}\big)^* = M_\epsilon^* \end{aligned}$$

Thus the elements M_ϵ are Hermitian $\forall \epsilon \neq 0$. By continuity in ϵ M_0 is also Hermitian. Moreover they are all invertible, and $M_1 = N_e$, so they are in fact strictly positive. We already knew that the element $\epsilon \to M_\epsilon$ of

$$C^{an}([-1,1]; \mathcal{H}^{an}(U)) \cong C^{an}([-1,1]\times U) \otimes_{\mathcal{A}} \mathcal{H}$$

is bounded, so we can construct its square root using holomorphic functional calculus in the Fréchet Q-algebra $C_b^{an}([-1,1]\times U)\otimes_{\mathcal{A}}\mathcal{H}$. This ensures that $\epsilon \to M_\epsilon^{1/2}$ is still analytic. Finally, for $\epsilon \neq 0$

$$\begin{aligned} \tilde\rho_\epsilon(h)^* &= \left(M_\epsilon^{1/2} \rho_\epsilon(h) M_\epsilon^{-1/2}\right)^* \\ &= M_\epsilon^{-1/2} \rho_\epsilon(h)^* M_\epsilon^{1/2} \\ &= M_\epsilon^{-1/2} \mu_\epsilon(\rho_\epsilon(h^*)) M_\epsilon^{1/2} \\ &= M_\epsilon^{1/2} \rho_\epsilon(h^*) M_\epsilon^{-1/2} = \tilde\rho_\epsilon(h^*) \end{aligned} \tag{5.56}$$

Again this extends to $\epsilon = 0$ by continuity. □

Let Rep $(C_r^*(\mathcal{R}, q))$ and Rep $(\mathcal{S}(\mathcal{R}, q))$ be the categories of finite dimensional representations of $C_r^*(\mathcal{R}, q)$ and of $\mathcal{S}(\mathcal{R}, q))$. We define

$$\begin{array}{lclcl} \mathrm{Rep}_U(C_r^*(\mathcal{R}, q)) & = & \mathrm{Rep}\,(C_r^*(\mathcal{R}, q)) & \cap & \mathrm{Rep}_U(\mathcal{H}(\mathcal{R}, q)) \\ \mathrm{Rep}_U(\mathcal{S}(\mathcal{R}, q)) & = & \mathrm{Rep}\,(\mathcal{S}(\mathcal{R}, q)) & \cap & \mathrm{Rep}_U(\mathcal{H}(\mathcal{R}, q)) \end{array} \tag{5.57}$$

Recall that $\overline{\mathcal{H}_\epsilon^t}$ is the residual algebra of $\mathcal{H}_\epsilon$ at $t \in T$, whose (finite dimensional) representations are precisely $\mathrm{Rep}_{W_0 t}(C_r^*(\mathcal{R}, q^\epsilon))$.

Lemma 5.17 *For $\epsilon \in [-1, 1]$ the map $\tilde{\rho}_\epsilon$ induces a "scaling map"*

$$\tilde{\sigma}_\epsilon : \mathrm{Rep}_{W_0 u}(\mathcal{H}) \to \mathrm{Rep}_{W_0 \sigma_\epsilon(u)}(\mathcal{H}_\epsilon)$$

which preserves unitarity and is a bijection if $\epsilon \neq 0$.

*For $\epsilon < 0$ $\tilde{\sigma}_\epsilon$ exchanges tempered and anti-tempered modules. For $\epsilon \geq 0$ $\tilde{\sigma}_\epsilon$ preserves (anti-)temperedness and $\tilde{\rho}_\epsilon$ descends to a *-homomorphism*

$$\overline{\rho_\epsilon} : \overline{\mathcal{H}_\epsilon^{\sigma_\epsilon(u)}} \to \overline{\mathcal{H}^u}$$

which is an isomorphism if $\epsilon > 0$.

Proof. If $\pi \in \mathrm{Rep}(\mathcal{H})$ and $\epsilon \neq 0$ then by construction $t \in T$ is an $\mathcal{A}$-weight of π if and only if $\sigma_\epsilon(t)$ is an $\mathcal{A}_\epsilon$-weight of $\pi \circ \tilde{\rho}_\epsilon$. The $\mathcal{A}_0$-weights of $\pi \circ \tilde{\rho}_0$ are all contained in $W_0 r \subset T_u$. Therefore

$$\tilde{\sigma}_\epsilon : \pi \to \pi \circ \tilde{\rho}_\epsilon \tag{5.58}$$

maps $\mathrm{Rep}_{W_0 u}(\mathcal{H})$ to $\mathrm{Rep}_{W_0 \sigma_\epsilon(u)}(\mathcal{H}_\epsilon)$, for any $u \in U$ and $\epsilon \in [-1, 1]$. Since $\tilde{\rho}_\epsilon$ is a *-homomorphism and a bijection (for $\epsilon \neq 0$) $\tilde{\sigma}_\epsilon$ has the desired properties for such ϵ.

Moreover for $x \in X$ we have $|\sigma_\epsilon(t)(x)| = |t(x)|^\epsilon$, which proves the assertions about temperedness. If $\epsilon \geq 0$ and π extends to $C_r^*(\mathcal{R}, q)$ then it is unitary and tempered by Proposition 5.6. Therefore $\pi \circ \tilde{\rho}_\epsilon$ is tempered and completely reducible, and Corollary 3.26 assures that it extends to $C_r^*(\mathcal{R}, q^\epsilon)$. This implies

$$\tilde{\rho}_\epsilon\big(\mathrm{Rad}_{\sigma_\epsilon(u), \epsilon}\big) \subset \mathrm{Rad}_u$$

so we get a *-homomorphism

$$\overline{\rho_\epsilon} : \mathcal{H}_\epsilon / \mathrm{Rad}_{\sigma_\epsilon(u), \epsilon} = \overline{\mathcal{H}_\epsilon^{\sigma_\epsilon(u)}} \to \mathcal{H}/\mathrm{Rad}_u = \overline{\mathcal{H}^u}$$

If $\epsilon > 0$ then we can follow the same reasoning for $\tilde{\rho}_\epsilon^{-1}$, so $\overline{\rho_\epsilon}$ is an isomorphism. □.

Assume once more that $0 \neq \epsilon \in [-1, 1]$. Then $r \to \sigma_\epsilon(r)$ is a bijection between the residual points for $(\mathcal{R}, q)$ and those for $(\mathcal{R}, q^\epsilon)$. The groupoid of intertwiners $\mathcal{W}$ is independent of q, and it acts on the set of $\mathcal{H}_\epsilon$-representations of the form $\pi(P, \delta', t)$ with δ' irreducible but not necessarily discrete series. The definitions (3.114) and (3.126) also makes sense in this case. If we realize the representation

$$\pi\big(P, \tilde\sigma_\epsilon(\delta), t\big) \quad \text{on} \quad \mathcal{H}\big(W^P\big) \otimes V_\delta \quad \text{as} \quad \mathrm{Ind}_{\mathcal{H}_\epsilon^P}^{\mathcal{H}_\epsilon}\big(\delta \circ \tilde\rho_\epsilon \circ \phi_{t,\epsilon}\big)$$

then we get homomorphisms

$$\begin{aligned} \mathcal{F}'_\epsilon : \mathcal{H}(\mathcal{R}, q^\epsilon) &\to \bigoplus_{(P,\delta)\in\Delta'} \mathcal{O}\big(T^P\big) \otimes \mathrm{End}\big(\mathcal{H}(W^P) \otimes V_\delta\big) \\ \mathcal{F}'_\epsilon(h)(P, \delta, t) &= \pi(P, \tilde\sigma_\epsilon(\delta), t)(h) \end{aligned} \tag{5.59}$$

(Notice that is also defined for $\epsilon = 0$.) The isotropy groups $\mathcal{W}_{\tilde\sigma_\epsilon(\delta)}$ for various ϵ's may be identified, so the image of (5.59) consists of sections that are invariant under a certain action of $\mathcal{W}_\delta$. We add a subscript ϵ to indicate which action we consider.

Recall from (5.43) that Γ_δ is a Schur extension of $\mathcal{W}_\delta$. There are uniquely determined $u_{\gamma,\epsilon} \in A_\delta^\times$ such that, just as in (5.46), we can write the associated Γ_δ-action as

$$\gamma_\epsilon(a) = u_{\gamma,\epsilon} \gamma a \gamma^{-1} u_{\gamma,\epsilon}^{-1} \qquad \text{in } A_\delta \rtimes \Gamma_\delta \tag{5.60}$$

Lemma 5.18 *The elements $u_{\gamma,\epsilon}$ depend analytically on ϵ and $u_{\gamma,0} = \lim_{\epsilon\to 0} u_{\gamma,\epsilon}$ exists. Moreover $u_{\gamma,\epsilon}$ is unitary $\forall \epsilon \in [-1, 1]$, $\gamma \in \Gamma_\delta$.*

Proof. Let γ be a lift of $kn \in \mathcal{W}_\delta$ and write $\xi = \gamma(P, \tilde\sigma_\epsilon(\delta), t) \in \Xi_{u,\epsilon}$. By definition (3.114) for $h_0 \in \mathcal{H}(W^P)$, $v \in V_\delta$

$$u_{\gamma,\epsilon}(\xi)(h_0 \otimes v) = h_0 \cdot_{q^\epsilon} \imath^o_{n^{-1},\epsilon}(t) \otimes \tilde\delta_{\gamma,\epsilon}(v) \tag{5.61}$$

where $\tilde\delta_{\gamma,\epsilon} \in U(V_\delta)$. More precisely, $\tilde\delta_{\gamma,\epsilon}$ is a multiple of a map $L_\epsilon := \tilde\delta_{kn,\epsilon} \in U(V_\delta)$ that satisfies

$$\delta(\tilde\rho_\epsilon h_1) = L_\epsilon^{-1} \delta(\rho_\epsilon \psi_k \psi_n h_1) L_\epsilon \qquad h_1 \in \mathcal{H}_{P,\epsilon} \tag{5.62}$$

This L_ϵ is only defined up to scalars, but we will show that these can be chosen such that $L_0 = \lim_{\epsilon\to 0} L_\epsilon$ exists. Since V_δ is a finite dimensional vector space, every automorphism of $\mathrm{End}(V_\delta)$ is inner, and $\mathrm{Aut}(\mathrm{End}(V_\delta)) \cong PGL(V_\delta)$. Because $GL(V_\delta) \to PGL(V_\delta)$ is a fiber bundle it suffices to show that L_0 exists as a projective linear map. Writing $h_1 = \rho_\epsilon^{-1} h_2$ and rearranging some terms transforms (5.62) into

$$\delta(M_\epsilon^{-1/2}) L_\epsilon \delta(M_\epsilon^{1/2}) \delta(h_2) \delta(M_\epsilon^{-1/2}) L_\epsilon^{-1} \delta(M_\epsilon^{1/2}) = \delta(\rho_\epsilon \psi_k \psi_n \rho_\epsilon^{-1} h_2)$$

If we can show that everything else in this equation is analytic in ϵ and has a well-defined limit as $\epsilon \to 0$, then the same must hold for $L_\epsilon \in PGL(V_\delta)$. By Theorem 5.16 we know this already for $M_\epsilon^{\pm 1/2}$. For any $f \in C^{an}(U)$ we have

$\rho_\epsilon \psi_k \psi_n \rho_\epsilon^{-1} f = \psi_k \psi_n f$ by the W_0-equivariance of σ_ϵ. For the simple reflection s associated to $\alpha \in F_1$ we have

$$\begin{aligned} \rho_\epsilon \psi_k \psi_n \rho_\epsilon^{-1} \big(N_s + q(s)^{-1/2} \big) &= \rho_\epsilon \psi_k \psi_n \big(q(s)^{(1-\epsilon)/2} (c_\alpha \circ \sigma_{1/\epsilon} c_{\alpha,\epsilon}^{-1} (N_s + q(s)^{-\epsilon/2}) \big) \\ &= \rho_\epsilon \psi_k \big(q(s)^{(1-\epsilon)/2} (c_{n\alpha} \circ \sigma_{1/\epsilon}) c_{n\alpha,\epsilon}^{-1} (N_{nsn^{-1}} + q(s)^{-\epsilon/2}) \big) \\ &= \psi_k \big(q(s)^{(1-\epsilon)/2} c_{n\alpha} (c_{n\alpha,\epsilon}^{-1} \circ \sigma_\epsilon) \big) \rho_\epsilon \big(N_{nsn^{-1}} + q(s)^{-\epsilon/2} \big) \end{aligned}$$

By Lemmas 5.14 and 5.15 the last expression has the required properties.

Now we turn our attention to the other parts of (5.61). For $\epsilon > 0$ Lemma 5.17 guarantees that $\tilde\sigma_\epsilon(\delta)$ is discrete series. Therefore $u_{\gamma,\epsilon}$ is unitary and $\imath^o_{n^{-1},\epsilon}$ cannot have a pole at t. From the explicit definition (3.120) we see that $\imath^o_{n^{-1},\epsilon}$ is regular at t for any $\epsilon \in [-1,1]$, and that $\lim_{\epsilon \to 0} \imath^o_{n^{-1},\epsilon}(t) = N_{n^{-1}}$. By Proposition 5.7 this implies

$$\lim_{\epsilon \to 0} h_0 \cdot_{q^\epsilon} \imath^o_{n^{-1},\epsilon}(t) = h_0 \cdot_{q^0} N_{n^{-1}}$$

Putting things together we conclude that L_ϵ, $\tilde\delta_{\gamma,\epsilon}$ and $u_{\gamma,\epsilon}$ are analytic in ϵ and have well-defined limits as $\epsilon \to 0$, all as projective linear maps. However, we already agreed that we may assume that this even holds for L_ϵ as a linear map. But by [37, §53] the way to lift this representation from $\mathcal{W}_\delta$ to Γ_δ is completely determined by the cocycle

$$\mathcal{W}_\delta \times \mathcal{W}_\delta \to \mathbb{C}^\times : (g_1, g_2) \to \tilde\delta_{g_1,\epsilon} \tilde\delta_{g_2,\epsilon} \tilde\delta^{-1}_{g_2^{-1} g_1^{-1},\epsilon}$$

This cocycle is continuous in ϵ and Γ_δ is finite, so the way to lift is independent of ϵ. Therefore $\lim_{\epsilon \to 0} \tilde\delta_{\gamma,\epsilon}$ and $\lim_{\epsilon \to 0} u_{\gamma,\epsilon}$ exist, as linear maps. It also follows that $u_{\gamma,\epsilon}$ is analytic in ϵ which, in combination with its unitarity $\forall \epsilon > 0$, shows that it is unitary $\forall \epsilon \in [-1,1]$. □

Lemma 5.19 *The pre-C^*-algebras $A_\delta^{\mathcal{W}_{\delta,\epsilon}} = A_\delta^{\Gamma_{\delta,\epsilon}}$ are all isomorphic, by isomorphisms that are piecewise analytic in $\epsilon, \epsilon' \in [-1,1]$.*

Proof. From Lemma 5.18 we get an analytic path of projections

$$[-1,1] \to A_\delta \rtimes \Gamma_\delta : \epsilon \to p_\delta(u_\epsilon) := |\Gamma_\delta|^{-1} \sum_{\gamma \in \Gamma_\delta} u_{\gamma,\epsilon} \gamma \tag{5.63}$$

Like in (5.48) the map

$$A_\delta^{\Gamma_{\delta,\epsilon}} \to p_\delta(u_\epsilon)(A_\delta \rtimes \Gamma_\delta) p_\delta(u_\epsilon) : a \to p_\delta(u_\epsilon) a p_\delta(u_\epsilon)$$

is an isomorphism of pre-C^*-algebras. If we apply [102, Lemma 1.15] we see that the $p_\delta(u_\epsilon)$ are all conjugate, by elements depending continuously on ϵ. To show analyticity we construct these elements explicitly, using the recipe of [10, Proposition 4.32]. For $\epsilon, \epsilon' \in [-1,1]$ consider the element

$$z(\delta, \epsilon, \epsilon') := (2 p_\delta(u'_\epsilon) - 1)(2 p_\delta(u_\epsilon) - 1) + 1 \in A_\delta \rtimes \Gamma_\delta$$

Clearly this is analytic in ϵ and ϵ' and

$$p_\delta(u_\epsilon) z(\delta,\epsilon,\epsilon') = 2 p_\delta(u'_\epsilon) p_\delta(u_\epsilon) = z(\delta,\epsilon,\epsilon') p_\delta(u_\epsilon)$$

Moreover if $\|\cdot\|$ is the norm of the enveloping C^*-algebra

$$C_\delta := C\big(T_u^P ; \mathrm{End}\big(\mathcal{H}(W^P) \otimes V_\delta\big)\big) \rtimes \Gamma_\delta$$

of $A_\delta \rtimes \Gamma_\delta$ and $\|p_\delta(u_\epsilon) - p_\delta(u'_\epsilon)\| < 2$ then

$$\begin{aligned} \|z(\delta,\epsilon,\epsilon') - 2\| &= \|4 p_\delta(u'_\epsilon) p_\delta(u_\epsilon) - 2 p_\delta(u_\epsilon) - 2 p_\delta(u_\epsilon)\| \\ &= \left\| -2\big(p_\delta(u_\epsilon) - p_\delta(u'_\epsilon)\big)^2 \right\| \quad < \quad 2 \end{aligned}$$

so $z(\delta,\epsilon,\epsilon')$ is invertible in C_δ. But $A_\delta \rtimes \Gamma_\delta$ is holomorphically closed in C_δ, so $z(\delta,\epsilon,\epsilon')$ is also invertible in this Fréchet algebra. Moreover, because the Fréchet topology on $A_\delta \rtimes \Gamma_\delta$ is stronger than the topology coming from $\|\cdot\|$, there exists an open interval I_ϵ containing ϵ such that $z(\delta,\epsilon,\epsilon')$ is invertible $\forall \epsilon' \in I_\epsilon$. For such ϵ' we construct the unitary element

$$u(\delta,\epsilon,\epsilon') := z(\delta,\epsilon,\epsilon') |z(\delta,\epsilon,\epsilon')|^{-1}$$

By construction the map

$$p_\delta(u_\epsilon)(A_\delta \rtimes \Gamma_\delta) p_\delta(u_\epsilon) \to p_\delta(u'_\epsilon)(A_\delta \rtimes \Gamma_\delta) p_\delta(u'_\epsilon) : x \to u(\delta,\epsilon,\epsilon') x u(\delta,\epsilon,\epsilon')^{-1}$$

is an isomorphism of pre-C^*-algebras. The composite map $A_\delta^{\Gamma_{\delta,\epsilon}} \to A_\delta^{\Gamma_{\delta,\epsilon'}}$ is given by

$$x \to |\Gamma_\delta| \left[u(\delta,\epsilon,\epsilon') p_\delta(u'_\epsilon) x p_\delta(u'_\epsilon) u(\delta,\epsilon,\epsilon')^{-1} \right]_e \tag{5.64}$$

which is analytic in ϵ and ϵ' because $p_\delta(u_\epsilon)$ is. Now we pick a finite cover $\{I_{\epsilon_i}\}_{i=1}^m$ of $[-1,1]$. Then for any ϵ, $\epsilon' \in [-1,1]$ an isomorphism between $A_\delta^{\Gamma_{\delta,\epsilon}}$ and $A_\delta^{\Gamma_{\delta,\epsilon'}}$ can be obtained by composing at most m isomorphisms of the type (5.64). □

The constructions in this section lead to the following

Corollary 5.20 *There exists a collection of injective *-homomorphisms*

$$\phi_\epsilon : \mathcal{H}(\mathcal{R}, q^\epsilon) \to \mathcal{S}(\mathcal{R}, q) \qquad \epsilon \in [-1,1]$$

such that

1. *for $\epsilon < 0$ the map*

$$\mathrm{Rep}(\mathcal{S}(\mathcal{R}, q)) \to \mathrm{Rep}(\mathcal{H}(\mathcal{R}, q^\epsilon)) : \pi \to \pi \circ \phi_\epsilon$$

is a bijection from tempered $\mathcal{H}$-representations to anti-tempered $\mathcal{H}_\epsilon$-representations

2. $\forall (P, \delta, t) \in \Xi_u$ *the representation* $\pi(P, \delta, t) \circ \phi_\epsilon$ *is equivalent with* $\pi_\epsilon(P, \tilde{\sigma}_\epsilon(\delta), t)$

3. ϕ_1 *is the canonical embedding*

4. $\phi_\epsilon(N_w) = N_w \; \forall w \in Z(W)$

5. $\forall h \in \mathcal{H}(\mathcal{R})$ *the function*

$$[-1, 1] \to \mathcal{S}(\mathcal{R}, q) : \epsilon \to \phi_\epsilon(h)$$

is piecewise analytic, and in particular analytic at 0.

Proof. By Lemma 5.18 the image of $\mathcal{F}'_0$ is invariant under the action of $\mathcal{W}_{\delta,0}$. So we can define

$$\phi_\epsilon = \mathcal{F}'^{-1} \circ \zeta_\epsilon \circ \mathcal{F}'_\epsilon \tag{5.65}$$

where $\zeta_\epsilon = \bigoplus_{(P,\delta)\in\Delta'} \zeta_{\epsilon,\delta}$ and

$$\zeta_{\epsilon,\delta} : A_\delta^{\mathcal{W}_{\delta,\epsilon}} \to A_\delta^{\mathcal{W}_\delta}$$

is the isomorphism from Lemma 5.19. Now 2 and 3 are valid by construction, 4 follows from the observation that the $Z(W)$-character of $\pi_\epsilon(P, \tilde{\sigma}_\epsilon(\delta), t)$ is equal to $t\big|_{Z(W)}$, for every $\epsilon \in [-1, 1]$, and 1 is a consequence of Lemma 5.17 and Proposition 3.17. Finally, for 5 we use (5.35), Theorem 5.16 and Lemma 5.19. From the proof of this lemma we see that we can arrange that $\epsilon \to \phi_\epsilon(h)$ is analytic at 0.

As concerns the injectivity of ϕ_ϵ, note that $\pi_\epsilon(P, \tilde{\sigma}_\epsilon(\delta_\emptyset), t) = I_{t,\epsilon}$ is a principal series representation for $\mathcal{H}_\epsilon$. So if $h \in \ker(\phi_\epsilon)$, then h acts as 0 on all unitary principal series. By Lemma 3.4 we must have $h = 0$. $\square$

We do not know whether $\phi_\epsilon(\mathcal{H}_\epsilon) \subset \mathcal{H}$, for two reasons : $\zeta_{\epsilon,\delta}$ need not preserve polynomiality, and not every polynomial section is in the image of $\mathcal{F}'$.

Theorem 5.21 *For* $\epsilon \in [0, 1]$ *there exist homomorphisms of pre-*C^**-algebras*

$$\begin{array}{cccc} \phi_\epsilon & : & \mathcal{S}(\mathcal{R}, q^\epsilon) & \to & \mathcal{S}(\mathcal{R}, q) \\ \phi_\epsilon & : & C_r^*(\mathcal{R}, q^\epsilon) & \to & C_r^*(\mathcal{R}, q) \end{array}$$

with the properties

1. ϕ_ϵ *is an isomorphism if* $\epsilon > 0$ *and* ϕ_0 *is injective*

2. $\forall (P, \delta, t) \in \Xi_u$ *the representation* $\pi(P, \delta, t) \circ \phi_\epsilon$ *is equivalent with* $\pi_\epsilon(P, \tilde{\sigma}_\epsilon(\delta), t)$

3. ϕ_1 *is the identity*

4. $\phi_\epsilon(h) = h \; \forall h \in \mathcal{S}(Z(W))$

5. $\epsilon \to \phi_\epsilon(h)$ *is continuous* $\forall h \in \mathcal{S}(\mathcal{R})$

Proof. For any $(P, \delta) \in \Delta$ the representation $\tilde{\sigma}_\epsilon(\delta)$, although not necessarily irreducible if $\epsilon = 0$, is certainly completely reducible, being unitary. Hence by Lemma 5.17 every irreducible constituent π_1 of $\tilde{\sigma}_\epsilon(\delta)$ is a direct summand of

$$\mathrm{Ind}_{\mathcal{H}_{\epsilon,P}^{P_1}}^{\mathcal{H}_{\epsilon,P}}(P_1, \delta_1, \phi_{t_1,\epsilon})$$

for certain $P_1 \subset P$, $\delta_1 \in \Delta_\epsilon$ and

$$t_1 \in \mathrm{Hom}_{\mathbb{Z}}\left((X_P)^{P_1}, S^1\right) = \mathrm{Hom}_{\mathbb{Z}}\left(X/(X \cap (P^\vee)^\perp + \mathbb{Q}P_1), S^1\right) \subset T_u$$

Consequently $\pi(P, \pi_1, t)$ is a direct summand of

$$\mathrm{Ind}_{\mathcal{H}_\epsilon^P}^{\mathcal{H}_\epsilon}\left(\mathrm{Ind}_{\mathcal{H}_{\epsilon,P}^{P_1}}^{\mathcal{H}_{\epsilon,P}}(\delta_1 \circ \phi_{t_1,\epsilon}) \circ \phi_{t,\epsilon}\right) = \pi_\epsilon(P_1, \delta_1, tt_1)$$

In particular every matrix coefficient of $\pi_\epsilon(P, \tilde{\sigma}_\epsilon(\delta), t)$ appears in the Fourier transform of $\mathcal{H}_\epsilon$, and (5.59) extends to the respective Schwartz and C^*-completions, as required. By Lemma 5.17 and (5.41) these maps are isomorphisms if $\epsilon > 0$.

In the same way as in the proof of Corollary 5.20 we can see that ϕ_0 remains injective: every irreducible tempered $\mathcal{H}(\mathcal{R}, q^0)$-representation is a quotient of a unitary principal series, so any element of $C_r^*(\mathcal{R}, q^0)$ that vanishes on all unitary principal series is 0. Furthermore properties 2. and 4. are direct consequences of Corollary 5.20.

Finally, if $x = \sum_{w \in W} x_w N_w \in \mathcal{S}(\mathcal{R})$ then this sum converges uniformly to x. For every partial sum x' the map $\epsilon \to \phi_\epsilon(x')$ is continuous by Corollary 5.20, so this also holds for x itself. $\square$

Although there were quite a few arbitrary choices involved in constructing ϕ_ϵ, the homotopy class of these maps is well-defined:

Lemma 5.22 *The construction of $(\phi_\epsilon)_{\epsilon \in [0,1]}$ is unique up to a homotopy of objects with the properties of Theorem 5.21.*

Proof. Let us inventorize all the choices we made in the above construction. Already in (5.41) we chose a set of representatives Δ' of $\Delta/\mathcal{W}$. This implicitly fixed realizations V_δ of the discrete series representations $\delta \in \Delta'$, but since we never used a basis of V_δ, the construction does not really depend on this vector space. Then we used intertwiners $\pi(g, \xi)$, $g \in \mathcal{W}_\delta$ that were defined only up to scalars, but this ambiguity dropped out when we lifted things to Γ_δ. Finally we chose a (monotonous) sequence $(\epsilon_i)_{i=0}^m$ such that $\epsilon_0 = \epsilon$, $\epsilon_m = 1$ and every isomorphism $\mathcal{S}(\mathcal{R}, q^{\epsilon_i}) \to \mathcal{S}(\mathcal{R}, q^{\epsilon_{i+1}})$ involved only one map of the type (5.64) for every δ.

Suppose we take another sequence $(\epsilon'_i)_{i=0}^{m'}$. Allowing some of the ϵ_i and ϵ'_j to coincide, we may assume that $m' = m$. Then we can continuously deform the first sequence into the second. Since the elements $u(\delta, \epsilon, \epsilon')$ depend analytically on ϵ and ϵ', this will give us a path of isomorphisms.

To see what happens when we use some Δ'' instead of Δ' is more difficult. It is clearly enough to investigate the effect on only one component of Ξ_u, corresponding to $(P,\delta) \in \Delta'$ and to $(P',\delta') \in \Delta''$. By the previous paragraph we may also restrict ourselves to ϵ and ϵ' with $|\epsilon-\epsilon'|$ "sufficiently" small. Then the original isomorphism is given by

$$\phi_{\epsilon,\epsilon'} : a \to |\Gamma_\delta| \left[p_\delta(u_{\epsilon'}) u(\delta,\epsilon,\epsilon') p_\delta(u_\epsilon) a p_\delta(u_\epsilon) u(\delta,\epsilon,\epsilon')^{-1} p_\delta(u_{\epsilon'}) \right]_e \tag{5.66}$$

and the alternative by

$$\phi'_{\epsilon,\epsilon'} : a' \to |\Gamma_{\delta'}| \left[u(\delta',\epsilon,\epsilon') p_{\delta'}(u_\epsilon) a' p_{\delta'}(u_\epsilon) u(\delta',\epsilon,\epsilon')^{-1} \right]_e \tag{5.67}$$

In both formulas $[\cdot]_e$ means taking the coefficient at the identity element in some group algebra. To compare $\phi_{\epsilon,\epsilon'}$ and $\phi'_{\epsilon,\epsilon'}$ we take $g \in \mathcal{W}_{\delta\delta'}$, $\xi = (P,\delta,t) \in \Xi_u$ and evaluate $\phi'_{\epsilon,\epsilon'} a(\xi)$, assuming that a and $\phi'_{\epsilon,\epsilon'}(a)$ are g-invariant:

$$\begin{array}{l}
\phi'_{\epsilon,\epsilon'} a(\xi) \quad = \quad g^{-1}(\phi'_{\epsilon,\epsilon'}(ga))(\xi) \quad = \\
\pi'_\epsilon(g^{-1},g\xi) \left(\phi'_{\epsilon,\epsilon'}(ga)\right)(g\xi)\, \pi'_\epsilon(g^{-1},g\xi)^{-1} \quad = \\
|\Gamma_{\delta'}|\, \pi'_\epsilon(g^{-1},g\xi) \left[u(\delta',\epsilon,\epsilon') p_{\delta'}(u_\eta) g(a) p_{\delta'}(u_\epsilon) u(\delta',\epsilon,\epsilon')^{-1} \right]_e (g\xi)\, \pi'_\epsilon(g^{-1},g\xi)^{-1} \quad = \\
|\Gamma_{\delta'}|\, \pi'_\epsilon(g^{-1},g\xi) \left[u(\delta',\epsilon,\epsilon')(g\xi) p_{\delta'}(u_\epsilon)(g\xi) \pi_\epsilon(g,\xi) a(\xi) \circ \right. \\
\quad \left. \pi_\epsilon(g,\xi)^{-1} p_{\delta'}(u_\epsilon)(g\xi) u(\delta',\epsilon,\epsilon')^{-1}(g\xi) \right]_e \pi'_\epsilon(g^{-1},g\xi)^{-1}
\end{array} \tag{5.68}$$

To compare Γ_δ and $\Gamma_{\delta'}$ we have to make the Schur extension (5.43) functorial. In general it is not even unique, but the recipe in [37, §53] always works. For the sake of the argument we might temporarily redefine the Schur extension to be the result of this construction. As a bonus it is easily seen that for $\gamma' \in \Gamma_{\delta'}$ the element $\gamma = g^{-1}\gamma' g \in \Gamma_\delta$ is well-defined. But then necessarily

$$\pi(g,\xi)^{-1} u_{\gamma'}(g\xi) \pi(g,\xi) = u_\gamma(\xi)$$

since both sides are proportional and satisfy the cocycle relations (5.44). In particular

$$\begin{array}{lcl}
p_{\delta'}(u)(g\xi)\pi(g,\xi) & = & \pi(g,\xi) p_\delta(u)(\xi) \\
\pi'_\epsilon(g^{-1},g\xi) u(\delta',\epsilon,\epsilon')(g\xi) \pi_\eta(g,\xi) & = & (2p_{\delta'}(u_{\epsilon'})(\xi) - 1) b(\xi) (2p_{\delta'}(u_\epsilon)(\xi) - 1) + b(\xi)
\end{array}$$

We denote the last expression by $u(b)(\xi)$, where $b(\xi) = \pi'_\epsilon(g^{-1},g\xi)\pi_\epsilon(g,\xi)$. In general elements of the type $u(b)$ in a C^*-algebra are invertible if

$$\|p_{\delta'}(u_\epsilon) - p_{\delta'}(u_{\epsilon'})\| + 4\,\|b - 1\| < 1$$

and they satisfy

$$p_{\delta'}(u_{\epsilon'}) u(b) = p_{\delta'}(u_{\epsilon'}) u(b) p_{\delta'}(u_\epsilon) = u(b) p_{\delta'}(u_\epsilon)$$

With this knowledge we can rewrite (5.68) as

$$\phi'_{\epsilon,\epsilon'} a(\xi) = |\Gamma_\delta| [p_\delta(u_{\epsilon'}) u(b) p_\delta(u_\epsilon) a p_\delta(u_\epsilon) u(b)^{-1} p_\delta(u_{\epsilon'})]_e(\xi)$$

but now e is an element of Γ_δ instead of $\Gamma_{\delta'}$. Comparing this to (5.66) we find that the only difference is that $u(\delta,\epsilon,\epsilon')$ has been replaced by this $u(b)$. The intertwiners $\pi_{\epsilon'}(g,\xi)$ depend analytically on ϵ', so we can always find a path from $u(b)$ to $u(\delta,\epsilon,\epsilon')$ consisting only of elements of the same type. (This step might require a subdivision of the interval between ϵ and ϵ', but that is no problem.) The corresponding isomorphisms form a path from $\phi'_{\epsilon,\epsilon'}$ to $\phi_{\epsilon,\epsilon'}$. $\square$

5.4 K-theoretic conjectures

We saw in Section 5.2 that the multiplication in $\mathcal{S}(\mathcal{R},q)$ varies continuously with q. Since a class in K-theory is rigid under small perturbations, it is natural to expect that the K-groups of $\mathcal{S}(\mathcal{R},q)$ are independent of q. We reformulate this conjecture in terms of the map ϕ_0 and show that it implies some other important conjectures.

Our main tools are the scaling maps ϕ_ϵ from Theorem 5.21. From Theorems 5.21 and 2.13, Lemma 5.22 and the homotopy invariance of K-theory we see that for all $\epsilon \in [0,1]$ the map

$$K_*(\phi_\epsilon) : K_*(C_r^*(\mathcal{R},q^\epsilon)) \cong K_*(\mathcal{S}(\mathcal{R},q^\epsilon)) \to K_*(C_r^*(\mathcal{R},q)) \cong K_*(\mathcal{S}(\mathcal{R},q)) \quad (5.69)$$

is natural. By (3.143) the same goes for

$$HP_*(\phi_\epsilon) : HP_*(\mathcal{S}(\mathcal{R},q^\epsilon)) \to HP_*(\mathcal{S}(\mathcal{R},q)) \quad (5.70)$$

Obviously, by Theorem 5.21 these maps are isomorphisms for $\epsilon > 0$, but whether this holds in general for $\epsilon = 0$ is not known.

Let $G(C_r^*(\mathcal{R},q))$ be the Grothendieck group of the additive category of finite dimensional $C_r^*(\mathcal{R},q)$-modules. There is a natural map

$$\begin{aligned} G(\phi_0) &: G\big(C_r^*(\mathcal{R},q)\big) \to G\big(C_r^*(W)\big) \\ G(\phi_0)(\pi,V) &= (\pi \circ \phi_0, V) \end{aligned} \quad (5.71)$$

For $U \subset T/W_0$ we introduce the two-sided ideals

$$\begin{aligned} J_U^s &= \{x \in \mathcal{S}(\mathcal{R},q) : \pi(P,W_P r,\delta,t)(x) = 0 \text{ if } t \in T_u^P,\ W_0 r t \in U\} \\ J_U^c &= \{x \in C_r^*(\mathcal{R},q) : \pi(P,W_P r,\delta,t)(x) = 0 \text{ if } t \in T_u^P,\ W_0 r t \in U\} \end{aligned} \quad (5.72)$$

By Theorem 5.21.2 ϕ_0 factors through

$$\phi_{W_0 t_0} : \mathcal{S}(W)/J^s_{W_0 t_0} \to \mathcal{S}(\mathcal{R},q)/J^s_{W_0 t_0 T_{rs}} \quad (5.73)$$

The induced map on K_0^+ can be regarded as a morphism of semigroups

$$K_0^+\big(\phi_{W_0 t_0}\big) : \mathrm{Rep}_{W_0 t_0}\big(C_r^*(W)\big) \to \mathrm{Rep}_{W_0 t_0 T_{rs}}\big(C_r^*(\mathcal{R}, q)\big) \tag{5.74}$$

The direct sum of these maps, over all $W_0 t_0 \in T_u / W_0$, is a homomorphism

$$K_{\mathrm{Rep}}(\phi_0) : G\big(C_r^*(W)\big) \to G\big(C_r^*(\mathcal{R}, q)\big) \tag{5.75}$$

This map is a bit weird, it does not always preserve dimensions, and it certainly is not an inverse of $G(\phi_0)$. For example, consider the case $R_0 = A_1$, X the root lattice and $q(s_0) = q(s_1) > 1$. Then

$$\big(K_{\mathrm{Rep}}(\phi_0)\pi\big) \circ \phi_0 = \pi$$

if π admits a central character $t \neq 1$. The only irreducible $\mathbb{C}W$-representations with central character $t = 1$ are the trivial and sign representations of W. However, there we see something strange: $K_{\mathrm{Rep}}(\phi_0)$ sends the trivial representation to the principal series $I_1 = \pi(\emptyset, \delta_\emptyset, 1)$, and it sends the sign representation to the direct sum of I_1 and the Steinberg representation of $\mathcal{H}(\mathcal{R}, q)$.

Theorem 5.23 *The following are equivalent:*

1. $G(\phi_0) \otimes \mathrm{id}_\mathbb{Q} : G\big(C_r^*(\mathcal{R}, q)\big) \otimes \mathbb{Q} \to G\big(C_r^*(W)\big) \otimes \mathbb{Q}$ *is a bijection*
2. $K_{\mathrm{Rep}}(\phi_0) \otimes \mathrm{id}_\mathbb{Q} : G\big(C_r^*(W)\big) \otimes \mathbb{Q} \to G\big(C_r^*(\mathcal{R}, q)\big) \otimes \mathbb{Q}$ *is a bijection*
3. $K_*(\phi_0) \otimes \mathrm{id}_\mathbb{Q} : K_*\big(C_r^*(W)\big) \otimes \mathbb{Q} \to K_*\big(C_r^*(\mathcal{R}, q)\big) \otimes \mathbb{Q}$ *is a bijection*
4. $HP_*(\phi_0) : HP_*(\mathcal{S}(W)) \to HP_*(\mathcal{S}(\mathcal{R}, q))$ *is a bijection*
5. $HH_0(\phi_0) : \mathcal{S}(W)/[\mathcal{S}(W), \mathcal{S}(W)] \to \mathcal{S}(\mathcal{R}, q)/[\mathcal{S}(\mathcal{R}, q), \mathcal{S}(\mathcal{R}, q)]$ *is an isomorphism of Fréchet spaces*

Proof. $1 \Leftrightarrow 2$. By construction it suffices to show this for $\phi_{W_0 t_0}$, for arbitrary $W_0 t_0 \in T_u / W_0$. But $\phi_{W_0 t_0}$ is just a homomorphism between finite dimensional semisimple algebras, so $K_0(\phi_0) \otimes \mathrm{id}_\mathbb{Q}$ and $G(\phi_0) \otimes \mathrm{id}_\mathbb{Q}$ are linear maps between finite dimensional vector spaces. With respect to the bases formed by irreducible presentations the matrices of these two maps are each others transpose. In particular one of them is bijective if and only if the other is.

$2 \Leftrightarrow 3$. Consider the projection

$$\begin{aligned} \mathrm{pr} &: \Xi_u / \mathcal{W} \to T_u / W_0 \\ \mathrm{pr} &\,(\mathcal{W}(P, W_P r, \delta, t)) = W_0 r_u t \end{aligned} \tag{5.76}$$

With this map we make $C_r^*(\mathcal{R}, q))$ into a $C(T_u / W_0)$-algebra. By (5.50) ϕ_0 is $C(T_u / W_0)$-linear. Triangulate T_u / W_0 such that every subset T_u^G / W_0 with $G \subset W_0$ becomes a subcomplex. In view of Proposition 2.21, 2 implies 3.

Contrarily, suppose that $K_{\mathrm{Rep}}(\phi_0) \otimes \mathrm{id}_{\mathbb{Q}}$ is not surjective. By definition there is a $t_0 \in T_u$ such that $K_0(\phi_{W_0 t_0}) \otimes \mathrm{id}_{\mathbb{Q}}$ is not surjective. However the canonical map

$$K_0(C_r^*(\mathcal{R}, q)) \to K_0\big(C_r^*(\mathcal{R}, q)/J^c_{W_0 t_0 T_{rs}}\big)$$

is always surjective. This can be seen as follows. Every component of X_u intersects $\mathrm{pr}^{-1}(W_0 t_0)$ in at most one W_0-orbit. If $[p_1] \in K_0\big(C_r^*(\mathcal{R}, q)/J^c_{W_0 t_0 T_{rs}}\big)$ then $[p_2] \in K_0(C_r^*(\mathcal{R}, q))$ maps to $[p_1]$ if and only if the value of p_2 on $\Xi_u \cap \mathrm{pr}^{-1}(W_0 t_0)$ is as prescribed by p_1. It follows from Theorem 3.25 that such a p_2 can always can be found. This shows that $K_*(\phi_0) \otimes \mathrm{id}_{\mathbb{Q}}$ and $K_*(\phi_0)$ are not surjective.

Now suppose that $K_{\mathrm{Rep}}(\phi_0) \otimes \mathrm{id}_{\mathbb{Q}}$ is not injective. Pick $W_0 t_0$ such that $K_0(\phi_{W_0 t_0})$ is not injective, with $|W_{0,t_0}|$ minimal for this property. Next pick $V, V' \in \mathrm{Rep}_{W_0 t_0}(C_r^*(W))$ such that $[V] - [V'] \in \ker K_0\big(\phi_{W_0 t_0}\big)$. Put

$$T' := \{t \in T_u : W_{0,t} \not\subset W_{0,t_0}\}$$

and introduce the ideals

$$\begin{aligned} I_0 &:= J^c_{T'} \subset C_r^*(W) \\ I_1 &:= J^c_{T' T_{rs}} \subset C_r^*(\mathcal{R}, q) \end{aligned}$$

Note that $\phi_0(I_0) \subset I_1$. Recall the description

$$C_r^*(W) \cong C\big(T_u; \mathrm{End}\, \mathbb{C}[W_0]\big)^{W_0} \tag{5.77}$$

from Lemma A.3, in combination with (2.105) and Theorem 3.15. These show that it is possible to find $m, n \in \mathbb{N}$ and projections $p, p' \in M_n(I_0^+)$ such that $p(t)$ and $p'(t)$ yield the W_{0,t_0}-modules mV and mV', for all $t \in T_u^{W_{0,t_0}} \setminus T'$. Now we insert $[p] - [p']$ in the commutative diagram

$$\begin{array}{ccc} K_0(I_0) & \to & K_0(C_r^*(W)) \\ \downarrow & & \downarrow \\ K_0(I_1) & \to & K_0(C_r^*(\mathcal{R}, q) \end{array}$$

By assumption

$$[\phi_0(p)] - [\phi_0(p')] = 0 \in K_0(I_1)$$

On the other hand, $[p]$ and $[p']$ are different on $T_u^{W_{0,t_0}}$, so by Theorem 2.22

$$Ch_{W_0}\big([p] - [p']\big) \neq 0 \in \check{H}^*\big(\widetilde{T_u}/W_0; \mathbb{C}\big)$$

Therefore $K_0(\phi_0) \otimes \mathbb{Q}$ and $K_0(\phi_0)$ are not injective.

3 $\Leftrightarrow$ 4 by Theorem 3.34.

1 $\Leftrightarrow$ 5. The localization of

$$HH_0(\mathcal{S}(\mathcal{R}, q)) = \mathcal{S}(\mathcal{R}, q)/[\mathcal{S}(\mathcal{R}, q), \mathcal{S}(\mathcal{R}, q)]$$

at $\mathcal{W}\xi \in \Xi_u/\mathcal{W}$ is a complex vector space whose dimension is the number of inequivalent irreducible constituents of $\pi(\xi)$. This gives a fibration of $\Xi_u/\mathcal{W}$, and by Theorem 3.25 $HH_0(\mathcal{S}(\mathcal{R},q))$ can be regarded as the set of global sections of the sheaf $\mathfrak{F}_q$ of smooth sections of this fibration. The direct image of $\mathfrak{F}_q$ under (5.76) is the sheaf of smooth sections of a fibration of T_u/W_0. The fiber at W_0t_0 is a vector space whose dimension is the number of irreducibles in $\mathrm{Rep}_{W_0t_0T_{rs}}(C_r^*(\mathcal{R},q))$.

The Fréchet space $HH_0(\mathcal{S}(W))$ admits a similar description it terms of a sheaf $\mathfrak{F}_1$, but with fibers of dimension the number of irreducibles in $\mathrm{Rep}_{W_0t_0}(C_r^*(W))$. Now ϕ_0 induces a morphism

$$\mathfrak{F}(\phi_0):\mathfrak{F}_1\to \mathrm{pr}_*(\mathfrak{F}_q)$$

of sheaves over T_u/W_0. If $G(\phi_0)\otimes \mathrm{id}_{\mathbb{Q}}$ is not bijective, then $\mathfrak{F}_1$ and $\mathrm{pr}_*(\mathfrak{F}_q)$ have different stalks, so $\mathfrak{F}(\phi_0)$ and $HH_0(\phi_0)$ cannot be isomorphisms. On the other hand, if $G(\phi_0)\otimes \mathrm{id}_{\mathbb{Q}}$ is bijective, then

$$\mathfrak{F}(\phi_0)(W_0t_0):\mathfrak{F}_1(W_0t_0)\to \mathrm{pr}_*(\mathfrak{F}_q)(W_0t_0)$$

is a bijection for every $W_0t_0\in T_u/W_0$. This implies that $\mathfrak{F}(\phi_0)$ is an isomorphism, see e.g. [47, Section II.1.6]. In particular $HH_0(\phi_0)=\mathfrak{F}(\phi_0)(T_u/W_0)$ is an isomorphism of Fréchet spaces. $\Box$

Conjecture 5.24 *The equivalent statements of Theorem 5.23 hold for every root datum and every positive label function.*

If q is an equal label function then by the Kazhdan-Lusztig classification (see page 132) and by Theorem 5.3 none of the vector spaces in Theorem 5.23 depends on q, up to natural isomorphisms. It is probable, though not a priori certain, that these isomorphisms can be realized with ϕ_0.

Actually, even for unequal labels strange things happen if Conjecture 5.24 does not hold. Suppose for example that $HH_0(\phi_0)$ is not injective. In that case there would exist an

$$x\in\mathcal{S}(W)\setminus[\mathcal{S}(W),\mathcal{S}(W)]\quad\text{such that}\quad\phi_0(x)\in[\mathcal{S}(\mathcal{R},q),\mathcal{S}(\mathcal{R},q)]$$

However, since ϕ_ϵ is an isomorphism $\forall\epsilon>0$, we would have

$$\phi_\epsilon^{-1}\phi_0(x)\in[\mathcal{S}(\mathcal{R},q^\epsilon),\mathcal{S}(\mathcal{R},q^\epsilon)]\quad\forall\epsilon>0 \tag{5.78}$$

But this is remarkable since

$$\epsilon\to\phi_\epsilon^{-1}\phi_0(x)\quad\text{and}\quad\epsilon\to z\cdot_{q^\epsilon}y-y\cdot_{q^\epsilon}z$$

are both continuous on $[0,1]$ $,\forall x,y,z\in\mathcal{S}(\mathcal{R})$. Maybe one can show that it is outright impossible, by a thorough study of conjugacy classes in affine Weyl groups. Related results for finite Coxeter groups can be found in [46, Sections 3.2 and 8.2].

Or suppose that $G(\phi_0)\otimes\mathrm{id}_{\mathbb{Q}}$ is not injective. Then there would exist $t_0\in T_u$ and $(\pi,V),(\pi',V')\in\mathrm{Rep}_{W_0t_0T_{rs}}(C_r^*(\mathcal{R},q))$ such that $\pi\circ\phi_0$ and $\pi'\circ\phi_0$ are equivalent $C_r^*(W)$-representations. The Euler-Poincaré pairing from (3.74) can help in this situation. Assume for the moment that $\mathcal{R}$ is semisimple. By Theorem 5.2 the finite dimensional semisimple subalgebras $\mathcal{H}(\mathcal{R}, I, q)$ are rigid under $q \to q^\epsilon$, so

$$\pi\big|_{\mathcal{H}(\mathcal{R},I,q)} \cong (\pi \circ \phi_0)\big|_{\mathcal{H}(\mathcal{R},I,q^0)} \cong (\pi' \circ \phi_0)\big|_{\mathcal{H}(\mathcal{R},I,q^0)} \cong \pi'\big|_{\mathcal{H}(\mathcal{R},I,q)} \tag{5.79}$$

Therefore $P_n(V)^\Omega \cong P_n(V')^\Omega$ for all n, and by Corollary 3.11

$$\mathrm{Eul}\,[V] = \mathrm{Eul}\,[V'] \in K_0(\mathcal{H})$$

Together with (3.77) this shows that $[V] - [V']$ is in the radical of the Euler-Poincaré pairing. However, we noticed on page 79 that the radical of EP is very large, so this does certainly not imply that π and π' are equivalent. The next theorem is very useful to overcome this problem. It is analogous to results of Meyer [91, Theorems 21 and 38] for Schwartz algebras of reductive p-adic groups.

Theorem 5.25 *Let $\mathcal{R}$ be any root datum and $V, V' \in \mathrm{Rep}(\mathcal{S}(\mathcal{R}, q))$. Then*

$$\big(\mathcal{S}(\mathcal{R}, q) \otimes_{\mathcal{H}(\mathcal{R},q)} P_*(V)^\Omega\,,\ \mathrm{id}_{\mathcal{S}(\mathcal{R},q)} \otimes d_*\big) \tag{5.80}$$

is a finitely generated resolution of V. This resolution consists of projective modules if $\mathcal{R}$ is semisimple. Moreover for such root data there are natural isomorphisms

$$\mathrm{Ext}^n_{\mathcal{H}(\mathcal{R},q)}(V, V') \cong \mathrm{Ext}^n_{\mathcal{S}(\mathcal{R},q)}(V, V') \qquad n \in \mathbb{N}$$

Proof. This has been proved recently by Opdam and the author. We can construct a suitable contracting homotopy operator of the differential complex

$$\big(P_*(V)^\Omega\,,\ d_*\big)$$

Using the temperedness of V we can show that this operator extends continuously to the complex (5.80). The details will be published elsewhere. $\square$.

Suppose that both V and V' are direct sums of discrete series representations. By Theorem 3.25 a discrete series module is projective in $\mathrm{Rep}\,(\mathcal{S}(\mathcal{R}, q))$, so with Theorem 5.25

$$\begin{aligned} EP\big([V] - [V'], [V] - [V']\big) &= \sum_{n=0}^{\infty} (-1)^n \dim \mathrm{Ext}^n_{\mathcal{S}(\mathcal{R},q)}\big([V] - [V'], [V] - [V']\big) \\ &= \dim \mathrm{Hom}_{\mathcal{S}(\mathcal{R},q)}\big([V] - [V'], [V] - [V']\big) \end{aligned} \tag{5.81}$$

Clearly this is positive whenever V and V' are inequivalent. This does not only imply $\pi \circ \phi_0 \not\cong \pi' \circ \phi_0$, but also that

$$G(\phi_0)\big([V] - [V']\big) \text{ is not in the radical of } EP. \tag{5.82}$$

Hence this element cannot be written as a sum of virtual representations that are induced from proper parabolic subalgebras. Because every irreducible tempered representation is a direct summand of a representation that is parabolically induced from a discrete series representation, this is an essential part of Conjecture 5.24.

In some important cases this actually suffices to prove the conjecture. Namely, using the theory of R-groups [40, 99] it can be shown that for certain labelled root data all representations of the form $\pi(P, \delta, t)$ with $(P, \delta, t) \in \Xi_u$ are irreducible. Using Theorems 3.19 and 5.25 we can apply an inductive argument to verify Conjecture 5.24 in such cases. See Section 6.7 for more details.

Let us have another look at Theorem 5.23. Clearly the first three statements can also be formulated with integral coefficients:

Proposition 5.26 *The following are equivalent:*

1. $G(\phi_0) : G\big(C_r^*(\mathcal{R}, q)\big) \to G\big(C_r^*(W)\big)$ *is an isomorphism*
2. $K_{\text{Rep}}(\phi_0) : G\big(C_r^*(W)\big) \to G\big(C_r^*(\mathcal{R}, q)\big)$ *is an isomorphism*
3. $K_*(\phi_0) : K_*\big(C_r^*(W)\big) \to K_*\big(C_r^*(\mathcal{R}, q)\big)$ *is an isomorphism*

Proof. This is completely analogous to the corresponding part of the proof of Theorem 5.23. $\square$

One of the motivations for considering these maps is that $K_*(\phi_0)$ is natural, in the sense that it can be constructed without really using ϕ_0. The idea is that small perturbations of invertibles or idempotents have no effect on classes in K-theory. By Theorem 2.27 $K_*(\mathcal{S}(W))$ is finitely generated. Using Theorem 2.12 we can find $k \in \mathbb{N}$ and a finite set of idempotents and invertibles in $M_k(\mathcal{S}(W))$ which generates $K_*(\mathcal{S}(W))$. Let $(u, q^0) \in M_k(\mathcal{S}(\mathcal{R}, q^0))$ be such an invertible. In view of Proposition 5.9 and the remark on page 146 there exists an $\epsilon_u > 0$ such that

$$(u, q^\epsilon) \in GL_k(\mathcal{S}(\mathcal{R}, q^\epsilon)) \quad \forall \epsilon < \epsilon_u$$

To handle idempotents in a similar way we need holomorphic functional calculus. Define the holomorphic function f_p on $\{z \in \mathbb{C} : \Re(z) \neq 1/2\}$ by

$$f_p(z) = \begin{cases} 1 & \text{if} \quad \Re(z) > 1/2 \\ 0 & \text{if} \quad \Re(z) < 1/2 \end{cases}$$

Note that $f_p(x)$ is idempotent whenever it is defined. Let $(e, q^0) \in M_k(\mathcal{S}(\mathcal{R}, q^0))$ be an idempotent. It is clear from Proposition 5.6 that $\exists \epsilon_e > 0$ such that

$$(e, q^\epsilon) - 1/2 - ai \in GL_k(\mathcal{S}(\mathcal{R}, q^\epsilon)) \quad \forall \epsilon < \epsilon_e, \forall a \in \mathbb{R}$$

In fact, by direct calculation one can show that this holds for all ϵ such that

$$\big\|\lambda(e, q^\epsilon) - \lambda(e, q^0)\big\|_{B(\mathfrak{H}(\mathcal{R}))} < \frac{1}{2 + 4\,\big\|\lambda(e, q^0)\big\|_{B(\mathfrak{H}(\mathcal{R}))}}$$

For such ϵ the idempotent $f_p(e, q^\epsilon)$ is well-defined.

Lemma 5.27 *The following equalities of K-theory classes hold for u and e as above.*

$$\begin{array}{rclcll}[(u,q^\epsilon)] & = & K_1(\phi_\epsilon^{-1}\phi_0)[(u,q^0)] & \in & K_1(\mathcal{S}(\mathcal{R},q^\epsilon)) & \forall \epsilon \in (0,\epsilon_u) \\ [f_p(e,q^\epsilon)] & = & K_0(\phi_\epsilon^{-1}\phi_0)[(e,q^0)] & \in & K_0(\mathcal{S}(\mathcal{R},q^\epsilon)) & \forall \epsilon \in (0,\epsilon_e)\end{array}$$

Proof. For any $x \in M_k(\mathbb{C}) \otimes \mathcal{S}(\mathcal{R})$ the map

$$[0,1] \to M_k(\mathbb{C}) \otimes \mathcal{S}(\mathcal{R}) : \epsilon \to \phi_\epsilon^{-1}\phi_0(x)$$

is continuous. Hence (u,q^{ϵ_1}) and $\phi_\epsilon^{-1}\phi_0(u,q^0)$ are homotopic in $GL_k(\mathcal{S}(\mathcal{R},q^\epsilon))$, for some small $\epsilon_1 > 0$. Clearly this implies that $\phi_\epsilon^{-1}\phi_{\epsilon_1}(u,q^{\epsilon_1})$ and $\phi_\epsilon^{-1}\phi_0(u,q^0)$ are homotopic $\forall \epsilon \in (0,\epsilon_u)$. But there also is a path from $\phi_\epsilon^{-1}\phi_{\epsilon_1}(u,q^{\epsilon_1})$ to (u,q^ϵ) along elements of the form $\phi_\epsilon^{-1}\phi_{\epsilon_2}(u,q^{\epsilon_2})$.

Similarly, by Corollary 5.10

$$[0,\epsilon_e) \to M_k(\mathbb{C}) \otimes \mathcal{S}(\mathcal{R}) : \epsilon \to f_p(e,q^\epsilon)$$

is continuous. According to [10, Proposition 4.3.2] there is a small $\epsilon_3 > 0$ such that $f_p(e,q^{\epsilon_3})$ and $\phi_{\epsilon_3}^{-1}\phi_0(e,q^0)$ are homotopic in $\mathrm{Idem}(M_k(\mathcal{S}(\mathcal{R},q^{\epsilon_3}))$. But then, as above for u, $\phi_\epsilon^{-1}\phi_0(e,q^0)$ and $f_p(e,q^\epsilon)$ are homotopic via $\phi_\epsilon^{-1}\phi_{\epsilon_3}(f_p(e,q^{\epsilon_3}))$. □

So we have a family of pre-C^*-algebras $\mathcal{S}(\mathcal{R},q))$ which are independent of q as Fréchet spaces, and whose multiplication depends continuously on q. Moreover, replacing q by q^ϵ with $\epsilon > 0$ sufficiently small, we may assume that the natural group homomorphism $K_*(\phi_0)$ can be constructed without using ϕ_0. Therefore it is not unreasonable to suspect the following.

Conjecture 5.28 *For any root datum $\mathcal{R}$ and positive label function q the map*

$$K_*(\phi_0) : K_*(\mathcal{S}(W)) \to K_*(\mathcal{S}(\mathcal{R},q))$$

is an isomorphism.

As mentioned, this conjecture stems from Baum, Connes and Higson [5], at least in the equal label case. Independently, Opdam [98, p. 533] stated it for unequal labels. In Chapter 6 we will verify Conjecture 5.28 for some classical root data.

Consider the root datum $\mathcal{R} \times \mathbb{Z}$ with the unique label function that extends q. By (3.90)

$$\mathcal{S}(\mathcal{R} \times \mathbb{Z}, q) \cong \mathcal{S}(\mathcal{R},q)\widehat{\otimes}\mathcal{S}(\mathbb{Z}) \cong C^\infty\big(S^1; \mathcal{S}(\mathcal{R},q)\big) \tag{5.83}$$

In Lemma 2.17 we constructed natural isomorphisms

$$K_0(\mathcal{S}(\mathcal{R} \times \mathbb{Z}, q)) \xleftarrow{\sim} K_*(\mathcal{S}(\mathcal{R},q)) \xrightarrow{\sim} K_1(\mathcal{S}(\mathcal{R} \times \mathbb{Z}, q)) \tag{5.84}$$

Therefore it suffices to prove Conjecture 5.24 either for K_0 and every $(\mathcal{R},q)$, or for K_1 and every $(\mathcal{R},q)$. Probably the K_1-case is easier, for two reasons. Firstly, invertibles are more flexible than idempotents. If we perturb them a little they remain invertible, so we can do without holomorphic functional calculus. Secondly, we can find a bound, uniform in q, on the size of the matrices that we need to represent all K_1-classes. In fact, by Proposition 2.16 we can bound the topological stable rank by

$$tsr(C_r^*(\mathcal{R}, q)) \leq |W_0|^2 \big(1 + \lfloor \dim T_u/2 \rfloor\big) \tag{5.85}$$

Now Theorems 2.13 and 2.15 show that

$$K_1(\mathcal{S}(\mathcal{R}, q)) \cong \pi_0\big(GL_n(\mathcal{S}(\mathcal{R}, q))\big) \quad \forall n \geq |W_0|^2 (1 + \mathrm{rk}(X)/2) \tag{5.86}$$

One way to attack Conjecture 5.28 goes approximately as follows. Pick a finite set of generators of $K_1(\mathcal{S}(\mathcal{R}, q))$. Find suitable representants $u_i \in GL_n(\mathcal{S}(\mathcal{R}, q))$, i.e. u_i should lie in

$$M_n(\mathbb{C}) \otimes \mathrm{span}\{N_w : \mathcal{N}(w) \leq M\}$$

with M "small" and $\mathrm{sp}(u_i)$ should lie in a "small" neighborhood of the unit circle in $\mathbb{C}$. If q is close to q^0 one may hope that to every u_i one can associate in an unambiguous way a unique homotopy class in $GL_n(\mathcal{S}(W))$. This class should be constructed by applying the isomorphisms ϕ_ϵ^{-1} and by perturbing u_i a little. In particular it should contain elements of $M_n(\mathbb{C}) \otimes \mathcal{S}(\mathcal{R})$ that are homotopic to u_i in $GL_n(\mathcal{S}(\mathcal{R}, q))$.

An analogue of Conjecture 5.28 does hold for noncommutative tori. Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$ be the standard compact n-dimensional torus. In this setting the underlying group W becomes $\mathbb{Z}^n$, and q is replaced by a skew-symmetric bilinear form θ on $\mathbb{Z}^n$. By taking iterated crossed products with $\mathbb{Z}$, one constructs a C^*-algebra which is a deformation of $C(\mathbb{T}^n)$ and is commonly denoted by A_θ. It has a holomorphically closed dense subalgebra $\mathcal{S}(\mathbb{Z}^n, \theta)$ which, as a Fréchet space, is naturally isomorphic to $\mathcal{S}(\mathbb{Z}^n)$. By deforming θ Elliott [43, Theorem 2.2] proved that there exists a natural group isomorphism

$$K_*(A_\theta) \xrightarrow{\sim} \bigwedge \mathbb{Z}^n \tag{5.87}$$

For $\theta = 0$ this can be interpreted geometrically as the classical Chern character

$$Ch : K^*(\mathbb{T}^n) \xrightarrow{\sim} H^*(\mathbb{T}^n; \mathbb{Z})$$

However, there are also quite big differences between noncommutative tori and affine Hecke algebras. Namely, the structure of A_θ is very different for θ rational or irrational, and an essential ingredient in Elliott's proof is the Pimsner-Voiculescu exact sequence, which is not available for crossed products with non-cyclic groups.

Chapter 6

Examples and calculations

The final chapter of this book is completely different from the others. We barely state or prove theorems here, we mostly make calculations.

In Chapters 3 and 5 we studied affine Hecke algebras in a very abstract way, almost without mentioning any examples. However the results in those chapters could hardly have been obtained without first checking, by hand, what happens in some simple exemplary cases. Basically we have two goals: we want have to examples of all the objects we introduced in Chapter 3, and we want to verify the conjectures made in Section 5.4 in some cases.

So we must devise a strategy to calculate the K-theory of the C^*-completion $C_r^*(\mathcal{R}, q)$ of an affine Hecke algebra with root datum $\mathcal{R}$ and label function q, and to find the homomorphisms (hopefully isomorphisms)

$$K_*(\phi_0) : K_*\big(C_r^*(\mathcal{R}, q^0)\big) \to K_*\big(C_r^*(\mathcal{R}, q)\big) \tag{6.1}$$

For q^0 we can use (2.105) and (3.91), which say that

$$K_*\big(C_r^*(W)\big) \otimes \mathbb{C} \cong K_*(C(T_u) \rtimes W_0) \otimes \mathbb{C} \cong \check{H}^*\big(\widetilde{T_u}; \mathbb{C}\big)^{W_0} \cong \check{H}^*\big(\widetilde{T_u}/W_0; \mathbb{C}\big) \tag{6.2}$$

Moreover, if $\check{H}^*\big(\widetilde{T_u}/W_0; \mathbb{Z}\big)$ is torsion free, then by Theorem 2.24

$$K_*\big(C_r^*(W)\big) = K_*\Big(C\big(T_u; \mathrm{End}\, \mathbb{C}[W_0]\big)^{W_0}\Big) \cong \check{H}^*\big(\widetilde{T_u}/W_0; \mathbb{Z}\big) \tag{6.3}$$

In general our procedure will involve the following steps.

1. Explicitly write down the root datum and the associated Weyl groups.
2. Determine the residual cosets and distinguish the different "genericity classes" of label functions.

   For every q there is, as noticed in (5.40), a canonical decomposition

   $$C_r^*(\mathcal{R}, q) = \bigoplus_P C_r^*(\mathcal{R}, q)_P \tag{6.4}$$

   where P runs over certain sets of simple roots.

3. List a good set of P's.

   For every chosen P we do the following:

4. Determine the root datum $\mathcal{R}_P$

5. Determine the discrete series of $\mathcal{H}(\mathcal{R}_P, q_P)$, and all the relevant intertwining operators.

   This is the only step where we can still encounter theoretical difficulties. The problem is that in general it is not known how many inequivalent discrete series representations there are. Fortunately we can decide this in the cases we consider.

6. Describe $C_r^*(\mathcal{R}, q)_P$ and its primitive ideal spectrum.

7. Calculate $K_*\big(C_r^*(\mathcal{R}, q)\big)_P$.

   Our main tools for this are excision and Proposition 2.21. In principle these will always lead to the answer, but it is tedious work that becomes unpractical in higher dimensions. On the other hand things become easier if we forget about torsion elements, for then we can use sheaf cohomology. For this reason will sometimes be satisfied with $K_*\big(C_r^*(\mathcal{R}, q)_P\big) \otimes_{\mathbb{Z}} \mathbb{C}$.

8. Find generating idempotents and invertibles, as explicit as possible.

9. Compare $K_*\big(C_r^*(\mathcal{R}, q)\big)$ and $K_*\big(C_r^*(\mathcal{R}, q^0)\big)$.

10. Determine $K_*(\phi_0)$ in terms of the given generators.

The results of these calculations are

- There are no examples for which (6.1) is known to be no isomorphism.
- (6.1) is an isomorphism for some root data of low rank
- (6.1) is an isomorphism the root data $\mathcal{R}(GL_n)$ and $\mathcal{R}(A_{n-1})^\vee$.

We will also see that these K-groups tend to be torsion free. This is due to the fact that the primitive ideal spaces of affine Hecke algebras look like quotients of tori by reflection groups, or direct products and unions of those. The root data we study all have free abelian K-groups, but whether this holds in general is hard to say.

6.1 A_1

Like in the theory of semisimple Lie algebras, the rank one root system A_1 plays an important role in the realm of Hecke algebras. Every Iwahori-Hecke algebra is in a sense built from such rank one Hecke algebras. There are two semisimple root data with $R_0 = A_1$, corresponding to the root lattice and the weight lattice. We will study the associated affine Hecke algebras in detail, and show that Conjecture 5.28 holds for these root data.

The notation $\mathcal{R}(A_1)$ will be reserved for X the root lattice. We start with the other case. Thus we consider the root datum $\mathcal{R}(A_1)^\vee = (X, Y, R_0, R_0^\vee, F_0)$ with

$$\begin{aligned}
&X = \mathbb{Z} \qquad Q = 2\mathbb{Z} \qquad X^+ = \mathbb{Z}_{\geq 0} \\
&Y = Q^\vee = \mathbb{Z} \\
&T = \mathbb{C}^\times \\
&R_0 = R_1 = \{\pm\alpha\} = \{\pm 2\} \\
&R_0^\vee = R_1^\vee = \big\{ \pm \alpha^\vee \big\} = \{\pm 1\} \\
&F_0 = \{\alpha\} \qquad W_0 = \{e, s_\alpha\} \\
&s_1 = s_\alpha : x \to -x \qquad s_0 = t_2 s_1 = t_1 s_1 t_{-1} : x \to 2 - x \\
&S_{\mathrm{aff}} = \{s_0, s_1\} \qquad W \neq W_{\mathrm{aff}} = \langle s_0, s_1 | s_0^2 = s_1^2 = e \rangle \\
&\Omega == \{e, \omega\} = \{e, t_1 s_1\}
\end{aligned}$$

For any label function q we have $q(s_0) = q(s_1) = q_{\alpha^\vee}$, and we denote this value simply by q. Then

$$c_\alpha = (1 - q^{-1}\theta_{-2})(1 - \theta_{-2})^{-1}$$

so generically the residual points are

$$q^{1/2},\ q^{-1/2},\ -q^{1/2},\ -q^{-1/2} \tag{6.5}$$

Hence there are only two essentially different cases, depending on whether q equals 1 or not.

- **group case q = 1**

From Theorem 3.15 we know that every irreducible representation is a direct summand of a unitary principal series $I_t = \pi(\emptyset, \delta_\emptyset, t)$. The underlying vector space of I_t is

$$\mathbb{C}[W_0] = \mathbb{C}T_e + \mathbb{C}T_{s_1}$$

and the intertwiner $\pi(s_1, \emptyset, \delta_\emptyset, t) : I_t \to I_{t^{-1}}$ is simply right multiplication by T_{s_1}. So with respect to the orthonormal basis

$$\big\{ 2^{-1/2}(T_e + T_{s_1}), 2^{-1/2}(T_e - T_{s_1}) \big\} \tag{6.6}$$

we have

$$\begin{aligned}
\mathcal{S}(W) &\cong \left\{ f \in C^\infty\big(S^1; M_2(\mathbb{C})\big) : f(t^{-1}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f(t) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \\
C_r^*(W) &\cong \big\{ f \in C\big([-1,1]; M_2(\mathbb{C})\big) : f(\pm 1) \text{ is diagonal} \big\}
\end{aligned} \tag{6.7}$$

The spectrum of these algebras is the non-Hausdorff space

$\mathrm{Prim}(C_r^*(W)) \quad \cong$ 

To calculate the K-theory we use the extension

$$0 \to C_0((-1,1); M_2(\mathbb{C})) \to C_r^*(W) \to \mathbb{C}^4 \to 0$$

defined by evaluation at 1 and -1. The associated exact hexagon is

$$\begin{array}{ccccc} 0 & \to & K_0(C_r^*(W)) & \to & \mathbb{Z}^4 \\ \uparrow & & & & \downarrow \\ 0 & \leftarrow & K_1(C_r^*(W)) & \leftarrow & \mathbb{Z} \end{array}$$

By comparing this with the standard extension

$$0 \to C_0((-1,1)) \to C(S^1) \to \mathbb{C} \to 0$$

we see that the vertical maps are surjective, so

$$\begin{aligned} K_0(C_r^*(W)) &\cong \mathbb{Z}^3 \\ K_1(C_r^*(W)) &= 0 \end{aligned} \tag{6.8}$$

We can even find explicit generating projections, namely

$$\begin{aligned} p_a &= (T_e + T_{s_1})/2 \\ p_b &= (T_e - T_{s_1})/2 \\ p_c &= T_e/2 - ((\theta_1 + \theta_{-1})T_{s_1} + i(\theta_{-1} - \theta_1)T_e)/4 \\ p_d &= T_e/2 + ((\theta_1 + \theta_{-1})T_{s_1} + i(\theta_{-1} - \theta_1)T_e)/4 \end{aligned} \tag{6.9}$$

The only relation between these generators is

$$[p_a] + [p_b] = [p_c] + [p_d] = [1] \in K_0(C_r^*(W))$$

- **generic, equal label case q $\neq$ 1**

- $P = \emptyset$

$$\begin{array}{llll} R_P = \emptyset & R_P^\vee = \emptyset & & \\ X^P = X & X_P = 0 & Y^P = Y & Y_P = 0 \\ T^P = T & T_P = \{1\} & K_P = \{1\} & \\ W^P = W(P,P) = \mathcal{W}_{PP} = W_0 & W_P = \{e\} & & \end{array}$$

Again there is a single intertwiner $\pi(s_1, \emptyset, \delta_\emptyset, t) : I_t \to I_{t^{-1}}$ and

$$\begin{aligned}
\imath^o_{s_1} &= (T_{s_1}(1-\theta_2) + (q-1)\theta_2)(q-\theta_2)^{-1} \\
\pi(s_1, \emptyset, \delta_\emptyset, 1) &= \pi(s_1, \emptyset, \delta_\emptyset, -1) = 1 \\
C_r^*(\mathcal{R}, q)_P &\cong C\big([-1,1]; M_2(\mathbb{C})\big) \\
\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) &\cong S^1/W_0 \cong [-1,1]
\end{aligned}$$

- $P = \{\alpha\}$

$$\begin{array}{l}
R_P = R_0 \qquad R_P^\vee = R_0^\vee \\
W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} \qquad W_P = W_0 \\
X^P = 0 \qquad X_P = X \qquad Y^P = 0 \qquad Y_P = Y \\
T^P = \{1\} \qquad T_P = T \qquad K_P = \{1\}
\end{array}$$

From Proposition 3.20.2 we know that there is exactly one discrete series representation for every orbit of residual points. So the spectrum of $C_r^*(\mathcal{R}, q)$ contains two isolated points, which we call δ_1 and δ_{-1}, by the sign of their central character.

$$C_r^*(\mathcal{R}, q)_P \cong \mathbb{C}^2$$

By brute calculation one finds the associated projectors

$$\begin{array}{lcll}
p_1 &=& \sum_{w \in W_{\mathrm{aff}}} (-q)^{\ell(w)} T_w (T_e - T_\omega) \left(\sum_{w\in W} q(w)^{-1} \right)^{-1} & \text{if} \quad q > 1 \\
p_1 &=& \sum_{w \in W} T_w \left(\sum_{w\in W} q(w)^{-1} \right)^{-1} & \text{if} \quad q < 1 \\
p_{-1} &=& \sum_{w \in W} (-q)^{\ell(w)} T_w \left(\sum_{w\in W} q(w) \right)^{-1} & \text{if} \quad q > 1 \\
p_{-1} &=& \sum_{w \in W_{\mathrm{aff}}} T_w (T_e - T_\omega) \left(\sum_{w\in W} q(w) \right)^{-1} & \text{if} \quad q < 1
\end{array} \tag{6.10}$$

Combining the results for $P = \emptyset$ and $P = \{\alpha\}$, the spectrum of $C_r^*(\mathcal{R}, q)$ becomes the Hausdorff space

$\mathrm{Prim}(C_r^*(W)) \quad \cong$ 

This implies

$$\begin{aligned}
K_0(C_r^*(\mathcal{R}, q)) &\cong \mathbb{Z}^3 \\
K_1(C_r^*(\mathcal{R}, q)) &= 0
\end{aligned} \tag{6.11}$$

Let p_0 be any rank one projector in $C([-1,1]; M_2(\mathbb{C}))$. Then (the classes of) p_{-1}, p_0 and p_1 generate $K_0(C_r^*(\mathcal{R}, q))$, and we have

$$[p_{-1}] + [p_1] + 2[p_0] = [1] \in K_0(C_r^*(\mathcal{R}, q))$$

Now we can compare the cases $q = 1$ and $q \neq 1$. Looking carefully at the behaviour near $t = 1$ and $t = -1$, we find that $K_0(\phi_0)$ is always an isomorphism. We describe this map completely with the following table.

$q < 1$	$q = 1$	$q > 1$
$p_0 + p_1 + p_{-1}$	p_a	p_0
p_0	p_b	$p_0 + p_1 + p_{-1}$
$p_0 + p_{-1}$	p_c	$p_0 + p_1$
$p_0 + p_1$	p_d	$p_0 + p_{-1}$

(6.12)

Let us move on to the type A_1 Hecke algebras with X the root lattice. This means that work with the root datum $\mathcal{R}(A_1)$:

$$\begin{aligned}
&X = Q = \mathbb{Z} \qquad X^+ = \mathbb{Z}_{\geq 0} \\
&Y = \mathbb{Z} \qquad Q^\vee = 2\mathbb{Z} \\
&T = \mathbb{C}^\times \\
&R_0 = \{\pm\alpha\} = \{\pm 1\} \qquad R_1 = \{\pm 2\} \\
&R_0^\vee = \{\pm\alpha^\vee = \{\pm 2\} \qquad R_1^\vee = \{\pm 1\} \\
&F_0 = \{\alpha\} \qquad W_0 = \{e, s_\alpha\} \\
&s_1 = s_\alpha : x \to -x \qquad s_0 = t_1 s_1 : x \to 1 - x \\
&S_{\text{aff}} = \{s_0, s_1\} \qquad W = W_{\text{aff}} = \langle s_0, s_1 | s_0^2 = s_1^2 = e\rangle
\end{aligned}$$

Now s_0 and s_1 are no longer conjugate, so $q_0 = q(s_0)$ and $q_1 = q(s_1)$ may be different. By definition

$$\begin{aligned}
&q_{\alpha^\vee} = q_0 \qquad q_{\alpha^\vee/2} = q_1 q_0^{-1} \\
&c_\alpha = (1 + q_1^{-1/2} q_0^{1/2} \theta_{-1})(1 - q_1^{-1/2} q_0^{-1/2} \theta_{-1})(1 - \theta_{-2})^{-1}
\end{aligned}$$

Generically the residual point are

$$q_0^{1/2} q_1^{1/2} \qquad q_0^{-1/2} q_1^{-1/2} \qquad -q_0^{1/2} q_1^{-1/2} \qquad -q_0^{-1/2} q_1^{1/2} \tag{6.13}$$

From this we see that there are four cases to study: the group case $q_0 = q_1 = 1$, the equal label case $q_0 = q_1 \neq 1$, the "inverse label" case $q_0 = q_1^{-1} \neq 1$ and the generic case.

- **group case $\mathbf{q_0 = q_1 = 1}$**

This is the same as the group case for X equal to the weight lattice of A_1.

- **equal label case $\mathbf{q_0 = q_1 = q \neq 1}$**

- $P = \emptyset$

$$R_P = \emptyset \qquad R_P^\vee = \emptyset$$
$$X^P = X \qquad X_P = 0 \qquad Y^P = Y \qquad Y_P = 0$$
$$T^P = T \qquad T_P = \{1\} \qquad K_P = \{1\}$$
$$W^P = W(P,P) = \mathcal{W}_{PP} = W_0 \qquad W_P = \{e\}$$
$$\imath^o_{s_1} = (T_{s_1}(1-\theta_1) + (q-1)\theta_1)(q-\theta_1)^{-1}$$

With respect to the orthonormal basis

$$\{(T_e + T_{s_1})(1+q)^{-1/2}, (qT_e - T_{s_1})(q+q^2)^{-1/2}\} \tag{6.14}$$

of $\mathcal{H}(W_0, q)$ we have

$$\pi(s_1, \emptyset, \delta_\emptyset, 1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \pi(s_1, \emptyset, \delta_\emptyset, -1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$C_r^*(\mathcal{R}, q)_P \cong \big\{ f \in C\big([-1,1]; M_2(\mathbb{C})\big) : f(-1) \text{ is diagonal} \big\}$$

$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \quad \cong$ 

- $P = \{\alpha\}$

$$R_P = R_0 \qquad R_P^\vee = R_0^\vee$$
$$X^P = 0 \qquad X_P = X \qquad Y^P = 0 \qquad Y_P = Y$$
$$T^P = \{1\} \qquad T_P = T \qquad K_P = \{1\}$$
$$W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} \qquad W_P = W_0$$

Obviously $-q_0^{1/2} q_1^{-1/2} = -q_0^{-1/2} q_1^{1/2} = -1$, so these points are not residual for this particular label function. On the other hand, by Proposition 3.20.2 there is a unique discrete series representation δ_1 with central character $q^{\pm 1}$. It has dimension 1, and the corresponding projection is

$$\begin{aligned} p_1 &= \sum_{w \in W} (-q)^{\ell(w)} T_w \left(\sum_{w \in W} q(w)^{-1} \right)^{-1} && \text{if} \quad q > 1 \\ p_1 &= \sum_{w \in W} T_w \left(\sum_{w \in W} q(w) \right)^{-1} && \text{if} \quad q < 1 \end{aligned} \tag{6.15}$$

We conclude that

$$C_r^*(\mathcal{R}, q) \cong \big\{ f \in C\big([-1,1]; M_2(\mathbb{C})\big) : f(-1) \text{ is diagonal} \big\} \oplus \mathbb{C}$$

$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)\big) \quad \cong$ 

Evaluating at -1 and at $q^{\pm 1}$ yields an extension

$$0 \to C_0\big([-1,1], \{-1\}; M_2(\mathbb{C})\big) \to C_r^*(\mathcal{R}, q) \to \mathbb{C}^3 \to$$

whose exact hexagon in K-theory is

$$\begin{array}{ccccc} 0 & \to & K_0(C_r^*(\mathcal{R},q)) & \to & \mathbb{Z}^3 \\ \uparrow & & & & \downarrow \\ 0 & \leftarrow & K_1(C_r^*(\mathcal{R},q)) & \leftarrow & 0 \end{array}$$

This shows that

$$\begin{aligned} K_0(C_r^*(\mathcal{R},q)) &\cong \mathbb{Z}^3 \\ K_1(C_r^*(\mathcal{R},q)) &= 0 \end{aligned} \tag{6.16}$$

Generating projections are

$$p_1 = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right) \qquad p_a = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0 \right) \qquad p_b = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0 \right)$$

Explicitly this works out to

$$\begin{array}{lcll} p_a & = & (T_e + T_{s_1})(1+q)^{-1} & \text{if} \quad q > 1 \\ p_a & = & (T_e + T_{s_1})(1+q)^{-1} - p_1 & \text{if} \quad q < 1 \\ p_b & = & (qT_e - T_{s_1})(1+q)^{-1} - p_1 & \text{if} \quad q > 1 \\ p_a & = & (T_e + T_{s_1})(1+q)^{-1} & \text{if} \quad q < 1 \end{array} \tag{6.17}$$

- **inverse label case $\mathbf{q = q_1 = q_0^{-1} \neq 1}$**
- $P = \emptyset$

$$\begin{array}{l} R_P = \emptyset \qquad R_P^\vee = \emptyset \\ X^P = X \qquad X_P = 0 \qquad Y^P = Y \qquad Y_P = 0 \\ T^P = T \qquad T_P = \{1\} \qquad K_P = \{1\} \\ W^P = W(P,P) = \mathcal{W}_{PP} = W_0 \qquad W_P = \{e\} \\ \imath^o_{s_1} = (T_{s_1}(1+\theta_1) + (1-q)\theta_1)(q+\theta_1)^{-1} \end{array}$$

With respect to the basis (6.14) we have

$$\pi(s_1, \emptyset, \delta_\emptyset, 1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \pi(s_1, \emptyset, \delta_\emptyset, -1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_r^*(\mathcal{R},q)_P \cong \big\{ f \in C\big([-1,1]; M_2(\mathbb{C})\big) : f(1) \text{ is diagonal} \big\}$$

$\mathrm{Prim}\big(C_r^*(\mathcal{R},q)_P\big) \quad \cong$ 

- $P = \{\alpha\}$

$$\begin{array}{l} R_P = R_0 \qquad R_P^\vee = R_0^\vee \\ X^P = 0 \qquad X_P = X \qquad Y^P = 0 \qquad Y_P = Y \\ T^P = \{1\} \qquad T_P = T \qquad K_P = \{1\} \\ W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} \qquad W_P = W_0 \end{array}$$

Now we have $q_0^{1/2} q_1^{1/2} = q_0^{-1/2} q_1^{-1/2} = 1$, so these points are not residual. There is a unique discrete series representation δ_{-1} with central character $-q^{\pm 1}$. Its projector is already a little more difficult to describe. For $w \in W$ write $\ell(w) = \ell_0(w) + \ell_1(w)$, where ℓ_1 counts the number of factors s_i in an reduced expression for w. Notice that this is well-defined only because there are no relations between s_0 and s_1 in W_{aff}.

$$\begin{aligned} p_{-1} &= \sum_{w \in W} (-q_1)^{\ell_1(w)/2} T_w \left(\sum_{w \in W} q^{-\ell(w)} \right)^{-1} \quad \text{if} \quad q > 1 \\ p_{-1} &= \sum_{w \in W} (-q_0)^{\ell_0(w)/2} T_w \left(\sum_{w \in W} q^{\ell(w)} \right)^{-1} \quad \text{if} \quad q < 1 \end{aligned} \tag{6.18}$$

Summarizing, we have

$$C_r^*(\mathcal{R}, q) \cong \mathbb{C} \oplus \big\{ f \in C\big([-1,1]; M_2(\mathbb{C})\big) : f(-1) \text{ is diagonal} \big\}$$

$\text{Prim}\big(C_r^*(\mathcal{R}, q)\big) \quad \cong$ 

Like in the equal case this leads to

$$\begin{aligned} K_0(C_r^*(\mathcal{R}, q)) &\cong \mathbb{Z}^3 \\ K_1(C_r^*(\mathcal{R}, q)) &= 0 \end{aligned} \tag{6.19}$$

and generating projections are

$$p_{-1} = \left(1, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) \qquad p_a = \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) \qquad p_b = \left(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

Now they are given by

$$\begin{aligned} p_a &= (T_e + T_{s_1})(1+q)^{-1} && \text{if} \quad q > 1 \\ p_a &= (T_e T_{s_1})(1+q)^{-1} - p_{-1} && \text{if} \quad q < 1 \\ p_b &= (qT_e - T_{s_1})(1+q)^{-1} - p_{-1} && \text{if} \quad q > 1 \\ p_b &= (qT_e - T_{s_1})(1+q)^{-1} && \text{if} \quad q < 1 \end{aligned} \tag{6.20}$$

- **generic case $\mathbf{q_0 \neq q_1 \neq q_0^{-1}}$**

- $P = \emptyset$

$$R_P = \emptyset \qquad R_P^\vee = \emptyset$$
$$X^P = X \qquad X_P = 0 \qquad Y^P = Y \qquad Y_P = 0$$
$$T^P = T \qquad T_P = \{1\} \qquad K_P = \{1\}$$
$$W^P = W(P,P) = \mathcal{W}_{PP} = W_0 \qquad W_P = \{e\}$$
$$\imath_{s_1}^o = (T_{s_1}(1+\theta_1) + (1-q)\theta_1)(q+\theta_1)^{-1}$$

In this case all unitary principal series representations are irreducible:

$$\pi(s_1, \emptyset, \delta_\emptyset, 1) = 1 \qquad \pi(s_1, \emptyset, \delta_\emptyset, -1) = 1$$
$$C_r^*(\mathcal{R}, q)_P \cong C\big([-1,1]; M_2(\mathbb{C})\big)$$
$$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \cong S^1/W_0 \cong [-1,1]$$

- $P = \{\alpha\}$

$$R_P = R_0 \qquad R_P^\vee = R_0^\vee$$
$$X^P = 0 \qquad X_P = X \qquad Y^P = 0 \qquad Y_P = Y$$
$$T^P = \{1\} \qquad T_P = T \qquad K_P = \{1\}$$
$$W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} \qquad W_P = W_0$$

The residual points were already listed in (6.13). By Proposition 3.20.2 there are exactly two inequivalent discrete series representations, δ_1 and δ_{-1}.

$$C_r^*(\mathcal{R}, q)_P \cong \mathbb{C}^2$$

To write down the corresponding projections we have to distinguish four cases.

$$p_1 = \sum_{w \in W} (-1)^{\ell(w)} q(w)^{-1} T_w \left(\sum_{w \in W} q(w)^{-1} \right)^{-1} \qquad \text{if } q_0 > q_1^{-1} \tag{6.21}$$

$$p_1 = \sum_{w \in W} T_w \left(\sum_{w \in W} q(w) \right)^{-1} \qquad \text{if } q_0 < q_1^{-1} \tag{6.22}$$

$$p_{-1} = \sum_{w \in W} (-q_1)^{\ell_1(w)/2} T_w \left(\sum_{w \in W} q_1^{-\ell_1(w)} q_0^{\ell_0(w)} \right)^{-1} \qquad \text{if } q_0 < q_1 \tag{6.23}$$

$$p_{-1} = \sum_{w \in W} (-q_0)^{\ell_0(w)/2} T_w \left(\sum_{w \in W} q_0^{-\ell_0(w)} q_1^{\ell_1(w)} \right)^{-1} \qquad \text{if } q_0 > q_1 \tag{6.24}$$

If we consider δ_1 and δ_{-1} only as representations of $\mathcal{H}(W_0, q)$, then in this list (6.21) and (6.23) are deformations of the sign representation, while (6.22) and (6.24) are deformations of the trivial representation.

We conclude that

$$C_r^*(\mathcal{R}, q) \cong \mathbb{C} \oplus C\big([-1,1]; M_2(\mathbb{C})\big) \oplus \mathbb{C}$$

$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)\big) \quad \cong$

Hence in the generic case also

$$\begin{aligned} K_0(C_r^*(\mathcal{R}, q)) &\cong \mathbb{Z}^3 \\ K_1(C_r^*(\mathcal{R}, q)) &= 0 \end{aligned} \tag{6.25}$$

Canonical generators are $[p_{-1}]$, $[p_1]$ and $[p_0]$, where p_0 is any rank one projector in $C\big([-1,1];M_2(\mathbb{C})\big)$.

So for this root datum the K-groups are independent of the label function. Moreover the various maps $K_0(\phi_0)$ all turn out to be isomorphisms. We list the images of the projections p_a, p_b, p_c and p_d below, in that order.

$q_0 = q_1 < 1$	$q_0^{-1} < q_1 > q_0$	$q_0 = q_1 > 1$
p_a	p_0	p_a
$p_b + p_{-1}$	$p_0 + p_1 + p_{-1}$	$p_b + p_1$
p_b	$p_0 + p_1$	$p_a + p_1$
$p_a + p_{-1}$	$p_0 + p_{-1}$	p_b
$q_1^{-1} > q_0 < q_1$	$q_0 = q_1 = 1$	$q_1^{-1} < q_0 > q_1$
$p_0 + p_1$	p_a	$p_0 + p_{-1}$
$p_0 + p_{-1}$	p_b	$p_0 + p_1$
p_0	p_c	$p_0 + p_1 + p_{-1}$
$p_0 + p_1 + p_{-1}$	p_d	p_0
$q_0 = q_1 < 1$	$q_0^{-1} > q_1 < q_0$	$q_0^{-1} = q_1 < 1$
$p_a + p_1$	$p_0 + p_1 + p_{-1}$	$p_a + p_{-1}$
p_b	p_0	p_b
p_a	$p_0 + p_{-1}$	$p_b + p_{-1}$
$p_b + p_1$	$p_0 + p_1$	p_a

(6.26)

6.2 GL_2

The simplest twodimensional root datum which is not a product of two onedimensionale root data is $\mathcal{R}(GL_2)$.

$$
\begin{aligned}
&X = \mathbb{Z}^2 \qquad Q = \{(n,-n) : n \in \mathbb{Z}\} \qquad X^+ = \{(m,n) \in \mathbb{Z}^2 : m \geq n\}\\
&Y = \mathbb{Z}^2 \qquad Q^\vee = \{(n,-n) : n \in \mathbb{Z}\}\\
&T = (\mathbb{C}^\times)^2 \qquad t = (t_1,t_2) = (t(1,0), t(0,1))\\
&R_0 = \{\pm\alpha\} = \{\pm(1,-1)\} = R_1\\
&R_0^\vee = \{\pm\alpha^\vee\} = \{\pm(1,-1)\} = R_1^\vee\\
&F_0 = \alpha \qquad W_0 = \{e, s_\alpha\}\\
&s_1 = s_\alpha : (m,n) \to (n,m) \qquad s_0 = t_\alpha s_\alpha = t_{(1,0)} s_1 t_{(-1,0)} : (m,n) \to (n+1, m-1)\\
&S_{\text{aff}} = \{s_0, s_1\} \qquad W \neq W_{\text{aff}} = \langle s_0, s_1 | s_0^2 = s_1^2 = e\rangle\\
&\Omega = \langle\omega\rangle = \langle t_{(1,0)} s_1\rangle \cong \mathbb{Z}
\end{aligned}
$$

For any label function q we have $q(s_0) = q(s_1) = q_{\alpha^\vee}$, so we call this value q. There are no residual point because $\mathcal{R}$ is not semisimple. We do have two residual cosets of dimension one, namely

$$\{t \in T : t_1^{-1} t_2 = q\} \quad \text{and} \quad \{t \in T : t_1^{-1} t_2 = q^{-1}\}$$

So there are only two really different cases, $q = 1$ and $q \neq 1$.

- **group case $\mathbf{q = 1}$**

As said before, we only need to look at unitary principal series representations. There is a single nonscalar intertwiner $\pi(s_1, \emptyset, \delta_\emptyset, t) : I_{(t_1,t_2)} \to I_{(t_2,t_1)}$ It is given by right multiplication with T_{s_1}, so with respect to the basis (6.6) we have

$$\mathcal{S}(W) \cong \left\{ f \in C^\infty\big(\mathbb{T}^2; M_2(\mathbb{C})\big) : f(t_2, t_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f(t_2, t_1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

Let M be the closed Möbius strip and ∂M its boundary. We see that

$$C_r^*(W) \cong \{f \in C(M; M_2(\mathbb{C})) : f(m) \text{ is diagonal if } m \in \partial M\}$$

Consider the ideal

$$A_1 = \{f \in C(M; M_2(\mathbb{C})) : f(m) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \text{ if } m \in \partial M\}$$

According to Proposition 2.21 the inclusion $A_1 \to C(M; M_2(\mathbb{C}))$ induces an isomorphism on K-theory. However, M is homotopy equivalent to a circle, so

$$K_0(A_1) \cong K_1(A_1) \cong \mathbb{Z}$$

Moreover $\partial M \cong S^1$, so from

$$0 \to A_1 \to C_r^*(W) \to C(\partial M) \to 0$$

we get

$$\begin{array}{ccccc} \mathbb{Z} & \to & K_0(C_r^*(W)) & \to & \mathbb{Z} \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \leftarrow & K_1(C_r^*(W)) & \leftarrow & \mathbb{Z} \end{array}$$

Now it is not difficult to see that the upper part of this hexagon is exact, and hence

$$\begin{aligned} K_0\big(C_r^*(W)\big) &\cong \mathbb{Z}^2 \\ K_1\big(C_r^*(W)\big) &\cong \mathbb{Z}^2 \end{aligned} \tag{6.27}$$

Generating projections and unitaries are

$$\begin{aligned} & p_a = (T_e + T_{s_1})/2 \\ & p_b = (T_e - T_{s_1})/2 \\ & \theta_{(0,1)} \\ & u = \theta_{(0,1)} p_a + \theta_{(0,-1)} p_b \end{aligned} \tag{6.28}$$

- **generic, equal label case q ≠ 1**
- $P = \emptyset$

$$
\begin{aligned}
&R_P = \emptyset \qquad R_P^\vee = \emptyset \\
&X^P = X \qquad X_P = 0 \qquad Y^P = Y \qquad Y_P = 0 \\
&T^P = T \qquad T_P = \{1\} \qquad K_P = \{1\} \\
&W^P = W(P,P) = \mathcal{W}_{PP} = W_0 \qquad W_P = \{e\} \\
&\imath^o_{s_1} = (T_{s_1}(1-\theta_{(1,-1)}) + (1-q)\theta_{(1,-1)})(q+\theta_{(1,-1)})^{-1}
\end{aligned}
$$

If $s_1(t) = t$ then $\pi(s_1, \emptyset, \delta_\emptyset, t) = 1$, so

$$
\begin{aligned}
&C_r^*(\mathcal{R},q)_P \cong C(M; M_2(\mathbb{C})) \\
&\mathrm{Prim}\big(C_r^*(\mathcal{R},q)_P\big) \cong T_u/W_0 \cong M
\end{aligned}
$$

- $P = \{\alpha\}$

$$
\begin{aligned}
&R_P = R_0 \qquad R_P^\vee = R_P \\
&X^P = X/\mathbb{Z}\alpha \cong \mathbb{Z} \qquad X_P = X/(R_P^\vee)^\perp \cong \mathbb{Z}\alpha/2 \\
&Y^P = Y \cap R_P^\perp = \mathbb{Z}(1,1) \qquad Y_P = Y \cap \mathbb{Q}R_P^\vee = \mathbb{Z}\alpha^\vee \\
&T^P = \{(t_1,t_1) : t_1 \in \mathbb{C}^\times\} \qquad T_P = \{(t_1,t_1^{-1}) : t_1 \in \mathbb{C}^\times\} \\
&K_P = \{(1,1),(-1,-1)\} = \{1, k_P\}
\end{aligned}
$$

The root datum $\mathcal{R}_P$ is isomorphic to $\mathcal{R}(A_1)^\vee$, so we an use the description of the discrete series on page 175. The representations $\pi(P, \delta_1, (t_1,t_1))$ and $\pi(P, \delta_1, (-t_1,-t_1))$ are intertwined by $\pi(k_P)$.

$$
\begin{aligned}
&C_r^*(\mathcal{R},q)_P \cong C(S^1) \\
&\mathrm{Prim}\big(C_r^*(\mathcal{R},q)_P\big) \cong (P, W_P(q^{1/2}, q^{-1/2}), \delta_1, T_u^P) \cong S^1
\end{aligned}
$$

The associated projector is

$$
\begin{aligned}
p_\alpha &= \sum_{w \in W_{\mathrm{aff}}} (-q)^{\ell(w)} T_w \left(\sum_{w \in W_{\mathrm{aff}}} q(w)^{-1} \right)^{-1} \quad \text{if} \quad q > 1 \\
p_\alpha &= \sum_{w \in W_{\mathrm{aff}}} T_w \left(\sum_{w \in W_{\mathrm{aff}}} q(w) \right)^{-1} \quad \text{if} \quad q < 1
\end{aligned} \tag{6.29}
$$

Adding these two summands we get

$$
\begin{aligned}
&K_0\big(C_r^*(\mathcal{R},q)\big) \cong \mathbb{Z}^2 \\
&K_1\big(C_r^*(\mathcal{R},q)\big) \cong \mathbb{Z}^2
\end{aligned} \tag{6.30}
$$

These groups are generated by the classes of the projections

$$
p_\alpha = \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right) \quad \text{and} \quad p_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 1 \right)
$$

and of the invertibles

$$u_\alpha = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathrm{id}_{S^1} \right) \quad \text{and} \quad u_0 = \left(\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}, 1 \right)$$

where $r : M \to S^1$ is a homotopy equivalence. Explicitly we may take

$$\begin{array}{rcll} u_\alpha & = & p_\alpha \theta_{(1,0)} + 1 - p_\alpha & \\ u_0 & = & \theta_{(1,0)}(1 - p_\alpha) + p_\alpha & \\ p_0 & = & (T_e + T_{s_1})(1+q)^{-1} & \text{if} \quad q > 1 \\ p_0 & = & (T_{s_1} - qT_e)(1+q)^{-1} & \text{if} \quad q < 1 \end{array} \tag{6.31}$$

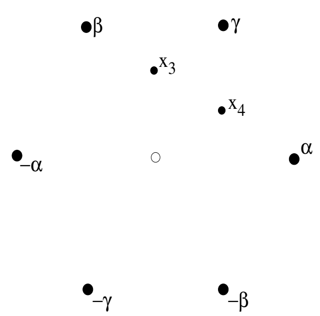
Hence for this root system the K-theory of $\mathcal{S}(\mathcal{R}, q))$ is independent of q. The group isomorphisms $K_0(\phi_0)$ are as follows:

$q < 1$	$q = 1$	$q > 1$
$p_0 + p_\alpha$	p_a	p_0
p_0	p_b	$p_0 + p_\alpha$
u_0	$\theta_{(1,0)}$	u_0
$u_0 u_\alpha$	u	$u_0 u_\alpha^{-1}$

(6.32)

6.3 A_2

Somewhat unusually we will not consider A_2 as embedded in $\mathbb{R}^3$, but only as a twodimensional object. There are two semisimple root data with R_0 of type A_2, depending on whether X is the root lattice or the weight lattice. The latter case is easier, so let us draw this root system together with the fundamental weights x_3 and x_4.



The root datum $\mathcal{R}(A_2)^\vee$ is described by

$$
\begin{aligned}
&X = \mathbb{Z}x_3 + \mathbb{Z}x_4 = \mathbb{Z}\big(0, 3^{-1/2}\big) + \mathbb{Z}\big(1/2, 3^{-1/2}/2\big) \qquad X^+ = \mathbb{N}x_3 + \mathbb{N}x_4 \\
&Q = \mathbb{Z}(1,0) + \mathbb{Z}\big(-1/2, \sqrt{3}/2\big) \qquad Y = Q^\vee = \mathbb{Z}(2,0) + \mathbb{Z}\big(1, \sqrt{3}\big) \\
&T = (\mathbb{C}^\times)^2 \qquad t = (t_3, t_4) = (t(x_3), t(x_4)) \\
&R_0 = \{\pm\alpha, \pm\beta, \pm\gamma\} = \{\pm(1,0), \pm\big(-1/2, \sqrt{3}/2\big), \pm\big(1/2, \sqrt{3}/2\big)\} = R_1 \\
&R_0^\vee = \big\{\pm\alpha^\vee, \pm\beta^\vee, \pm\gamma^\vee\big\} = \{\pm(1,0), \pm\big(-1/2, \sqrt{3}/2\big), \pm\big(1/2, \sqrt{3}/2\big)\} = R_1^\vee \\
&F_0 = \{\alpha, \beta\} \qquad W_0 = \langle s_\alpha, s_\beta | s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^3 = e\rangle \cong S_3 \\
&s_1 = s_\alpha : (n, m) \to (-n, m) \quad s_2 = s_\beta : \big(n + m/2, m\sqrt{3}/2\big) \to \big(m + n/2, n\sqrt{3}/2\big) \\
&s_0 = t_\gamma s_\gamma : \big(n + m/2, m\sqrt{3}/2\big) \to \big((1+n-m)/2, (1-n-m)\sqrt{3}/2\big) \\
&S_{\text{aff}} = \{s_0, s_1, s_2\} \qquad W \neq W_{\text{aff}} = \langle s_0, W_0 | s_0^2 = (s_0 s_1)^3 = (s_0 s_2)^3 = e\rangle \\
&\Omega = \{e, \omega_1, \omega_2\} = \{e, t_{x_3} s_\beta s_\alpha, t_{x_3} s_\alpha s_\beta\}
\end{aligned}
$$

For any label function q we have

$$q(s_0) = q(s_1) = q(s_2) = q(s_3) = q_{\alpha^\vee} = q_{\beta^\vee} = q_{\gamma^\vee}$$

so we denoted this value simply by q.

$$c_\eta = (1 - q^{-1}\theta_{-\eta})(1 - \theta_{-\eta})^{-1}$$

for $\eta \in \{\alpha, \beta, \gamma\}$. Generically there are 6 tempered residual circles and 18 residual points. Representatives for the W_0-conjugacy classes are

$$(q, q) \quad (q\zeta, q\zeta^2) \quad (q\zeta^2, q\zeta) \quad \text{and} \quad \big\{(q^{1/3}t_4^2, q^{2/3}t_4) : t_4 \in \mathbb{T}\big\} \tag{6.33}$$

where $\zeta = e^{2\pi i/3}$ is a root of unity.

- **group case $\mathbf{q = 1}$**

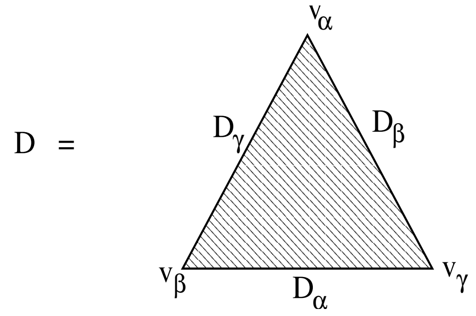
  In the compact torus T_u there are three W_0-invariant points:

$$(1,1) \qquad (\zeta, \zeta^2) \qquad (\zeta^2, \zeta) \tag{6.34}$$

Furthermore we have the following circles with nontrivial stabilizers:

$$
\begin{aligned}
&\big\{(t_4^2, t_4) : t_4 \in \mathbb{T}\big\} \qquad s_\alpha \\
&\big\{(t_3, t_3^2) : t_3 \in \mathbb{T}\big\} \qquad s_\beta \\
&\big\{(t_3, t_3^{-1}) : t_3 \in \mathbb{T}\big\} \quad s_\gamma
\end{aligned}
$$

These circles are conjugate under W_0. Therefore $\mathrm{Prim}(\mathcal{S}(\mathcal{R}, q))$ is a non-Hausdorff triangle whose interior is Hausdorff, whose edges are doubled and whose vertices are triple points. Let us call the underlying Hausdorff quotient space T_u/W_0 D, and its edges D_α, D_β and D_γ, indicating their stabilizer.



There is no need to indicate which of the points (6.34) is v_α, as all configurations occur, for a suitable choice of a fundamental domain.

$$C_r^*(W) \cong \big\{ f \in C\big(D; \mathrm{End}(\mathbb{C}[W_0])\big) : T_{s_\eta} f(t) T_{s_\eta} = f(t) \;\forall t \in D_\eta \;, \\ f(v_\eta) \in \mathrm{End}(\mathbb{C}[W_0])^{W_0} \;\forall \eta \in \{\alpha, \beta, \gamma\} \big\}$$

Consider the extension

$$\begin{aligned} & 0 \to C_0\big(D, \partial D; \mathrm{End}(\mathbb{C}[W_0])\big) \to C_r^*(W) \to A_1 \to 0 \\ & A_1 = \big\{ f \in C\big(\partial D; \mathrm{End}(\mathbb{C}[W_0])\big) : T_{s_\eta} f(t) T_{s_\eta} = f(t) \;\forall t \in D_\eta \;, \\ & \qquad f(v_\eta) \in \mathrm{End}(\mathbb{C}[W_0])^{W_0} \;\forall \eta \in \{\alpha, \beta, \gamma\} \big\} \end{aligned} \tag{6.35}$$

In the vertices v_η thet A_1-representation $\mathbb{C}[W_0]$ is a direct sum of three irreducibles: the trivial W_0-representation, the sign representation of W_0 and the defining (reflection) representation of W_0 (with multiplicity two). Likewise, on the edge D_η $\mathbb{C}[W_0]$ is the direct sum of a part corresponding to the trivial representation of $\{e, s_\eta\}$ and a part corresponding to the trivial representation of $\{e, s_\eta\}$. Both summands have dimension three. Evaluating the reflection representations at the vertices gives an extension

$$\begin{aligned} & 0 \to A_2^2 \to A_1 \to M_2(\mathbb{C})^3 \to 0 \\ & A_2 = \{ f \in C(\partial D; M_3(\mathbb{C})) : f(v_\eta) \in \mathbb{C} \oplus O_2 \} \end{aligned} \tag{6.36}$$

where O_2 is the 2×2 zero matrix. By Propostion 2.21 the inclusion $A_2 \to C(\partial D; M_3(\mathbb{C}))$ induces an isomorphism on K-theory, so we have an exact hexagon

$$\begin{array}{ccccc} \mathbb{Z}^2 & \to & K_0(A_1) & \to & \mathbb{Z}^3 \\ \uparrow & & & & \downarrow \\ 0 & \leftarrow & K_1(A_1) & \leftarrow & \mathbb{Z}^2 \end{array}$$

The vertical maps are 0, so

$$\begin{aligned} K_0(A_1) &\cong \mathbb{Z}^5 \\ K_1(A_1) &\cong \mathbb{Z}^2 \end{aligned}$$

Plugging this into (6.35) we get

$$\begin{array}{ccccc} \mathbb{Z} & \to & K_0\big(C_r^*(W)\big) & \to & \mathbb{Z}^5 \\ \uparrow & & & & \downarrow \\ \mathbb{Z}^2 & \leftarrow & K_1\big(C_r^*(W)\big) & \leftarrow & 0 \end{array}$$

In terms of suitable generators the left vertical map becomes addition, and therefore

$$\begin{aligned} K_0\big(C_r^*(W)\big) &\cong \mathbb{Z}^5 \\ K_1\big(C_r^*(W)\big) &\cong \mathbb{Z} \end{aligned} \tag{6.37}$$

Define the projections

$$\begin{array}{rcl} p_{\text{triv}} & = & \frac{1}{6} \sum_{w \in W_0} T_w \\ p_{\text{sign}} & = & \frac{1}{6} \sum_{w \in W_0} (-1)^{\ell(w)} T_w \end{array}$$

We can unambiguously define classes of projections p_3, p_4, p_5 by the requirements

$$p_i(\zeta^j, \zeta^{3-j}) \left\{ \begin{array}{llll} = & p_{\text{triv}} + p_{\text{sign}} & \text{if} & i \neq j \mod 3 \\ \perp & p_{\text{triv}} + p_{\text{sign}} & \text{if} & i = j \mod 3 \end{array} \right.$$

Then $K_0\big(C_r^*(W)\big)$ is generated by

$$[p_{\text{triv}}] \qquad [p_{\text{sign}}] \qquad [p_3] \qquad [p_4] \qquad [p_5]$$

and a generator for $K_1\big(C_r^*(W)\big)$ is

$$[u] = [p_{\text{triv}} N_{x_3} p_{\text{triv}} + p_{\text{sign}} N_{-x_3} p_{\text{sign}} + T_e - p_{\text{triv}} - p_{\text{sign}}]$$

- **generic case $\mathbf{q \neq 1}$**

- $P = \emptyset$

$$\begin{array}{llll} R_P = \emptyset & R_P^\vee = \emptyset & & \\ X^P = X & X_P = 0 & Y^P = Y & Y_P = 0 \\ T^P = T & T_P = \{1\} & K_P = \{1\} & \\ W^P = W(P,P) = \mathcal{W}_{PP} = W_0 & W_P = \{e\} & & \\ \imath^o_{s_eta} = (T_{s_\eta}(1-\theta_\eta) + (q-1)\theta_\eta)(q-\theta_\eta)^{-1} & \eta \in \{\alpha, \beta, \gamma\} & & \end{array}$$

If $s_\eta(t) = t$ then $\imath^o_{s_eta} = 1$, and there are no points with stabilizer $\{e, s_1 s_2, s_2 s_1\}$, so

$$\begin{aligned} &C_r^*(\mathcal{R}, q)_P \cong C(D; M_6(\mathbb{C})) \\ &\text{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \cong T_u / W_0 \cong D \end{aligned}$$

- $P = \{\alpha\}$

$$R_P = \{\pm\alpha\} \qquad R_p^\vee = \{\pm\alpha^\vee\}$$
$$X^P = X/\mathbb{Z}\alpha \cong \mathbb{Z}x_4 \qquad X_P = X/Z\curvearrowleft_3 \cong \mathbb{Z}\alpha/2$$
$$Y^P = \mathbb{Z}(0, 2\sqrt{3}) \qquad Y_P = \mathbb{Z}\alpha^\vee$$
$$T^P = \{(t_4^2, t_4) : t_4 \in \mathbb{T}\} \quad T_P\{(1, t_4 : t_4 \in \mathbb{T}\} \quad K_P = \{(1,1),(1,-1)\} = \{1, k_P\}$$
$$W_p = \{e, s_\alpha\} \quad W^P = \{e, s_\beta, s_\alpha s_\beta\} \quad W(P,P) = \{e\} \quad \mathcal{W}_{PP} = K_P$$

The root datum $\mathcal{R}_P$ is isomorphic to $\mathcal{R}(A_1)^\vee$, so we can use the analysis on page 174. The representations $\pi(P, \delta_1(1, t_4))$ and $\pi(P, \delta_{-1}, (1, -t_4))$ are intertwined by $\pi(k_P)$.

$$C_r^*(\mathcal{R}, q)_P \cong C(S^1; M_3(\mathbb{C}))$$
$$\mathrm{Prim}(C_r^*(\mathcal{R}, q)_P) \cong \big(P, W_P(q^{1/3}, q^{2/3}), \delta_1, T_u^P\big) \cong S^1$$

The corresponding central idempotent is $p_\alpha + p_\beta + p_\gamma$, where

$$\begin{aligned} p_\eta &= \sum_{w \in W_\eta} (-q)^{\ell(w)} T_w \Big(\textstyle\sum_{w \in W_\eta} q(w)^{-1}\Big)^{-1} && \text{if} \quad q > 1 \\ p_\eta &= \sum_{w \in W_\eta} T_w \Big(\textstyle\sum_{w \in W_\eta} q(w)\Big)^{-1} && \text{if} \quad q < 1 \end{aligned} \tag{6.38}$$

and $W_\eta = \langle s_\eta, t_\eta \rangle \subset W_{\mathrm{aff}}$.

- $P = \{\beta\}$

This subset of F_0 is conjugate to $\{\alpha\}$ by $s_\alpha s_\beta$.

- $P = \{\alpha, \beta\}$

$$R_P = R_0 \qquad R_P^\vee = R_0^\vee$$
$$X^P = 0 \qquad X_P = X \qquad Y^P = 0 \qquad Y_P = Y$$
$$T^P = \{1\} \qquad T_P = T \qquad K_P = \{1\}$$
$$W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} \qquad W_P = W_0$$

The residual points (6.33) all carry exactly one inequivalent discrete series representation. We have $C_r^*(\mathcal{R}, q) \cong \mathbb{C}^3$ with central idempotent

$$\begin{aligned} p_{\alpha,\beta} &= \sum_{w \in W_{\mathrm{aff}}} (-q)^{\ell(w)} T_w \left(\sum_{w \in W_{\mathrm{aff}}} q(w)^{-1} \right)^{-1} && \text{if} \quad q > 1 \\ p_{\alpha,\beta} &= \sum_{w \in W_{\mathrm{aff}}} T_w \left(\sum_{w \in W_{\mathrm{aff}}} q(w) \right)^{-1} && \text{if} \quad q < 1 \end{aligned} \tag{6.39}$$

The projections for the specific points are

$$\begin{aligned} p_{(1,1)} &= (T_e + T_{\omega_1} + T_{\omega_2})\, p_{\alpha,\beta}/3 \\ p_{(\zeta,\zeta^2)} &= (T_e + \zeta T_{\omega_1} + \zeta^2 T_{\omega_2})\, p_{\alpha,\beta}/3 \\ p_{(\zeta^2,\zeta)} &= (T_e + \zeta^2 T_{\omega_1} + \zeta T_{\omega_2})\, p_{\alpha,\beta}/3 \end{aligned} \tag{6.40}$$

Notice that they agree on $\mathcal{H}(W_{\mathrm{aff}}, q)$.

We conclude that the spectrum of $C_r^*(\mathcal{R}, q))$ is the Hausdorff space

$$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)\big) \cong D \sqcup S^1 \sqcup 3 \text{ points}$$

Since D is compact and contractible this implies

$$\begin{aligned} K_0\big(C_r^*(\mathcal{R}, q)\big) &\cong \mathbb{Z}^5 \\ K_1\big(C_r^*(\mathcal{R}, q)\big) &\cong \mathbb{Z} \end{aligned} \tag{6.41}$$

Let p_0 and p_1 be rank one projectors in $C_r^*(\mathcal{R}, q)_\emptyset$ and $C_r^*(\mathcal{R}, q)_{\{\alpha\}}$. Then p_0, p_1, $p_{(1,1)}, p_{(\zeta,\zeta^2)}$ and $p_{(\zeta^2,\zeta)}$ generate $K_0\big(C_r^*(\mathcal{R}, q)\big)$, while T_{x_3} generates $K_1\big(C_r^*(\mathcal{R}, q)\big)$.

Once again, the K-theory turns out to be independent of the parameters. In terms of all the above generators, the group isomorphisms $K_*(\phi_0)$ are as follows.

$q < 1$	$q = 1$	$q > 1$
$p_0 + p_1 + p_{(1,1)} + p_{(\zeta,\zeta^2)} + p_{(\zeta^2,\zeta)}$	p_{triv}	p_0
p_0	p_{sign}	$p_0 + p_1 + p_{(1,1)} + p_{(\zeta,\zeta^2)} + p_{(\zeta^2,\zeta)}$
$2p_0 + p_1 + p_{(\zeta,\zeta^2)} + p_{(\zeta^2,\zeta)}$	p_3	$2p_0 + p_1 + p_{(\zeta,\zeta^2)} + p_{(\zeta^2,\zeta)}$
$2p_0 + p_1 + p_{(1,1)} + p_{(\zeta^2,\zeta)}$	p_4	$2p_0 + p_1 + p_{(1,1)} + p_{(\zeta^2,\zeta)}$
$2p_0 + p_1 + p_{(1,1)} + p_{(\zeta,\zeta^2)}$	p_5	$2p_0 + p_1 + p_{(1,1)} + p_{(\zeta,\zeta^2)}$
T_{x_3}	u	$T_{x_3}^{-1}$

(6.42)

As promised, we also discuss the root datum $\mathcal{R}(A_2)$, where X is the root lattice.

$$\begin{aligned}
&X = Q = \mathbb{Z}(1,0) + \mathbb{Z}\big(1/2, \sqrt{3}/2\big) \\
&X^+ = \{\big(n + m/2, m\sqrt{3}/2\big) : 0 \leq m \leq 2n \leq 4m\} \\
&Y = \mathbb{Z}\big(0, 2/\sqrt{3}\big) + \mathbb{Z}\big(1, 1/\sqrt{3}\big) \qquad Q^\vee = \mathbb{Z}(1,0) + \mathbb{Z}\big(1/2, \sqrt{3}/2\big) \\
&R_0 = \{\pm\alpha, \pm\beta, \pm\gamma\} = \{\pm(1,0), \pm\big(-1/2, \sqrt{3}/2\big), \pm\big(1/2, \sqrt{3}/2\big)\} = R_1 \\
&R_0^\vee = \big\{ \pm\alpha^\vee, \pm\beta^\vee, \pm\gamma^\vee \big\} = \{\pm(1,0), \pm\big(-1/2, \sqrt{3}/2\big), \pm\big(1/2, \sqrt{3}/2\big)\} = R_1^\vee \\
&T = (\mathbb{C}^\times)^2 \qquad t = (t_1, t_2) = (t(\alpha), t(\beta)) \\
&F_0 = \{\alpha, \beta\} \qquad W_0 = \langle s_\alpha, s_\beta | s_\alpha^2 = s_\beta^2 = (s_\alpha s_\beta)^3 = e\rangle \cong S_3 \\
&s_1 = s_\alpha : (n,m) \to (-n, m) \quad s_2 = s_\beta : (n + m/2, m\sqrt{3}/2) \to (m + n/2, n\sqrt{3}/2) \\
&s_0 = t_\gamma s_\gamma : \big(n + m/2, m\sqrt{3}/2\big) \to \big((1 + n - m)/2, (1 - n - m)\sqrt{3}/2\big) \\
&S_{\mathrm{aff}} = \{s_0, s_1, s_2\} \qquad W = W_{\mathrm{aff}} = \langle s_0, W_0 | s_0^2 = (s_0 s_1)^3 = (s_0 s_2)^3 = e\rangle \\
&q(s_0) = q(s_1) = q(s_2) = q(s_3) = q_{\alpha^\vee} = q_{\beta^\vee} = q_{\gamma^\vee} := q
\end{aligned}$$

Generically there are 6 residual points and 6 tempered residual circles. Borh form a single

W_0-conjugacy class, typical examples being

$$(q^{-1}, q^{-1}) \quad \text{and} \quad \{(q^{-1}, t_2 : t_2 \in \mathbb{T}\}$$

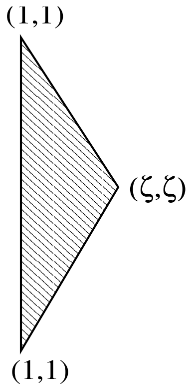
As usual we distinguish the cases $q = 1$ and $q \neq 1$.

- **group case q = 1**

The following subtori of T_u have nontrivial stabilizers:

$$\begin{array}{ll} \{(1, t_2) : t_2 \in \mathbb{T}\} & \{e, s_\alpha\} \\ \{(t_1, 1) : t_1 \in \mathbb{T}\} & \{e, s_\beta\} \\ \{(t_1, t_1^{-1}) : t_1 \in \mathbb{T}\} & \{e, s_\gamma\} \\ (\zeta, \zeta), (\zeta^2, \zeta^2) & \{e, s_\alpha s_\beta, s_\beta s_\alpha\} \\ (1, 1) & W_0 \end{array}$$

The following part T' of T_u is a fundamental domain for the action of W_0.

T' = 

The left edge is $T_u^{s_\alpha}$, and to get T_u/W_0 we only have to identify the other two edges by means of a rotation around (ζ, ζ). Then $C_r^*(W)$ consists of all $f \in C\big(T'; \mathrm{End}(\mathbb{C}[W_0])\big)$ such that

1. $T_{s_\alpha} f(t) T_{s_\alpha} = f(t)$ if $t \in T_u^{s_\alpha}$
2. $f(1, 1) \in \big(\mathrm{End}(\mathbb{C}[W_0])\big)^{W_0}$
3. $f(s_\beta s_\alpha (t_1, t_1) = T_{s_\beta s_\alpha} f(t_1 t_1) T_{s_\beta s_\alpha} \quad \forall t_1 \in \exp(\pi i [0, 2/3])$

If $t \in T_u^{s_\alpha}$ then $f(t)$ stabilizes $\mathbb{C}[W_0]^{s_\alpha}$, so there are extensions

$$0 \to A_2 \to C_r^*(W) \to A_1 \to 0 \tag{6.43}$$

$$0 \to A_3 \to A_1 \to \mathbb{C} \to 0 \tag{6.44}$$

$$A_1 = \left\{ f \in C\Big(T_u^{s_\alpha}; \mathrm{End}\big(\mathbb{C}[W_0]^{s_\alpha}\big)\Big) : f(1,1) = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$$

$$A_2 = \{ f \in C_r^*(W) : f(1,1) \in \mathbb{C} \oplus O_5,\ f(t) \in M_3(\mathbb{C}) \oplus O_3 \ \forall t \in T_u^{s_\alpha} \}$$

$$A_3 = \{ f \in A_1 : f(1,1) \in O_1 \oplus M_2(\mathbb{C}) \}$$

Here O_n denotes the $n \times n$ zero matrix. By Proposition 2.21 the inclusions

$$\begin{aligned} A_2 &\to A_4 := \{ f \in C\big(T'; \mathrm{End}(\mathbb{C}[W_0])\big) : \ 3. \text{ holds } \} \\ A_3 &\to C\big(T_u^{s_\alpha}; \mathrm{End}\big(\mathbb{C}[W_0]^{s_\alpha}\big)\big) \end{aligned} \tag{6.45}$$

induce isomorphisms on K-theory. With the help of Lemma 2.26 we find that

$$\begin{array}{rclrcl} K_0(A_4) & \cong & \mathbb{Z} & K_1(A_4) & = & 0 \\ K_0(A_3) & \cong & \mathbb{Z} & K_1(A_3) & \cong & \mathbb{Z} \\ K_0(A_2) & \cong & \mathbb{Z}^3 & K_1(A_2) & = & 0 \\ K_0(A_1) & \cong & \mathbb{Z}^2 & K_1(A_1) & \cong & \mathbb{Z} \end{array}$$

From (6.43) we get an exact hexagon

$$\begin{array}{ccccc} \mathbb{Z}^3 & \to & K_0\big(C_r^*(W)\big) & \to & \mathbb{Z}^2 \\ \uparrow & & & & \downarrow \\ \mathbb{Z} & \leftarrow & K_1\big(C_r^*(W)\big) & \leftarrow & 0 \end{array}$$

The upper row is exact, so

$$\begin{aligned} K_0\big(C_r^*(W)\big) &\cong \mathbb{Z}^5 \\ K_1\big(C_r^*(W)\big) &\cong \mathbb{Z} \end{aligned} \tag{6.46}$$

It is rather difficult to write down explicit generators, so we only indicate what they look like. Consider the projections

$$\begin{array}{rcl} p_{\text{triv}} & = & \frac{1}{6} \sum_{w \in W_0} T_w \\ p_{\text{sign}} & = & \frac{1}{6} \sum_{w \in W_0} (-1)^{\ell(w)} T_w \\ p_{\text{rot}} & = & (T_e + \zeta T_{s_\alpha s_\beta} + \zeta^2 T_{s_\beta s_\alpha})/3 \end{array}$$

With these we can define classes of projections $[p_\zeta]$ and $[p_{\zeta^2}]$ by the conditions

$$\begin{aligned} & p_\zeta\big(T_u^{s_\alpha}\big) = p_{\zeta^2}\big(T_u^{s_\alpha}\big) = p_{\text{triv}} \\ & p_\zeta(\zeta,\zeta)(p_{\text{triv}} + p_{\text{sign}} + p_{\text{rot}}) = 0 \\ & p_{\zeta^2}(\zeta,\zeta) p_{\text{rot}} = p_{\zeta^2}(\zeta,\zeta) \end{aligned}$$

These five classes of projections generate $K_0\big(C_r^*(W)\big)$, and a generator for $K_1\big(C_r^*(W)\big)$ is

$$u = p_{\text{triv}} \theta_\beta \, p_{\text{triv}} + p_{\text{sign}} \theta_{-\beta} \, p_{\text{sign}} + T_e - p_{\text{triv}} - p_{\text{sign}}$$

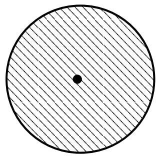
- **generic case $\mathbf{q \neq 1}$**
- $P = \emptyset$

$$\begin{aligned} & R_P = \emptyset \qquad R_P^\vee = \emptyset \\ & X^P = X \qquad X_P = 0 \qquad Y^P = Y \qquad Y_P = 0 \\ & T^P = T \qquad T_P = \{1\} \qquad K_P = \{1\} \\ & W^P = W(P,P) = \mathcal{W}_{PP} = W_0 \qquad W_P = \{e\} \\ & \imath_{s_\eta}^o = (T_{s_\eta}(1 - \theta_\eta) + (q-1)\theta_\eta)(q - \theta_\eta)^{-1} \qquad \eta \in \{\alpha, \beta, \gamma\} \end{aligned}$$

If $s_\eta(t) = t$ then $\imath^o_{s_eta}(t) = 1$. There are two points in T_u whose stabilizer is not generated by reflections: (ζ, ζ) and (ζ^2, ζ^2). Let A_4 be as in (6.45). Then

$$C_r^*(\mathcal{R}, q)_P \cong A_4$$

$$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \quad \cong$$



which is supposed to depict T_u/W_0 with in the center, instead of just $W_0(\zeta, \zeta)$, a triple non-Hausdorff point. By Lemma 2.26 we have

$$K_0\big(C_r^*(\mathcal{R}, q)_P\big) \cong \mathbb{Z}^3$$
$$K_1\big(C_r^*(\mathcal{R}, q)_P\big) = 0$$

The class of a projection in this algebra is completely determined by its value at (ζ, ζ), and over there we only have to say which representation of $\{e, s_1 s_2, s_2 s_1\}$ it gives. So we have generators p_0, p_1, p_2 with

$$p_i(\zeta, \zeta)\imath^o_{s_1 s_2}(\zeta, \zeta) = \zeta^i p_i(\zeta, \zeta)$$

- $P = \{\alpha\}$

$$\begin{aligned}
&R_P = \{\pm\alpha\} \quad R_P^\vee = \{\pm\alpha^\vee\} \\
&X^P = X/\mathbb{Z}\alpha \cong \mathbb{Z}\beta \quad X_P = X/\mathbb{Z}(0, 2) \cong \mathbb{Z}(\alpha/2) \\
&Y^P = \mathbb{Z}(0, 2/\sqrt{3}) \qquad Y_P = \mathbb{Z}\alpha^\vee = \mathbb{Z}(2, 0) \\
&T^P = \{(1, t_2) : t_2 \in \mathbb{C}^\times\} \qquad T_P = \{(t_2^{-2}, t_2) : t_2 \in \mathbb{C}^\times\} \\
&K_P = \{(1, 1), (1, -1)\} = \{1, k_P\} \\
&W_P = \{e, s_\alpha\} \quad W^P = \{e, s_\beta, s_\alpha s_\beta\} \quad W(P, P) = \{e\} \quad \mathcal{W}_{PP} = K_P
\end{aligned}$$

The root datum $\mathcal{R}_P$ is isomorphic to $\mathcal{R}(A_1)^\vee$, so we already know its discrete series. The representations $\pi(P, \delta_1, (1, t_2))$ and $\pi(P, \delta_{-1}, (1, -t_2))$ are interwined by $\pi(k_P)$, so

$$\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \cong (P, W_P(q, q^{1/2}), \delta_1, T_u^P) \cong S^1$$
$$C_r^*(\mathcal{R}, q)_P \cong C(S^1; M_3(\mathbb{C}))$$

The central idempotent for this component is $p_\alpha + p_\beta + p_\gamma$, as given by (6.39).

- $P = \{\beta\}$

This subset of F_0 is conjugate to $\{\alpha\}$ by $s_\alpha s_\beta$.

- $P = \{\alpha, \beta\}$

$$R_P = R_0 \quad R_P^\vee = R_0^\vee$$
$$X^P = 0 \quad X_P = X$$
$$Y^P = 0 \quad Y_P = Y$$
$$T^P = \{1\} \quad T_P = T \quad K_P = \{1\}$$
$$W_P = W_0 \quad W^P = W(P,P) = \mathcal{W}_{PP} = \{e\}$$

Up to W_0-conjugacy there is a single residual point, so by Proposition 3.20.2 there is a unique discrete series representation δ. We have $C_r^*(\mathcal{R},q)_P \cong \mathbb{C}$, and the corresponding projector is

$$\begin{array}{lcll} p_\delta & = & \sum_{w\in W} (-q)^{\ell(w)} T_w \Big(\sum_{w \in W} q(w)^{-1} \Big)^{-1} & \text{if} \quad q > 1 \\ p_\delta & = & \sum_{w\in W} T_w \Big(\sum_{w\in W} q(w) \Big)^{-1} & \text{if} \quad q < 1 \end{array} \tag{6.47}$$

On the whole we found that

$$\begin{aligned} K_0\big(C_r^*(\mathcal{R},q)\big) &\cong \mathbb{Z}^5 \\ K_1\big(C_r^*(\mathcal{R},q)\big) &\cong \mathbb{Z} \end{aligned} \tag{6.48}$$

which is the same as for $q = 1$. Generators are the invertible θ_β and the rank one projectors p_0, p_1, p_2, p_δ and $p_{\{\alpha\}}$.

The isomorphisms $K_*(\phi_0)$ take the following form:

$q < 1$	$q = 1$	$q > 1$
$p_0 + p_{\{\alpha\}} + p_\delta$	p_{triv}	p_0
p_0	p_{sign}	$p_0 + p_{\{\alpha\}} + p_\delta$
$2p_2 + p_{\{\alpha\}}$	p_{rot}	$2p_2 + p_{\{\alpha\}}$
$p_1 + p_{\{\alpha\}} + p_\delta$	p_ζ	p_1
$p_2 + p_{\{\alpha\}} + p_\delta$	p_{ζ^2}	p_2
θ_β	u	θ_β^{-1}

(6.49)

6.4 B_2

The rank two root system B_2 is probably the best testcase for Conjecture 5.28. Because there are roots of different lengths the conjecture is not yet known, and at the same time the calculations are still manageable. Moreover some interesting phenomena already occur for these affine Hecke algebras, like residual points that carry several inequivalent discrete series representations.

We will only consider the root datum $\mathcal{R}(B_2)^\vee$ where X is the weight lattice:

$$X = \mathbb{Z}^2 \quad Q = \{(m,n) \in \mathbb{Z}^2 : n+m \text{ is even } \} \quad X^+ = \{(m,n) \in \mathbb{Z}^2 : n \geq m \geq 0\}$$
$$Y = Q^\vee = \mathbb{Z}^2$$
$$T = (\mathbb{C}^\times)^2 \qquad t = (t_1, t_2) = (t(1,0), t(0,1))$$
$$R_0 = R_1 = \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4\} = \{\pm(2,0), \pm(-1,1), \pm(1,1), \pm(0,2)\}$$
$$R_0^\vee = R_1^\vee = \{\pm\alpha_1^\vee, \pm\alpha_2^\vee, \pm\alpha_3^\vee, \pm\alpha_4^\vee\} = \{\pm(1,0), \pm(-1,1), \pm(1,1), \pm(0,1)\}$$
$$F_0 = \{\alpha_1, \alpha_2\} \quad W_0 = \langle s_1, s_2 | s_1^2 = s_2^2 = (s_1 s_2)^4 = e\rangle \cong D_4$$
$$s_i = s_{\alpha_i} \quad s_0 = t_{\alpha_3} s_{\alpha_3} = (t_{(1,0)} s_2 s_1) s_2 (t_{(1,0)} s_2 s_1)^{-1} : (m,n) \to (1-n, 1-m)$$
$$S_{\text{aff}} = \{s_0, s_1, s_2\} \quad W \neq W_{\text{aff}} = \langle s_0, W_0 | s_0^2 = (s_0 s_2)^2 = (s_0 s_1)^4 = e\rangle$$
$$\Omega = \{e, t_{(1,0)} s_1\} = \{e, \omega\}$$
$$q_1 := q(s_1) = q_{\alpha_1^\vee} = q_{\alpha_3^\vee} \qquad q_2 := q(s_2) = q(s_0) = q_{\alpha_2^\vee} = q_{\alpha_4^\vee}$$
$$c_{\alpha_i} = (1 - q_{\alpha_i^\vee}^{-1} \theta_{-\alpha_i})(1 - \theta_{-\alpha_i})^{-1} \quad i = 1,2,3,4$$

Generically there are 24 tempered residual circles and 40 residual points. Representatives are

$$\left\{(q_1^{1/2}, t_2) : t_2 \in \mathbb{T}\right\} \tag{6.50}$$
$$\left\{(-q_1^{1/2}, t_2) : t_2 \in \mathbb{T}\right\} \tag{6.51}$$
$$\left\{(q_2^{1/2} t_1, q_2^{-1/2} t_1) : t_1 \in \mathbb{T}\right\} \tag{6.52}$$
$$\begin{gathered}(q_1^{-1/2}, q_1^{1/2} q_2^{-1}),\ (q_1^{1/2}, q_1^{-1/2} q_2^{-1}),\ (-q_1^{-1/2}, -q_1^{1/2} q_2^{-1}), \\ (-q_1^{1/2}, -q_1^{-1/2} q_2^{-1}),\ (-q_1^{-1/2}, q_1^{1/2})\end{gathered} \tag{6.53}$$

It turns out that there are five classes of parameters with the same level of genericity. The first three are easily found from (6.50) - (6.52): $q_1 = 1 = q_2$, $q_1 \neq 1 = q_2$ and $q_1 = 1 \neq q_2$. Furthermore we have the generic class and the four special lines

$$q_1 = q_2 \neq 1,\ q_1 = q_2^{-1} \neq 1,\ q_1 = q_2^2 \neq 1,\ q_1 = q_2^{-2} \neq 1 \tag{6.54}$$

- **group case $\mathbf{q_1 = q_2 = 1}$**

From (6.2) and (6.3) we see that it pays off to determine the extended quotient

$$\widetilde{T_u}/W_0 \cong \bigsqcup_{\langle w\rangle \in \langle W_0\rangle} T_u^w / Z_{W_0}(w)$$

There are five conjugacy classes, namely

$$\{e\},\ \{s_1, s_4\},\ \{s_2, s_3\},\ \{s_1 s_2, s_2 s_1\},\ \{s_1 s_2 s_1 s_2\}$$

We pick the elements written first as representatives.

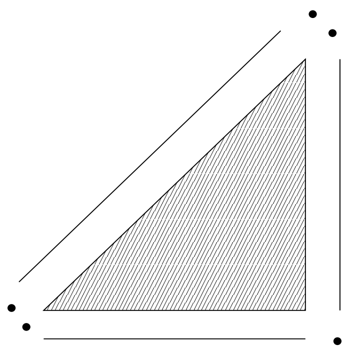
w	$Z_{W_0}(w)$	T_u^w	$T_u^w/Z_{W_0}(w)$
e	W_0	T_u	$\{(t_1,t_2)\in[0,1]^2 : t_1 \geq t_2\}$
s_1	$\{e, s_1, s_4, s_1s_4\}$	$\{(\pm1, t_2)\in T_u\}$	$\{\pm1\}\times[-1,1]$
s_2	$\{e, s_2, s_3, s_2s_3\}$	$\{(t_1,t_1)\in T_u\}$	$[-1,1]$
s_1s_2	$\langle s_1s_2\rangle$	$\{(1,1),(-1,-1)\}$	$\{(1,1),(-1,-1)\}$
$(s_1s_2)^2$	W_0	$\{(\pm1,\pm1),(\pm1,\mp1)\}$	$\{(1,1),(-1,-1),(1,-1)\}$

Since all components of this space are contractible, with (6.3) we find that

$$\begin{aligned} K_0\big(C_r^*(W)\big) &\cong \check{H}^*\big(\widetilde{T_u}/W_0;\mathbb{Z}\big)\cong\mathbb{Z}^9 \\ K_1\big(C_r^*(W)\big) &= 0 \end{aligned} \tag{6.55}$$

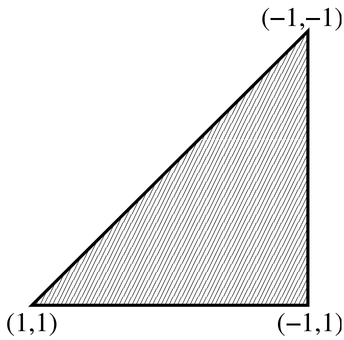
We may visualize the extended quotient as

$\widetilde{T_u}/W_0 \quad \cong$



In the same way we have

$\mathrm{Prim}\big(C_r^*(W)\big) \quad \cong$



where the multiplicities of the non-Hausdorff points at the edge can be read off from the previous picture, by collapsing everything on the triangle.

To find generating projections for the K-theory we first have a closer look at W_0. This group has four one-dimensional representations, let us call them ϵ_i.

These representations and the corresponding projections are

ϵ_i	$\epsilon_i(s_1)$	$\epsilon(s_2)$	p_i
ϵ_0	1	1	$\frac{1}{8}(T_e + T_{s_1s_2} + T_{s_2s_1} + T_{(s_1s_2)^2} + T_{s_1} + T_{s_2} + T_{s_3} + T_{s_4})$
ϵ_1	-1	1	$\frac{1}{8}(T_e + T_{s_2} + T_{s_3} + T_{(s_1s_2)^2} - T_{s_1} - T_{s_4} - T_{s_1s_2} - T_{s_2s_1})$
ϵ_2	1	-1	$\frac{1}{8}(T_e + T_{s_1} + T_{s_4} + T_{(s_1s_2)^2} - T_{s_2} - T_{s_3} - T_{s_1s_2} - T_{s_2s_1})$
ϵ_3	-1	-1	$\frac{1}{8}(T_e + T_{s_1s_2} + T_{s_2s_1} + T_{(s_1s_2)^2} - T_{s_1} - T_{s_2} - T_{s_3} - T_{s_4})$

(6.56)

The remaining irreducible W_0-representation ρ_4 has dimension two, it defines W_0 as a reflection group. Let $p_4 \in \mathbb{C}[W_0]$ be rank one projector for that representation, i.e.

$$\rho_4(p_4) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_i(p_4) = 0 \quad i = 0, 1, 2, 3$$

Note that p_4 has rank two in $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[W_0])$.

We also have to consider the stabilizer of $(-1, 1) \in T$, the subgroup $\{e, s_1, s_4, (s_1s_2)^2\} \cong D_2$. Let us list its irreducible representations, which all have dimension one:

ϵ	$\epsilon(s_1)$	$\epsilon(s_4)$	$\mathrm{Ind}_{\langle s_1,s_4\rangle}^{W_0}(\epsilon)$	$p(\epsilon)$
ϵ_{++}	1	1	$\epsilon_0 \oplus \epsilon_2$	p_0
ϵ_{+-}	1	-1	ρ_4	p_{+-}
ϵ_{-+}	-1	1	ρ_4	p_{-+}
ϵ_{--}	-1	-1	$\epsilon_1 \oplus \epsilon_3$	p_3

(6.57)

In the last column we indicate an element of $\mathrm{End}_{\mathbb{C}}(\mathbb{C}[W_0])$ that acts as a rank one projector for that representation of $\langle s_1, s_4\rangle$. For ϵ_{+-} and ϵ_{-+} such an element cannot be found in $\mathbb{C}[W_0] \cong \mathrm{End}_{W_0}(\mathbb{C}[W_0])$, so we refrain from giving an explicit formula.

Now we can indicate generators $p_0, \ldots, p_8$ for $K_0(C_r^*(W))$. The last four classes of projections are defined by their values in three special points.

p	$p(1,1)$	$p(-1,-1)$	$p(-1,1)$
p_5	p_0	p_1	p_{-+}
p_6	p_2	p_3	p_{+-}
p_7	$p_0 + p_3$	p_4	$p_0 + p_3$
p_8	$p_0 + p_3$	p_4	$p_{+-} + p_{-+}$

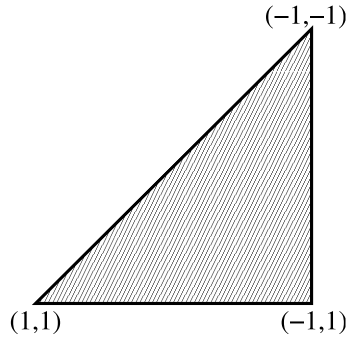
- **generic case**
- $P = \emptyset$

$$R_P = \emptyset \qquad R_P^\vee = \emptyset$$
$$X^P = X \qquad X_P = 0 \qquad Y^P = Y \qquad Y_P = 0$$
$$T^P = T \qquad T_P = \{1\} \qquad K_P = \{1\}$$
$$W^P = W(P, P) = \mathcal{W}_{PP} = W_0 \qquad W_P = \{e\}$$
$$\imath^o_{s_i} = (T_{s_i}(1 - \theta_{\alpha_i}) + (q(s_i) - 1)\theta_{\alpha_i})(q(s_i) - \theta_{\alpha_i})^{-1} \qquad i = 1, 2, 3, 4$$
$$\imath^o_{s_i}(t) = 1 \text{ if } t \in T^{s_i}$$

The W_0-stabilizer of any $t \in T$ is generated by reflections, so all the unitary representations $\pi(\emptyset, \delta_\emptyset, t)$ are irreducible.

$$C_r^*(\mathcal{R}, q)_P \cong C(T_u/W_0; M_8(\mathbb{C}))$$

$$\text{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) = T_u/W_0 \cong$$



Since T_u/W_0 is contractible we conclude that

$$\begin{array}{rcl} K_0\big(C_r^*(\mathcal{R}, q)_P\big) & \cong & \mathbb{Z} \\ K_1\big(C_r^*(\mathcal{R}, q)_P\big) & = & 0 \end{array}$$

A generator is

$$p_\emptyset = \frac{1}{8} e_\emptyset \sum_{w \in W_0} q(w) T_w$$

- $P = \{\alpha_1\}$

$R_P = \{\pm\alpha_1\} \qquad R_P^\vee = \{\pm\alpha_1^\vee\}$

$X^P = X/\mathbb{Z}(\alpha_1/2) \cong \mathbb{Z}\alpha_4/2 \quad X_P = X/\mathbb{Z}(\alpha_4/2) \cong \mathbb{Z}\alpha_1/2$

$Y^P = \mathbb{Z}\alpha_4^\vee \qquad Y_P = \mathbb{Z}\alpha_1^\vee$

$T^P = \{(1, t_2) : t_2 \in \mathbb{C}^\times\} \quad T_P = \{(t_1, 1) : t_1 \in \mathbb{C}^\times\} \quad K_P = \{1\}$

$W_P = \{e, s_1\} \quad W^P = \{e, s_2, s_4, s_1 s_3\} \quad W(P,P) = \mathcal{W}_{PP} = \{e, s_4\}$

$\imath_{s_4}^o = (T_{s_4}(1 - \theta_{\alpha_4}) + (q_1 - 1)\theta_{\alpha_4})(q_1 - \theta_{\alpha_4})^{-1}$

The algebra $\mathcal{H}_P$ is isomorphic to $\mathcal{H}(\mathcal{R}(A_1)^\vee, q_1)$. Hence it has two discrete series representations, with central characters $\big(q_1^{\pm 1/2}, 1\big)$ and $\big(-q_1^{\pm 1/2}, 1\big)$. Clearly $\imath_{s_4}^o(t) = 1$ if $t \in \big(T_u^P\big)^{s_4}$, so

$$C_r^*(\mathcal{R}, q)_P \cong C([0,1]; M_4(\mathbb{C}))^2$$

$$\text{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \cong$$

$$K_0\big(C_r^*(\mathcal{R}, q)_P\big) \cong \mathbb{Z}^2$$
$$K_1\big(C_r^*(\mathcal{R}, q)_P\big) = 0$$

Let us denote the canonical generators by $[p^+_{\alpha_1}]$ and $[p^-_{\alpha_1}]$. Notice that as $\mathcal{H}(W_0, q)$-representations the $\pi(P, \delta, t)$ are deformations of $\mathrm{Ind}_{W_P}^{W_0}(\mathrm{sign}_{W_P})$ if $q_1 > 1$ and of $\mathrm{Ind}_{W_P}^{W_0}(\mathrm{triv}_{W_P})$ if $q_1 < 1$.

- $P = \{\alpha_2\}$

$$
\begin{aligned}
&R_P = \{\pm\alpha_2\} \qquad R_P^\vee = \{\pm\alpha_2^\vee\}\\
&X^P = X/\mathbb{Z}\alpha_2 \cong \mathbb{Z}\alpha_1/2 \qquad X_P = X/\mathbb{Z}\alpha_3 \cong \mathbb{Z}\alpha_2/2\\
&Y^P = \mathbb{Z}\alpha_3^\vee \qquad Y_P = \mathbb{Z}\alpha_2^\vee\\
&T^P = \{(t_1, t_1) : t_1 \in \mathbb{C}^\times\} \qquad T_P = \{(t_2^{-1}, t_2) : t_2 \in \mathbb{C}^\times\}\\
&K_P = \{(1,1), (-1,-1)\} = \{1, k_P\}\\
&W_P = \{e, s_2\} \quad W^P = \{e, s_1, s_4, s_3 s_1\} \quad W(P,P) = \{e, s_3\}\\
&\imath^o_{s_3} = (T_{s_3}(1 - \theta_{\alpha_3}) + (q_2 - 1)\theta_{\alpha_3})(q_2 - \theta_{\alpha_3})^{-1}\\
&\imath^o_{s_3}(t) = 1 \text{ if } t \in \big(T_u^P\big)^{s_3}
\end{aligned}
$$

The algebra $\mathcal{H}_P$ is isomorphic to $\mathcal{H}(\mathcal{R}(A_1)^\vee, q_2)$, so it has two discrete series representations δ_+ and δ_-. Their central characters are respectively $(q_2^{\pm 1/2}, q_2^{\mp 1/2})$ and $(-q_2^{\pm 1/2}, -q_2^{\mp 1/2})$. However $\pi(k_P)$ intertwines $\pi(P, \delta_+, (t_1, t_1))$ and $\pi(P, \delta_-, (-t_1, -t_1))$, so in the spectrum we get only one component (P, δ_+, T_u^P). The intertwiner $\pi(s_3)$ acts as a reflection on T_u^P and $\pi(s_3, P, \delta_+, t) = 1$ whenever $s_3(t) = t$. Therefore

$$
\begin{aligned}
&C_r^*(\mathcal{R}, q)_P \cong C([0,1]; M_4(\mathbb{C}))\\
&\mathrm{Prim}\big(C_r^*(\mathcal{R}, q)_P\big) \cong (P, \delta_+, T_u^P)/W(P,P) \cong [0,1]\\
&K_0\big(C_r^*(\mathcal{R}, q)_P\big) \cong \mathbb{Z}\\
&K_1\big(C_r^*(\mathcal{R}, q)_P\big) = 0
\end{aligned}
$$

There is a canonical generator $[p_{\alpha_2}] \in K_0\big(C_r^*(\mathcal{R}, q)_P\big)$. The type of $\pi(P, \delta_+, t)$ as a representation of $\mathcal{H}(W_0, q)$ is easily determined: for $q_2 < 1$ it is a deformation of $\mathrm{Ind}_{W_P}^{W_0}(\mathrm{triv}_{W_P})$ while for $q_2 > 1$ it is a deformation of $\mathrm{Ind}_{W_P}^{W_0}(\mathrm{sign}_{W_P})$.

- $P = \{\alpha_1, \alpha_2\}$

$$
\begin{aligned}
&R_P = R_0 \qquad R_P^\vee = R_0^\vee\\
&X^P = 0 \qquad X_P = X \qquad Y^P = 0 \qquad Y_P = Y\\
&T^P = \{1\} \qquad T_P = T \qquad K_P = \{1\}\\
&W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} \qquad W_P = W_0
\end{aligned}
$$

Generically all the residual points (6.53) are in orbits consisting of $|W_0| = 8$ points. Proposition 3.20.1 tells us that every such W_0-orbit is the central character of precisely one discrete series representation. The discrete series representation

δ_5 with central character $W_0(q_1^{-1/2}, -q_1^{-1/2})$ is the easiest to describe. It has dimension two, and as a W_0-representation $\delta_5 \circ \phi$ is equivalent to $\epsilon_1 \oplus \epsilon_3$ (for $q_1 > 1$) or to $\epsilon_0 \oplus \epsilon_2$ (for $q_1 < 1$). For the other residual points we have to distinguish more relations between the parameters. Let $\delta_1, \delta_2, \delta_3, \delta_4$ be the discrete series representations with respective central characters

$$W_0\big(q_1^{1/2}, q_1^{-1/2}q_2\big)\,, W_0\big(q_1^{1/2}, q_1^{1/2}q_2\big)\,, W_0\big(-q_1^{1/2}, -q_1^{-1/2}q_2\big)\,, W_0\big(-q_1^{1/2}, -q_1^{1/2}q_2\big)$$

We list the type of $\delta_i \circ \phi_0$ as a representation of $W_0 \subset \mathcal{S}(W)$, for different q's :

	δ_1	δ_2	δ_3	δ_4
$1 < q_1^{1/2} < q_2 < q_1$	ρ_4	ϵ_3	ρ_4	ϵ_3
$q_1^{-1} < q_2 < q_1^{-1/2} < 1$	ϵ_1	ρ_4	ϵ_1	ρ_4
$1 < q_1^{-1/2} < q_2 < q_1^{-1}$	ϵ_2	ρ_4	ϵ_2	ρ_4
$q_1 < q_2 < q_1^{1/2} < 1$	ρ_4	ϵ_0	ρ_4	ϵ_0
$q_1^{-1/2} < q_2 < q_1^{1/2} > 1$	ϵ_1	ϵ_3	ϵ_1	ϵ_3
$q_2^{-1} < q_1 < q_2 > 1$	ϵ_2	ϵ_3	ϵ_2	ϵ_3
$1 > q_1^{1/2} < q_2 < q_1^{-1/2}$	ϵ_2	ϵ_0	ϵ_2	ϵ_0
$1 > q_2 < q_1 < q_2^{-1}$	ϵ_1	ϵ_0	ϵ_1	ϵ_0

(6.58)

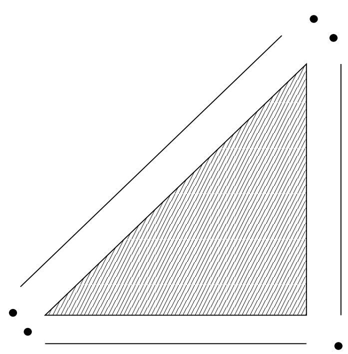
It can be shown by direct calculation that this table still gives all discrete series representations if either q_1 or q_2 (but not both) equals 1. This not true for the special parameters (6.54) however. Nevertheless, for all the parameters under consideration here

$$\begin{aligned} K_0\big(C_r^*(\mathcal{R},q)_P\big) &\cong \mathbb{Z}^5 \\ K_1\big(C_r^*(\mathcal{R},q)_P\big) &= 0 \end{aligned}$$

We denote the generating projections by $p(\delta_i)$, $i = 1,2,3,4,5$.

On the whole we found that $C_r^*(\mathcal{R},q)$ is Morita-equivalent to the commutative C^*-algebra with spectrum

$$\mathrm{Prim}\big(C_r^*(\mathcal{R},q)\big) \quad \cong$$



and that

$$\begin{aligned} K_0\big(C_r^*(\mathcal{R},q)\big) &\cong \mathbb{Z}^9 \\ K_1\big(C_r^*(\mathcal{R},q)\big) &= 0 \end{aligned} \qquad (6.59)$$

- $\mathbf{q_1 \neq 1 = q_2}$

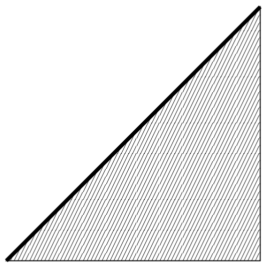
- $P = \emptyset$

$$\begin{array}{llll} R_P = \emptyset & R_P^\vee = \emptyset & & \\ X^P = X & X_P = 0 & Y^P = Y & Y_P = 0 \\ T^P = T & T_P = \{1\} & K_P = \{1\} & \\ W^P = W(P,P) = \mathcal{W}_{PP} = W_0 & W_P = \{e\} & & \\ \imath^o_{s_1} = (T_{s_1}(1-\theta_{\alpha_1}) + (q_1 - 1)\theta_{\alpha_1})(q_1 - \theta_{\alpha_1})^{-1} & \imath^o_{s_2} = T_{s_2} & & \end{array}$$

Since $\imath^o_{s_1}(t) = 1$ if $s_1(t) = t$, all the nonscalar selfintertwiners of unitary principal series representations come from s_2 or its conjugates. We see that $\pi(\emptyset, \delta_\emptyset, t)$ is irreducible unless $t \in T_u^{s_2} \cup T_u^{s_3}$, in which case it is the direct sum of two inequivalent subrepresentations. Hence

$$C_r^*(\mathcal{R},q)_P \;\cong\; \big\{ f \in C\big(T_u/W_0; M_8(\mathbb{C})\big) : f\big(T_u^{s_2}\big) \in M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \big\}$$

$$\mathrm{Prim}\big(C_r^*(\mathcal{R},q)_P\big) \quad \cong$$



In this picture the diagonal edge should be regarded as consisting of double points. The algebra is diffeotopy equivalent to $M_4(\mathbb{C})^2$, so

$$\begin{array}{rcl} K_0\big(C_r^*(\mathcal{R},q)_P\big) & \cong & \mathbb{Z}^2 \\ K_1\big(C_r^*(\mathcal{R},q)_P\big) & = & 0 \end{array}$$

Generators are for example

$$p_0 := \frac{1}{8} e_\emptyset \sum_{w \in W_0} q(w) T_w \text{ and } p_3 := \frac{1}{8} e_\emptyset \sum_{w \in W_0} (-1)^{\ell(w)} T_w$$

- $P = \{\alpha_1\}$

This is identical to $P = \{\alpha_1\}$ in the generic case.

- $P = \{\alpha_2\}$

Here $\mathcal{H}_P \cong \mathcal{H}(\mathcal{R}(A_1)^\vee, q_2) = \mathbb{C}[W(A_1)]$. As we saw before, this algebra has no discrete series representations, so there is no component in the spectrum of $\mathcal{S}(\mathcal{R},q)$ corresponding to this P.

- $P = \{\alpha_1, \alpha_2\}$

$$\begin{array}{llll} R_P = R_0 & R_P^\vee = R_0^\vee & & \\ X^P = 0 & X_P = X & Y^P = 0 & Y_P = Y \\ T^P = \{1\} & T_P = T & K_P = \{1\} & \\ \multicolumn{3}{l}{W^P = W(P,P) = \mathcal{W}_{PP} = \{e\}} & W_P = W_0 \end{array}$$

Some residual points confluence when $q_2 \to 1$, and only three orbits remain. Two of those consist of four points, represented by $\big(q_1^{-1/2}, q_1^{1/2}\big)$ and $\big(-q_1^{-1/2}, -q_1^{-1/2}\big)$. The last orbit still contains 8 different points, for example $\big(-q_1^{-1/2}, q_1^{1/2}\big)$. By Proposition 3.20.1 there is exactly one discrete series representation δ_5 with central character $W_0\big(-q_1^{-1/2}, q_1^{1/2}\big)$. It has dimension two and restricted to $\mathcal{H}(W_0, q)$ it is a deformation of the W_0-representations $\epsilon_1 \oplus \epsilon_3$ or $\epsilon_0 \oplus \epsilon_2$, depending on whether $q_1 > 1$ or $q_1 < 1$. We calculated the other discrete series representations already in (6.58). For $q_1 > 1$ we have the onedimensional representations

δ	$\delta(T_{s_1})$	$\delta(T_{s_2})$	$\delta(\theta_x)$
δ_a	-1	1	$(q_1^{-1/2}, q_1^{-1/2})(x)$
δ_b	-1	-1	$(q_1^{-1/2}, q_1^{-1/2})(x)$
δ_c	-1	1	$(-q_1^{-1/2}, -q_1^{-1/2})(x)$
δ_d	-1	-1	$(-q_1^{-1/2}, -q_1^{-1/2})(x)$

(6.60)

On the other hand, for $q_1 < 1$ we can define onedimensional representations by

δ	$\delta(T_{s_1})$	$\delta(T_{s_2})$	$\delta(\theta_x)$
δ_a	q_1	1	$(q_1^{1/2}, q_1^{1/2})(x)$
δ_b	q_1	-1	$(q_1^{1/2}, q_1^{1/2})(x)$
δ_c	q_1	1	$(-q_1^{1/2}, -q_1^{1/2})(x)$
δ_d	q_1	-1	$(-q_1^{1/2}, -q_1^{1/2})(x)$

(6.61)

This leads to

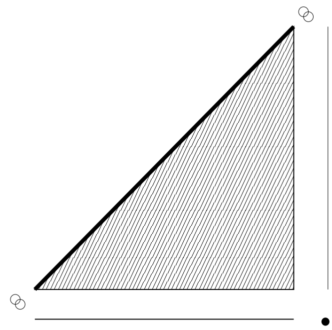
$$\begin{array}{rcl} C_r^*(\mathcal{R}, q)_P & \cong & M_2(\mathbb{C}) \oplus \mathbb{C}^4 \\ K_0\big(C_r^*(\mathcal{R}, q)_P\big) & \cong & \mathbb{Z}^5 \\ K_1\big(C_r^*(\mathcal{R}, q)_P\big) & = & 0 \end{array}$$

The generators of $K_0\big(C_r^*(\mathcal{R}, q)_P\big)$ corresponding to rank one projectors in these representations are denoted by $p(\delta_v)$, $v \in \{a, b, c, d, 5\}$.

Combining all P's we find

$$K_0\big(C_r^*(\mathcal{R},q)_P\big) \cong \mathbb{Z}^5$$
$$K_1\big(C_r^*(\mathcal{R},q)_P\big) = 0$$

$$\mathrm{Prim}\big(C_r^*(\mathcal{R},q)\big) \quad \cong$$



- $\mathbf{q_1 = 1 \neq q_2}$
- $P = \emptyset$

$$\begin{array}{llll} R_P = \emptyset & R_P^\vee = \emptyset & & \\ X^P = X & X_P = 0 & Y^P = Y & Y_P = 0 \\ T^P = T & T_P = \{1\} & K_P = \{1\} & \\ W^P = W(P,P) = \mathcal{W}_{PP} = W_0 & W_P = \{e\} & & \\ \imath_{s_1}^o = T_{s_1} & \imath_{s_2}^o = (T_{s_2}(1-\theta_{\alpha_2}) + (q_2-1)\theta_{\alpha_2})(q_2-\theta_{\alpha_2})^{-1} & & \end{array}$$

Since $\imath_{s_2}^o(t) = 1$ whenever $t \in T^{s_2}$, all the nonscalar self-intertwiners of principal series representations come from s_1, $s_4 = s_2 s_1 s_2$ and $s_1 s_4$. This implies that we should divide the points $t \in T_u$ in five classes.

1. $s_1(t) \neq t \neq s_4(t)$
   Here we do not encounter nonscalar self-intertwiners, so $\pi(\emptyset, \delta_\emptyset, t)$ is irreducible.

2. $s_1(t) = t \neq s_4(t)$
   For such t $\pi(\emptyset, \delta_\emptyset, t)$ splits into two summands, which as $\mathcal{H}(W_0,q)$-representations are deformations of $\mathrm{Ind}_{\{e,s_1\}}^{W_0}(\mathrm{triv})$ and of $\mathrm{Ind}_{\{e,s_1\}}^{W_0}(\mathrm{sign})$.

3. $s_1(t) \neq t = s_4(t)$
   Just as in 2. $\pi(\emptyset, \delta_\emptyset, t)$ is the direct sum of two irreducible subrepresentations, which can be regarded as deformations of $\mathrm{Ind}_{\{e,s_4\}}^{W_0}(\mathrm{triv})$ and of $\mathrm{Ind}_{\{e,s_4\}}^{W_0}(\mathrm{sign})$.

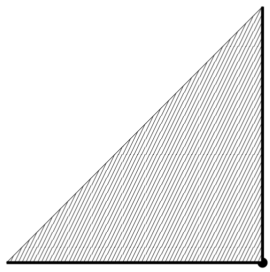
4. $(1,1)$ and $(-1,-1)$
   These points are W_0-invariant, so s_1 and s_4 are conjugate in $W_{0,t}$. Hence $\pi(s_1, \emptyset, \delta_\emptyset, t)$ is essentially the only independent intertwiner, and $\pi(\emptyset, \delta_\emptyset, t)$

can be decomposed in only two irreducible parts. These are deformations of the W_0-representations $\epsilon_1 \oplus \epsilon_3 \oplus \rho_4$ and $\epsilon_0 \oplus \epsilon_2 \oplus \rho_4$.

5. $(1,-1)$ and $(-1,1)$
These points make up one W_0-orbit in T_u, their stabilizer being $\{e, s_1, s_4, s_1 s_4\}$. Here the intertwiners $\pi(e), \pi(s_1), \pi(s_4), \pi(s_1 s_4)$ are all linearly independent, so $\pi(\emptyset, \delta_\emptyset, t)$ splits into no less than four irreducible summands, corresponding to the irreducible representations of $\{e, s_1, s_4, s_1 s_4\} \cong (\mathbb{Z}/2\mathbb{Z})^2$.

We visualize this as

$$\mathrm{Prim}\big(C_r^*(\mathcal{R},q)_P\big) \cong$$



where the lower right corner depicts the fourfold non-Hausdorff point $(-1,1)$. This algebra is diffeotopy-equivalent to its fiber over $(-1,1)$, so

$$\begin{aligned} K_0\big(C_r^*(\mathcal{R},q)_P\big) &\cong \mathbb{Z}^4 \\ K_1\big(C_r^*(\mathcal{R},q)_P\big) &= 0 \end{aligned}$$

Generators are the rank one projectors

$$p_0 = \frac{1}{8}\sum_{w \in W_0} q(w) T_w \ , \ p_3 = \frac{1}{8}\sum_{w \in W_0} (-1)^{\ell(w)} T_w \ , \ p_{-+} \ , \ p_{+-}$$

the last two being defined in the same way as the homonymous (classes of) projections in (6.57).

- $P = \{\alpha_1\}$

Here $\mathcal{H}_P \cong \mathcal{H}(\mathcal{R}(A_1)^\vee, q_1) \cong \mathbb{C}[W(A_1)]$, so we do not find any discrete series representations to induce.

- $P = \{\alpha_2\}$

This is identical to $P = \{\alpha_2\}$ for generic labels.

- $P = \{\alpha_1, \alpha_2\}$

$$\begin{array}{llll} R_P = R_0 & R_P^\vee = R_0^\vee & & \\ X^P = 0 & X_P = X & Y^P = 0 & Y_P = Y \\ T^P = \{1\} & T_P = T & K_P = \{1\} & \\ W^P = W(P,P) = \mathcal{W}_{PP} = \{e\} & W_P = W_0 & & \end{array}$$

In the limit $q_1 \to 1$ many residual points confluence, and some lose the residuality. There remain only two W_0-orbits of four points, from which we pick the representatives $(1, q_2)$ and $(-1, -q_2)$. The other discrete series representations were already constructed in (6.58). They have dimension one, and for $q_2 > 1$:

$$\begin{array}{cccr}
\delta & \delta(T_{s_1}) & \delta(T_{s_2}) & \delta(\theta_x) \\ \hline
\delta_a & 1 & -1 & (1, q_2^{-1})(x) \\
\delta_b & -1 & -1 & (1, q_2^{-1})(x) \\
\delta_c & 1 & -1 & (-1, -q_2^{-1})(x) \\
\delta_d & -1 & -1 & (-1, -q_2^{-1})(x)
\end{array} \tag{6.62}$$

On the other hand, for $q_2 < 1$ we find:

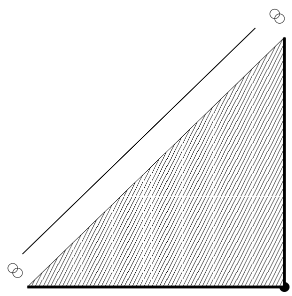
$$\begin{array}{cccr}
\delta & \delta(T_{s_1}) & \delta(T_{s_2}) & \delta(\theta_x) \\ \hline
\delta_a & 1 & q_2 & (1, q_2^{-1})(x) \\
\delta_b & -1 & q_2 & (1, q_2^{-1})(x) \\
\delta_c & 1 & q_2 & (-1, -q_2^{-1})(x) \\
\delta_d & -1 & q_2 & (-1, -q_2^{-1})(x)
\end{array} \tag{6.63}$$

This summand of $C_r^*(\mathcal{R}, q)$ is actually commutative:

$$\begin{array}{lcl}
C_r^*(\mathcal{R}, q)_P & \cong & \mathbb{C}^4 \\
K_0\big(C_r^*(\mathcal{R}, q)_P\big) & \cong & \mathbb{Z}^4 \\
K_1\big(C_r^*(\mathcal{R}, q)_P\big) & = & 0
\end{array}$$

Thus the spectrum of $C_r^*(\mathcal{R}, q)$ is the non-Hausdorff space

$$\text{Prim}\big(C_r^*(\mathcal{R}, q)\big) \cong$$



and its K-groups are

$$\begin{array}{lcl}
K_0\big(C_r^*(\mathcal{R}, q)\big) & \cong & \mathbb{Z}^9 \\
K_1\big(C_r^*(\mathcal{R}, q)\big) & = & 0
\end{array} \tag{6.64}$$

- $\mathbf{q_1 = q_2 \neq 1,\ q_1 = q_2^{-1} \neq 1,\ q_1 = q_2^2 \neq 1,\ q_1 = q_2^{-2} \neq 1}$

These parameters are a bit too tricky to analyse with techniques we used so far. To determine the spectra of the corresponding C^*-algebras one can use the precise results on R-groups in [122]. It turns out that there are only three inequivalent discrete series representations. On the other hand, compared to the generic case two of the representations $\pi(P, \delta, t)$ with $|P| = 1$ become reducible.

Let us make up the balance for the root datum $\mathcal{R}(B_2)^\vee$. The K_1-groups vanish for all label functions, and the K_0-groups are all free abelian of rank 9. We did not give all the generating projections explicitly, but we have enough information to determine the maps $K_0(\phi_0)$. In the next table we list the images of the generators p_i of $K_0(\mathcal{S}(W))$. Assuming that all the calculations in this section are correct, the table shows that Conjecture 5.28 is valid for $\mathcal{R}(B_2)^\vee$.

We will not discuss the root datum $\mathcal{R}(B_2)$ exhaustively, because there too are many different label functions. The group case is as in this section, and the equal label case is well understood, as described in Section 5.1. We will study the generic label case in Section 6.7.

$1<q_1^{-1/2}<q_2<q_1^{-1}$	$q_1=1<q_2$	$1<q_1^{1/2}<q_2<q_1$
$p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p(\delta_5)$ $p_\emptyset$ $p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p_{\alpha_2}+ p(\delta_1)+p(\delta_3)+p(\delta_5)$	p_0 p_3 $p_0+p_{\alpha_2}+p(\delta_a)+p(\delta_c)$	$p_\emptyset$ $p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p(\delta_5)$ $p_\emptyset+p_{\alpha_2}$
$p_\emptyset+p_{\alpha_2}$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p(\delta_2)+p(\delta_4)$ $p_\emptyset+p^+_{\alpha_1}$	$p_3+p_{\alpha_2}+p(\delta_b)+p(\delta_d)$ $p_{-+}+p_{+-}+p_{\alpha_2}$ p_{-+}	$p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p_{\alpha_2} +p(\delta_2)+p(\delta_4)+p(\delta_5)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p(\delta_1)+p(\delta_3)$ $p_\emptyset+p^-_{\alpha_1}$
$p_\emptyset+p^+_{\alpha_1}+p_{\alpha_2}+p(\delta_1)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p(\delta_4)+p(\delta_5)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p(\delta_4)$	$p_{+-}+p_{\alpha_2}+p(\delta_a)+p(\delta_d)$ $p_0+p_3+p_{\alpha_2}+p(\delta_b)$ $p_{+-}+p_{-+}+p_{\alpha_2}+p(\delta_b)$	$p_\emptyset+p^-_{\alpha_1}+p_{\alpha_2}+p(\delta_4)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p_{\delta_2}+p(\delta_3)+p(\delta_5)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p_{\delta_2}+p(\delta_3)$
$q_1<1=q_2$	$q_1=1=q_2$	$q_1>1=q_2$
$p_0+p^+_{\alpha_1}+p^-_{\alpha_1}+ p(\delta_a)+p(\delta_c)+p(\delta_5)$ p_0 $p_3+p^+_{\alpha_1}+p^-_{\alpha_1}+ p(\delta_b)+p(\delta_d)+p(\delta_5)$	p_0 p_1 p_2	p_0 $p_0+p^+_{\alpha_1}+p^-_{\alpha_1}+ p(\delta_a)+p(\delta_c)+p(\delta_5)$ p_3
p_3 $p_0+p_3+p^+_{\alpha_1}+p^-_{\alpha_1}$ $p_0+p^+_{\alpha_1}+p(\delta_a)$	p_3 p_4 p_5	$p_3+p^+_{\alpha_1}+p^-_{\alpha_1}+ p(\delta_b)+p(\delta_d)+p(\delta_5)$ $p_0+p_3+p^+_{\alpha_1}+p^-_{\alpha_1}$ $p_0+p^-_{\alpha_1}+p(\delta_c)$
$p_3+p^+_{\alpha_1}+p(\delta_b)$ $p_0+p_3+p^+_{\alpha_1}+ p^-_{\alpha_1}+p(\delta_a)+p(\delta_5)$ $p_0+p_3+p^+_{\alpha_1}+ p^-_{\alpha_1}+p(\delta_a)$	p_6 p_7 p_8	$p_3+p^-_{\alpha_1}+p(\delta_d)$ $p_0+p_3+p^+_{\alpha_1}+ p^-_{\alpha_1}+p(\delta_b)+p(\delta_5)$ $p_0+p_3+p^+_{\alpha_1}+ p^-_{\alpha_1}+p(\delta_b)$
$q_1<q_2<q_1^{1/2}<1$	$q_1=1>q_2$	$q_1^{-1}<q_2<q_1^{-1/2}<1$
$p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p_{\alpha_2} +p(\delta_1)+p(\delta_4)+p(\delta_5)$ $p_\emptyset+p_{\alpha_2}$ $p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p(\delta_5)$	$p_0+p_{\alpha_2}+p(\delta_a)+p(\delta_c)$ $p_3+p_{\alpha_2}+p(\delta_b)+p(\delta_d)$ p_0	$p_\emptyset+p_{\alpha_2}$ $p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p_{\alpha_2} +p(\delta_1)+p(\delta_3)+p(\delta_5)$ $p_\emptyset$
$p_\emptyset$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p(\delta_1)+p(\delta_3)$ $p_\emptyset+p^+_{\alpha_1}+p_{\alpha_2}+p(\delta_1)$	p_3 $p_{+-}+p_{-+}$ $p_{-+}+p_{\alpha_2}+p_(\delta_a)+p(\delta_d)$	$p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p(\delta_5)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p(\delta_2)+p(\delta_4)$ $p_\emptyset+p^-_{\alpha_1}+p_{\alpha_2}+p(\delta_3)$
$p_\emptyset+p^+_{\alpha_1}$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+p_{\alpha_2} +p_(\delta_1)+p(\delta_3)+p(\delta_5)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p_(\delta_1)+p(\delta_3)$	p_{+-} $p_0+p_3+p_{\alpha_2}+p(\delta_a)$ $p_{+-}+p_{-+}+p_{\alpha_2}+p(\delta_a)$	$p_\emptyset+p^-_{\alpha_1}$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p(\delta_4)+p(\delta_5)$ $2p_\emptyset+p^+_{\alpha_1}+p^-_{\alpha_1}+ p_{\alpha_2}+p(\delta_4)$

6.5 GL_n

After the calculations with twodimensional root data we move on to higher ranks. The easiest root data to study are those associated with the reductive group GL_n. The way right way to do this was indicated in [17]. From [103, Lemma 5.3] we know that the topological K-groups of these affine Hecke algebras are free abelian, and according to Theorem 5.3 they do not depend on q. Because of the higher dimensionality we do not provide explicit generators for these K-groups anymore. Nevertheless, by a different argument we show that Conjecture 5.28 holds for these root data.

From now on many things will be parametrized by partitions and permutations, so let us agree on some notations. We write partitions in decreasing order and abbreviate $(x)^3 = (x, x, x)$. A typical partition looks like

$$\mu = (\mu_1, \mu_2, \dots, \mu_d) = (n)^{m_n} \cdots (2)^{m_2} (1)^{m_1} \tag{6.65}$$

where some ot the multiplicities m_i may be 0. By $\mu \vdash n$ we mean that that the weight of μ is

$$|\mu| = \mu_1 + \dots + \mu_d = n$$

The number of different μ_i's (i.e. the number of blocks in the diagram of μ) will be denoted by $b(\mu)$ and the dual partition (obtained by reflecting the diagram of μ) by $\mu^\vee$. Sometimes we abbreviate

$$\begin{aligned} g &= \gcd(\mu) = \gcd(\mu_1, \dots, \mu_d) \\ \mu! &= \mu_1! \mu_2! \cdots \mu_d! \end{aligned} \tag{6.66}$$

With a such partition μ of n we associate the permutation

$$\sigma(\mu) = (12 \cdots \mu_1)(\mu_1 + 1 \cdots \mu_1 + \mu_2) \cdots (n + 1 - \mu_d \cdots n) \in S_n$$

As is well known, this gives a bijection between partitions of n and conjugacy classes in the symmetric group S_n. The centralizer $Z_{S_n}(\sigma(\mu))$ is generated by the cycles

$$((\mu_1 + \dots + \mu_i + 1)(\mu_1 + \dots + \mu_i + 2) \cdots (\mu_1 + \dots + \mu_i + \mu_{i+1}))$$

and the "permutations of cycles of equal length", for example if $\mu_1 = \mu_2$:

$$(1\, \mu_1 + 1)(2\, \mu_1 + 2) \cdots (\mu_1\, 2\mu_1) \tag{6.67}$$

Using the second presentation of μ this means that

$$Z_{S_n}(\sigma(\mu)) \cong \prod_{l=1}^{n} (\mathbb{Z}/l\mathbb{Z})^{m_l} \rtimes S_{m_l}$$

Let us recall the definition of $\mathcal{R}(GL_n)$:

$$\begin{aligned}
&X = \mathbb{Z}^n \quad Q = \{x \in X : x_1 + \cdots + x_n = 0\}\\
&X^+ = \{x \in \mathbb{Z}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}\\
&Y = \mathbb{Z}^n \quad Q^\vee = \{y \in Y : y_1 + \cdots + y_n = 0\}\\
&T = (\mathbb{C}^\times)^n \quad t = (t(e_1), \ldots, t(e_n)) = (t_1, \ldots, t_n)\\
&R_0 = R_1 = \{e_i - e_j \in X : i \neq j\}\\
&R_0^\vee = R_1^\vee = \{e_i - e_j \in Y : i \neq j\}\\
&F_{\text{aff}} = \{\alpha_i^\vee = e_i - e_{i+1}\} \cup \{1 - \alpha_0^\vee = 1 - (e_1 - e_n)\}\\
&s_i = s_{\alpha_i} \quad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{-\alpha_1} s_{\alpha_0} t_{\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee , x \rangle \alpha_0\\
&W_0 = \langle s_1, \cdots, s_{n-1} | s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e : |i-j| > 1 \rangle \cong S_n\\
&S_{\text{aff}} = \{s_0, s_1, \ldots s_{n-1}\}\\
&W_{\text{aff}} = \langle s_0, W_0 | s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \rangle\\
&W = W_{\text{aff}} \rtimes \Omega \quad \Omega = \langle t_{e_1} (1\, 2 \cdots n) \rangle \cong \mathbb{Z}
\end{aligned}$$

Because all roots of R_0 are conjugate, s_0 is conjugate to any $s_i \in S_{\text{aff}}$. Hence for any label function we have

$$q(s_0) = q(s_i) = q_{\alpha_i^\vee} := q$$

The W_0-stabilizer of a point $\big((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}\big) \in T$ is isomorphic to $S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_d}$.

- **group case $\mathbf{q = 1}$**

By (6.2) we have

$$K_*\big(C_r^*(W)\big) \otimes \mathbb{C} \cong \check{H}^*\big(\widetilde{T_u}/S_n; \mathbb{C}\big) \cong \bigoplus_{\mu \vdash n} \check{H}^*\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)); \mathbb{C}\big)$$

Therefore we want to determine $T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$. If μ is as in (6.65) then

$$\begin{aligned}
&T^{\sigma(\mu)} = \{(t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d} \in T\}\\
&T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) \cong (\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}
\end{aligned} \tag{6.68}$$

where S_{m_l} acts on $(\mathbb{C}^\times)^{m_l}$ by permuting the coordinates. To handle this space we use the following nice, elementary result, a proof of which can be found for example in [17, p. 97].

Lemma 6.1 *For any $m \in \mathbb{N}$ there is an isomorphism of algebraic varieties*

$$(\mathbb{C}^\times)^m/S_m \cong \mathbb{C}^{m-1} \times \mathbb{C}^\times$$

It follows that $T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$ has the homotopy type of $\mathbb{T}^{b(\mu)}$. The latter space has torsion free cohomology, so by (6.3)

$$K_*\big(C_r^*(W)\big) \cong \check{H}^*\big(\widetilde{T_u}/S_n;\mathbb{Z}\big) \cong \bigoplus_{\mu\vdash n} \check{H}^*\big(\mathbb{T}^{b(\mu)};\mathbb{Z}\big) \cong \bigoplus_{\mu\vdash n} \mathbb{Z}^{2^{b(\mu)}} \tag{6.69}$$

- **generic, equal label case q** $\neq 1$
- $P_\mu = F_0 \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \ldots, \alpha_{n-\mu_d}\}$

Inequivalent subsets of F_0 are parametrized by partitions μ of n. For the typical partition (6.65) we put

$$\begin{aligned}
&R_{P_\mu} \cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R_{P_\mu}^\vee \\
&X^{P_\mu} \cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d \\
&X_{P_\mu} \cong \big(\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n)\big)^{m_n} \times \cdots \times \big(\mathbb{Z}^2/\mathbb{Z}(e_1 + e_2)\big)^{m_2} \\
&Y^{P_\mu} = \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n) \\
&Y_{P_\mu} = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\} \\
&T^{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} \in T\} \\
&T_{P_\mu} = \{t \in T : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\} \\
&K_{P_\mu} = \{t \in T^{P_\mu} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\} \\
&W_{P_\mu} \cong (S_n)^{m_n} \times \cdots \times (S_2)^{m_2} \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1} \\
&\mathcal{W}_{P_\mu P_\mu} = K_{P_\mu} \rtimes W(P_\mu, P_\mu) \qquad Z_{S_n}(\sigma(\mu)) = W(P_\mu, P_\mu) \ltimes \prod_{l=1}^n (\mathbb{Z}/l\mathbb{Z})^{m_l}
\end{aligned}$$

The W_{P_μ}-orbits of residual points for $\mathcal{H}_{P_\mu}$ are parametrized by

$$K_{P_\mu}\big((q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \ldots, q^{(1-\mu_1)/2}) \cdots (q^{(\mu_d-1)/2}, q^{(\mu_d-3)/2}, \ldots, q^{(1-\mu_d)/2})\big)$$

This set is obviously in bijection with K_{P_μ}, and indeed the intertwiners $\pi(k), k \in K_{P_\mu}$ act on it by multiplication. Together with Proposition 3.20.2 this implies

$$\begin{aligned}
&\bigcup_{\Delta_{P_\mu}} \big(P_\mu, \delta, T^{P_\mu}\big)/K_{P_\mu} \cong T^{P_\mu} \\
&\bigcup_{\Delta_{P_\mu}} \big(P_\mu, \delta, T^{P_\mu}\big)/\mathcal{W}_{P_\mu P_\mu} \cong T^{P_\mu}/W(P_\mu, P_\mu) = T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))
\end{aligned}$$

It a point $t \in T^{P_\mu}$ has a nontrivial stabilizer in $W(P_\mu P_\mu) \cong \prod_{l=1}^n S_{m_l}$, then this isotropy group is generated by transpositions. From (6.67) we see that every such transposition $w \in W_0$ can be written as a product of mutually commuting reflections s_α with $c_\alpha^{-1}(t) = 0$. By (3.122) this gives $\imath_w^o(t) = 1$, and since dim $\delta = 1$ also

$$\pi(w, P_\mu, \delta, t) = 1 \quad \text{if} \quad w(t) = t \tag{6.70}$$

So the action of $\mathcal{W}_{P_\mu P_\mu}$ on

$$C\Big(\bigsqcup_{\Delta_{P_\mu}} T_u^{P_\mu}; M_{n!/\mu!}(\mathbb{C})\Big)$$

is essentially only on $\bigsqcup_{\Delta_{P_\mu}} T_u^{P_\mu}$ and the conjugation part doesn't really matter. In particular we deduce that

$$C_r^*(\mathcal{R},q) \cong \bigoplus_{\mu\vdash n} M_{n!/\mu!}\Big(C\Big(\bigsqcup_{\Delta_{P_\mu}} T_u^{P_\mu}\Big)\Big) \cong \bigoplus_{\mu\vdash n} M_{n!/\mu!}\left(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\right) \tag{6.71}$$

Similar results were obtained by completely different methods in [93]. Just as in the group case it follows that

$$K_*\big(C_r^*(\mathcal{R},q)\big) \cong \bigoplus_{\mu\vdash n} K^*\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\big) \cong \bigoplus_{\mu\vdash n} K^*\left(\mathbb{T}^{b(\mu)}\right) \cong \bigoplus_{\mu\vdash n} \mathbb{Z}^{2^{b(\mu)}} \tag{6.72}$$

As promised, we show that Conjecture 5.28 holds in this particular case.

Theorem 6.2

$$K_*(\phi_0) : K_*\big(C_r^*(\mathcal{R}(GL_n),q^0)\big) \to K_*\big(C_r^*(\mathcal{R}(GL_n),q)\big)$$

is an isomorphism.

Proof. We assume that $q > 1$. If instead $q < 1$ then we only have to modify our argument by replacing the sign representations everywhere by trivial representations.

Let $J_i^c \subset C_r^*(\mathcal{R},q)$ be the norm completion of the ideal $J_i \subset \mathcal{S}(\mathcal{R},q)$ from (3.144). For every i there is a unique partition μ_i such that

$$J_{i-1}^c/J_i^c \cong C(T_u^{P_i};\mathrm{End}V_i)^{\mathcal{W}_i} \cong M_{n!/\mu_i!}C\big(T_u^{\sigma(\mu_i)}/Z_{S_n}(\sigma(\mu_i))\big)$$

The induced homomorphism

$$f_i : \phi_0^{-1}(J_{i-1}^c)/\phi_0^{-1}(J_i^c) \to J_{i-1}/J_i$$

is injective. If we would know that $K_*(f_i)$ is an isomorphism for every i, then the theorem would follow with Lemma 2.3.

We know already that the number of components of $\mathrm{Prim}(C_R^*(\mathcal{R},q))$ equals the number of components of $\mathrm{Prim}(C_r^*(W)) \cong \widetilde{T_u}/W_0$. Therefore it suffices to construct, for every i, a projection $p \in \phi_0^{-1}(J_{i-1}^c)/\phi_0^{-1}(J_i^c)$ such that $f_i(p) \in J_{i-1}^c/J_i^c$ has rank one.

We can do this because we know precisely what all discrete series look like. We may assume that the central character of (δ_i,V_i) lies in T_{rs}, so that the central character of $\pi(P_i,\delta_i,t)\circ\phi_0$ is $W_0 t$. As a W_0-representation $\pi(P_i,\delta_i,t)\circ\phi_0$ is equivalent with $\mathrm{Ind}_{W_{P_i}}^{W_0}(\epsilon_{W_{P_i}})$, where ϵ_{W_P} denotes the sign representation of W_P.

The ideal $\phi_0^{-1}(J_{i-1}^c)$ does not annihilate this representation, but it is contained in the kernel of the representation

$$\pi(P,\delta,t') \text{ for } \quad P \supset P_i = P_{\mu_i} \text{ and } \quad t' \in T_u^P \subset T^{P_i} = T^{\sigma(\mu_i)}.$$

As W_0-representation we have $\pi(P,\delta,t') \circ \phi_0 \cong \mathrm{Ind}_{W_P}^{W_0}(\epsilon_{W_P})$. Hence the "stalk" of $\phi_0^{-1}(J_{i-1}^c)/\phi_0^{-1}(J_i^c)$ at $W_0 t$ contains

$$\bigcap_{P \supsetneq P_i, t \in T_u^P} \ker \mathrm{Ind}_{W_P}^{W_0}(\epsilon_{W_P}) \Big/ \ker \mathrm{Ind}_{W_{P_i}}^{W_0}(\epsilon_{W_{P_i}})$$

which is a subquotient algebra of $\mathbb{C}[W_0]$. Therefore we can find a suitable p already in $\mathbb{C}[W_0]$: pick a projection of minimal rank in $\mathbb{C}[W_0]$ which is in

$$\bigcap_{P \supsetneq P_i} \ker \mathrm{Ind}_{W_P}^{W_0}(\epsilon_{W_P}) \text{ , but not in } \quad \ker \mathrm{Ind}_{W_{P_i}}^{W_0}(\epsilon_{W_{P_i}}).$$

Then $\phi_0(p)$ will act as a rank one projector on $\pi(P_i,\delta_i,t)$. $\square$

6.6 A_{n-1}

It is known from Theorem 5.3 that the periodic cyclic homology of type A affine Hecke algebras does not depend on q, but it will still be insightful to determine the spectra of these algebras. The title of this section is A_{n-1} instead of A_n because we want to consider everything as a quotient or a subset of $\mathbb{Z}^n$. If we require that our root datum is semisimple, then the easiest case is when X is the weight lattice. This is completely analogous to the GL_n-case, we can even show in the same way that Conjecture 5.28 holds. The calculations are also manageable when X is the root lattice. Intermediate lattice however would require quite some extra bookkeeping, so we do not study those. Throughout this section we assume that $n > 2$, because the root system A_1 has slightly different properties.

The root datum $\mathcal{R}(A_{n-1})^\vee$ is defined as

$$
\begin{aligned}
& X = \mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots e_n) \cong Q + ((e_1 + \cdots + e_n)/n - e_n) \\
& Q = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\} \quad X^+ = \{x \in X : x_1 \geq x_2 \geq \cdots \geq x_n\} \\
& Y = Q^\vee = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_n = 0\} \\
& T = \{t \in (\mathbb{C}^\times)^n : t_1 \cdots t_n = 1\} \quad t = (t(e_1), \ldots, t(e_n)) = (t_1, \ldots, t_n) \\
& R_0 = R_1 = \{e_i - e_j \in X : i \neq j\} \\
& R_0^\vee = R_1^\vee = \{e_i - e_j \in Y : i \neq j\} \\
& F_0 = \{\alpha_i = e_i - e_{i+1}\} \quad \alpha_0 = e_1 - e_n \\
& s_i = s_{\alpha_i} \quad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{-\alpha_1} s_{\alpha_0} t_{\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee \,, x \rangle \alpha_0 \\
& W_0 = \langle s_1, \cdots, s_{n-1} | s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ if} |i-j| > 1 \rangle \cong S_n \\
& S_{\text{aff}} = \{s_0, s_1, \ldots, s_{n-1}\} \\
& W_{\text{aff}} = \langle s_0, W_0 | s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \rangle \\
& W = W_{\text{aff}} \rtimes \Omega \quad \Omega = \langle t_{e_1 - (e_1 + \cdots e_n)/n} (12 \cdots n) \rangle \cong \mathbb{Z}/n\mathbb{Z}
\end{aligned}
$$

Because all roots are conjugate, s_0 is conjugate to any $s_i \in S_{\text{aff}}$, and for any label function

$$q(s_0) = q(s_i) = q_{\alpha_i^\vee} = q$$

The W_0-stabilizer of $\big((t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d}\big)$ is isomorphic to $S_{\mu_1} \times \cdots \times S_{\mu_d}$. Generically there are $n!n$ residual points, and they all satisfy $t(\alpha_i) = q$ or $t(\alpha_i) = q^{-1}$ for $1 \leq i < n$. There residual points form n conjugacy classes unless $q = 1$, in which case T itself is the only residual coset.

- **group case $\mathbf{q = 1}$**

According to (6.2) we have

$$K_*\big(C_r^*(W)\big) \otimes \mathbb{C} \cong \check{H}^*\big(\widetilde{T_u}/S_n; \mathbb{C}\big) \cong \bigoplus_{\mu \vdash n} \check{H}^*\big(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)); \mathbb{C}\big)$$

Pick a partition μ of n and write it as in (6.65).

$$
\begin{aligned}
T^{\sigma(\mu)} &= \{(t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d} \in T\} \\
&\cong \{(t_1)^{\mu_1} (t_{\mu_1+1})^{\mu_2} \cdots (t_n)^{\mu_d} \in (\mathbb{C}^\times)^n\}/\mathbb{C}^\times \times \{(e^{2\pi i k/n})^n : 0 \leq k < g\} \\
T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) &\cong \left((\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}\right)/\mathbb{C}^\times \times \\
&\qquad \{(e^{2\pi i k/n})^n : 0 \leq k < g\}
\end{aligned}
$$

where $\mathbb{C}^\times$ acts diagonally. By Lemma 6.1 each factor $(\mathbb{C}^\times)^{m_i}/S_{m_i}$ is homotopy equivalent to a circle. The induced action of $S^1 \subset \mathbb{C}^\times$ on this direct product of circles identifies with a direct product of rotations. Hence $T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))$ is homotopy equivalent with $\mathbb{T}^{b(\mu)-1} \times \{\gcd(\mu) \text{ points}\}$. The cohomology of this space has no torsion, so by (6.3)

$$
\begin{aligned}
& K_*\big(C_r^*(W)\big) \cong \check{H}^*\big(\widetilde{T_u}/S_n; \mathbb{Z}\big) \cong \mathbb{Z}^{d(n)} \\
& d(n) = \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1}
\end{aligned}
\tag{6.73}
$$

- **generic, equal label case** $\mathbf{q} \neq 1$
- $P_\mu = F_0 \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \dots, \alpha_{n-\mu_d}\}$

Inequivalent subsets of F_0 are parametrized by partitions μ of n. For the typical partition (6.65) we put

$$\begin{aligned}
&R_{P_\mu} \cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R_{P_\mu}^\vee \\
&X^{P_\mu} \cong \big(\mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d\big)/\mathbb{Z}(e_1 + \cdots + e_n)/g \\
&X_{P_\mu} \cong \big(\mathbb{Z}^n/\mathbb{Z}(e_1 + \cdots + e_n)\big)^{m_n} \times \cdots \times \big(\mathbb{Z}^2/\mathbb{Z}(e_1 + e_2)\big)^{m_2} \\
&Y^{P_\mu} = \{y \in \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n) : y_1 + \cdots + y_n = 0\} \\
&Y_{P_\mu} = \{y \in Y : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\} \\
&T^{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} \in T : t_1^{\mu_1/g} \cdots t_n^{\mu_d/g} = 1\} \\
&T_{P_\mu} = \{t \in T : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\} \\
&K_{P_\mu} = \{t \in T^{P_\mu} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\} \\
&W_{P_\mu} \cong (S_n)^{m_n} \times \cdots \times (S_2)^{m_2} \qquad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1} \\
&\mathcal{W}_{P_\mu P_\mu} = K_{P_\mu} \rtimes W(P_\mu, P_\mu) \qquad Z_{S_n}(\sigma(\mu)) = W(P_\mu, P_\mu) \ltimes \prod_{l=1}^n (\mathbb{Z}/l\mathbb{Z})^{m_l}
\end{aligned}$$

The W_{P_μ}-orbits of residual points for $\mathcal{H}_{P_\mu}$ are represented by the points

$$\begin{aligned}
\big((q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \dots, q^{(1-\mu_1)/2}) \cdots (q^{(\mu_d-1)/2}, q^{(\mu_d-3)/2}, \dots, q^{(1-\mu_d)/2})\big) \cdot \\
\big((e^{2\pi i k_1/\mu_1})^{\mu_1} \cdots (e^{2\pi i k_d/\mu_d})^{\mu_d}\big) \quad , \quad 0 \leq k_i < \mu_i \quad (6.74)
\end{aligned}$$

These points are in bijection with $K_{P_\mu} \times \mathbb{Z}/\gcd(\mu)\mathbb{Z}$. Also $T^{\sigma(\mu)}$ consists of exactly gcg(μ) components, one of which is T^{P_μ}. Together with Proposition 3.20.2 this leads to

$$\begin{aligned}
&\bigcup_{\Delta_{P_\mu}} \big(P_\mu, \delta, T^{P_\mu}\big)/K_{P_\mu} \cong T^{P_\mu} \times \mathbb{Z}/\gcd(\mu)\mathbb{Z} \cong T^{\sigma(\mu)} \\
&\bigcup_{\Delta_{P_\mu}} \big(P_\mu, \delta, T^{P_\mu}\big)/\mathcal{W}_{P_\mu P_\mu} \cong T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))
\end{aligned}$$

If a point $t \in T^{P_\mu}$ has a nontrivial stabilizer in $W(P_\mu, P_\mu) \cong \prod_{l=1}^n S_{m_l}$ then this stabilizer is generated by transpositions. From (6.67) we see that every such transposition $w \in W(P_\mu, P_\mu)$ can be written as a product of mutually commuting reflections s_α with $\alpha \in R_0$ and $c_\alpha^{-1}(t) = 0$. So by (3.122) $\imath_w^o(t) = 1$ and since the discrete series representation δ is onedimensional, also $\pi(w, P_\mu, \delta, t) = 1$. On the other hand, K_{P_μ} permutes the components of $\bigcup_{\Delta_{P_\mu}} \left(P_\mu, \delta, T^{P_\mu}\right)$ faithfully, so the action of $\mathcal{W}_{P_\mu P_\mu}$ on

$$C\left(\bigcup_{\Delta_{P_\mu}} T^{P_\mu}; M_{n!/\mu!}(\mathbb{C})\right)$$

is essentially only on the underlying space. Therefore

$$\begin{aligned} C_r^*(\mathcal{R}, q) &\cong \bigoplus_{\mu \vdash n} M_{n!/\mu!}\Big(C\Big(\bigsqcup_{\Delta_{P_\mu}} T_u^{P_\mu}\Big)\Big) \cong \bigoplus_{\mu \vdash n} M_{n!/\mu!}\left(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\right) \\ K_*\big(C_r^*(\mathcal{R}, q)\big) &\cong \bigoplus_{\mu \vdash n} K^*\left(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu))\right) \cong \bigoplus_{\mu \vdash n} K^*\left(\mathbb{T}^{b(\mu)-1}\right) \cong \mathbb{Z}^d \end{aligned} \tag{6.75}$$

where $d = \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1}$.

We conclude that for $\mathcal{R}(A_{n-1})^\vee$ the K-theory of $C_r^*(\mathcal{R}, q)$ does not depend on q, and is a free abelian group.

Theorem 6.3

$$K_*(\phi_0) : K_*\big(C_r^*\big(\mathcal{R}(A_{n-1})^\vee, q^0\big)\big) \to K_*\big(C_r^*\big(\mathcal{R}(A_{n-1})^\vee, q\big)\big)$$

is an isomorphism.

Proof. This is completely analogous to Theorem 6.2. The essential common properties of these two root data are that the reduced C^*-algebras are Morita equivalent to commutative C^*-algebras, and that we have explicit descriptions of the discrete series representations of the algebras $\mathcal{H}_P$. □

The analogy with $\mathcal{R}(GL_n)$ is significantly weaker for the root datum $\mathcal{R}(A_{n-1})$:

$$\begin{aligned}
& X = Q = \{x \in \mathbb{Z}^n : x_1 + \cdots + x_n = 0\} \quad X^+ = \{x \in X : x_1 \geq x_2 \geq \cdots \geq x_n\} \\
& Q^\vee = \{y \in \mathbb{Z}^n : y_1 + \cdots + y_n = 0\} \\
& Y = \mathbb{Z}^n / \mathbb{Z}(e_1 + \cdots + e_n) \cong Q^\vee + ((e_1 + \cdots + e_n)/n - e_1) \\
& T = (\mathbb{C}^\times)^n / \mathbb{C}^\times \quad t = (t_1, \ldots, t_n) = (t(e_1), \ldots, t(e_n)) \\
& R_0 = R_1 = \{e_i - e_j \in X : i \neq j\} \\
& R_0^\vee = R_1^\vee = \{e_i - e_j \in Y : i \neq j\} \\
& F_0 = \{\alpha_i = e_i - e_{i+1} : 1 \leq i < n\} \quad \alpha_0 = e_1 - e_n \\
& s_i = s_{\alpha_i} \quad s_0 = t_{\alpha_0} s_{\alpha_0} = t_{-\alpha_1} s_{\alpha_0} t_{\alpha_1} : x \to x + \alpha_0 - \langle \alpha_0^\vee, x \rangle \alpha_0 \\
& W_0 = \langle s_1, \cdots, s_{n-1} | s_i^2 = (s_i s_{i+1})^3 = (s_i s_j)^2 = e \text{ if} |i-j| > 1 \rangle \cong S_n \\
& S_{\text{aff}} = \{s_0, s_1, \ldots, s_{n-1}\} \quad \Omega = \{e\} \\
& W = W_{\text{aff}} = \langle s_0, W_0 | s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^3 = (s_0 s_{n-1})^3 = e \text{ if } 2 \leq i \leq n-2 \rangle \\
& q(s_0) = q(s_i) = q_{\alpha_i^\vee} := q
\end{aligned}$$

For $q \neq 1$ there are $n!$ residual points. They form one W_0-orbit, and a typical residual point is

$$\left(q^{(1-n)/2}, q^{(3-n)/2}, \dots, q^{(n-1)/2}\right)$$

To determine the isotropy group of points of T we have to be careful. In general the W_0-stabilizer of

$$\left((t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2}\cdots(t_n)^{\mu_d}\right) \in T$$

is isomorphic to

$$S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_d} \subset W_0$$

However, in some special cases the diagonal action of $\mathbb{C}^\times$ on $(\mathbb{C}^\times)^n$ gives rise to extra stabilizers. Let r be a divisor of n, $k \in (\mathbb{Z}/r\mathbb{Z})^\times$ and $\lambda = (\lambda_1, \dots, \lambda_l)$ a partition of n/r. The isotropy group of

$$\left((t_1)^{\lambda_1}(e^{2\pi ik/r}t_1)^{\lambda_1}\cdots(e^{-2\pi ik/r}t_1)^{\lambda_1}(t_{r\lambda_1+1})^{\lambda_2}\cdots \right. \\ \left. (e^{-2\pi ik/r}t_{r\lambda_1+1})^{\lambda_2}\cdots(e^{-2\pi ik/r}t_n)^{\lambda_l}\right) \quad (6.76)$$

is isomorphic to

$$S^r_{\lambda_1} \times S^r_{\lambda_2} \times \cdots \times S^r_{\lambda_l} \rtimes \mathbb{Z}/r\mathbb{Z} \quad (6.77)$$

Explicitly the subgroup $\mathbb{Z}/r\mathbb{Z}$ is generated by

$$(1\ \lambda_1{+}1\ 2\lambda_1{+}1\cdots(r{-}1)\lambda_1{+}1)(2\ \lambda_1{+}2\ 2\lambda_1{+}2\cdots(r{-}1)\lambda_1{+}2)\cdots(\lambda_1\ 2\lambda_1\cdots r\lambda_1) \\ \cdots(n{+}1{-}r\lambda_d\ n{+}1{+}(1{-}r)\lambda_d\cdots n{+}1{+}(r{-}1)\lambda_d)(n{+}(1{-}r)\lambda_d\ n{+}(2{-}r)\lambda_d\cdots n) \quad (6.78)$$

and it acts on every factor $S^r_{\lambda_j}$ in (6.77) by cyclic permutations.

- **group case $\mathbf{q = 1}$**

As we saw before

$$K_*\left(C_r^*(W)\right) \otimes \mathbb{C} \cong \check{H}^*\left(\widetilde{T_u}/S_n; \mathbb{C}\right) \cong \bigoplus_{\mu \vdash n} \check{H}^*\left(T_u^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)); \mathbb{C}\right)$$

For the typical partition μ we have

$$T^{\sigma(\mu)} = \{(t_1)^{\mu_1}(t_{\mu_1+1})^{\mu_2}\cdots(t_n)^{\mu_d}\}/\mathbb{C}^\times \times \{t : t(e_j) = e^{2\pi ijk/g},\ 0 \leq k < g\} \quad (6.79)$$

which is the disjoint union of $g = \gcd(\mu)$ complex tori of dimension $m_n + m_{n-1} + \cdots + m_1 - 1$.

$$T^{\sigma(\mu)}/Z_{S_n}(\sigma(\mu)) \cong \left((\mathbb{C}^\times)^{m_n}/S_{m_n} \times \cdots \times (\mathbb{C}^\times)^{m_1}/S_{m_1}\right)/\mathbb{C}^\times \times \\ \{t : t(e_j) = e^{2\pi ijk/g},\ 0 \leq k < g\} \quad (6.80)$$

Curiously enough these sets are diffeomorphic to the corresponding sets for $\mathcal{R}(A_{n-1})^\vee$, a phenomenon for which the author does not have a good explanation. Anyway, we do take advantage of this by reusing our deduction that (6.80) is homotopy equivalent with $\mathbb{T}^{b(\mu)-1} \times \{\gcd(\mu) \text{ points}\}$. Just as in (6.73) we conclude that

$$\begin{aligned} K_*\big(C_r^*(W)\big) &\cong \check{H}^*\big(\widetilde{T_u}/S_n;\mathbb{Z}\big) \cong \mathbb{Z}^{d(n)} \\ d(n) &= \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1} \end{aligned} \tag{6.81}$$

- **generic case q ≠ 1**

This is noticeably different from the generic cases for $\mathcal{R}(GL_n)$ and $\mathcal{R}(A_{n-1}^\vee)$ because $C_r^*(\mathcal{R}(A_{n-1},q))$ is not Morita equivalent to a commutative C^*-algebra. Of course the inequivalent subsets of F_0 are still parametrized by partitions μ of n.

- $P_\mu = F_0 \setminus \{\alpha_{\mu_1}, \alpha_{\mu_1+\mu_2}, \dots, \alpha_{n-\mu_d}\}$

$$\begin{aligned}
&R_{P_\mu} \cong (A_{n-1})^{m_n} \times \cdots \times (A_1)^{m_2} \cong R_{P_\mu}^\vee \\
&X^{P_\mu} \cong \{x \in \mathbb{Z}(e_1 + \cdots + e_{\mu_1})/\mu_1 + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mu_d : \\
&\quad x_1 + \cdots + x_n = 0\} \\
&X_{P_\mu} \cong \{x \in \mathbb{Z}^{\mu_1}/\mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}^{\mu_d}/\mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n) : \\
&\quad x_1 + \cdots + x_n \in g\mathbb{Z}/g\mathbb{Z}\} \\
&Y^{P_\mu} \cong \mathbb{Z}(e_1 + \cdots + e_{\mu_1}) + \cdots + \mathbb{Z}(e_{n+1-\mu_d} + \cdots + e_n)/\mathbb{Z}(e_1 + \cdots + e_n) \\
&Y_{P_\mu} \cong \{y : y_1 + \cdots + y_{\mu_1} = \cdots = y_{n+1-\mu_d} + \cdots + y_n = 0\}/\mathbb{Z}(e_1 + \cdots e_n) \\
&T^{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d}\}/\mathbb{C}^\times \\
&T_{P_\mu} = \{t : t_1 t_2 \cdots t_{\mu_1} = \cdots = t_{n+1-\mu_d} \cdots t_n = 1\}/\{z \in \mathbb{C} : z^g = 1\} \\
&K_{P_\mu} = \{(t_1)^{\mu_1} \cdots (t_n)^{\mu_d} : t_1^{\mu_1} = \cdots = t_n^{\mu_d} = 1\}/\{z \in \mathbb{C} : z^g = 1\} \\
&W_{P_\mu} \cong S_n^{m_n} \times S_{n-1}^{m_{n-1}} \times \cdots \times S_2^{m_2} \quad W(P_\mu, P_\mu) \cong S_{m_n} \times \cdots \times S_{m_2} \times S_{m_1}
\end{aligned}$$

Note that

$$T^{\sigma(\mu)} = T^{P_\mu} \times \{t : t(e_j) = e^{2\pi i j k/g},\ 0 \leq k < g\}$$

The W_{P_μ}-orbits of residual points for $\mathcal{H}_{P_\mu}$ are represented by the points of

$$K_{P_\mu}\big(q^{(\mu_1-1)/2}, q^{(\mu_1-3)/2}, \dots q^{(1-\mu_1)/2}, q^{(\mu_2-1)/2}, \dots, q^{(\mu_d-1)/2}, \dots, q^{(1-\mu_d)/2}\big)$$

Hence the intertwiners $\pi(k)$ with $k \in K_{P_\mu}$ permute the elements of Δ_{P_μ} faithfully, and

$$\bigsqcup_{\Delta_{P_\mu}} (P_\mu, \delta, T^{P_\mu})/K_{P_\mu} \cong T^{P_\mu} = \Big(T^{\sigma(\mu)}\Big)_1$$

where $(\cdot)_1$ means the connected component containing $(1, 1, \dots, 1) = 1 \in T$. In (6.70) we saw that the intertwiners for $\mathcal{R}(GL_n), q \neq 1$ have the property

$$w(t) = t \Rightarrow \pi(w, P_\mu, \delta, t) = 1$$

This implies that in our present setting we can have $w(t) = t$ and $\pi(w, P_\mu, \delta, t) \neq 1$ only if $w(t) = t$ does not hold without taking the action of $\mathbb{C}^\times$ into account. Let us classify such $w \in W(P_\mu, P_\mu)$ and $t \in T^{P_\mu}$ up to conjugacy. For a divisor r of $g^\vee := \gcd(\mu^\vee)$ we have the partition

$$\mu^{1/r} := (nr)^{m_n/r} \cdots (2r)^{m_2/r} (r)^{m_1/r}$$

Notice that

$$b(\mu^{1/r}) = b(\mu) = b(\mu^\vee)$$

There exists a $\sigma \in S_n$ which is conjugate to $\sigma(\mu^{1/r})$ and satisfies $\sigma^r = \sigma(\mu)$. We construct a particular such σ as follows. If $r = g^\vee$ then (starting from the left) replace every block

$$(d+1\ d+2 \cdots d+m)(d+1+m \cdots d+2m) \cdots (d+(g^\vee-1)m \cdots d+g^\vee m)$$

of $\sigma(\mu)$ by

$$(d{+}1\ d{+}1{+}m \cdots d{+}1{+}(g^\vee{-}1)m\ 2\ d{+}2{+}m \cdots d{+}2{+}(g^\vee{-}1)m\ d{+}3 \cdots d{+}g^\vee m)$$

We denote the resulting element by $\sigma(\mu)^{1/g^\vee}$ and for general $r|g^\vee$ we define

$$\sigma(\mu)^{1/r} := \big(\sigma(\mu)^{1/g^\vee}\big)^{g^\vee/r}$$

Consider the cosets of subtori

$$T^{P_\mu} r, k := \big(T^{\sigma(\mu)^{1/r}}\big)_1 \big((1)^{g^\vee \mu_1/r} (e^{2\pi i k/r})^{g^\vee \mu_{1+g^\vee/r}/r} \cdots (e^{-2\pi i k/r})^{g^\vee \mu_d/r}\big) \quad k \in \mathbb{Z}$$

If $\gcd(k, r) = 1$ then the generic points of $T^{P_\mu} r, k$ have $W(P_\mu, P_\mu)$-stabilizer

$$\langle W_{P_\mu}\,,\ \sigma(\mu)^{1/r} \rangle \cap W(P_\mu, P_\mu) \cong \mathbb{Z}/r\mathbb{Z}$$

Note that for $r'|g^\vee$

$$T^{P_\mu}_{r',k} \subset T^{P_\mu}_{r,k} \qquad \text{if } r|r' \tag{6.82}$$

If a point $t \in T^{P_\mu}_{r,k}$ does not lie on any $T^{P_\mu}_{r',k'}$ with $r' > r$, then its $W(P_\mu, P_\mu)$-stabilizer may still be larger than $\mathbb{Z}/r\mathbb{Z}$. However, it is always of the form

$$S^r_{\lambda_1} \times \cdots \times S^r_{\lambda_l} \rtimes \mathbb{Z}/r\mathbb{Z}$$

By an argument like on pages 207 and 211 one can show that the intertwiners $\pi(w, P_\mu, \delta, t)$ are scalar for $w \in S^r_{\lambda_1} \times \cdots \times S^r_{\lambda_l}$ and nonscalar for $w \in (\mathbb{Z}/r\mathbb{Z}) \setminus \{e\}$. Because $\mathbb{Z}/r\mathbb{Z}$ is cyclic this implies that $\pi(P_\mu, \delta, t)$ is the direct sum of exactly r inequivalent irreducible representations. For a more systematic discussion of such matters we refer to [40].

Different choices of $\sigma(\mu)^{1/r}$ or of $k \in (\mathbb{Z}/r\mathbb{Z})^\times$ lead to conjugate subvarieties of T^{P_μ}, so we have a complete discription of $\mathrm{Prim}\big(C^*_r(\mathcal{R}, q)_{P_\mu}\big)$. To calculate the K-theory of this algebra we use Theorem 2.24, which says that (modulo torsion) it is isomorphic to

$$H^*_{W(P_\mu,P_\mu)}\big(T^{P_\mu}_u; \mathcal{L}_u\big) \cong \check{H}^*\big(T^{P_\mu}/W(P_\mu, P_\mu); \mathcal{L}_u^{W(P_\mu,P_\mu)}\big)$$

We know from [62] that we can endow $T_u^{P_\mu}$ with the structure of a finite $W(P_\mu, P_\mu)$-CW-complex, such that every $T_{u,r,k}^{P_\mu}$ is a subcomplex. The local coefficient system $\mathcal{L}_u$ is not very complicated: $\mathcal{L}_u(B) \cong \mathbb{Z}^r$ if and only if B ∂B consists of generic points in a conjugate of $T_{u,r,k}^{P_\mu}$. In suitable coordinates the maps $\mathcal{L}_u(B \to B')$ are all of the form

$$\mathbb{Z}^r \to \mathbb{Z}^{r/d} : (x_1, \ldots, x_r) \to (x_1 + x_2 + \cdots + x_d, \ldots, x_{1+r-d} + \cdots + x_r)$$

Hence the associated sheaf is the direct sum of several subsheaves $\mathfrak{F}_r^\mu$ one for each divisor r of $\gcd(\mu^\vee)$. The support of $\mathfrak{F}_r^\mu$ is

$$W(P_\mu, P_\mu) T_{u,r,1}^{P_\mu} / W(P_\mu, P_\mu) \cong T_u^{P_{\mu^{1/r}}} / Z_{S_n}(\sigma(\mu^{1/r}))$$

and on that space it has constant stalk $\mathbb{Z}^{\phi(r)}$. Here ϕ is the Euler ϕ-function, i.e.

$$\phi(r) = \#\{m \in \mathbb{Z} : 0 \leq m < r : \gcd(m, r) = 1\} = \#(\mathbb{Z}/r\mathbb{Z})^\times$$

This the rank of $\mathfrak{F}_r^\mu$ because in every point of $T_{u,r,1}$ we have r irreducible representations, but the ones corresponding to numbers that are not coprime with r are already accounted for by the sheaves $\mathfrak{F}_{r'}^\mu$ with $r'|r$. Now we can calculate

$$\begin{aligned}
\check{H}^*\big(T^{P_\mu}/W(P_\mu, P_\mu); \mathcal{L}_u^{W(P_\mu,P_\mu)}\big) &\cong \bigoplus_{r|\gcd(\mu^\vee)} \check{H}^*\big(T^{P_\mu}/W(P_\mu, P_\mu); \mathfrak{F}_r^\mu\big) \\
&\cong \bigoplus_{r|\gcd(\mu^\vee)} \check{H}^*\big(T_u^{P_{\mu^{1/r}}}/Z_{S_n}(\sigma(\mu^{1/r})); \mathbb{Z}^{\phi(r)}\big) \\
&\cong \bigoplus_{r|\gcd(\mu^\vee)} \check{H}^*\big(\mathbb{T}^{b(\mu^{1/r})-1}; \mathbb{Z}^{\phi(r)}\big) \\
&\cong \bigoplus_{r|\gcd(\mu^\vee)} \mathbb{Z}^{\phi(r) 2^{b(\mu^{1/r})-1}} \\
&= \bigoplus_{r|\gcd(\mu^\vee)} \mathbb{Z}^{\phi(r) 2^{b(\mu^\vee)-1}} = \mathbb{Z}^{\gcd(\mu^\vee) 2^{b(\mu^\vee)-1}}
\end{aligned} \tag{6.83}$$

By Theorem 2.24 $K_*\big(C_r^*(\mathcal{R}, q)_{P_\mu}\big)$ must also be a free abelian group of rank $\gcd(\mu^\vee) 2^{b(\mu^\vee)-1}$.

Summing over partitions μ of n we find that $K_*\big(C_r^*(\mathcal{R}, q)\big)$ is a free abelian group of rank

$$\sum_{\mu \vdash n} \gcd(\mu^\vee) 2^{b(\mu^\vee)-1} = \sum_{\mu \vdash n} \gcd(\mu) 2^{b(\mu)-1}$$

We conclude that for the root datum $\mathcal{R}(A_{n-1})$

$$K_*\big(C_r^*(\mathcal{R}, q)\big) \cong K_*\big(C_r^*(W)\big) \tag{6.84}$$

6.7 B_n

The root systems of type B_n are more complicated than those of type A_n because there are roots of different lengths. This implies that the associated root data allow label functions which have three independent parameters. Detailed information about the representations of type B_n affine Hecke algebras is available from [121].

We will compare the C^*-algebras for generic labelled root data with the reduced C^*-algebra of the affine Weyl group of type B_n. Consider the root datum $\mathcal{R}(B_n)$ where X is the root lattice:

$$X = Q = \mathbb{Z}^n \quad X^+ = \{x \in X : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}$$
$$Y = \mathbb{Z}^n \quad Q^\vee = \{y \in Y : y_1 + \cdots + y_n \text{ even}\}$$
$$T = (\mathbb{C}^\times)^n \quad t = (t_1, \ldots, t_n) = (t(e_1), \ldots, t(e_n))$$
$$R_0 = \{x \in X : \|x\| = 1 \text{ or } \|x\| = \sqrt{2}\} \quad R_1 = \{x \in X : \|x\| = 2 \text{ or } \|x\| = \sqrt{2}\}$$
$$R_0^\vee = \{x \in X : \|x\| = 2 \text{ or } \|x\| = \sqrt{2}\} \quad R_1^\vee = \{x \in X : \|x\| = 1 \text{ or } \|x\| = \sqrt{2}\}$$
$$F_0 = \{\alpha_i = e_i - e_{i+1} : i = 1, \ldots, n-1\} \cup \{\alpha_n = e_n\} \quad \alpha_0 = e_1$$
$$s_i = s_{\alpha_i} \quad s_0 = t_{\alpha_0} s_{\alpha_0} : x \to x + \langle \alpha_0^\vee, x \rangle \alpha_0$$
$$W_0 = \langle s_1, \ldots, s_n | s_j^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = (s_{n-1} s_n)^4 = e : i \leq n-2, |i-j| > 1 \rangle$$
$$S_{\text{aff}} = \{s_0, s_1, \ldots, s_{n-1}, s_n\} \quad \Omega = \{e\}$$
$$W = W_{\text{aff}} = \langle W_0, s_0 | s_0^2 = (s_0 s_i)^2 = (s_0 s_1)^4 = e : i \geq 2 \rangle$$

For a generic label function we have different labels $q_0 = q(s_0)$, $q_1 = q(s_i)$, $1 \leq i < n$ and $q_2 = q(s_n)$. For completeness we mention that

$$q_{\alpha_i^\vee} = q_1 \quad q_{\alpha_n^\vee} = q_0 \quad q_{\alpha_n^\vee/2} = q_2 q_0^{-1}$$

The finite reflection group $W_0 = W_0(B_n)$ is naturally isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. If $\mu \vdash n$ then the W_0-stabilizer of

$$\big((1)^{\mu_1} (-1)^{\mu_2} (t_{\mu_1+\mu_2+1})^{\mu_3} \cdots (t_n)^{\mu_d}\big) \in T$$

is isomorphic to

$$W_0(B_{\mu_1}) \times W_0(B_{\mu_2}) \times S_{\mu_3} \times S_{\mu_d}$$

- **group case $\mathbf{q_0 = q_1 = q_2 = 1}$**

In view of (6.2) we want to determine the extended quotient $\widetilde{T}/W_0$. Therefore we start with the classification of conjugacy classes in W_0. We already know that the quotient of W_0 by the normal subgroup $(\mathbb{Z}/2\mathbb{Z})^n$ of sign changes is isomorphic to S_n, and that conjugacy classes in S_n are parametrized by partitions of n. So we wonder what the different conjugacy classes in $(\mathbb{Z}/2\mathbb{Z})^n \sigma(\mu)$ are for $\mu \vdash n$.

To handle this we introduce the some notations. Assume that $|\mu| + |\lambda| = n$ and $|\mu| + |\lambda| + |\rho| = n'$.

$$\begin{array}{lcl} \nu_I & = & \prod_{i\in I} s_{e_i} \quad I \subset \{1,\ldots,n\} \\ I_\lambda & = & \{1, 1+\lambda_1, 1+\lambda_1+\lambda_2, \cdots\} \quad \lambda = (\lambda_1,\lambda_2,\lambda_3,\ldots\} \\ \sigma'(\lambda) & = & \nu_{I_\lambda}\sigma(\lambda) \in W_0(B_{|\lambda|}) \\ \sigma(\mu,\lambda) & = & \sigma(\mu)\,(m \to m - |\lambda| \bmod n)\,\sigma'(\lambda)\,(m \to m + |\lambda| \bmod n) \\ \sigma(\mu,\lambda,\rho) & = & \sigma(\mu,\lambda)\,(m \to m - |\rho| \bmod n')\,\sigma'(\rho)\,(m \to m + |\rho| \bmod n') \end{array} \tag{6.85}$$

Let $I \subset \{1,\ldots,m\}$ and $J \subset \{m+1,\ldots,2m\}$. It is easily verified that $\nu_I(1\,2\cdots m)$ is conjugate to $\mu_J(m+1\, m+2 \cdots 2m)$ if and only if $|I|+|J|$ is even. Therefore the conjugacy classes in W_0 are parametrized by ordered pairs of partitions of total weight n. Explicitly (μ,λ) corresponds to $\sigma(\mu,\lambda)$ as in (6.85). The set $T^{\sigma(\mu,\lambda)}$ and the group $Z_{W_0(B_n)}(\sigma(\mu,\lambda))$ are both the direct product of the corresponding objects for the blocks of μ and λ, i.e. for the parts $(m,m,\ldots,m)$. The centralizer of $\sigma((m)^k)$ in $W_0(B_{km})$ is generated by $(1\ 2\cdots m)$, $\nu_{\{1,2,\ldots,m\}}$ and the transpositions of cycles.

$$(am+1\ am+m+1)(am+2\ am+m+2)\cdots(am+m\ am+2m) \qquad 0 \leq a \leq k-2 \tag{6.86}$$

It follows that

$$\begin{array}{lcl} Z_{W_0(B_{km})}(\sigma((m)^k)) & \cong & W_0(B_k) \\ \left((\mathbb{C}^\times)^{km}\right)^{\sigma((m)^k)} & = & \left\{\left((t_1)^m(t_{m+1})^m\cdots(t_{km+1-m})^m\right) : t_i \in \mathbb{C}^\times\right\} \\ \left(\mathbb{T}^{km}\right)^{\sigma((m)^k)}/Z_{W_0(B_{km})}(\sigma((m)^k)) & \cong & [-1,1]^k/S_k \end{array} \tag{6.87}$$

Now consider the following element of $W_0(B_{km})$:

$$\sigma'((m)^k) = \nu_{\{1,m+1,\ldots,km+1-m\}}\,(1\ 2\cdots m)(m+1\cdots 2m)\cdots(km+1-m\cdots km)$$

It has only 2^k fixpoints, namely

$$\left((\pm1)^m(\pm1)^m\cdots(\pm1)^m\right)$$

The centralizer of $\sigma'((m)^k)$ is generated by $\nu_{\{1\}}(1\ \ 2\cdots m)$, $\nu_{\{1,2,\ldots,m\}}$ and the elements (6.86). Hence

$$\begin{array}{lcl} Z_{W_0(B_{mk})}(\sigma'((m)^k)) & \cong & W_0(B_k) \\ \left(\mathbb{T}^{km}\right)^{\sigma'((m)^k)}/Z_{W_0(B_{mk})}(\sigma'((m)^k)) & \cong & \{(1)^{am}(-1)^{(k-a)m} : 0 \leq a \leq k\} \end{array} \tag{6.88}$$

Now we can see what $T_u^{\sigma(\mu,\lambda)}/Z_{W_0}(\sigma(\mu,\lambda))$ looks like. Its number of components $N(\lambda)$ depends only on λ, and all these components are mutually homeomorphic contractible orbifolds, the shape and dimension being determined by μ. More precisely, for every block of μ of width k we get a factor $[-1,1]^k/S_k$, and for every block of λ of width l we must multiply the number components by $l+1$.

Alternatively, we can obtain the same space (modulo the action of W_0) as

$$\begin{aligned} T_u^{\sigma(\mu,\lambda)}/Z_{W_0}(\sigma(\mu,\lambda)) &= \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} \Big(T_u^{\sigma(\mu,\lambda_1,\lambda_2)}/Z_{W_0}(\sigma(\mu,\lambda_1,\lambda_2))\Big)_c \\ &= \bigsqcup_{\lambda_1\cup\lambda_2=\lambda} \Big(\mathbb{T}^{|\mu|}\Big)^{\sigma(\mu)}/Z_{W_0(B_{|\mu|})}(\sigma(\mu))\,(-1)^{|\lambda_1|}\,(1)^{|\lambda_2|} \end{aligned} \tag{6.89}$$

where the subscript c is supposed to indicate that we take only the connected component containing the point $\big((1)^{|\mu|}(-1)^{|\lambda_1|}(1)^{|\lambda_2|}\big)$.

In effect we parametrized the components of the extended quotient $\widetilde{T_u}/W_0$ by ordered triples of partitions $(\mu,\lambda_1,\lambda_2)$ of total weight n, and every such components is contractible. Denote the number of ordered k-tuples of partitions of total weight n by $\mathcal{P}(k,n)$.

$$\check{H}^*\big(\widetilde{T_u}/W_0;\mathbb{Z}\big) = \check{H}^0\big(\widetilde{T_u}/W_0;\mathbb{Z}\big) \cong \mathbb{Z}^{\mathcal{P}(3,n)}$$

By (6.3) also

$$K_*\big(C_r^*(W)\big) = K_0\big(C_r^*(W)\big) \cong \mathbb{Z}^{\mathcal{P}(3,n)} \tag{6.90}$$

- **generic case**

- $P_\mu = F_0 \setminus \{\alpha_{\mu_1},\alpha_{\mu_1+\mu_2},\dots,\alpha_{|\mu|}\}$

The inequivalent subsets of F_0 are parametrized by partitions μ of weight at most n.

$$\begin{aligned}
&R_{P_\mu} \cong (A_{n-1})^{m_n}\times\cdots\times(A_1)^{m_2}\times B_{n-|\mu|}\\
&R^\vee_{P_\mu} \cong (A_{n-1})^{m_n}\times\cdots\times(A_1)^{m_2}\times C_{n-|\mu|}\\
&X^{P_\mu} \cong \mathbb{Z}(e_1+\cdots+e_{\mu_1})/\mu_1+\cdots+\mathbb{Z}(e_{|\mu|+1-\mu_d}+\cdots+e_{|\mu|})/\mu_d\\
&X_{P_\mu} \cong (\mathbb{Z}^n/\mathbb{Z}(e_1+\cdots+e_n))^{m_n}\times\cdots\times(\mathbb{Z}^2/(\mathbb{Z}(e_1+e_2))^{m_2}\times\mathbb{Z}^{n-|\mu|}\\
&Y^{P_\mu} = \mathbb{Z}(e_1+\cdots+e_{\mu_1})+\cdots+\mathbb{Z}(e_{|\mu|+1-\mu_d}+\cdots+e_{|\mu|})\\
&Y_{P_\mu} = \{y\in\mathbb{Z}^n : y_1+\cdots+y_{\mu_1}=\cdots=y_{|\mu|+1-\mu_d}+\cdots+y_{|\mu|}=0\}\\
&T^{P_\mu} = \{(t_1)^{\mu_1}(t_{\mu+1})^{\mu_2}\cdots(t_{|\mu|})^{\mu_d}(1)^{n-|\mu|} : t_i\in\mathbb{C}^\times\}\\
&T_{P_\mu} = \{t\in(\mathbb{C}^\times)^n : t_1\cdots t_{\mu_1}=t_{\mu_1}\cdots t_{\mu_1+\mu_2}=\cdots=t_{|\mu|+1-\mu_d}\cdots t_{|\mu|}=1\}\\
&K_{P_\mu} = \{t\in T^{P_\mu} : t_1^{\mu_1}=\cdots=t_{|\mu|}^{\mu_d}=1\}\\
&W_{P_\mu} \cong S_n^{m_n}\times\cdots\times S_2^{m_2}\times W_0(B_{n-|\mu|})\\
&W(P_\mu,P_\mu) \cong W_0(B_{m_n})\times\cdots\times W_0(B_{m_1})
\end{aligned}$$

We see that $\mathcal{R}_{P_\mu}$ is the product of various root data of type A_m and one factor $\mathcal{R}(B_{n-|\mu|})$. Hence by (3.42) $\mathcal{H}_{P_\mu}$ is the tensor product of a type A part and a type B part. From our study of $\mathcal{R}(A_{n-1})$ we recall that the discrete series representations of the type A part of $\mathcal{H}_{P_\mu}$ are in bijection with K_{P_μ}. From [58, Proposition 4.3] and [98, Appendix A.2] we know that the residual points for $B_{n-|\mu|}$ are parametrized by ordered pairs (λ_1, λ_2) of total weight $n - |\mu|$. The unitary part of such a residual point is in the component we indicated in (6.89). Let $RP(\mathcal{R}, q)$ denote the collection of residual points for the pair $(\mathcal{R}, q)$.

$$\begin{aligned} \bigsqcup_{t \in RP(\mathcal{R}_{P_\mu}, q_{P_\mu})} tT_u^{P_\mu}/\mathcal{W}_{P_\mu P_\mu} &\cong \bigsqcup_{t \in RP(\mathcal{R}(B_{n-|\mu|}, q))} tT_u^{P_\mu}/W(P_\mu, P_\mu) \\ &\cong T_u^{P_\mu}/Z_{W_0(B_{|\mu|})}(\sigma(\mu)) \times \bigsqcup_{(\lambda_1,\lambda_2):|\lambda_1|+|\lambda_2|=n} (-1)^{|\lambda_1|}(1)^{|\lambda_2|} \end{aligned} \tag{6.91}$$

This space is diffeomorphic to the extended quotient described on page 219. By Theorem 3.19 every point is the central character of at least one irreducible $C_r^*(\mathcal{R}, q)$-representation. Therefore it is natural to compare Conjecture 5.24 with the following statements.

1) Every parabolically induced $C_r^*(\mathcal{R}(B_n)^\vee, q)$-representation $\pi(P, \delta, t)$ is irreducible.

2) $\mathrm{Prim}\big(C_r^*(\mathcal{R}(B_n)^\vee, q)\big)$ is naturally in bijection with (6.91).

Opdam and Slooten [99] have announced a proof of 1). Extending the results from [122] they can show that all R-groups are trivial in this situation. In view of the above calculations and (5.82), 1) and Theorem 5.25 together imply 2). Probably 2) can also be derived from the recent work of Kato [74, 75].

Theorem 6.4 *Let q be a generic positive label function on the root datum $\mathcal{R}(B_n)$. Then Conjecture 5.24 holds. In particular there are precisely $\mathcal{P}(2, n)$ inequivalent discrete series representations, one for each W_0-orbit of residual points.*

Proof. This theorem was predicted in [99, §8.1]. The proof relies on 1) and on Theorem 5.25, whose proofs will appear elsewhere. As said, via (5.82) these results imply 2) and in particular the statement about the discrete series.

For every $W_0 t_0 \in T_u/W_0$ the image of $G(\phi_{W_0 t_0})$ is spanned by the modules

$$G(\phi_0)\pi(P, W_P r, \delta, t) = \pi_0(P, W_P r_u, \tilde{\sigma}_0(\delta), t) \quad \text{with} \quad (P, \delta) \in \Delta,\ W_0 r_u t = W_0 t_0 \tag{6.92}$$

By 2) the number of such modules equals the number of irreducible W-representations with central character $W_0 t_0$. So we only have to show that the elements (6.92) are linearly independent in $G(C_r^*(W))$. Abbreviate

$$E_0 = \mathrm{Rep}_{W_0 t_0}\big(C_r^*(W)\big) \otimes_{\mathbb{Z}} \mathbb{C}$$

Fix a set $\mathcal{P}'$ of representatives for the action of W_0 on the power set of F_0. For $P, Q \in \mathcal{P}'$ we write

$$P \leq Q \text{ if } \exists w \in W_0 : wP \subset Q$$

Pick $t_1 \in W_0 t_0$ such that

$$P_1 := \{\alpha \in F_1 : \theta_\alpha(t_1) = 1\}$$

is maximal. We may assume that the set P_0 of short roots corresponding to P_1 is an element of $\mathcal{P}'$. From the proof of Theorem 3.15 we see that the standard pairing between representations of the isotropy group W_{0,t_1} gives an inner product on E_0.

Now we can decompose E_0 into subspaces corresponding to the $P \in \mathcal{P}'$. Let E_1^P be the span of the virtual representations in E_0 whose character only allows continuous deformations along T_u^P/W_0, not along other directions on T_u/W_0. We put

$$E_0^P = E_1^P \cap \Big(\sum_{Q>P} E_1^Q \Big)^\perp$$

From Theorem 3.15, (6.89), (6.91) and (2.106) (for HH^0) we deduce

$$\dim E_0^P = \#\{(P, W_P r, \delta, t) \in \Xi_u : W_0 r_u t = W_0 t_0\} \tag{6.93}$$

Notice that this is a simple version of [99, Proposition 6.6]. Upon applying (5.82) to the affine Hecke algebra $\mathcal{H}_P$ (which has generic labels), we see that

$$G(\phi_0) \otimes \mathrm{id}_{\mathbb{C}} \big(\mathrm{span}\{(P, W_P r, \delta, t) \in \Xi_u : W_0 r_u t = W_0 t_0\} \big) \subset E_1^P \tag{6.94}$$

and that $G(\phi_0) \otimes \mathrm{id}_{\mathbb{C}}$ is injective on this domain. Moreover by (6.93) the image in (6.94) intersects $\sum_{Q>P} E_1^Q$ only in 0. Therefore $G(\phi_0)$ is injective. $\square$

Appendix A

Crossed products

We collect some well-known results on crossed products of algebras by compact groups. We do this in the large category of m-algebras, but it is straightforward to see that they also hold for C^*-algebras, if we assume that the action is *-preserving. So let A be an m-algebra, G a compact group (with its normalized Haar measure) and

$$\begin{aligned} \alpha &: G \times A \to A \\ \alpha(g,a) &= \alpha_g(a) \end{aligned} \tag{A.1}$$

a continuous action of G on A by algebra homomorphisms. Recall that the crossed product $A \rtimes_\alpha G$ is the vector space $C(G; A)$ with multiplication defined by

$$(f \cdot f')(g') = \int_G f(g)\alpha_g(f'(g^{-1}g'))dg \tag{A.2}$$

This is again an m-algebra. Notice that if $a \in A$ and δ_g is the δ-function concentrated at $g \in G$, then $a\delta_g$ is an element of the multiplier algebra $\mathcal{M}(A \rtimes_\alpha G)$. Explicitly, multiplying by this element is defined as

$$(f \cdot a\delta_g)(g') = f(g'g^{-1})\alpha_{g'g^{-1}}(a) \tag{A.3}$$

$$(a\delta_g \cdot f)(g') = a\alpha_g(f(g^{-1}g')) \tag{A.4}$$

Similarly, if N is a closed subgroup of G then any $\psi \in C^*(N)$ is also a multiplier of $A \rtimes_\alpha G$, defined naturally by

$$(f \cdot \psi)(g') = \int_N f(g'n^{-1})\psi(n)dn \tag{A.5}$$

$$(\psi \cdot f)(g') = \int_N \psi(n)\alpha_n(f(n^{-1}g'))dn \tag{A.6}$$

The next result is useful in connection with projective representations.

Lemma A.1 *Let $\{e\} \to N \to G \to H \to \{e\}$ be a short exact sequence of compact groups, all equipped with their normalized Haar measures. Assume that $N \subset \ker\alpha$,* *so that α descends to H. Let $p_N \in C^*(N)$ be the constant function with value 1, considered as an idempotent in $\mathcal{M}(A \rtimes_\alpha G)$.*

a) p_N is central and $p_N(A \rtimes_\alpha G) \cong (A \rtimes_\alpha H)$

Assume now that moreover N is finite and the extension from H to G by N central. Then we let $\widehat{N}$ be the dual of the abelian group N and we consider every (one-dimensional) character χ of N as an idempotent $p_\chi \in \mathcal{M}(A \rtimes_\alpha G)$. In particular $p_N = p_\chi$ for the trivial character $\chi \equiv 1$.

b) p_χ is also central and $A \rtimes_\alpha G = \bigoplus_{\chi \in \widehat{N}} p_\chi (A \rtimes_\alpha G)$

Proof. For any $b \in A \rtimes_\alpha G$ and $g' \in G$ we have

$$\begin{aligned}(b \cdot p_\chi)(g') &= \int_G b(g)\alpha_g(p_\chi(g^{-1}g'))dg \\ &= \int_N b(g'g^{-1})\chi(g^{-1})|N|^{-1}dg \\ &= \int_N |N|^{-1}\chi(g^{-1})b(g^{-1}g')dg \\ &= \int_G p_\chi(g)\alpha_g(b(g^{-1}g'))dg \;=\; (p_\chi \cdot b)(g')\end{aligned} \tag{A.7}$$

The third equality holds if N is central or if $\chi \equiv 1$ and N only normal, that is, in all the cases we need. So p_χ indeed commutes with $A \rtimes_\alpha G$ and $p_\chi(A \rtimes_\alpha G)$ is a subalgebra. Now the statement b) follows from two standard equalities in the representation theory of finite groups:

$$\sum_{\chi \in \hat{N}} p_\chi = 1 \qquad p_\chi \cdot p_{\chi'} = \delta_{\chi\chi'} p_\chi \tag{A.8}$$

Writing $\pi : G \to H$, it is easily verified that

$$\pi^* : A \rtimes_\alpha H \to A \rtimes_\alpha G : f \to f \circ \pi \tag{A.9}$$

is a monomorphism of m-algebras, so let us determine its image. Since

$$\begin{aligned}(p_N \cdot \pi^* f)(g') &= \int_G p_N(g)\alpha_g(\pi^* f(g^{-1}g'))dg \\ &= \int_N p_N(g) f(\pi(g^{-1}g'))dg \\ &= \int_N |N|^{-1} f(\pi g')dg \;=\; \pi^* f(g')\end{aligned} \tag{A.10}$$

the image is contained in $p_N(A \rtimes_\alpha G)$. From the above expressions for bp_N it follows immediately that anything of this type is N-biinvariant, so that it descends to an element of $A \rtimes_\alpha H$. Hence the image of π^* is exactly $p_N(A \rtimes_\alpha G)$ □

Suppose now that A is unital and we are given $p \in C(G; A^{\times})$ with the properties

$$p(gg') = p(g)\alpha_g(p(g')) \tag{A.11}$$

$$p(e) = 1 \tag{A.12}$$

Then p is an idempotent in $A \rtimes_\alpha G$ and we can define a new action β of G on A by

$$\beta_g(a) = p(g)\alpha_g(a)p(g)^{-1} \tag{A.13}$$

The following description of the invariant algebra

$$A^{\beta(G)} := \{a \in A : \beta_g(a) = a \; \forall g \in G\} \tag{A.14}$$

is essentially due to Rosenberg [109].

Lemma A.2

$$A^{\beta(G)} \cong p(A \rtimes_\alpha G)p$$

Proof. We will show that the obvious map

$$\begin{aligned} &\phi : A^{\beta(G)} \to p(A \rtimes_\alpha G)p \\ &\phi(a) = p(a\delta_e)p \end{aligned} \tag{A.15}$$

is an isomorphism. Clearly ϕ is linear and continuous. If $a, a' \in A^\beta$ then

$$\phi(a) = p(a\delta_e)p = p(a\delta_e) = (a\delta_e)p \tag{A.16}$$

$$\phi(a)\phi(a') = p(a\delta_e)pp(a'\delta_e)p = p(a\delta_e)(a'\delta_e)p = p(aa'\delta_e)p = \phi(aa') \tag{A.17}$$

so ϕ turns out to be injective and multiplicative. Next we observe that for any $g \in G$

$$p(g)\delta_g \cdot p = p = p \cdot p(g)\delta_g \tag{A.18}$$

This gives, for $b \in p(A \rtimes_\alpha G)p$,

$$b(g) = (p(g)\delta_g \cdot b)(g) = p(g)\alpha_g(b(e)) \tag{A.19}$$

$$b(g) = (b \cdot p(g)\delta_g)(g) = b(e)p(g) \tag{A.20}$$

Comparing these expressions we see that $b(e) \in A^\beta$. Therefore ϕ is bijective, with continuous inverse $b \to b(e)$. □

Assume now that G is finite. Then evaluating integrals of A-valued functions on G is easy, so we do not need any topology on A to define $A \rtimes_\alpha G$. Moreover we agree to use the counting measure on G, even though it is not normalized. The crossed product thus obtained also appears in another way:

Lemma A.3 *Let $(\mathbb{C}[G], \rho)$ be the right regular representation of G and A any complex algebra. Endow $A \otimes \mathrm{End}(\mathbb{C}[G] \otimes \mathbb{C}^n)$ with the G-action $g(a) := \rho(g)\alpha_g(a)\rho(g^{-1})$, where α stands also for the action of G on A tensored with the identity. Then*

$$\big(A \otimes \mathrm{End}(\mathbb{C}[G] \otimes \mathbb{C}^n)\big)^G \cong M_n(A \rtimes_\alpha G)$$

Proof. The left hand side is isomorphic to $M_n\left(\left(A \otimes \mathrm{End}(\mathbb{C}[G])\right)^G\right)$, so it suffices to prove the case $n = 1$.

For $a \in A$ and $g, h \in G$ define $L(a \otimes g)(h) = \alpha_{h^{-1}g^{-1}}(a) \otimes gh$. It is easily checked that this extends to an algebra homomorphism $L : A \rtimes_\alpha G \to \left(A \otimes \mathrm{End}(\mathbb{C}[G])\right)^G$. I claim that $L' : b \to \sum_{g \in G} b(g^{-1})_e \otimes g$ is the inverse of L. It is clear that $L'L = \mathrm{Id}$, so we only have to show that $L(L'b) = b$ for any $b \in \left(B \otimes \mathrm{End}(\mathbb{C}[G])\right)^G$.

$$\begin{aligned}
L(L'b)(h) &= \sum_{g \in G} \alpha_{h^{-1}g^{-1}}(b(g^{-1})_e) \otimes gh \\
&= \alpha_{h^{-1}} \alpha_{g^{-1}}(b)(g^{-1})_e \otimes gh \\
&= \sum_{g \in G} \alpha_{h^{-1}} \left(\rho(g) b \rho(g^{-1})(g^{-1})\right)_e \otimes gh \\
&= \sum_{g \in G} \alpha_{h^{-1}} \left(b(e) g^{-1}\right)_e \otimes gh \\
&= \alpha_{h^{-1}}(b)(e) h \\
&= \rho(h^{-1}) \alpha_{h^{-1}}(b) \rho(h)(h) \ = \ b(h)
\end{aligned} \tag{A.21}$$

This holds for any $h \in G$, so indeed $L' = L^{-1}$ □

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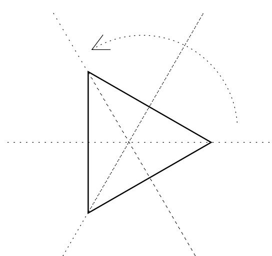
Samenvatting

De afgelopen jaren is mij vaak gevraagd wat ik nou eigenlijk onderzoek. Op deze vraag heb ik inmiddels een voorraadje antwoorden uitgeprobeerd, die in zekere zin allemaal wel correct waren. Niettemin bleek dat sommige antwoorden aanzienlijk meer begrip en waardering oogsten dan andere. Eén persoon besloot zelfs geheel af te zien van verdere communicatie nadat ik haar de titel van dit proefschrift had verteld.

Daarom lijkt het me wel een goed idee om in ieder geval de wellicht enigszins cryptische zinsnede *Periodiek cyclische homologie van affiene Hecke algebra's* toe te lichten. Dit wordt dan ook niet zozeer een samenvatting van mijn onderzoek, als wel een relatief gezellige wandeling langs de randen van de oneindig dimensionale ruimten waarin ik mij gewoonlijk begeef.

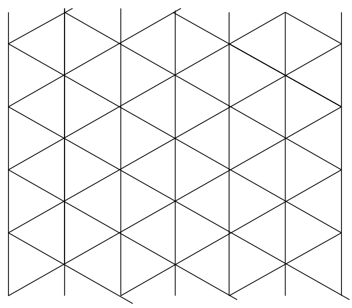
Laten we beginnen bij groepen. Met groepen kun je de symmetrieën beschrijven van uiteenlopende dingen, zoals een voetbal, een velletje papier, een molecuul, een differentiaalvergelijking of ruimte-tijd, maar ook van simpele figuren als een lijn, een kubus of een zevenhoek. Een eenvoudige groep, die een rol speelt in dit boek, bestaat uit de zes symmetrieën van een gelijkzijdige driehoek.



We zien drie spiegelingen (in de stippellijnen) en rotaties over 120° en over 240°. De laatste symmetrie is de afbeelding die alles op zijn plek laat. We zien direct dat *groep* binnen de wiskunde iets heel anders betekent dan in het dagelijks leven. In abstracto is het een verzameling waarvan je de elementen kunt samenstellen, waarbij aan bepaalde voorwaarden voldaan moet zijn.

De bovenstaande groep heeft allerlei bijzondere eigenschappen. Bijvoorbeeld, als je een willekeurige lijn neemt die door het middelpunt van de driehoek gaat, en je past daarop een symmetrie van de driehoek toe, dan krijg je weer een lijn die door dat middelpunt gaat. Men zegt daarom wel dat deze groep bestaat uit lineaire afbeeldingen.

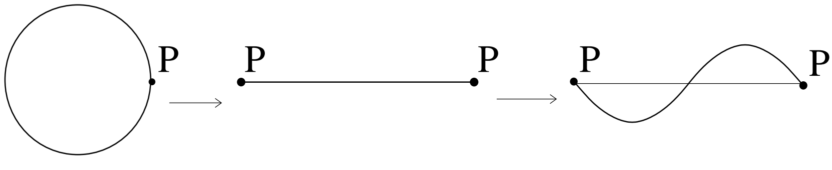
Beschouw het rooster dat is opgebouwd uit gelijkzijdige driehoeken. (Zie de kaft voor een artistiekere impressie van dit rooster, waarvoor mijn dank uitgaat naar Bill Wenger.)



De symmetriegroep van dit (oneindig grote) figuur bevat oneindig veel elementen, onder andere spiegelingen en rotaties, maar ook verschuivingen. Stel dat je een punt in dit rooster kiest, en een lijn door dat punt. Als je zomaar een symmetrie van het rooster toepast krijg je zeker weer een lijn, maar er is geen garantie dat die nog steeds door het gekozen punt gaat. Daarom noemt men dit soort symmetrieën geen lineaire afbeelingen, maar *affiene* afbeeldingen.

Met deze hele opzet kunnen we iets doen waar wiskundigen dol op zijn, we kunnen het zaakje generaliseren. In dat geval vervangen we de driehoek door en ingewikkelder kristal, en het driehoekige rooster door een rooster van hogere dimensie. De symmetriegroep van het kristal is een (eindige) Weyl groep, en de symmetriegroep van het rooster heet een affiene Weyl groep.

Als je een groep goed wil begrijpen is het van groot belang om zijn zogeheten representaties te kennen. Dit illustreren we aan de hand van ander voorbeeld, de cirkel. Enerzijds kan deze worden beschouwd als de groep van rotaties om zijn eigen middelpunt. Anderszijds kunnen we de cirkel opvatten als een snaar, en dan kan hij trillen. De representaties van de cirkelgroep corresponderen precies met de trillingen van de cirkelvormige snaar, waarbij we één specifiek punt P vasthouden.



Er is een grondtoon, waarvan de golflengte exact de lengte van de snaar is. De boventonen hebben een golflengte die een geheel aantal keren in de snaar past.

Elke trilling is te maken als een geschikte combinatie van dergelijke harmonische trillingen.

In wiskundig jargon betekent dit dat de harmonische trillingen corresponderen met de irreducibele representaties. Irreducibele representaties zijn een soort bouwsteentjes waarmee je elke representatie kunt maken. Ze zijn zelf niet verder op te delen.

Grof gezegd is een algebra een verzameling waarbinnen je kunt optellen en vermeningvuldigen. Zo vormen de gehele getallen een algebra. Een wat lastiger voorbeeld zijn de 2 × 2-matrices met reële coëfficienten:

$$M_2(\mathbb{R}) := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} : x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}$$

Dit soort matrices kan je coördinaatsgewijs optellen:

$$\begin{pmatrix} 1 & 0 \\ \pi & 4 \end{pmatrix} + \begin{pmatrix} 3/4 & -5/9 \\ -1 & -4 \end{pmatrix} = \begin{pmatrix} 7/4 & -5/9 \\ \pi - 1 & 0 \end{pmatrix}$$

Een matrix is op te vatten als een lineaire afbeelding. Als we punten in het vlak schrijven als vectoren $\begin{pmatrix} x \\ y \end{pmatrix}$, dan stuurt een matrix dus een vector naar een andere vector:

$$\begin{pmatrix} 3/4 & -5/9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 3a/4 \\ -a \end{pmatrix} \qquad \begin{pmatrix} 3/4 & -5/9 \\ -1 & -4 \end{pmatrix} \begin{pmatrix} 0 \\ b \end{pmatrix} = \begin{pmatrix} -5b/9 \\ -4b \end{pmatrix}$$

De standaardmanier om matrices te vermenigvuldigen correspondeert met het samenstellen van afbeeldingen, bijvoorbeeld

$$\begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 6 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$$

Al met al maakt dit $M_2(\mathbb{R})$ tot een algebra over de reële getallen $\mathbb{R}$. Merk op dat bepaalde gebruikelijke rekenregels voor getallen niet meer gelden voor matrices. We zijn gewend dat $7 \cdot 9 = 9 \cdot 7 = 63$. Zelfs zonder dat we de uitkomst weten kunnen we met zekerheid zeggen dat

$$4676013 \cdot 2369655 = 2369655 \cdot 4676013$$

In feite geldt $x \cdot y = y \cdot x$ voor alle reële getallen x en y. Niettemin zien we hierboven dat er matrices A en B bestaan zodat

$$A \cdot B \neq B \cdot A$$

Men zegt dan dat A en B niet commuteren en dat de algebra $M_2(\mathbb{R})$ niet commutatief is.

Wat is nu het verband tussen algebra's en groepen? Vanuit een groep kunnen we een algebra construeren die alle eigenschappen van de groep reflecteert. Zo hebben algebra's ook representaties, en de representaties van een groep komen overeen met de representaties van de bijbehorende groepsalgebra.

Een aanzet tot de algebra's in de titel van dit boek werd gegeven door de Duitse wiskundige Erich Hecke, die leefde van 1887 tot 1947. Hecke hield zich vooral bezig met getaltheorie, bijvoorbeeld met p-adische getallen. Hier is p een priemgetal, bijvoorbeeld 5. De verzameling 5-adische getallen geven we aan met $\mathbb{Q}_5$. Een typisch 5-adisch getal ziet eruit als een decimale expansie in de verkeerde richting:

$$x = \cdots 42130012044.113 \in \mathbb{Q}_5$$

Omdat $p = 5$ komen alleen de symbolen 0, 1, 2, 3 en 4 in deze schrijfwijze van x voor. We dienen x te interpreteren als

$$x = 3 \cdot 5^{-3} + 1 \cdot 5^{-2} + 1 \cdot 5^{-1} + 4 \cdot 5^0 + 4 \cdot 5^1 + 0 \cdot 5^2 + 2 \cdot 5^3 + \cdots$$

Als y een ander 5-adisch getal is, dan zijn $x+y$ en $x \cdot y$ zonder al te veel problemen te bepalen met behulp van de regeltjes

$$\begin{array}{ccccc} a5^n & + & b5^n & = & (a+b)5^n \\ a5^n & \cdot & c5^m & = & (a \cdot c)5^{n+m} \end{array}$$

voor gehele getallen a, b, c, n en m. Om de coëfficient van $x \cdot y$ bij 5^n uit te rekenen hebben we slechts kleine stukjes van de expansies van x en y nodig. Zelfs delen is mogelijk met p-adische getallen, omdat p een priemgetal is.

Met p-adische getallen kunnen we weer groepen bouwen. Een simpel voorbeeld van zo'n p-adische groep is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Q}_5 \, , a \cdot d - b \cdot c = 1 \right\}$$

De samenstelling in deze groep is de gebruikelijke matrixvermeningvuldiging, maar dan uitgevoerd met 5-adische getallen. Dergelijke p-adische groepen spelen een belangrijke rol in verschillende gebieden van de wiskunde. Men zou graag alle irreducibele representaties van zo'n groep klassificeren, maar dat is erg lastig. Het blijkt dat een belangrijk deel van de representatietheorie van een p-adische groep valt uit te drukken met een zekere algebra. Zo een algebra is een generalisatie van een type algebra's dat Hecke indertijd vanuit een iets andere hoek heeft gedefinieerd en bestudeerd, vandaar dat ze onder de naam *Hecke algebra's* door het leven gaan.

Hecke algebra's zijn er in soorten en maten. Ik ben vooral geïnteresseerd in Hecke algebra's die sterk gerelateerd zijn aan Weyl groepen. In feite zijn deze Hecke algebra's te beschouwen als deformaties van Weyl groepen. De groepsalgebra die bij een Weyl groep hoort heeft zoals zoals gezegd grotendeels dezelfde eigenschappen als de groep zelf. Als je die groepsalgebra op een geschikte manier vervormt krijg je een Hecke algebra.

Doe je dit met een eindige Weyl groep, dan krijg je een Hecke algebra van eindige dimensie. Hoewel *eindig dimensionaal* in eerste instantie nog niet zo eenvoudig klinkt, is het vast makkelijker dan *oneindig dimensionaal.* In ieder geval begrijpt men eindig dimensionale Hecke algebra's heel goed.

Echter, als je een affiene Weyl groep deformeert krijg je een affiene Hecke algebra, en die heeft oneindige dimensie. Affiene Hecke algebra's zijn veel ingewikkelder dan Hecke algebra's van eindige dimensie. Toch is dat niet zo'n ramp. Men gebruikt affiene Hecke algebra's onder andere om tot een beter begrip te komen van de gecompliceerde representatietheorie van p-adische groepen, en daar zou weinig van te verwachten zijn als ze te eenvoudig waren. Het blijkt dat affiene Hecke algebra's aan de ene kant diepzinnig genoeg zijn om tot nieuwe inzichten te leiden, en aan de andere kant makkelijk genoeg om er prettig mee te kunnen werken. Dus een affiene Hecke algebra heeft precies de goede moeilijkheid om hem tot een interessant studieobject te maken.

Zonet zagen we de analogie tussen trillingen van een snaar en representaties van een groep. Als we dit uitbreiden correspondeert een algebra niet meer met één snaar, maar met een snaarinstrument, bijvoorbeeld een piano. Een representatie van die algebra wordt dan een toon die je met die piano kunt voortbrengen. Op deze manier kunnen we een irreducibele representatie identificeren met een zuivere toon van de piano.

Het uiteindelijke doel van mijn promotieonderzoek was om alle irreducibele representaties van een algemene affiene Hecke algebra te bepalen. Het ligt voor de hand om ze eerst maar eens te tellen. Helaas laten ze zich niet zo gemakkelijk tellen, want het zijn er oneindig veel. Net zo heeft het weinig zin om alle zuivere tonen van een piano te tellen, want dat zijn er ook oneindig veel. Een beter idee is daarom om alle grondtonen van de snaren van de piano te tellen, dat vertelt je bijvoorbeeld al hoeveel snaren je piano heeft.

In de context van algebra's ligt dat wat subtieler, daar heet de geschikte manier om grondtonen te tellen *periodiek cyclische homologie*. "Homologie" komt uit het Grieks en betekent zoveel als "studie van gelijkheid". Dat gaat ongeveer als volgt. Stel dat je twee objecten, bijvoorbeeld twee algebra's, wilt vergelijken. Kies een geschikte methode (een homologietheorie) om aan een algebra iets relatief eenvoudigs toe te kennen, bijvoorbeeld een simpel type groep, een rijtje getallen of zelfs een rijtje groepen. Dat heet dan de homologie van de algebra. Als je twee algebra's in essentie hetzelfde zijn zullen ze dezelfde homologie hebben. Daarentegen, als ze verschillende homologie hebben dan zijn de algebra's niet hetzelfde, en ook niet ongeveer.

Op de kaft van dit boek zie je duidelijk periodieke en cyclische verschijnselen. Dat is geen toeval, maar de etymologische achtergrond van de term *periodiek cyclische homologie* is anders. De periodiek cyclische homologie van een algebra A is een rijtje groepen:

$$\ldots, HP_{-2}(A),\ HP_{-1}(A),\ HP_0(A),\ HP_1(A),\ HP_2(A),\ HP_3(A), \ldots$$

Dit is periodiek in de zin dat voor elk geheel getal n geldt

$$HP_n(A) = HP_{n+2}(A)$$

Het cyclische zit wat dieper verstopt, dat heeft te maken met hoe de groepen

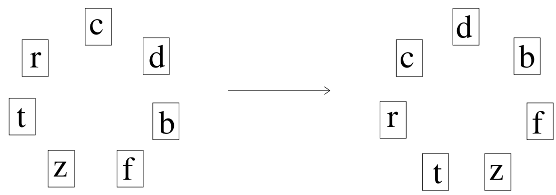
$HP_n(A)$ expliciet geconstrueerd worden. Stel dat we zeven hokjes hebben, die allemaal gevuld zijn met een letter:

f	b	d	c	r	t	z

Nu schuiven we elke letter één hokje naar rechts, en de meest rechtse letter stoppen we in het vrijgekomen linker hokje:

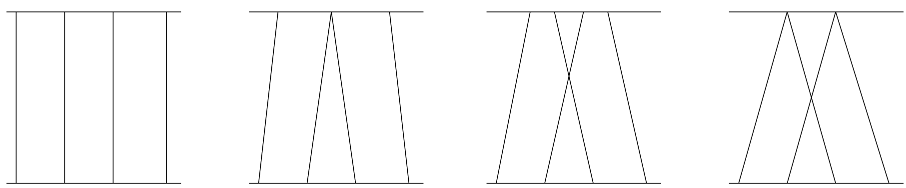
z	f	b	d	c	r	t

Dit kunnen we nog wat suggestiever tekenen:



Nu is het wel duidelijk waarom dit een cyclische permutatie heet. Zulke permutaties worden gebruikt in de definitie van periodiek cyclische homologie.

Het is nog niet zo eenvoudig om de periodiek cyclische homologie van een affiene Hecke algebra ook daadwerkelijk uit te rekenen. Daartoe grijpen we terug op de affiene Weyl groep waarvan het een deformatie is. Als we weer denken aan piano's en snaren betekent deze deformatie dat we wat gaan sleutelen aan de snaren: wat langer of iets korter, een stukje dichter bij elkaar. Hoewel het in muzikaal opzicht barbaars is zouden we zelfs sommige snaren aan elkaar vast kunnen knopen. Zo'n deformatie kan er schematisch uitzien als



Men vermoedt dat de representatietheorie van een affiene Weyl groep niet essentieel verandert onder deze deformaties. Dit vermoeden wordt ondersteund door diepzinnige stellingen die zeggen dat het in bepaalde belangrijke gevallen klopt. Het is een belangrijk vermoeden, want hiermee kan men representaties van een affiene Hecke algebra herleiden tot representaties van een affiene Weyl groep, en die zijn allemaal al lang bekend.

In dit proefschrift heb ik bewezen dat dit vermoeden equivalent is met een ogenschijnlijk zwakkere uitspraak, namelijk dat de periodiek cyclische homologie van de groepsalgebra van een affiene Weyl groep niet verandert als je die algebra vervormt tot een affiene Hecke algebra. Grof gezegd betekent de sterke versie van dit vermoeden dat je met elk van de boven getekende pianootjes evenveel verschillende tonen kan voortbrengen. De zwakke versie zegt zo ongeveer dat al die piano's evenveel grondtonen hebben.

Verder worden in dit boek onder andere een aantal nieuwe technieken geïntroduceerd om de periodiek cyclische homologie van een redelijk algemeen type algebra uit te rekenen. Mede daardoor kunnen de bovenstaande vermoedens nu bewezen worden in veel nieuwe gevallen.

Curriculum vitae

De auteur werd geboren op 5 februari 1979 in Amsterdam. Zijn middelbare schooltijd bracht hij door aan het Montessori Lyceum Amsterdam, alwaar hij in juni 1997 voor zijn VWO-examen slaagde.

In september 1997 begon hij als student aan de Universiteit van Amsterdam. In 1998 met behaalde hij cum laude een dubbele propedeuse wiskunde en natuurkunde, waarop hij besloot zich verder te concentreren op de wiskunde. Hij schreef zijn scriptie, getiteld *Lie algebra cohomology and Macdonald's conjectures*, onder de hoede van prof. dr. Eric Opdam. Hierop studeerde hij in september 2002 cum laude af. In november 2002 trad hij in dienst bij de Universiteit van Amsterdam als assistent in opleiding (AIO), eveneens onder begeleiding van Eric Opdam. In deze hoedanigheid heeft hij vier jaar lang gewerkt aan dit proefschrift en werkcolleges verzorgd voor studenten van verschillende exacte studies. Momenteel is de auteur werkzaam aan de Universiteit van Amsterdam als onderzoeker en docent wiskunde.

Min of meer parallel met zijn wiskundige activiteiten liep de schaakcarrière van de auteur. Hij reikte diverse keren tot een gedeelde tweede plaats in Nederlandse jeugdkampioenschappen schaken: t/m 12 jaar (1991), t/m 14 jaar (1992), t/m 16 jaar (1994) en t/m 20 jaar (1998). In 2000 voldeed hij aan de voorwaarden voor de titel Internationaal Meester. Hij speelde schaaktoernooien in Nederland, België, Frankrijk, Andorra, Engeland, Ierland, Denemarken, Duitsland, Zwitserland, Liechtenstein en in de Dominicaanse Republiek. Een hoogtepunt kwam in 2003, toen hij topscorer werd van de hoofdklasse van de Nederlandse schaakcompetitie.

Reeds enige jaren houdt de auteur zich ook actief bezig met een andere sport, badminton. Sinds december 2005 is hij penningmeester van de vereniging US badminton.