Cosmological Inflation and Quintessence

Paul Friedel
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Master thesis
Supervisor:
Prof. Dr. J. Smit
Instituut voor Theoretische Fysica
Universiteit van Amsterdam
Cover: Perturbations to the background radiation as seen by the WMAP satellite. Dipole caused by the movement of the earth has been subtracted. From the NASA/WMAP Science Team, http://map.gsfc.nasa.gov.
Cosmological Inflation and Quintessence

Abstract
The standard Hot Big Bang scenario of the evolution of the universe faces some problems which can be solved using the inflation concept. Inflation (accelerating expansion of the early universe) can be modelled by assuming that the energy density in the early universe is dominated by a quantum field, slowly rolling in its potential. At present, the universe is believed to be again in an accelerating phase. The smallness of this acceleration compared to naïve dimensional estimates rises the question as what the origin of the present acceleration may be (the quintessence problem).

In this thesis some solutions for inflation and quintessence are discussed. Contrary to the inflationary case, the quintessence problem cannot be solved in a compelling way using a quantum field in a potential.
“Who ordered the inflaton?”
Free after Rabi.
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Preface

Cosmology, inflation and quintessence

Perhaps, cosmology is the oldest form of pure science. Long before the Greeks built their culture, astronomers gazed the sky wondering what could be the meaning of the events taking place above their heads, but as a science, cosmology really came to prospering after the formulation of general relativity by Einstein in 1914. Our dynamically evolving universe can only be studied well using relativistic gravity.

During the twentieth century, the Hot Big Bang (HBB) model of the early universe was developed. This model faces some serious problems, which mainly have to do with the causal structure (or rather, the apparent absence thereof) in the universe. The solution to these problems is provided by inflation, a period of accelerated expansion of the universe. This acceleration influences the structure in just the right way to solve the causal problems.

In the last decades, an additional problem of the ‘standard model’ of cosmology was discovered; the expansion of the universe is accelerating right now. While inflation solves the problem in the early universe, it should be absent at present, for inflation ‘blows apart’ the universe, blocking all possible structure formation. The tiny amount of inflation that we measure right now is small enough not to interfere with standard cosmology, but large enough to be measured. Many cosmologists feel that this cannot be coincidental and several solutions to this so-called quintessence problem are posed in the literature. In this thesis, some of these solutions will be discussed.

While early-time inflation is understood quite well, we will reach the conclusion that there is no really satisfying solution to the quintessence problem yet.

The tool that is most often used in describing inflation and quintessence models is quantum field theory, the ground state energy of field theories providing the ‘force’ to drive the accelerated expansion of the universe. The observations of the Cosmic Microwave Background Radiation (CMBR), a relic from the Big Bang, constitute an important experimental check to many of the ideas treated in this thesis.

Outline of this thesis

In the first chapter, a short introduction to cosmology is given, mainly on a historical basis. The second chapter discusses the mathematical tools needed to describe inflation and quintessence. The third chapter presents a specific model of the universe, which serves mainly to clarify the matter from the second chapter.

After these three introductory chapters, chapter four presents general results that apply to inflation. The link with observations is also presented in this chapter. The fifth chapter goes into more details regarding the quintessence problem and possible solutions.

In the last chapter, conclusions are drawn and a short outlook for the future is presented.
Preface

Acknowledgements

I would like to express my gratitude towards my supervisor, Jan Smit. His critical questions have forced me to be absolutely clear about even the most difficult subjects. Without Jan, this work would have been much less thorough.

Secondly, I would like to thank my fellow students at the institute for many fruitful discussions and much fun during lunch time.

Last but not least, I am indebted to Karla, my family and my friends for their moral support.

Paul Friedel
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1. Cosmology, inflation and quintessence

1.1 Introduction

Cosmology is concerned with the study of the universe at a large scale. We are not primarily interested in subjects like stellar formation, galaxy dynamics or even clusters of galaxies, we rather look at distances that are comparable to the size of the visible universe, that is, distances of the order of billions of light years. At such large distances, the exact nature of the constituent ‘particles’ (galaxies) that make up the universe is no longer relevant, we may treat the energy content of the universe as a continuous distribution. In other words, we treat the universe as if it were filled with a fluid. It isn’t, of course, but on the large scales we talk about, the fluid description serves as an excellent approximation.

The main objective of cosmology is to gather an understanding of the nature of the universe as a whole: why does the universe look like we see it today? Looking far into the universe means looking into the past, and this automatically leads to the study of the evolution of our universe.

The universe we see is to large extent isotropic (provided we look on large enough scales). If we make the reasonable assumption that our location in the universe is in no way special, this isotropy implies that the universe is homogeneous also. In the next section we will give a general overview of cosmic phenomenology and discuss this point further.

1.2 The universe we see

The earth forms together with the other eight planets and the sun our solar system. The solar system itself is situated in the outer regions of the galaxy that we call the Milky Way. The Milky Way, in its turn, is part of the Local Group, a cluster of about 30 galaxies. Going to even larger length scales, we see that the Local Group is part of the Virgo supercluster. At the largest scales, we observe that the superclusters seem to be organised in filamentary structures throughout the universe.

The distance from the earth to the sun is about 8 light minutes, the radius of the Milky Way is about sixty thousand light years, and filamentary structures (like the Great Wall) can have sizes of the order of 100 Million light years. If we compare this to the size of the observable universe (of the order of 10 Billion light years), we see that these structures are still very small compared to cosmologically interesting length scales. (We must remember that a scale difference of $10^2$ in length, means a scale difference of $10^6$ in volume.) The distribution of these structures seems to be isotropic to quite a large extent, which implies that the universe is also homogeneous. In our fluid model of the
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universe we can therefore use a perfect fluid description. This important statement lies at the basis of modern cosmology. Without the perfect fluid description, the Einstein equations are practically impossible to solve.

Modern cosmology rests on four pillars. First of all, we need a relativistic theory of gravity, since the visible universe is so large that the propagation speed of light cannot be ignored. This theory is of course the theory of general relativity, developed by Einstein in 1914.

Three observations have further shaped the understanding of the universe we have today. The first of these observations is the fact that astronomical objects have a redshift that is proportional to their distance. Of course it is no surprise that all luminous objects exhibit a Doppler shift. Since in general all objects are in motion with respect to the earth, the wavelength of the light we detect will be influenced by that motion. The thing that is surprising, is that the overwhelming majority of the objects we observe exhibit a redshift, and not a blueshift, that is, all objects are moving away from us. Moreover, since the redshift increases with the distance of the objects, we can conclude that the recession speed increases with distance. We can account for these facts by assuming that the universe itself expands uniformly. This recession was observed for the first time by Edwin Hubble in 1929. We must note that bound systems, such as our galaxy, do not experience this expansion because of the local gravitational attraction. This is the reason why we can actually measure the redshift. If our local system would expand also, this would imply a stretching of our measurement apparatus and it would be impossible to measure the expansion. The expansion velocity of our universe is given by the Hubble rate, which gives the increase in recession speed in km/s for every Megaparsec. Its size is often specified as

\[ H = h \times 100 \text{ km/s/Mpc}, \]  

(1.1)

where the number \( h \) is about 0.7. This corresponds to a relative expansion of about \( 2.3 \times 10^{-18} \text{ s}^{-1} \). A lot of measurable quantities in space depend on the recession speed and the measured values are often given as a number times a dependence on \( h \).

The second important observation is the existence of an almost isotropic background of photon radiation, having a black body spectrum with a temperature of about 3 K. This Cosmic Microwave Background Radiation (CMBR) was accidentally discovered by Penzias and Wilson in 1965. The CMBR is the relic from the time that the universe became transparent to photons. The existence of the background radiation is compelling evidence in favour of the Hot Big Bang (HBB) theory.

The third observation deals with the relative abundances of the light elements (up to \( ^7\text{Li} \)). The measured abundances fit very well with theoretical predictions of HBB models. We cannot account for the relative abundances using stellar evolution models, since stellar production of light elements inevitably brings along production of heavier elements, e.g. carbon and nitrogen. There is no evidence that the primordial abundance of the light elements is also accompanied by an abundance of heavier elements.

Some empirical data regarding cosmological quantities is given in table 1.1. The density parameters are a measure of the relative importance of various components to the total energy density. \( \Omega_i \) is defined by \( \Omega_i \equiv \rho_i / \rho_{\text{crit}} \), with \( \rho_{\text{crit}} \) the critical density, which corresponds to a perfectly flat universe. We note that the value of \( \Omega_{\text{tot}} \) implies that the universe is almost (perhaps exactly) flat, this will be discussed in the next chapter.

\[ ^7\text{Li} \] The objects that have a blueshift are all situated relatively near to our galaxy. The fact that they move towards us can be explained by local motions within our clusters of galaxies.


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<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hubble rate</td>
<td>$H_0 = 71^{+4}_{-3}$ km s$^{-1}$ Mpc$^{-1}$</td>
</tr>
<tr>
<td>CMBR temperature</td>
<td>$T_0 = 2.725 \pm 0.002$ K</td>
</tr>
<tr>
<td>Age of the universe</td>
<td>$t_0 = 13.4 \pm 0.3$ Gyr</td>
</tr>
<tr>
<td>Radiation density parameter$^a$</td>
<td>$\Omega_{\text{rad}} = 4.8^{+1.3}_{-0.9} \times 10^{-5}$</td>
</tr>
<tr>
<td>Baryonic matter</td>
<td>$\Omega_{\text{bar}} = 0.044 \pm 0.004$</td>
</tr>
<tr>
<td>Total matter</td>
<td>$\Omega_{\text{tot}} = 0.29 \pm 0.07$</td>
</tr>
<tr>
<td>Cosmological constant</td>
<td>$\Omega_{\Lambda} = 0.73 \pm 0.04$</td>
</tr>
</tbody>
</table>

| Total density parameter          | $1.02 \pm 0.02$ |

$^a$Value taken from [100]

Table 1.1: Present-day values of some cosmological parameters. Values taken from [76, 78].

1.3 From the Big Bang until now

In this section, an overview of the history of our universe according to the Big Bang scenario is given. On details, models found in the literature may differ from the outline sketched below, but the overall picture is the same for most models.

We believe the universe to be some 13.5 billion years old. The Hot Big Bang model assumes that the universe began in a state of very high energy density, and started to expand. This cooled the energy content. At very high energies, we assume that there is a full unification between the electromagnetic, weak and strong forces. This era is called the GUT-era (GUT=Grand Unified Theory). After the universe cooled below $10^{16}$ GeV, GUT broke down and at 100 GeV, or $t = 10^{-10}$ s, the unification between the electromagnetic and the weak forces also broke down. The universe was then filled with a plasma. At 100 MeV, this plasma started condensing, forming hadrons.

Around 1 MeV, the interaction rate of neutrinos with the rest of the matter in the universe had dropped so low, that they effectively decoupled. We expect that there still is a sea of neutrinos from this age, which by now have a temperature of about 2 K. This sea cannot be detected by present means, because neutrinos interact very weakly with ordinary matter. Around this neutrino decoupling time, nucleosynthesis started. The protons and neutrons combined to form light nuclei, but the temperature was still high enough to prohibit the formation of neutral atoms and the universe was filled with a plasma, not transparent to photons.

Because of the continuous expansion of the universe, the temperature dropped further and the electrons and nuclei combined to form complete atoms. This happened at a temperature of about 1 eV, much lower than the ionisation energy of hydrogen. This is because there are far more photons in the universe than baryons. The photons in the high energy ‘tail’ of the photon distribution were thus effective in ionising hydrogen down to an energy of 1 eV. Because the interaction rate between photons and atoms is far less than that between photons and nuclei, the photons decoupled from matter during recombination: the universe became transparent. These decoupled photons constitute the CMBR we observe today. Since the time of decoupling (also called the time of last scattering) the photons have travelled freely. Somewhat before the decoupling of the photons, the energy density of the photons was equal to the energy density of the non-relativistic matter content of the universe. At present, the photon density is but a very small fraction of the matter density.

After photon decoupling, a period of large-scale structure formation was entered, which lasts up to now. A summary of the history of the universe is shown in figure 1.1.
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<table>
<thead>
<tr>
<th>Energy Scale</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plank time</td>
<td>$10^{-37}$ s</td>
</tr>
<tr>
<td>GUT breaking</td>
<td>$10^{-37}$ s</td>
</tr>
<tr>
<td>QG transition</td>
<td>$10^{-6}$ s</td>
</tr>
<tr>
<td>Neutrino decoupling</td>
<td>$10^{-3}$ s</td>
</tr>
<tr>
<td>Nucleosynthesis</td>
<td>$15$ min</td>
</tr>
<tr>
<td>Recombination</td>
<td>$3 \times 10^{-4}$ eV</td>
</tr>
<tr>
<td>Large scale structure formation</td>
<td>$10^7$ eV</td>
</tr>
</tbody>
</table>

Figure 1.1: The history of the universe, approximate logarithmic energy scale. See e.g. [9, 103].

1.4 Problems with the HBB-model

The HBB-model seems very successful. The primordial explosion explains why the universe is expanding, the observed abundances of light elements fit into detailed models of nucleosynthesis, and the background radiation is but a relic from the time the universe became transparent to photons. The model also leaves us with some major problems, though. Several observations cannot be explained within the HBB-framework [103].

Horizon problem

As we have seen, the universe expands. This expansion can be described using a *scale factor* $a(t)$. This factor determines the scale of the universe at all times. To find the physical distance between to points, we calculate their distance in coordinates moving along with the expansion and multiply by the scale factor. It is possible to calculate the evolution of the scale factor, given the contents of the universe. For a universe in which the energy density is dominated by non-relativistic matter, which is the case for by far the largest time in the history of our universe, the scalefactor will be shown to be (see table 2.1)

$$a(t) \propto t^{2/3}.$$  \hspace{1cm} (1.2)

The time has been set to zero at the time of the Big Bang, when the scale factor was zero and the density infinite.

With help of the scale factor, we can calculate the distance that a photon can travel through the universe if it is emitted at time $t = 0$. This so-called *horizon distance*\(^2\) is given by

$$d_H(t) = a(t) \int_0^t \frac{dt'}{a(t')}.$$  \hspace{1cm} (1.3)

For a matter dominated universe, we find

$$d_H = 3t.$$  \hspace{1cm} (1.4)

\(^2d_H\), which corresponds to the forward light-cone of an event at time zero, is more accurately called the particle horizon, to distinguish it from the backward light-cone, which is called the event horizon.
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Figure 1.2: Light rays travelling through a curved space will be deflected from Euclidean straight lines, as shown in this schematic. This influences angles as seen by an observer (the dotted lines). Shown in the middle is a space-time with positive curvature, the right diagram shows the effect of negative curvature.

If we do not assume that the universe was homogeneous at the very start, this horizon distance is the largest possible size of any homogeneous region, since homogeneity of any region implies that all parts of the region have at least been in causal contact at some moment in the past. The CMBR was emitted at $t = t_{ls} = 3.8 \times 10^5 \text{ yr}$, corresponding to a horizon distance of $d_H(t_{ls}) = 1.14 \text{ Myr}$. On the other hand, we know that the age of the universe is $t_0 = 13.4 \text{ Gyr}$. This means that the surface of last scattering presently lies at a distance $d_H(t_0) = 40.2 \text{ Gyr}$. But these two figures are in contradiction.

At the time of last scattering, the universe had a size of $a(t_{ls}) \approx a(t_0)/1100$. We can now calculate the circumference of the surface of last scattering. It is given by

$$C_{ls} = 2\pi d_H(t_0)/1100 = 230 \text{ Myr}. \quad (1.5)$$

We have just seen that a causally connected region at last scattering can only be as large as 1.14 Myr, corresponding to an angle of just $1.7^{\circ}$ on the surface of last scattering. But the CMBR is homogeneous over the whole sky. This homogeneity cannot be accounted for if we start with a non-homogeneous initial situation and dominance of non-relativistic matter. If we incorporate the effect of radiation dominance in the early universe, the figures do not change significantly and the problem still exists. There are two possibilities to correct this error: either the universe was homogeneous on large scales from the very start, or the HBB scenario is not correct.

Structure formation

The CMBR is not quite homogeneous. Deviations from a perfect black body spectrum of about one part in $10^5$ are measured. Taking one look into the sky at night (or day!) directly reveals that the matter content in the universe is certainly not homogeneous to such an extent on smaller scales. A natural question to ask is then what causes the inhomogeneities. Where does the structure in the CMBR come from, and in what way has the structure we described in section 1.2 arisen?

Flatness

Although it is perfectly possible to construct homogeneous space-times having a non-zero curvature, the universe we live in has a curvature that is very close to zero, and perhaps even exactly zero. This can be inferred from the angular spectrum of the CMBR, since the presence of curvature would influence the light rays from the CMBR, as illustrated in figure 1.2. The effect of curvature would be to shift the positions of the peaks in the spectrum of the CMBR in figure 4.1 and 4.2, but no influence of such a
1. Cosmology, inflation and quintessence

curvature term can be detected. The question now is: why is the universe (almost) flat? Or, in other words, why can we use Euclidean geometry in describing the universe?

A fact that makes this question even more pregnant is that the effect of curvature in the universe will grow with time. This means that the universe should have been even less curved in the past.

Absence of thermal relics

Grand Unified Theories generally predict a large abundance of exotic objects like magnetic monopoles. These objects should still be observable today. See, for example [12]. In reality, no such objects are found. Where are they?

1.5 Inflation

The problems above can be solved by invoking one powerful concept: inflation. The idea of inflation is that at one stage in the evolution of the universe, before the standard Big Bang, there has been a period of accelerated expansion. Alan Guth [104] was among the first to propose this idea and he coined the term inflation. A detailed account of the historical development of the inflation concept can be found in [12].

Inflation works fairly simple. If the universe expands with a scale factor that grows more rapidly than $t$ (exponentially fast expansion, for example, falls into this class), a very small region, initially thermalized, can easily grow to encompass our entire visible universe at last scattering. The mutual thermalisation of the ‘apparent decoupled regions’ from the last section is then of course no surprise.

The flatness problem is solved by the fact that the accelerated expansion ‘blows away’ the curvature, like inflating a balloon to a larger volume erases the wrinkles and flattens the surface. In a similar way all thermal relics are diluted.

The exact formulation of these two arguments will be postponed until next chapter. Later on, we will also treat in detail how the structure formation problem is solved. The solution to the problem of structure formation is now one of the main reasons for acceptance of the inflation theory, but this was recognised as such only after Guth proposed his model.

1.6 Unsolved by inflation: the quintessence problem

At this moment, observational evidence (see, e.g. [96, 97]) indicates that the universe is inflating right now. It is clear that inflation solves a lot of problems in the early universe, but the presence of accelerated expansion at this moment poses a large problem: we do not know the origin of the present acceleration and we do not know why this acceleration has the size we observe.

We can describe accelerating behaviour effectively by adding a cosmological constant to the Einstein equations (possible origins of such a constant will be discussed later). The problem with this is, that such a constant should be extremely small compared to unification scales, to account for the energy density associated with the constant. Secondly, the cosmological constant starts dominating the energy density of the universe very quickly and if we want it to dominate right now, it should have been tremendously

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3It is not always clear what is meant by the Big Bang. Many authors mean the beginning of the universe (whatever that means exactly) if they talk about the Big Bang. In this work, the Big Bang here means the whole evolution from right after inflation until the beginning of structure formation.
1. Cosmology, inflation and quintessence

small in the early universe. The *quintessence*\(^4\) problem, as it is called, actually consists of two problems.

The temporal fine-tuning: why is the cosmological constant starting to dominate the energy density right at this moment?
The magnitude fine-tuning: why is the energy density of the cosmological constant so small?

Solving one of these questions often means that the other question automatically is answered as well and we will often refer to these two problems together as the fine-tuning or the quintessence problem.

Most cosmologists are reluctant to put in the cosmological constant by hand, since it involves setting its value just right to solve the fine-tuning problem. It is possible to invoke dynamical mechanisms which are hoped to cope in a natural way with the smallness and timing problem. Although many working models have been built that address this problem without using a cosmological constant term, there is no consensus regarding the solution of this problem. The difficulty of building and interpreting these so-called quintessence models will be treated extensively in later chapters.

Many authors mean a specific class of solutions to the fine-tuning problem whenever they talk about quintessence. In this work, quintessence encompasses all possible solutions to the problem, whatever their nature.

\(^4\)Quinta Essentia = The Fifth Element, the name stresses the unusual nature of the physics responsible for quintessence.
1. Cosmology, inflation and quintessence
2. Calculational tools

In this chapter, we will develop the necessary calculational tools to give a quantitative description of the universe. In the first section, we will treat the Einstein equations, the Robertson-Walker metric and the Friedmann equations, which follow from the former two concepts. In later sections, we will again take a look at the inflation and quintessence problem and discuss possible solutions in some more detail. The last two sections present some more details on inflation that are needed in order to appreciate chapter 3.

2.1 The Einstein equations and some simple solutions

The starting point for a quantitative description of cosmology are of course the Einstein equations:

\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 8\pi G N T_{\mu\nu}. \]  

(2.1)

The Einstein tensor \( G_{\mu\nu} \) is formed out of the metric \( g_{\mu\nu} \) and its first and second order derivatives. Throughout this thesis, we will use Planck units. We will set

\[ \hbar = c = 1 \]  

(2.2)

and we define the reduced Planck mass

\[ M_P = \sqrt{\frac{\hbar c}{8\pi G N}} = 2.436 \times 10^{18} \text{ GeV}. \]  

(2.3)

Note that this quantity differs by a factor of \( \sqrt{8\pi} \) from the usual Planck mass, denoted by \( m_P \). The reduced version is used because it conveniently absorbs cumbersome numerical factors. Of course we now also have a Planck length, Planck time etc. In appendix A some values and conversions are given. We further use the conventions

\[ \eta_{\mu\nu} = \text{diag} (-1,1,1,1) \]  

(2.4)

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\eta} (g_{\nu\eta,\mu} + g_{\mu\eta,\nu} - g_{\nu\mu,\eta}) \]  

(2.5)

\[ R^\kappa_{\lambda\mu,\nu} = \Gamma^\kappa_{\lambda\nu,\mu} - \Gamma^\kappa_{\lambda\mu,\nu} + \Gamma^\eta_{\lambda\kappa,\mu} \Gamma^\kappa_{\nu\eta} - \Gamma^\eta_{\lambda\mu,\nu} \Gamma^\kappa_{\nu\eta} \]  

(2.6)

\[ R_{\mu\nu} = R^\eta_{\mu\eta\nu} \]  

(2.7)

\[ R = g^{\mu\nu} R_{\mu\nu}. \]  

(2.8)

A comma denotes differentiation

\[ f,\mu = \partial_{\mu} f = \frac{\partial f}{\partial x^\mu}. \]  

(2.9)

The quantity \( \Gamma^\lambda_{\mu\nu} \) is called the affine connection. It defines the covariant (gauge invariant) derivative, denoted by ‘\( ;\mu \)’ or \( D_\mu \), see appendix A. It is a function of the metric
2. Calculational tools

\( g_{\mu\nu}, R^{\kappa}_{\lambda\mu\nu} \) is the Riemann curvature tensor, \( R_{\mu\nu} \) is the Ricci tensor and \( R \) is called the curvature scalar. The combination of \( R_{\mu\nu}, R \) and \( g_{\mu\nu} \) in equation 2.1 is the most general divergenceless object we can make using only the metric and its derivatives up to second order (see e.g. [1]). It is perfectly possible to construct a divergenceless tensor using higher order derivatives. These factors can, however, be neglected at small distances \( l \) compared to the typical variation distance of the gravitational field \( L \) because higher order factors contain an extra derivative of the metric. The relative importance of these terms will thus be suppressed by a factor \( l/L \).

2.1.1 The energy-momentum tensor

The right hand side of equation (2.1) is the energy-momentum tensor, or stress tensor. The stress tensor has divergence 0, that is \( T_{\mu\nu} = 0 \). It should, of course, since the Einstein tensor also has divergence zero, but it also is a statement of local energy-momentum conservation.

We have seen in section 1.2 that we may take the universe to be filled with a perfect fluid. In a Minkowskian (local inertial) frame of reference, we then have the stress tensor

\[
T_{\mu\nu} = \text{diag} (\rho, p, p, p) \quad \text{(rest frame)},
\]

with \( \rho \) the energy density of the fluid and \( p \) the pressure.

The factor \( \Lambda \) in the Einstein equations is the cosmological constant. We will mostly write

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = M_p^{-2} T_{\mu\nu}^{(\Lambda)},
\]

including the contribution of \( \Lambda \) in the stress tensor. From now on we will suppress the label \( (\Lambda) \), from the context it will be clear whether or not the cosmological constant is included in the stress tensor. At this moment we will not bother what the physical nature of the cosmological constant may be. We just treat it like a parameter in our equations. In the rest frame, we can figure out what the stress tensor for the cosmological constant will be:

\[
T_{\mu\nu}^{(\Lambda)} = -\Lambda M_p^2 \eta_{\mu\nu} = \Lambda M_p^2 \text{diag} (1, -1, -1, -1) = \text{diag} (\rho_\Lambda, p_\Lambda, p_\Lambda, p_\Lambda). \tag{2.12}
\]

The pressure of the cosmological constant is negative! The presence of a nonzero \( \Lambda \)-term in the equations will therefore cause an accelerated expansion, provided that the contributions from matter and radiation in the energy-momentum tensor are not dominant.

2.1.2 The Robertson-Walker metric

The fact that space is homogeneous not only determines the form of \( T_{\mu\nu} \), but it also fixes the functional form of the metric \( g_{\mu\nu} \). For a homogeneous metric in 3+1 dimensions, we have the Robertson-Walker (RW) line element. This is the most general line element that is compliant with the demand for homogeneity.\(^1\) In spherical coordinates:

\[
\text{d}s^2 = g_{\mu\nu}^{\text{RW}} \text{d}x^\mu \text{d}x^\nu = -\text{d}t^2 + a(t)^2 \left( \frac{\text{d}r^2}{1 - kr^2} + r^2 \text{d}\Omega^2 \right), \tag{2.13}
\]

\(^1\)In [1] a very general deviation of this fact is given. It is simpler just to consider a general line element \( \text{d}s^2 = g_{\mu\nu} \text{d}x^\mu \text{d}x^\nu \). Isotropy implies that we can write the element as \( \text{d}s^2 = a(r, t)\text{d}t^2 + b(r, t)\text{d}(\text{d}x\text{d}x) + c(r, t)(\text{d}r\text{d}r) + d(r, t)\text{d}x^2 \), which we may rewrite as \( \text{d}s^2 = -\text{d}t^2 + a(t)^2 (f(r')\text{d}r'^2 + r'^2 \text{d}\Omega^2) \). The primed symbols are now functions of the original variables \( (r, t) \). We fix the function \( f \) by noting that homogeneity implies a constant curvature \( k \) throughout space and thus \( f = (1 - kr'^2)^{-1} \). The function \( e \), finally, can be identified with the scale factor of the universe.
2. Calculational tools

and the metric is thus

\[ g^{RW}_{\mu\nu} = \text{diag} \left( -1, \frac{a^2}{1 - kr^2}, a^2 r^2, a^2 r^2 \sin^2 \theta \right) . \]  (2.14)

The factor \( a(t) \) that multiplies the space-part of the metric is the already mentioned scale factor. The scale factor is usually defined to be unity at the present time, \( a_0 \equiv a(t = t_0) = 1 \). (2.15)

The relative expansion velocity of the universe as found by Hubble is given by

\[ H = \frac{\dot{a}}{a} , \]  (2.16)

where the overdot indicates the time-derivative.

An often-used way of indicating the scale factor is the use of redshift. The Doppler-shift \( z \) of photons travelling in an expanding volume is

\[ z = \frac{\lambda_{\text{observed}} - \lambda_{\text{emitted}}}{\lambda_{\text{emitted}}} = a^{-1} - 1 \rightarrow a = (1 + z)^{-1} . \]  (2.17)

The factor \( k \) in the RW-metric determines the topology of space-time. For \( k = 0 \) the space component has the Euclidean form, we have flat space. \( k > 0 \) or \( k < 0 \) gives positive or negative curvature, respectively. We often speak of a flat, closed or an open universe. We may always rescale the coordinate \( r \) such that \( k \) is 0, 1 or \(-1\).

From (2.14) we find the connections

\[
\begin{align*}
\Gamma^0_{ij} & = H g_{ij} \\
\Gamma^1_{1j} & = \frac{kr}{1 - kr^2} \\
\Gamma^1_{j1} & = -r(1 - kr^2) \sin^2 \theta \\
\Gamma^2_{12} & = 1/r \\
\Gamma^2_{22} & = -\sin \theta \cos \theta \\
\Gamma^2_{33} & = -\sin \theta \cos \theta \\
\Gamma^3_{23} & = \tan^{-1} \theta ,
\end{align*}
\]  (2.18)

the other components being zero. For the curvature tensor we have

\[ R_{00} = -3\ddot{a}/a . \]  (2.19)

\[ R_{ij} = g_{ij}(2k + 2\dot{a}^2 + a\ddot{a}) . \]  (2.20)

\[ R = 6 \left( \frac{k}{a^2} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \right) = 6 \left( \frac{k}{a^2} + \dot{H} + 2H^2 \right) . \]  (2.21)

\[ ^3R = 6k/a^2 . \]  (2.22)

The quantity \(^3R\) is the spatial curvature which is calculated by taking into account only the space coordinates of \( R^\kappa_{\lambda\mu\nu} \). From the expression for \( R \) it is clear that the curvature-term \( k/a^2 \) decreases as the universe expands, regardless of the topology of space-time.

Conformal time

For many purposes, it is convenient to rewrite the RW-element in a different way, pulling the scale factor in front of the time-time-component:

\[ ds^2 = a(\eta)^2 \left( -d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) . \]  (2.23)
2. Calculational tools

The new time variable $\eta$ is defined by

$$a\,d\eta = dt.$$  \hfill (2.24)

$\eta$ is called conformal time and the usual time variable $t$ is called coordinate time. In this thesis, conformal time will often be used. Derivatives with respect to $\eta$ are denoted by a prime, and we have

$$\dot{f} = \frac{1}{a} f'.$$  \hfill (2.25)

This relation makes it easy to transform expressions from coordinate to conformal time and vice versa. A useful relation concerning the Hubble rate is

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{\dot{a}}{a} = a \dot{H}.$$  \hfill (2.26)

In this thesis, a ‘calligraphic’ $\mathcal{H}$ will always denote the Hubble rate in conformal time and a ‘standard’ $H$ will denote the Hubble rate in coordinate time. In this chapter, some important formulas will be given in coordinate as well as in conformal time.

Another possibility to parameterise time, is to use the scale factor itself as the time variable. As long as the scale factor is continuously increasing (which is the case in the history our universe), there exists a bijective mapping between time and the scale factor. A particular useful relation is

$$\frac{df}{dN} = \frac{df}{d\ln a} = \frac{1}{H} \dot{f}.$$  \hfill (2.27)

2.1.3 The Friedmann equation

Now the metric and the stress tensor are determined, it pays to take another look at the Einstein equations. The expressions found for the curvature tensor are independent of the choice of space coordinates. Therefore, it is permitted to use cartesian coordinates in the Einstein equations and the stress tensor takes the form (2.10). Taking the 00-component of 2.1 the Friedmann equation can be derived:

$$H^2 + \frac{k}{a^2} = \frac{1}{3M_P^2} \rho_{\text{tot}}.$$  \hfill (2.28)

Or, using conformal time:

$$a^2 \mathcal{H}^2 + \frac{k}{a^2} = \frac{1}{3M_P^2} \rho_{\text{tot}}.$$  \hfill (2.29)

$\rho_{\text{tot}}$ is the total energy density, including a possible contribution from the cosmological constant $\rho_\Lambda \equiv \Lambda M_P^2$. From the $ij$-component of the Einstein equations:

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{1}{M_P^2} p_{\text{tot}},$$  \hfill (2.30)

$$\frac{1}{a^2} (\mathcal{H}^2 - \mathcal{H}' + k) = -\frac{1}{M_P^2} p_{\text{tot}},$$  \hfill (2.31)

including $p_\Lambda \equiv -\Lambda M_P^2 = -\rho_\Lambda$. From here on, we will drop the label $\text{tot}$, always meaning the total density or pressure, including a contribution from $\Lambda$, unless indicated otherwise.
2. Calculational tools

From these two equations, a number of useful relations can be deduced.

\[
\dot{H} = -\frac{1}{2M_p^2}(\rho + p) + \frac{k}{a^2}
\]  
(2.32)

\[
\frac{\dot{a}}{a} = \dot{H} + H^2 = -\frac{1}{M_p^2}(p/2 + \rho/6)
\]  
(2.33)

\[
\frac{1}{a^2}(\dot{\mathcal{H}} - \ddot{H}) = -\frac{1}{2M_p^2}(\rho + p) + \frac{k}{a^2}
\]  
(2.34)

\[
\frac{1}{a^2}\dot{\mathcal{H}} = -\frac{1}{2M_p^2}(p/2 + \rho/6).
\]  
(2.35)

The Friedmann equation can be written in a more enlightening way. Taking \(\frac{\rho}{\rho_{\text{crit}}} = \frac{3H^2M_p^2}{k}\) implies \(k = 0\) in equation (2.28). The critical density is not constant, but depends on the Hubble rate, and thus on time. Its present value is about \(9.5 \times 10^{-27}\) kg/m\(^3\), which corresponds to approximately six protons per cubic meter. The relative density parameter is defined by

\[
\Omega_i = \frac{\rho_i}{\rho_{\text{crit}}},
\]  
(2.36)

The subscript indicates the fluid component we talk about. Introducing

\[
\rho_k \equiv 3kM_p^2a^{-2},
\]  
(2.37)

the Friedmann equation becomes

\[
1 + \Omega_k = \Omega_\Lambda + \Omega_{\text{mat}} + \Omega_{\text{rad}}.
\]  
(2.38)

Particles moving with a velocity comparable to the speed of light contribute to \(\rho_{\text{rad}}\), non-relativistic matter contributes to \(\rho_{\text{mat}}\). \(\Omega_k\) is measured to be very close to zero. This does not tell us anything about the topology of space-time. There may be a constant positive or negative curvature, the only thing that we know, is that the contribution of the curvature to the energy density is very small. Looking at the remaining density parameters, there seems to be a problem. From 1.1, the baryonic matter turns out to account for some 5% of the critical density, while observations of galaxy dynamics indicate that about 30% of the critical density is provided by non-relativistic matter [6]. Thus, there must be a dark matter component that cannot be seen with optical telescopes, but that nonetheless contributes to the non-relativistic matter in the universe. The nature of this dark matter is subject of much debate, but we will not be concerned with the discussion in this thesis. From the table, it is also clear that the radiation-density is for all practical purposes negligible. Therefore, approximately 70% of the critical density must be provided by the cosmological constant. These 70% are often denoted by the name dark energy, while the 25% ‘missing’ matter are called dark matter. One should not confuse these names.

2.1.4 Solutions to the Friedmann equation

Now that we have derived the Friedmann equation, we would like to find solutions to it. Fortunately, it is quite easy to find solutions, as long as the energy density in the universe is dominated by a single fluid (radiation or matter, for instance). To construct
2. Calculational tools

<table>
<thead>
<tr>
<th>Dominance</th>
<th>( w = )</th>
<th>( \rho = )</th>
<th>( a(t) \propto )</th>
<th>( H = )</th>
</tr>
</thead>
<tbody>
<tr>
<td>rad. dom.</td>
<td>( \frac{1}{3} )</td>
<td>( \rho_0 (a/\alpha_0)^{-4} )</td>
<td>( t^{-1/2} )</td>
<td>( (2t)^{-1} )</td>
</tr>
<tr>
<td>matt. dom.</td>
<td>0</td>
<td>( \rho_0 (a/\alpha_0)^{-3} )</td>
<td>( t^{2/3} )</td>
<td>( (\frac{4}{3}t)^{-1} )</td>
</tr>
<tr>
<td>cosm. const.</td>
<td>( -1 )</td>
<td>( \Lambda M_p^2 )</td>
<td>( \exp(\sqrt{\Lambda/3} t) )</td>
<td>( \sqrt{\Lambda/3} )</td>
</tr>
<tr>
<td>kination</td>
<td>1</td>
<td>( \rho_0 (a/\alpha_0)^{-6} )</td>
<td>( t^{1/3} )</td>
<td>( (3t)^{-1} )</td>
</tr>
</tbody>
</table>

Table 2.1: Solutions to the Friedmann equation for radiation, matter, cosmological constant dominance and kination.

solutions of the equations, it is practical to start with the energy-momentum tensor. Divergencelessness implies

\[
T^{0\mu}_{;\mu} = 0 \Rightarrow \frac{d}{dt} (a^3 [p + \rho]) = \dot{p} a^3 \quad \text{or} \quad (2.39)
\]

\[
\frac{d}{dt} (pa^3) = -p \frac{d}{dt} a^3. \quad (2.40)
\]

This is just the continuity equation for a comoving volume. The equation of state for a fluid species component \( i \) is assumed to be

\[
p_i = w_i \rho_i, \quad (2.41)
\]

which leads to

\[
\rho_i = \rho_i \left( \frac{a}{\alpha_0} \right)^{-3(1+w_i)}. \quad (2.42)
\]

The value of \( w \) is different for each component, and it is by no means obvious how to use the above equation to describe a mixture of components. It is very easy, however, to calculate what happens in the case that one particular component dominates the energy density. For relativistic matter (radiation) \( w = 1/3 \), for non-relativistic matter, the pressure is negligible compared to the energy density, so \( w = 0 \). The explanation for the different scaling behaviour of matter and radiation lies in the redshift. On top of the expected dilution of the energy density \(( \sim a^{-3} )\), the radiation wavelength increases and the energy thus decreases. This causes the extra factor \( a^{-1} \).

From (2.12) it follows that \( w_\Lambda = -1 \). For \( k = 0 \), some limiting cases are displayed in table 2.1. From equation (2.33) follows that if \( w < -1/3 \), accelerated expansion will occur. If we demand that the velocity of sound must not exceed the speed of light, we must have

\[
v_{s,i}^2 = \frac{\partial P_i}{\partial \rho_i} \leq 1 \quad (2.43)
\]

and the equation of state is therefore constrained to lie in the range

\[
w \leq 1. \quad (2.44)
\]

All solutions for the scale factor are continuously increasing. A dominating cosmological constant even leads to exponential expansion. For positive \( k \) there also exist solutions that recollapse: after reaching a maximum value, the scale factor starts to decrease again. Because the different component densities depend on the scale factor in a different way, the relative contributions of the components to the energy density will change in time. A radiation dominated universe will evolve towards a matter dominated universe. The relative matter density will in its turn be overtaken by \( \rho_\Lambda \). A curious fact is that the energy density of the cosmological constant does not decrease at all!
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2.2 How inflation solves the problems with the HBB-model

In this section, the problems posed in section 1.4 will again be investigated. Inflation will be shown to provide a natural solution to them.

Horizon problem

The horizon problem can be solved if we take into account inflation. Simply said: inflation blows up any initially thermalized region, no matter how small, to a monstrous size, easily large enough to encompass our entire visible universe. There is then no extraordinary homogeneity needed at the start of the universe.

This argument can be made quantitative by considering the comoving Hubble length \((aH)^{-1}\). Suppose that the universe is in a phase of accelerated expansion. We state

\[
a(t) \propto t^p
\]

with \(p > 1\) for accelerated expansion and \(p < 1\) for decelerated expansion (e.g. matter or radiation dominance). The comoving Hubble length is then given by

\[
(aH)^{-1} = \frac{1}{pt^{1-p}}.
\]

For an accelerating scale factor, the comoving Hubble length decreases, and for a decelerating scale factor, it increases in time.

The Hubble length sets a natural length scale for all cosmological processes, which enters in the cosmological equations, see for example equation (2.64). Processes that occur on length scales much larger than the Hubble length are suppressed relative to processes on smaller scales. For example, the cosmological perturbations which we will encounter later, stop evolving as soon as their length scales grow larger than the Hubble length. The crossing of the Hubble scale for a length scale under consideration is thus an important event. Because the comoving Hubble length decreases and increases during inflation and deceleration, fixed length scales will overtake the Hubble length scale during inflation and get smaller than the comoving Hubble length during deceleration. This effect is illustrated in figure 2.1. In this thesis, we will denote the process of growing larger and smaller than the size of the comoving Hubble length by Hubble scale crossing. Unfortunately, Hubble crossing is often denoted by horizon crossing, since apart from a factor of order unity, the horizon is equal to the Hubble length for radiation and matter dominance. We will not use this language, since the only relevant quantity we should look at is the Hubble scale. The horizon will play no further role in this work and we will only consider the Hubble scale.

Structure formation

As will be discussed in the next section, inflation is believed to be caused by a self-interacting quantum field. This field will exhibit vacuum fluctuations and because the length scales of the fluctuations leave the Hubble scale during inflation, they will ‘freeze in’ and become classical. At Hubble scale re-entry, the fluctuations form the seeds for structure formation. Regions with a higher density will accumulate matter and regions with a lower density will lose some of their energy content to the higher density regions according to the Jeans-mechanism. A review on these matters can be found in e.g. [64, 67, 70].
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Figure 2.1: A comoving region of space is shown shaded in the diagrams. During inflation, the comoving Hubble length (dotted) decreases. During deceleration, the comoving Hubble length increases again. The white line shows a comoving length scale. During inflation, it leaves the Hubble scale, and during deceleration, it re-enters the Hubble scale. In reality, the effect is much more dramatic, as the growth of the scale factor typically is of the order of tens of e-folds.

Flatness
From the expression for the ‘energy density’ of the curvature term (2.37), one sees that the curvature term will eventually dominate over the matter contribution to the energy density, due to the $a^{-2}$ dependence of $\rho_k$ compared to $\rho_{m/r} \propto a^{-3}/a^{-4}$ for matter or radiation. Since the universe is nearly flat today, it should have even been closer to critical density in early times. Inflation accomplishes this by blowing up the scale factor to huge proportions, effectively setting $k$ to zero.

Absence of thermal relics
If inflation happens right after the GUT-era, the thermal relics problem is trivially solved. All unwanted relics are simply blown away by the dilution of the energy density caused by the inflation.

2.3 The origin of inflation

The logical question to ask after posing the inflation concept is of course, “Where does it come from?” There are basically three possible answers to this question.

1. Renormalization group
   In quantum field theory, many observable quantities will depend on the energy scale you probe the system with. A well-known example of this is the value of the electron charge. Due to shielding effects of virtual electron-positron pairs, the charge of the electron will seem to be larger if the probing energy is larger and the probing length scale smaller. Another example is asymptotic freedom within the strong interactions, the effect being opposite: the coupling tends to zero at small distances.

   There is really no reason why the ground state energy (GSE), could not exhibit such scaling behaviour. If it does, setting the GSE to zero at all scales using renormalization techniques is not an option, since this could be done but at one energy scale. The GSE would automatically be non-zero at other energy scales.

   The GSE behaves as a cosmological constant, which can be seen if its contribution to the energy and pressure are calculated. The renormalization behaviour of the
2. Inflation field
Another possibility to generate inflation is to introduce a scalar field, \( \sigma \),\(^2\) which dominates the energy density in the early universe. If the potential energy of the field is large compared to its kinetic energy, the field behaves as a fluid with \( w = -1 \), and if the situation is reversed, the field behaves as a fluid with \( w = +1 \). This will be shown in the next section. There is now a possible solution to the inflation problem. Let’s take a look at figure 2.2. If the field (appropriately called the inflaton field) starts in the left regime of the potential, without much kinetic energy, its potential energy will dominate, thus causing inflation. The inflaton starts rolling down the hill and its kinetic energy starts to dominate, thus entering a phase of kination. After the inflationary period, the energy density in the universe is virtually zero, and there must be a mechanism to reheat the universe. This is often established by adding a coupling between the inflaton and other fields. This coupling makes it possible for the inflaton field to transfer its energy to the other fields during oscillations in the potential well. This process is called reheating. The fields in turn decay into the constituents of the standard model, refilling the universe with matter.

3. Higher order curvature terms
The last option to generate inflation is to change the laws of gravity. In section 2.1, the possibility of introducing higher order curvature terms was already hinted at. For processes at small length scales compared to the variation scale of the gravitational field, introducing higher order terms does not influence the physics, but at large scales, these terms may cause inflation. It is possible to rewrite the

---

\(^2\)We use \( \sigma \) instead of the conventional \( \phi \) to denote the field, since the symbol \( \phi \) also occurs frequently in perturbation theory. Using \( \sigma \) for the field avoids confusion.
2. Calculational tools

effect of higher order curvature terms in terms of an inflaton model. An example of this procedure can be found in [11].

In this thesis, the main concern are field models for inflation.

2.4 A solution to the quintessence problem?

Now that we have seen some possible mechanisms that generate inflation in the early universe, is it also possible to apply those mechanisms to the quintessence problem? The answer is yes, to some extent. First, we will treat the three cases from the last section again.

1. Renormalization group

Just as in the inflation scenario, we can apply RG-methods to quintessence. If the renormalization turns out such that the ground state energy is larger for large length scales, the quintessence problem could be solved. The present acceleration of the expansion of the universe would then be a quantum effect which manifests itself but at large scales.

It is possible to model this behaviour, but the problem with this is that there is still no consensus on how the calculations should be performed. Some proposals can be found in [42, 43, 44, 45, 46, 47, 48, 49].

2. Quintessence field

An inflaton-like field, called a quintessence field in this context, can of course also be used in the late-time universe. There are two distinct cases here. First, we can choose the minimum of the inflaton potential not to be exactly zero. The inflaton field comes to rest in the minimum, and from that moment, it will behave as a cosmological constant. The inflaton field and the quintessence field are one and the same in this case. Although this is a quite elegant way to generate inflation at present, it does not solve the fine-tuning problem at all. This procedure is in fact closely related to the ground state energy discussed above.

Secondly, it is possible to introduce an altogether new scalar field, which moves in a potential. During the evolution of the universe, the field may roll fast or more slowly, constantly changing its own equation of state. The dynamics must be such that the field comes to a near stand still exactly at present, to fulfill its role as a cosmological constant. We will treat some possibilities for introducing these kind of potentials in chapter 5.

In the next chapter, we will treat an explicit model, in which the inflation potential and the quintessence potential are linked together in the origin, combining the two concepts. We will see, however, that such a combination is not very valuable, since the inflation and quintessence part of the potential effectively decouple, resulting in two essentially independent fields.

3. Changing gravity

A change in the gravitational equations is also sometimes proposed to solve the quintessence problem. Brane world models, for example, could effectively modify the Einstein equations in the right way to solve the quintessence problem.

But... the real quintessence problem lies not so much in the generation of acceleration, but in the numerical fine-tuning that is necessary. The energy density should come
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out all-right, and in the second place, the timing should be such that the acceleration becomes dominant right now. If the acceleration would have started too early, there would not have been enough time for structure formation. If the acceleration starts too late, we would not observe it today.

In the quintessence case, we will also be concerned mostly with the field concept. Before we can go on and calculate what happens in cosmological scenarios, we must calculate the equation of motion for the field. We will do this in the next section.

2.5 Equation of motion for a scalar field

To understand the dynamics of the inflaton or the quintessence field, the equation of motion for the scalar field must be computed. The starting point for this calculation is the inflaton Lagrangian

\[ \mathcal{L}_\sigma = \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\sigma) \right). \]  

(2.47)

The potential term can, a priori, be anything. The only assumption is that \( V \) is sufficiently smooth to be able to perform the necessary calculations. We ignore the possible coupling to other fields. Varying the action gives

\[ \delta S = \int_x \delta \mathcal{L}_\sigma \]  

(2.48)

\[ = \int_x \left[ \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V \right) \delta \sqrt{-g} - \frac{1}{2} \sqrt{-g} \partial_\mu \sigma \partial_\nu \sigma \delta g^{\mu\nu} \right] \]  

(2.49)

\[ = \int_x \left\{ \frac{1}{2} \sqrt{-g} \left[ \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V \right) g^{\kappa\lambda} + \partial_\mu \partial_\nu g^{\kappa\mu} g^{\lambda\nu} \right] \delta g_{\kappa\lambda} \right\} \]  

(2.50)

\[ T^{\kappa\lambda} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\kappa\lambda}} \]  

(2.51)

\[ = \partial^\kappa \sigma \partial^\lambda \sigma - g^{\kappa\lambda} \left( \frac{1}{2} (\partial \sigma)^2 + V \right). \]  

(2.52)

For a homogeneous field in a homogeneous background, with Robertson-Walker metric, this expression simplifies to

\[ \partial_i \sigma = 0, \quad \partial_0 \sigma = \dot{\sigma} \]  

(2.53)

\[ T^{\mu\nu} = g^{\mu\nu}_{RW} \left( \frac{1}{2} \dot{\sigma}^2 - V \right) + \delta^{\mu\nu} \delta^{00} \dot{\sigma}^2. \]  

(2.54)

Comparing this with 2.10 yields

\[ p = \frac{1}{2} \dot{\sigma}^2 - V, \quad \rho = \frac{1}{2} \dot{\sigma}^2 + V. \]  

(2.55)

From these expressions, it is directly clear that dominance of the potential energy leads to \( w = -1 \), and dominance of the kinetic energy leads to \( w = +1 \).

\[ ^3 \text{In the remainder of this chapter, we will often speak of an inflaton field, implying that the results we derive apply also to quintessence fields, which behave in the same way, basically.} \]
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The variation of the action with respect to $\sigma$ gives the Euler-Lagrange equation in a curved background:

$$S = \int x \sqrt{-g} \left( -\frac{1}{2} (\partial \sigma)^2 - V \right)$$

(2.56)

$$= -\int x \left[ \frac{1}{2} g_{\mu \nu} (\sqrt{-g} \partial^\mu \sigma) - \frac{1}{2} \partial^\mu \sigma \partial_\mu \sqrt{-g} - \frac{1}{2} \sqrt{-g} \partial^2 \sigma + \sqrt{-g} V \right]$$

(2.57)

$$= \int x \sqrt{-g} \left[ \frac{1}{2} \sigma \partial_\mu (\sqrt{-g} \partial^\mu \sigma) - \sqrt{-g} V \right]$$

(2.58)

$$= \int x \sqrt{-g} \left[ \frac{1}{2} \sigma D_\mu \partial^\mu \sigma - V \right]$$

(2.59)

$$\frac{\delta S}{\delta \sigma} = 0 \rightarrow D_\mu \partial^\mu \sigma - V'(\sigma) = 0.$$  

(2.60)

Here, a prime is used to denote the derivative with respect to the argument of the potential, $\sigma$. This should not cause confusion, since the potential is not explicitly dependent on time. In the remainder of this section, and in the following section, a prime will always denote differentiation with respect to $\sigma$. Using the RW-metric, the covariant derivative can be evaluated explicitly, leading to

$$D_\mu \partial^\mu \sigma = \partial^2 \sigma + \Gamma^\mu_{\nu \rho} \partial^\mu \sigma$$

(2.61)

$$= -\dot{\sigma} - \Gamma^\mu_{0 \nu} \dot{\sigma}$$

(2.62)

$$= -\dot{\sigma} - 3H \dot{\sigma}$$

(2.63)

$$\rightarrow \dot{\sigma} + 3H \dot{\sigma} + V'(\sigma) = 0.$$  

(2.64)

This expression can also be found directly by feeding the relations (2.55) into equation (2.40). In Minkowski space-time, which has $H = 0$, the equation of motion reduces to the standard form $\dot{\sigma} = -V'$.

The Friedmann equation and the evolution equation for $H$ (2.28,2.32) become

$$H^2 = \frac{1}{3M_p^2} \left( \frac{1}{2} \dot{\sigma}^2 + V \right) - \frac{k}{a^2}$$

(2.65)

$$\dot{H} = -\frac{\dot{\sigma}^2}{2M_p^2} + \frac{k}{a^2}$$

(2.66)

This set of equations is the basis for performing inflation and quintessence calculations. From now on, the curvature term $k$ will be set to zero.

2.6 The slow-roll approximation

For performing actual calculations, the set (2.64, 2.65) is most often used. In general, it is not possible to find an exact solution for this system. To facilitate calculations during the inflationary era, (or the era for which $w \approx -1$ in quintessence models) the slow-roll approximation is used.

In this approximation, we assume that the field has little kinetic energy, such that the potential energy dominates the expression for the Hubble rate. This constraint, imposed on (2.65), leads to

$$H^2 \approx \frac{V}{3M_p^2}.$$  

(2.67)
Further, if we want the accelerated expansion to last for a considerable time, the acceleration of the field should be small. Imposing this restriction on (2.64) leads to

\[ 3H\dot{\varphi} \approx -V'. \]  

(2.68)

We can state the restrictions in an other way, clarifying the role of the potential in the dynamics. For the first restriction, we have

\[ \frac{\dot{\varphi}^2}{6M_p^2} \ll H^2 \]  

(2.69)

\[ \Rightarrow \frac{1}{3} M_p^2 \left( \frac{-3H\dot{\varphi}}{3M_p^2 H^2} \right)^2 \ll 1 \]  

(2.70)

\[ \frac{1}{3} M_p^2 \left( \frac{V'}{V} \right)^2 \ll 1. \]  

(2.71)

The second restriction can be written as (assuming that \( \dot{\varphi} \) and \( \ddot{\varphi} \) have the same sign)

\[ \ddot{\varphi} \ll 3H\dot{\varphi} \]  

(2.72)

\[ \Rightarrow \frac{1}{3} \left( \frac{H\dot{\varphi}}{H^2\dot{\varphi}} - \frac{\dot{\varphi}^2}{2M_p^2 H^2} \right) \ll 1 \]  

(2.73)

\[ \frac{1}{3} M_p^2 \left( \frac{3(H\dot{\varphi} + H)}{3M_p^2 H^2} \right) \ll 1 \]  

(2.74)

\[ \frac{1}{3} M_p^2 \left( \frac{-V''}{V} \right) \ll 1. \]  

(2.75)

In the second step, we used validity of equation (2.69). Choosing a mutually opposite sign for \( \dot{\varphi} \) and \( \ddot{\varphi} \) leads to an overall minus sign in (2.75).

If we define the slow-roll parameters

\[ \epsilon \equiv \frac{M_p^2}{2} \left( \frac{V'}{V} \right)^2, \quad \eta \equiv M_p^2 \frac{V''}{V}, \]  

(2.76)

we may state the slow-roll conditions by

\[ \epsilon \ll 1, \quad |\eta| \ll 1. \]  

(2.77)

These conditions are often used the other way around. Setting the slow-roll parameters small, should lead to inflation. That this is justified follows if we rewrite \( \epsilon \), with help of (2.67) and (2.68):

\[ \epsilon = \frac{-\dot{H}}{H^2}. \]  

(2.78)

From equation (2.33) we see that inflation arises if \( \epsilon < 1 \). This is only a rough estimate, though. Since we assume the validity of the slow-roll conditions in deriving this result, it may not be exact if \( \epsilon \) is very close to unity. Moreover, even if \( \epsilon > 1 \), accelerated expansion can occur.

The conditions only constrain the form of the potential. Suitable initial conditions (\( \dot{\varphi} \) very small) could lead to inflation. Fortunately, this is but for a short time, since in very steep parts of a potential, the field will start to roll ever faster, eventually leading to kination dominance.
2. Calculational tools

Although it seems that the slow-roll conditions are not very accurate at predicting inflation, they do a pretty good job. Violation of slow-roll during inflation cannot last for a very long time. Since inflation typically lasts very long (as measured in Hubble times), the error is just a minor correction.

The reason why we want to use the slow-roll parameters, is that the expressions we find concerning the spectrum of the background radiation can be expressed in terms of them. This makes slow-roll a very powerful tool in comparing theory with data.

Hamilton-Jacobi form of slow-roll

There is another way to define the slow-roll parameters, such that inflation can be defined exactly in terms of them. To do this we rewrite the field equations, using $\sigma$ as the independent variable. Differentiating (2.65) with respect to $t$, and substituting the result in (2.64) gives us\(^4\)

\[
\dot{\sigma} = -2M_p^2 H'(\sigma).
\] (2.79)

Back substitution of this relation in (2.65) gives the so-called Hamilton-Jacobi form of the Friedmann equation

\[
H'^2 - \frac{3}{2M_p^2} H^2 = -\frac{1}{2M_p^2} V.
\] (2.80)

This relation makes clear that it is possible to rewrite every inflationary concept in terms of a potential. Given the evolution of the scale factor $a(t)$, we can—at least numerically—find the Hubble rate ($\dot{a}/a$) as a function of time. The next step is solving (2.66) for $\dot{\sigma}$, (2.79) for $H(\sigma)$ and then the potential can be explicitly reconstructed from (2.80).

The slow-roll parameters in the Hamilton-Jacobi formulation are

\[
\epsilon_H = 2M_p^2 \left( \frac{H'}{H} \right)^2
\] (2.81)

\[
\eta_H = 2M_p^2 \frac{H''}{H}.
\] (2.82)

In the slow-roll regime, we have

\[
\epsilon_H \rightarrow \epsilon, \quad \eta_H \rightarrow \eta - \epsilon.
\] (2.83)

We can rewrite $\epsilon_H$ as

\[
\epsilon_H = -\frac{d \ln H}{d \ln a} = 1 - \frac{a \ddot{a}}{a^2}.
\] (2.84)

We thus have the exact correspondence

\[
\ddot{a} > 0 \Leftrightarrow \epsilon_H < 1.
\] (2.85)

In some respects, this will make the description of inflation in terms of the Hamilton-Jacobi parameters more accurate and better to handle. Moreover, the quantity $H$ is directly measurable, while the potential $V$ is but a theoretical tool. Since the use of the ‘normal’ slow-roll parameters has become common, we will stick to it for the greatest part of this thesis.

\(^4\)Again, the prime denotes differentiation with respect to $\sigma$.  
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2.7 The attractor nature of inflation

The slow-roll approximation reduces the order of the differential equations. Lowering the order of a set of differential equations (without increasing the number of equation) would normally mean that information is lost. In the case of inflation, however, this is not a problem. The evolution of the field will always tend towards the slow-roll regime. In other words, regardless of initial conditions, the field will slow down until the slow-roll solution is approached. Only in the latest stages of inflation, when the potential is steep enough to provide the necessary acceleration, the slow-roll approximation breaks down.

This phenomenon is easily recognised if we take a closer look at the equation of motion for the field. It looks just like an anharmonic oscillator with a friction term proportional to $H$. If the velocity of the field down the potential is large, the friction will cause it to decelerate, provided the potential is not very steep at that point. The field slows down until the slow-roll attractor is reached.

This can be seen very nicely in an example potential. We use the same potential that we will also use in chapter 3. The inflation potential for this model is given by

$$V = \lambda \sigma^4.$$  \hspace{1cm} (2.86)

The coupling constant $\lambda$ must be of order $10^{-14}$, to give the right spectrum of the background radiation, from equation (4.22). The field typically starts out at several Planck-masses: $\sigma = -nM_P$, $n = \mathcal{O}(10)$. This does not mean that the energy density is of the Planck order. Because of the smallness of $\lambda$, the energy density is small enough that we do not need Planck scale physics. In figure 2.3, a phase-portrait of the dynamics is given. In the left panel, starting values for the derivative of the field have been chosen extraordinarily large, of the order of the Planck scale. This is extraordinary, because these high values do give an energy density of the Planck scale. The high values for $\dot{\sigma}$ give a huge friction term, and the field velocity drops quickly. The detailed picture, to the right, shows beautifully that all trajectories ‘tag on’ to the slow-roll solution, which is a straight line through the origin, with equation (found by solving (2.67, 2.68))

$$\frac{\dot{\sigma}}{M_p^2} = -\frac{4}{3} \sqrt{3\lambda} \frac{\sigma}{M_p^2}. \hspace{1cm} (2.87)$$

Although $\epsilon = 1$ for $\sigma = -2\sqrt{2}$, the slow-roll approximation seems to be valid even for much smaller values of $\sigma$.

The argument given above can be made somewhat more precise. Suppose we have an inflating solution $H_0$, which satisfies the Friedmann equation in the Hamilton-Jacobi form:

$$H_0^2 = \frac{3}{2M_p^2} \frac{V}{2M_p^2}, \hspace{1cm} (2.88)$$

Adding a small perturbation $\delta H$ and linearising gives

$$H_0'\delta H' - \frac{3}{2M_p^2} H_0 \delta H = 0, \hspace{1cm} (2.89)$$

and the solution is

$$\delta H = \delta H_i \exp \left\{ -\frac{3}{2M_p^2} \int_{\sigma_i}^\sigma d\sigma \frac{H_0}{H_0^2} \right\}. \hspace{1cm} (2.90)$$
2. Calculational tools

![Figure 2.3: Phase-portrait for the evolution of the inflaton field. Left: large starting values for the field and its derivative evolve rapidly to slow-roll. The slow-roll solution is indistinguishable from the horizontal axis. Right: close up of the dynamics close to the slow-roll regime. The small scale of the field velocity is set by the smallness of the coupling constant $\lambda$ of the potential. If the field starts rolling to the left, up the potential, it turns around to join the attractor solution.]

Using the slow-roll parameter, we can set a bound on the evolution of $\delta H$.

$$\epsilon_H < 1$$  \hspace{1cm} (2.91)

$$\frac{|H_0|}{H_0} < \sqrt{2}M_p$$  \hspace{1cm} (2.92)

$$\delta H < \delta H_i \exp \left\{- \frac{3\sqrt{2}}{M_p} (\sigma - \sigma_i)\right\}. \hspace{1cm} (2.93)$$

We see that the perturbation is exponentially suppressed, and pure inflationary behaviour is left, thus erasing the dependence on the initial conditions $\sigma_{\text{init}}$ and $\dot{\sigma}_{\text{init}}$. The slow-roll solution is left after the initial conditions have been wiped out.

We have now seen something very important. The attractor nature of the slow-roll solution guarantees that we do not have to worry about initial conditions. That means that inflation is a truly robust concept, with only a very weak dependence on the initial conditions. As we will see later on, the exact potential that is used for inflation also does not matter much. This makes the inflation potential a generic and powerful concept. On the other hand, the absence of severe constraints leaves just theoretical arguments to judge the validity of inflationary models. In the near future, observations will probably somewhat constrain the possibilities.

More on the attractor nature of inflation may be found in [22, 23, 8].

2.8 The amount of inflation

The amount of inflation in most inflation models is huge. To facilitate calculations, the number of e-folds is introduced. This is the logarithm of the growth of the scale factor between some initial time and the end of inflation.

$$N = \ln \frac{a_f}{a_i} = \int_{t_i}^{t_f} H dt \approx \frac{1}{M_p^2} \int_{\sigma_i}^{\sigma_f} \frac{V}{V'} d\sigma. \hspace{1cm} (2.94)$$
2. Calculational tools

The field value at the end of inflation can be determined using the slow-roll conditions. It is quite easy to make an estimate for the number of e-folds in a particular model of inflation. To do so, we consider a comoving scale $k^{-1}$. Scales are, as in this case, often given as an inverse wave number. At some point during inflation this scale leaves the Hubble scale, that is, $k^{-1} > (aH)^{-1}$. The number we are interested in is the number of e-folds of inflation after the scale $k^{-1}$ leaves the Hubble scale until the end of inflation. This is also closely related to the number of e-folds after inflation, until the scale re-enters the Hubble scale, as will be shown in the next section. Introducing a lot of subscripts, we have

$$N(k) = \ln \frac{a_e}{a_k} = \ln \left( \frac{a_e}{a_{reh}} \frac{a_{eq} a_0 H_0 H_k}{a_0 H_k H_0} \right).$$

(2.95)

The subscripts denote:
- $e$: end of inflation
- $k$: scale $k^{-1}$ leaves Hubble scale
- $reh$: reheating
- $eq$: matter-radiation equality
- $0$: present.

If we know the evolution of the scale factor in all epochs, we can calculate this quantity. For the era between reheating and matter-radiation equality, the universe is radiation dominated and from equality until present, the evolution is matter dominated. The era between the end of inflation and reheating is more difficult to assign a scale factor to.

Suppose that the field starts oscillating after the end of inflation, transferring its energy to ordinary matter. If the oscillations are rapid enough that we may ignore the effect of damping during a single oscillation, the potential energy and the kinetic energy are equal on the average. This leads to an average pressure of $p_\sigma = 0$ and the equation of state is thus

$$w_\sigma = 0,$$

(2.96)

leading to an energy dependence of the scale factor of

$$a \propto \rho^{-1/3}.$$

(2.97)

The situation may be different, however. Just after inflation ends, the field may become kination dominated until the radiation density takes over. This leads to an equation of state

$$w_\sigma = 1$$

(2.98)

and the scale factor is

$$a \propto \rho^{-1/6}.$$  

(2.99)

This is the situation that we will encounter in chapter 3.

If we use the evolution of the scale factors as described above and in table 2.1, we can rewrite the expression for $N$ (denoting the scale factor dependence of the field just

---

5In this calculation, we should compare the frequency of the oscillations with the Hubble rate. Indeed, just after inflation ends, the Hubble rate drops to a very low value. The oscillations will thus in practice be very fast compared to the Hubble rate, which governs the damping in the equations.
2. Calculational tools

after inflation by \( l \):

\[
N = \ln \left( \frac{\rho_{\text{reh}}^{1/4}}{\rho_{\text{eq}}^{1/4}} \frac{1}{1 + z_{\text{eq}}} \frac{a_0 H_0}{a_k H_k} \right) + \ln \left( \frac{10^{16} \text{ GeV} \rho_{\text{eq}}^{1/4}}{m_{\text{Pl}} H_0} \right) + \ln \left( \frac{V_e^{1/4}}{V_k^{1/4}} \right) + \ln \left( \frac{10^{16} \text{ GeV} \rho_{\text{eq}}^{1/4}}{m_{\text{Pl}} H_0} \right).
\]

(2.100)

We can simplify the above expression considerably, since \( \rho_{\text{eq}} \) is known in terms of the present radiation density:

\[
\rho_{\text{eq}}^{1/4} = \rho_{\text{rad},0}^{1/4} (1 + z_{\text{eq}}) = (\Omega_{\text{rad},0} 3 H_0^2 m_{\text{Pl}}^2)^{1/4} (1 + z_{\text{eq}}).
\]

(2.101)

Substitution leads to

\[
N(k) \approx 60 + \ln \frac{a_0 H_0}{k} + \ln \frac{V_e^{1/4}}{10^{16} \text{ GeV}} + \ln \frac{V_k^{1/4}}{V_e^{1/4}} + \ln \frac{V_k^{1/4}}{V_e^{1/4}}.
\]

(2.102)

The last three terms are determined by the characteristics of the inflation model. The value of \( l \) is either 6 or 3. In typical GUT-motivated inflation models, the last three terms will amount to but a small correction. Thus, a typical number for the amount of inflation after the present Hubble length left the Hubble scale during inflation is about 60. This figure serves as a starting value from which we can compute \( a/a_0 \) for different values of the wave number \( k \). As cosmological interesting length scales lie approximately between one and a couple of thousands of Megaparsecs, this implies a region

\[
\Delta N = \ln 10^4 = 9
\]

(2.103)

in which all cosmological scales leave the Hubble scale. This figure is relatively small compared to \( N(k = a_0 H_0) = 60 \). Moreover, the effects of slow-roll violation occur but in the latest stages of inflation. We therefore may assume that the slow-roll regime applies when all interesting length scales cross the Hubble scale and that the slow-roll parameters are approximately constant during this crossing.

2.8.1 The number of e-folds after inflation

Now that we can calculate the number of e-folds after a scale leaves the Hubble scale until the end of inflation, we also want to find the amount of expansion until the scale re-enters the Hubble scale. We will show now that these two numbers are in fact nearly equal. This symmetry is caused by the fact that the universe has been radiation dominated for many Hubble times time after inflation ended. To derive this estimate, we consider a universe with a scale factor given by

\[
a = \begin{cases} 
a_0 e^{H_t} & \text{for } t < t_e \cr a_0 e^{H_{t_e}} \left[ \frac{H}{p} (t - t_e) + 1 \right]^p & \text{for } t > t_e \end{cases}
\]

(2.104)

\[\text{In this equation, we should put in a factor 2, since the total density at matter-radiation equality is of course twice the radiation density. For simplicity, we ignore this factor, since it does not contribute a significant amount to the number of e-folds.}\]
Inflation ends at time $t = t_e$ and the constants are chosen as to make the scale factor continuous up to the first derivative. The value of $p$ determines the evolution of the scale factor after inflation. For radiation dominance it equals $p = 1/2$ and for matter dominance it is $p = 2/3$. In any case, $p$ should be smaller than unity to ensure that the scale factor decelerates. From the given scale factor, we find the derivative

$$\dot{a} = \begin{cases} H_a(t) & t < t_e \\ H_0 e^{H(t_e)} \left( \frac{H}{p}(t - t_e) + 1 \right)^{p-1} & t > t_e \end{cases}$$

A scale $k$ is equal to the Hubble scale if

$$k = \frac{a}{H} = \dot{a}.$$ (2.108)

We define $t_L$ to be the time at which the scale leaves the Hubble scale during inflation and $t_R$ to be the time at which the scale re-enters the Hubble scale after inflation. We have thus $t_L < t_e < t_R$.

If we take a particular scale at $t = t_L$ and at $t = t_R$, the derivative of the scale factor should be equal:

$$H_0 e^{H(t_L)} = H_0 e^{H(t_R)} \left( \frac{H}{p}(t_R - t_e) + 1 \right)^{p-1}.$$ (2.109)

From this equation, we calculate the scale factor at Hubble scale exit and re-entry:

$$a_L = a_0 e^{H(t_L)}$$ (2.110)

$$a_R = a_0 e^{H(t_R)} \left( e^{H(t_L - t_R)} \right)^{\frac{1}{p-1}}.$$ (2.111)

The amount of inflation after Hubble scale exit and before re-entry is now

$$\frac{a_e}{a_L} = e^{H(t_e - t_L)}$$ (2.112)

$$\frac{a_R}{a_e} = e^{H(t_e - t_L)} \frac{1}{t_e - t_R}$$ (2.113)

and the number of e-folds is given by

$$\Delta N_L = H(t_e - t_L)$$ (2.114)

$$\Delta N_L = H(t_e - t_L) \frac{p}{1-p}.$$ (2.115)

If the universe is radiation dominated, $p = 1/2$ and the two numbers are exactly equal. In reality, the universe is matter dominated in its later stages, which leads to a slightly smaller value for the number of e-folds after inflation than before inflation.

We have now built a large enough body of knowledge to attack some real-world scenarios. In the next chapter, a specific inflation and quintessence model will be treated to get a grip on the concepts involved and to gain a better understanding of the way the theory that has been brought about in this chapter works in practice. In the chapters after that, with the ideas from chapter 3 in mind, we will turn to a more general treatment of inflation and quintessence.
2. Calculational tools
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

3.1 The $\sigma^{\pm 4}$ model

We will now turn to an explicit model for the inflation and the quintessence potential. The extensive treatment of a sample-model will clarify the most important ideas and calculations. This chapter therefore serves as an instructive introduction to inflation and quintessence models. In later chapters, we will look at general properties of classes of models.

One of the simplest models for inflation is the self-interacting scalar field with a $\sigma^4$ potential. The $\sigma^4$ interaction is the simplest extension to the free field, and is particularly easy to work with. A very important property is that this model is renormalizable.

An often used model for quintessence is the inverse monomial model. In this case, we take the $\sigma^{-4}$ model, which is not renormalizable. The $\sigma^{-4}$ potential will have to be derived as an effective potential for some deeper theory. The explicit model we will treat is a model discussed by Peebles and Vilenkin [33]. The model poses the existence of a potential given by

$$V = \begin{cases} \lambda(\sigma^4 + M^4) & \sigma < 0 \\ \frac{\lambda M^4}{\sigma^4 + M^4} & \sigma > 0. \end{cases}$$ (3.1)

The coupling constant $\lambda$ must be of order $10^{-14}$ to give the right CMBR-spectrum, see (4.22). The mass $M$ of the field will turn out to be of the order of $10^6$ GeV.

The dynamics of the model are as follows. At the beginning of inflation, the field starts out several Planck-masses to the left of the origin. This may seem large, but the energy density will be much lower than the Planck-density, so there will be no Planck-scale physics involved. Since the mass of the field is about twelve orders of magnitude smaller than the Planck-mass, we can safely ignore the $M^4$-term in the potential, except perhaps very close to the origin of the potential. The slow-roll parameters become

$$\epsilon = 8 \left( \frac{\sigma}{M_P} \right)^{-2}$$ (3.2)

$$\eta_{\sigma < 0} = \frac{3}{2} \epsilon$$ (3.3)

$$\eta_{\sigma > 0} = \frac{5}{2} \epsilon.$$ (3.4)

If the field is far to the left of the origin, the slow-roll approximation will surely apply, and the field energy is dominated by the potential, causing exponential inflation. If the field gets closer to the origin, the slow-roll conditions will eventually be violated. As the field acquires more kinetic energy, a period of kination will begin, with an energy dependence on the scale factor of $\rho_\sigma \propto a^{-6}$, leading to $a \propto t^{1/3}$. The transition of
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

A potential dominated area to a kinetic energy dominated era will cause gravitational particle production, to be discussed in appendix B. In this model, no explicit coupling between the inflaton field and the Standard Model is present. The gravitational particle production occurs just because the metric changes during the transition from potential dominance to kinetic energy dominance. The produced radiation density will fall off as $\rho_{\text{rad}} \propto a^{-4}$ and will therefore start to dominate over the field density (scaling with $a^{-6}$). This leads to expansion with $a \propto t^{1/2}$. As the universe cools, it will become matter dominated ($\rho_{\text{mat}} \propto a^{-3}, a \propto t^{2/3}$). The radiation and matter dominance will cause extra friction in the field equation, since $H_{\text{rad}}, H_{\text{mat}} > H_\sigma$. This will cause the field velocity to decrease, eventually leading to slow roll again. That this must happen follows as well from the slow-roll conditions. The field will then again start to dominate the energy density in the universe. A new, endless, period of inflation is then entered. We will now take a closer look at this various stages, calculating analytically what is going on in the different regimes. We will also make a numerical calculation of the entire period from inflation until the present.

It might be helpful to take a look ahead at figures 3.2 and 3.3 on pages 42-43 to get an idea of the numerical results. It is then easier to understand the brief treatment of the various eras in the coming sections.

3.2 Evolution through different eras

3.2.1 Potential term dominates

We start out with the field far to the left of the origin and the slow-roll conditions do apply. This implies that we must solve the set of equations (2.67, 2.68):

$$H^2 = \frac{V}{3M_P^2}, \quad 3H\dot{\sigma} = -V\dot{\sigma}. $$

These equations lead to (neglecting $M$)

$$\frac{\dot{\sigma}}{M_P^2} = -\frac{4}{3}\sqrt{3\lambda} \frac{\sigma}{M_P}, $$

The solution is given by

$$\frac{\sigma}{M_P} = \frac{\sigma_{\text{init}}}{M_P} e^{-\frac{4}{3}\sqrt{3\lambda} M_P t}, $$

starting at $\sigma = \sigma_{\text{init}}$ at time $t = 0$. This leads to a Hubble rate

$$H = \frac{M_P}{3} \sqrt{3\lambda} \left( \frac{\sigma_{\text{init}}}{M_P} \right)^2 e^{-\frac{4}{3}\sqrt{3\lambda} M_P t}. $$

Calculating the acceleration of the scale factor, we have

$$\ddot{a} = a(\dot{H} + H^2) $$

and there will be accelerated expansion if the absolute value of $\dot{H}$ is smaller than the value of $H^2$. This is guaranteed for (using equation (2.66))

$$\frac{\abs{\dot{H}}}{H^2} = 8 \left( \frac{\sigma}{M_P} \right)^{-2} < 1. $$
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

This condition is exactly the slow-roll condition $\epsilon < 1$.

As the field rolls to the right, the slow-roll conditions are eventually violated. The expansion becomes less rapid and the field speeds up, which causes a transition to a kination dominated stage. This happens for

$$\sigma_e \approx -M_P \quad \rho_{\sigma,e} \approx \lambda M_P^4 \quad H_e \approx M_P \sqrt{\frac{\lambda}{3}}$$

(3.12)

The subscript $e$ refers to the end of inflation. Formally, the slow-roll conditions are violated first for $\epsilon = \frac{1}{8} M_P$, but from numerical calculations, accelerated expansion turns out to last until the value stated in (3.12). The transition to a kination dominated expansion causes particle production. This has to do with the fact that it is not possible to choose a reference frame in which the vacuum in the expanding De Sitter space-time is also the vacuum of the kination dominated space-time. Thus, the De Sitter vacuum appears to be filled with particles to an observer in later eras. In appendix B, we see that the density of the formed particles is given by

$$\rho_{\text{part}} = \rho_{\text{rad}} \approx 0.01 g_* H_e^4 \left( \frac{a}{a_e} \right)^{-4}.$$  

(3.13)

In this derivation, all possible degrees of freedom are treated as scalar fields. The number $g_*$ in the above expression is just this number. We will take this number to be approximately 100. Since the density of the produced particles is much less than the density in the field, a period of kination dominance indeed follows, radiation dominance does not happen directly after inflation.

3.2.2 Kinetic term dominates

In the next phase, when the kinetic term is dominant, we may neglect the potential, and we have the set of equations

$$\ddot{\sigma} = -3H\dot{\sigma} \quad , \quad H^2 = \frac{\dot{\sigma}^2}{6M_P^2}.$$  

(3.14)

The Hubble rate is entirely determined by the energy density of the field, since the radiation content of the universe is still negligible. The solutions for the field, the scale factor, the Hubble rate and the densities from these equations are

$$\sigma = \sqrt{6} M_P (N - N_e) + \sigma_e$$  

(3.15)

$$a = a_e (3(t - t_e) H_e + 1)^{1/3}$$  

(3.16)

$$H = H_e e^{3(N_e - N)}$$  

(3.17)

$$\rho_{\sigma} = 3M_P^2 H_e^2 e^{6(N_e - N)}$$  

(3.18)

$$\rho_{\text{rad}} = 0.01 g_* H_e^4 e^{4(N_e - N)}$$  

(3.19)

$$\rho_{\text{rad}} / \rho_{\sigma} = \frac{0.01 g_* H_e^2}{3M_P^2} e^{2(N - N_e)}.$$  

(3.20)

We have introduced the number of e-folds, defined by $N = \ln a$. From now on, we will mainly use this quantity instead of the scale factor.

During this era, we neglect the potential term, and therefore, the results are not dependent on the exact shape of the potential. The potential does, however, play a role
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

in the transition to radiation dominance. Since the production of matter quanta and the expansion of the universe are causing friction in the equations of motion, the field will slow down and eventually a period of potential dominance will again commence. We must have radiation dominance before this happens, since the universe we now live in clearly has been in a radiation dominated state. Comparing the kinetic and potential energy of the field gives

$$ V \approx \frac{\lambda M^8}{\sigma^4} \approx \lambda M^8 (36 M_P^4 (N_e - N)^4)^{-1} \quad (3.21) $$

$$ \sigma^2 / 2 = 3 M_P^2 H^2 \approx \lambda M_P^4 e^{6(N_e - N)}. \quad (3.22) $$

Equating these expressions gives ($N_{inf}$ denotes the number of e-folds at new potential dominance)

$$ M / M_P = \left(36 (N_{inf} - N_e)^4 e^{6(N_e - N_{inf})}\right)^{1/8}. \quad (3.23) $$

Radiation dominance (x denotes the end of kination and the start of radiation), on the other hand, starts when

$$ M / M_P < \left(36 \ln^4 \left(\frac{3}{\sqrt{0.01 g_* H_e^2}}\right) \left(\frac{3}{\sqrt{0.01 g_* H_e^2}}\right)^6\right)^{1/8} \approx 2.6 \times 10^6, \quad (3.25) $$

for $\lambda = 10^{-14}$. It seems not very likely that this bound will be violated.

At the end of kinetic dominance, we have approximately

$$ N_x - N_e \approx \ln \left(\frac{3 M_P^2}{\sqrt{0.01 g_* H_e^2}}\right) \approx \ln \left(\frac{3}{\sqrt{0.01 g_* H_e^2}}\right) \approx 3 \times 10^7 \quad (3.26) $$

$$ \sigma_x = M_P \sqrt{6 \ln \left(\frac{3}{\sqrt{0.01 g_* H_e^2}}\right) + \sigma_e} \approx 41 M_P \quad (3.27) $$

$$ \rho_{\sigma x} = \rho_{rad x} = 3 H_x^2 M_P^4 \left(\frac{3 M_P^2}{\sqrt{0.01 g_* H_e^2}}\right)^{-3} \approx \frac{\lambda^4 (0.01 g_*)^3 M_P^4}{729} \approx 1.4 \times 10^{-59} M_P^4 \quad (3.28) $$

$$ H_x = H_e (a_x/a_e)^{-3} = \frac{H_e^2 (0.01 g_*)^{3/2}}{3^{3/2} M_P^3} \approx 2.1 \times 10^{-30} M_P. \quad (3.29) $$

Next, an era of radiation dominance commences.

**Thermalisation**

In the section above, we assumed that an era of radiation dominance starts after the kination dominated phase. For the standard evolution to happen, however, we need to apply the equation of state $w_{rad} = 1/3$. For this equation to hold, the particles that were created after inflation must be thermalized. This is not necessarily the case. By a rough estimate, we will now show that the thermalisation time scale is much shorter
than the transition time scale from kination to radiation dominance. This means that thermalisation is accomplished quickly enough that we may assume that thermalisation does not play an important role in the dynamics.

For thermalisation to occur, the interaction rate should at least be equal to the Hubble rate:

\[ H = nA_s, \tag{3.30} \]

where \( A_s \) is the scattering cross-section. For scattering through gauge-bosons, we have

\[ A_s \sim \alpha^2 \varepsilon^{-2}. \tag{3.31} \]

\( \alpha \) is the coupling constant and \( \varepsilon \) is the energy per particle. We know:

\[ \frac{n}{\rho_{\text{rad}}} = \frac{100}{g_c \alpha^2} \approx 10 - 100. \tag{3.36} \]

If we assume that the interaction time scale can be used as a measure for the thermalisation time scale, we can compare this estimate with (3.26). This shows that thermalisation will be complete long before the radiation density starts to dominate.

### 3.2.3 Radiation dominates

Let us continue our description of the succession of events at the point where kination has ended and radiation dominance commences. At the beginning of radiation dominance, we may still neglect the potential term for the field, leading to the equations

\[ \ddot{a} = -3H \dot{a}, \tag{3.37} \]

\[ H^2 = \frac{1}{3M_P^2} \rho_{\text{rad}} = \frac{\rho_{\text{rad},x}}{3M_P^2} \left( \frac{a}{a_x} \right)^{-4}. \tag{3.38} \]

Again, the equations can be solved easily, giving

\[ \left( \frac{a}{a_x} \right) = \sqrt[4]{\frac{4 \rho_{\text{rad},x}}{3M_P^2}} (t - t_x) M_P + 1 \tag{3.39} \]

\[ H = \sqrt[4]{\frac{\rho_{\text{rad},x}}{3M_P^2}} e^{2(N - N_x)} \tag{3.40} \]

\[ \sigma = \sigma_x + \sqrt{\frac{3M_P^2}{\rho_{\text{rad},x}}} \dot{\sigma}(t = t_x)(1 - e^{N_x - N}). \tag{3.41} \]
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

The value of $\dot{\sigma}_x \equiv \dot{\sigma}(t = t_x)$ can be calculated with help of the field solution (3.15). It is found to be

$$\dot{\sigma}_x = \frac{\sqrt{2}}{81} (0.01 g_*)^{3/2} \lambda^2 M_P^2 \approx 10^{-30} M_P^2. \quad (3.42)$$

The field effectively stops running very soon after radiation starts to dominate the energy density. The field ‘freezes in’ at the value $\sigma_f \approx \sigma_x$ and it starts behaving as a cosmological constant: the energy density remains the same and at some time in the future, it will overtake the energy density of the radiation and matter content of the universe.

3.2.4 Kinetic dominance to radiation dominance

The transition from kinetic dominance to radiation dominance, which we assumed to be sudden in the last paragraph, can be calculated exactly if the potential energy of the field is ignored [20]. The kinetic energy density of the field is given by (from table 2.1)

$$\rho_\sigma = 3M_P^2 e^{6(N_\ast - N)} \quad , \quad \dot{\sigma} = \sqrt{6} H_\sigma M_P e^{3(N_\ast - N)}. \quad (3.43)$$

The energy density of the radiation is given by (3.13);

$$\rho_{\text{rad}} \approx H_\sigma^4 e^{4(N_\ast - N)} \quad (3.44)$$

and the Friedmann equation becomes

$$H^2 = \dot{N}^2 = H_\sigma^2 e^{6(N_\ast - N)} + \frac{H_\sigma^4}{3M_P^2} e^{4(N_\ast - N)}. \quad (3.45)$$

It is possible to solve this equation implicitly, giving $t$ as function of $N$, but we will not need to do this. We rather rewrite the equation to obtain

$$\frac{\partial t}{\partial N} = \left( H_\sigma \sqrt{1 + \frac{H_\sigma^2}{3M_P^2} e^{2(N_\ast - N)}} \right)^{-1} e^{3(N_\ast - N) - 1}. \quad (3.46)$$

It is now possible to obtain the field value as a function of $N$.

$$\sigma = \sigma_\ast + \int_{t_\ast}^{t} dt \dot{\sigma} \quad (3.47)$$

$$= \sigma_\ast + \int_{N_\ast}^{N} dN \dot{\sigma} \frac{\partial t}{\partial N} \quad (3.48)$$

$$= \sigma_\ast + \sqrt{6} M_P \int_{N_\ast}^{N} dN \left( 1 + \frac{H_\sigma^2}{3M_P^2} e^{2(N_\ast - N)} \right)^{-1/2} \quad (3.49)$$

$$= \sigma_\ast + \sqrt{6} M_P \left\{ \text{atanh} \left[ \left( 1 + \frac{H_\sigma^2}{3M_P^2} \right)^{-1/2} \right] \right\}.$$ 

$$-\text{atanh} \left[ \left( 1 + \frac{H_\sigma^2}{3M_P^2} e^{2(N_\ast - N)} \right)^{-1/2} \right]. \quad (3.50)$$

From this expression, the energy density can in turn be derived. The results, however, are not very enlightening, because of the presence of the arctangens hyperbolicus in...
the equations. The results have been used, though, in the compilation of the figures in section 3.4. It is instructive to derive the kination and radiation dominated limits of expression (3.50). For small scale factors, \( \exp \left( 2(N - N_e) \right) \ll 3M_P^2/H_e^2 \), the field behaves as

\[
\sigma = \sigma_e + \sqrt{6} M_P \frac{1}{\sqrt{1 + \lambda/9}} (N - N_e) + \mathcal{O} \left( (N - N_e)^2 \right) \quad (3.51)
\]

\[
\approx \sigma_e + \sqrt{6} M_P (N - N_e). \quad (3.52)
\]

This corresponds exactly to kination dominated expansion.

For large scale factors, \( \exp \left( 2(N - N_e) \right) \gg 3M_P^2/H_e^2 \), the field is given by

\[
\sigma = \sigma_e + \sqrt{6} M_P \tanh \left[ \left( 1 + \frac{H_e^2}{3M_P^2} \right)^{-1/2} \right] \approx 43M_P. \quad (3.53)
\]

The freezing value is thus equal to \( \sigma_f \approx 43M_P \), just a little bit more than the value at the beginning of radiation dominance. We will now continue our \(-\text{crude}\) description of the succession of events.

### 3.2.5 Matter dominates

After the radiation dominated era, a matter dominated era starts. According to [78], the redshift at matter-radiation equality was \( 3.45 \times 10^3 \). This means, together with a present radiation density of \( \rho_{\text{rad,0}} = 5.6 \times 10^{-125} M_P^4 \), a radiation density at matter-radiation equality of

\[
\rho_{\text{rad,eq}} = 7.9 \times 10^{-111} M_P^4. \quad (3.54)
\]

With equation (3.13), we find that

\[
N_{\text{eq}} - N_e = \ln \left( \rho_{\text{rad,eq}}^{-1/4} H_e^{-1} \right) \approx 46.7. \quad (3.55)
\]

Together with the extra inflation after matter-radiation equality we have a total number of e-folds of approximately 55, for the present Hubble scale. From \( N = N_{\text{eq}} \) onwards, we have matter dominance and thus

\[
\rho_{\text{part}} = \rho_{\text{mat}} = \rho_{\text{rad,eq}} e^{3(N_{\text{eq}} - N)}. \quad (3.57)
\]

The field still remains in its freezing value and not much interesting regarding the evolution of the field happens in this era.

### 3.2.6 Potential term dominates again

The last era that we will describe is the new era of potential dominance. We will have potential dominance again if \( \rho_{\sigma} \sim \rho_{\text{mat}} \). Since the field has come to a standstill at \( \sigma \approx \sigma_s \), we have

\[
\rho_{\sigma} = \lambda M^8 \sigma_s^{-4} = \rho_{\text{mat}} = \rho_{\text{rad,eq}} e^{3(N_{\text{eq}} - N)}. \quad (3.58)
\]

At present, the field density and the matter density are roughly equal, we can thus solve the equation above for the mass:

\[
M \approx \left( \frac{\rho_{\text{mat,0}}}{\lambda \sigma_s^{-4}} \right)^{1/8} = \sqrt{\sigma_s} \left( \frac{\rho_{\text{mat,0}}}{\lambda} \right)^{1/8} \approx 10^6 \text{ GeV}. \quad (3.59)
\]
This confirms our claim about the inflaton mass made in the first section of this chapter.

If the potential term of the field dominates the energy density for the second time, it will continue doing so forever. This is because the slow-roll conditions hold better if the field continues rolling down the potential. At the beginning of the evolution, the slow-roll conditions were on the contrary violated because of the movement through the potential.

Equation (3.50) suggests that the field indeed has an absolute maximum value to which it freezes in, but we neglected the potential energy of the field. Will the picture be the same if we do take into account the potential? We will now make an explicit calculation of the evolution of the field after the potential term dominates again. If the field dominates the energy density, we have the set

\[ H^2 = \frac{V}{3M_P^2} \]  
\[ 3H\dot{\sigma} = -V' \]  

(3.60)  
(3.61)

describing the field evolution, but this time the potential is given by \( V = \lambda M^8 \sigma^{-4} \). For \( \sigma \) we have the exact solution

\[ \left( \frac{\sigma}{\sigma_f} \right)^4 = 1 + \frac{16}{3}\sqrt{3}\lambda \left( \frac{M}{\sigma_f} \right)^4 M_P(t-t_f). \]  

(3.62)

We assume that the slow-roll constants are small enough, so that we can use the equations above, from some time \( t_f \) onwards. The field value is then \( \sigma_f \). From this point, the slow-roll approximation only becomes better as the field rolls away from the origin. In our case, \( \sigma_f \) can be taken to be the freezing value from (3.53). Obviously, the field has not stopped running, but the running is very slow. Substituting the values from \( M \) and \( \lambda \), we find that it takes the field \( 2 \times 10^{37} \) times the present age of the universe to change by one percent! Of course, this running may be neglected.

### 3.3 Mass-dependence of the model

The inflation and quintessence model that we used in the analysis in this chapter is of great conceptual beauty, since it incorporates the effects of inflation and quintessence into one single potential. A crucial element is the use of gravitational particle production. If the field would interact with other fields to transfer its energy, as in most inflation scenarios, the energy density would be lost and there would be no possibility for the field to act as quintessence in the present era.

This conceptual beauty, however, may be misleading, for there is really no hard connection between the inflation potential and the quintessence potential. First of all, the shape of the potential does not matter between the end of inflation and the present era. This means that the two parts of the potential are in practice disconnected. The whole era before radiation dominance could be summarised by choosing just two initial conditions at the start of radiation dominance: the potential density and the radiation density.

Secondly, there are two parameters to be fixed in the potential. The scale of the inflation part of the potential, set by \( \lambda \), and the mass term. The mass term comes into play in determining the exact moment of new potential dominance. The two parameters are to a great extent independent of each other. As an example, we consider a
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

generalisation of the model, given by

$$V = \begin{cases} 
\lambda \left( \phi^{\alpha} M^{4-\alpha} + M^4 \right) & \phi < 0 \\
\frac{\lambda M^{4+\alpha}}{\phi^{\alpha} + M^4} & \phi > 0
\end{cases}. \quad (3.63)$$

If we introduce the parameters

$$M_1 \equiv \lambda^{1/(4-\alpha)} M \quad (3.64)$$
$$M_2 \equiv \lambda^{1/(4+\alpha)} M \quad (3.65)$$

we may write the potential as

$$V = \begin{cases} 
\sigma^{\alpha} M_1^{4-\alpha} & \sigma \ll 0 \\
\sigma^{-\alpha} M_2^{4+\alpha} & \sigma \gg 0
\end{cases}. \quad (3.66)$$

if we remember that the field values are typically much larger than the mass term.

The mass terms $M_1$ and $M_2$ are not completely independent. To see this, we consider the generalisation of equation (3.59).

$$M_2 = (\sigma_x)^{\alpha/(4+\alpha)} \rho_{\text{mat},0}^{1/(4+\alpha)} \quad (3.67)$$

Since the value of $\sigma_x$ depends on $\lambda$, so too does $M_2$. This dependence is eventually caused by the value of the Hubble rate directly after inflation, see equation (3.12). There is, however, no reason why we could not consider this value just as an initial condition for the quintessence field. Moreover, the dependence on $\lambda$ is very weak, as we see from equation (3.27).

Incorporating the $\lambda$-dependence explicitly, we have

$$M_2 \approx \frac{\sigma_x^{\alpha/(4+\alpha)} \rho_{\text{mat},0}^{1/(4+\alpha)}}{\sqrt{6}} \left[ \ln \left( \frac{3}{\sqrt{0.01 \lambda s}} \right) \right]^{\alpha/(4+\alpha)}/M_P. \quad (3.68)$$

A graph of $M_2$ is given in figure 3.1. The weak dependence on the coupling constant $\lambda$ can be clearly seen: the upper line corresponds to $\lambda = 10^{-28}$, the middle line corresponds to $\lambda = 10^{-14}$ and the lower line is calculated by setting $\lambda = 1$. A difference of 28 orders! The mass is limited to a couple of times the Planck mass. For small $\alpha$ the mass may be extremely small, in the order of meV.

A crucial ingredient for this analysis is that we may ignore the mass term in the denominator of the RHS-potential. For large $\alpha$ and thus large $M$ this may not be the case. The relation (3.68) gives confidence that we may however neglect the mass term in the denominator, since the value of $\sigma_x$ will be about the same as that of $M_2$. The correction will thus amount to not more than a factor of order unity.

From this example, it is clear that there is an intrinsic difficulty in constructing models that address both inflation and quintessence. There are two observational parameters that constrain all models. Firstly, the potential should be appropriately normalised to fit the CMBR-data. Secondly, the onset of the potential energy dominance at present should be timed with a reasonable accuracy. This means that there will in general be at least two free parameters in the model that have to be fitted to the data.

Since the exact shape of the potential does not matter during kination and radiation/matter dominance, the inflation regime and the quintessence regime will decouple. We can therefore ask the question if it is of any use whatsoever to construct potentials addressing both inflation and quintessence. If someone were to find a potential that would fit both observational constraints using only one free parameter, then we would have a serious candidate for a unified model. Such a model, however, has not been found yet.
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

Figure 3.1: The dependence of the field mass $M_2$ on $\alpha$ is determined by the condition that at present time the energy density of the field and non-relativistic mass are equal. Note that the field mass has a clear maximum, which is a couple of times the Planck mass. The dependence on $\lambda$ is very weak. The three lines all lie close to each other, even though the outer lines are calculated using totally unrealistic values of $\lambda$. From top to bottom, the value for $\lambda$ is given by $10^{-28}$, $10^{-14}$ and 1.
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

3.4 Computational approach to the model

In the second section of this chapter, we tried to describe analytically what is going on during the evolution of the field and the radiation/matter density. In doing so, we paid little attention to the transitions between the various eras. Except for the transition between kination and radiation dominance, the transitions cannot be calculated exactly. To obtain a more detailed and complete overview of the evolution, it is necessary to revert to computational methods.

The equation of motion for the field, combined with the Friedmann equation and the equation for the evolution of the Hubble rate fortunately constitutes a fairly simple set of equations, which can be solved with a simple computing routine.

The results of the computation are displayed in figures 3.2 and 3.3. Before we start to interpret the diagrams, it is wise to take a look at the time scale of the figures. The time scale is set by the logarithm of the scale factor. Because the expansion was much faster in the early universe than it is now, this scale is misleading if one wants to know the time dependence of the quantities. However, since the Hubble scale sets a natural length and time-scale to all cosmological processes, the scale factor is perhaps the purest way to represent the time ordering. In any case, the use of coordinate time in the diagrams would render most features unrecognisable. To get an idea of the coordinate time that elapses in the diagrams, the following comparison may be useful.

$$\log_{10}(1 + z)$$

<table>
<thead>
<tr>
<th>$\log_{10}(1 + z)$</th>
<th>Coordinate time</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>$2.5 \times 10^{-38}$ s</td>
</tr>
<tr>
<td>20</td>
<td>$1.5 \times 10^{-20}$ s</td>
</tr>
<tr>
<td>10</td>
<td>$5.4 \times 10^{-4}$ s</td>
</tr>
<tr>
<td>0</td>
<td>$1.5 \times 10^{17}$ s = 4.8 Gyr</td>
</tr>
</tbody>
</table>

The discrepancy between the coordinate time for $z = 0$ and the actual age of the universe is probably caused by the fact that the field mass and the amount of particle production have been chosen fairly roughly in the computational routine. This causes an inaccuracy in the freezing value of the field. Since $t = t_0$ has been defined by setting $\Omega_\sigma = 0.7$, this error is also transferred to the value of the time at which $z = 0$ in the model. Fine-tuning should help to solve this timing problem, but this is of no theoretical importance.

The upper panel of figure 3.2 shows the energy density of the field and the matter density. The matter density just after the end of inflation suddenly jumps from zero to its starting value. This is not what happens in reality of course, but the methods used in calculating $\rho_{rad,c}$ in appendix B can only be applied to a full transition from De Sitter space to a non-vacuum dominated space-time. It is thus not possible to calculate the matter density just before the end of inflation. In any case the transition between the inflationary era and the kination era takes place very rapidly, and a numerical calculation of the matter density would yield an almost vertical line.

At redshift $10^{17}$, the kination density is overtaken by the radiation density, which dominates nearly until present. At a redshift of approximately $10^{8}$, the kination density drops below the potential density of the field, resulting in a constant density which will last forever. The lower panel shows the value of the field. The kination era is clearly recognisable. The rolling of the field stops abruptly as the radiation density starts to dominate.

The upper diagram of figure 3.3 shows the density parameter of the matter content, including radiation. The condition $\Omega_m = 0.3$ is used to determine the point of zero redshift. The lower panel, finally, shows the equation of state of the inflaton field. The variation between $w = -1$ during potential dominated eras and $w = 1$ for the kination
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

Figure 3.2: Dynamics of the $\sigma^{\pm 4}$ field.
Top: energy density of the field (dotted) and energy density of the matter.
Bottom: value of the field.
The dots indicate three important events. From left to right, they depict the end of inflation, end of kination, and matter-radiation equality.
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

Figure 3.3: Dynamics of the $\sigma^{\pm 4}$ field.
Top: relative contribution of matter to the total energy density.
Bottom: equation of state for the field. Note that what looks like a sharp edge at $z \approx 10^{25}$ is actually a smooth line!
The dots indicate three important events. From left to right, they depict the end of inflation, end of kination, and matter-radiation equality.
3. A worked example: $\sigma^4$ inflation and $\sigma^{-4}$ quintessence

dominated era is obvious.
4. Inflation and the background radiation

Now that we have seen how inflation works in a specific model we can turn to some general characteristics of inflation. In particular, we will confront inflation models with the data gathered by the WMAP satellite. Although our main focus lies with the WMAP data, many observations on the CMBR have been made in recent years, the most famous project being the COBE satellite [73] which was launched in 1989.

In this chapter we will freely make use of the results on perturbation theory presented in appendix C. This appendix only serves as an aid to memory. Those interested in perturbation theory should consult the references in the appendix.

4.1 The WMAP mission

The WMAP (Wilkinson Microwave Anisotropy Probe) satellite [75] is a NASA-project to measure the temperature and the polarisation of the Cosmic Microwave Background Radiation (CMBR) with a never-before achieved precision. The perturbations in the CMBR, caused by the fluctuations of the inflaton field can be determined with high accuracy from the data. The temperature spectrum is shown in figure 4.1. On the horizontal axis, the two point correlation function of the temperature fluctuations is displayed. The normalisation follows from equation (C.15). The horizontal axis shows the multipole, a measure of the angular distance. A small angular size corresponds to large \( l \) and vice versa. Three features are obvious from the graph. On the left, for \( l \) smaller than about one hundred, a flat plateau exists (the normalisation ensures that this part of the graph is indeed a horizontal line), to the right of this plateau, a series of peaks can be seen. For high multipoles (small angles), the spectrum tends to zero. We will now explain why these features arise.

The flat plateau is there, because for \( l < 100 \), the scales are so big that they had not yet entered the Hubble scale at decoupling of matter and radiation. Perturbations on these scales have been ‘frozen’ during the time they were outside the Hubble scale. We can thus observe the spectrum as it looked when these scales left the horizon. This can be seen as follows. If \( x_{ls} \) is the comoving distance to the surface of last scattering (the CMBR-surface), the comoving size of structures corresponding to an angular size \( \alpha \) is \( r = \alpha x_{ls} \), for small angles in a flat universe. The corresponding wave number is \( k = 2\pi/r = 2\pi/(\alpha x_{ls}) \). Furthermore, the \( l \)th Legendre polynomial in \( \cos \alpha \) from expression (C.14) has a maximum for \( \alpha \approx \pi/l \) and thus dominates the expansion around that value for \( \alpha \). Putting this together gives as a rough estimate

\[
k \approx \frac{2l}{x_{ls}} \quad (4.1)
\]

In section 1.4 we saw that the present Hubble length corresponds to an angular size of
4. Inflation and the background radiation

\[ \frac{(l+1)C_{l}}{\mu K^{2}} \]

![Figure 4.1: The temperature perturbation spectrum. The flat plateau for small \( l \), or large angles, is used to test inflation predictions. A perfectly flat spectrum \( P \propto k^{0} \) corresponds to a horizontal line. Figure courtesy of the NASA/WMAP Science Team, http://map.gsfc.nasa.gov](image)

\( \alpha = 1.7^\circ \) at the last scattering surface. We find for \( l \)

\[ \alpha = 1.7^\circ \rightarrow l = \frac{\pi}{\alpha} \approx 100. \]  \hspace{1cm} (4.2)

For \( l > 100 \), the scales were already inside the Hubble scale at decoupling. Being within the Hubble scale the perturbations started evolving. In gravitational wells (higher concentration of matter), the gravitational attraction caused an extra accumulation of this matter. This caused a pressure which in its turn reverses the accumulation. Thus acoustic oscillations are started. It depends on the time of Hubble scale entry how many oscillations can be made in the plasma before decoupling takes place. As soon as decoupling takes place, the oscillations stop, since matter does cluster while photons resist clustering. The peaks in the spectrum correspond to scales that could accomplish just a half integer numbers of oscillations, for at the extremum of an oscillation, the density and thus the temperature of the plasma is maximally perturbed. Because the correlation function squares the perturbations, a higher temperature and a lower temperature both give rise to peaks in the spectrum.

At large \( l \) (small length scales) the spectrum goes to zero. This is caused by Silk damping. The process of decoupling did not take place instantaneously. This means that the photons had a finite mean free path and were still frequently scattered during the process of decoupling. At large length scales compared to the mean free path, this effect has no impact, but on smaller scales the scattering smoothed out the density perturbations by energy redistribution.

To probe the inflationary dynamics, we should look at the characteristics of the plateau, that is, small values of \( l \). In this region, the inflation era can be probed, since at these scales, the spectrum has not evolved after Hubble crossing. Since the Hubble rate is close to constant during inflation the spectrum should look very simple: the conditions during Hubble exit of different scales were very much the same. Therefore, the chance of observing a perturbation does not depend on the size of that perturbation.
4. Inflation and the background radiation

This implies that the spectrum is almost flat, that is, not strongly dependent on \( k \). The spectrum of a general perturbation is defined in (C.3):

\[ P_g(k) = \frac{k^3}{2\pi^2} \langle |g_k|^2 \rangle, \tag{4.3} \]

with \( \langle |g_k|^2 \rangle \) the expectation value of the perturbation. A nearly flat spectrum can be approximated by

\[ P \propto k^{n-1}, \tag{4.4} \]

with \( n \approx 1 \). In appendix C, we show that the perturbation spectrum of the CMBR is indeed very close to being flat. The derivation given there is both accurate and very general, but it is possible to derive the flatness of the spectrum directly from the inflation dynamics, assuming slow-roll, which we will do in the next section.

4.2 The spectrum and slow roll

4.2.1 The scalar spectrum

From equation (C.82), we have

\[ P_R = \left( \frac{H^2}{2\pi^2} \right)^2. \tag{4.5} \]

This quantity is the power spectrum of the curvature perturbation, which is directly related to the spectrum of the field perturbation. If we are in the slow-roll regime, we can rewrite the curvature perturbation using

\[ 3H^2 = \frac{V}{\dot{V}} \tag{4.6} \]
\[ 3H^2 M_p^2 = V \tag{4.7} \]

and we find

\[ P_R(k) = \frac{1}{12\pi^2} \frac{V^3}{M_p^{6} V'^2} = \frac{1}{24\pi^2} \frac{V}{M_p^2}. \tag{4.8} \]

We define the spectral index by

\[ n(k) - 1 = \frac{d \ln P_R}{d(\ln k)}. \tag{4.9} \]

During slow-roll, we have approximately \( a = a_i e^{Ht} \), with \( H \) constant, and at Hubble scale exit

\[ d(\ln k) = \frac{da}{a} + \frac{dH}{H} \tag{4.10} \]
\[ = Hdt + \frac{dH}{H}. \tag{4.11} \]
\[ 2HdH = \frac{dV}{3M_p^2} \Rightarrow \frac{dH}{H} = \frac{V'}{2V} d\sigma \tag{4.12} \]

Further,

\[ \dot{\sigma} = \frac{-V'}{3H} \Rightarrow Hdt = \frac{-3H^2}{V'} d\sigma = - \frac{V}{M_p^2 V'} d\sigma \tag{4.13} \]
\[ 2HdH \Rightarrow \frac{dH}{H} = \frac{V'}{2V} d\sigma \tag{4.14} \]
4. Inflation and the background radiation

and therefore

$$\frac{d}{d\ln k} = \left( \frac{1}{\epsilon - 1} \right) M_p^2 V' \frac{d}{d\sigma}. \quad (4.15)$$

For the spectral index we find (using (4.8)) to first order in slow-roll

$$n(k) - 1 = \frac{d}{d\ln k} \left( \ln P_R \right) = 2 \left( M_p^2 \frac{V''}{V} \right) - 6 \left( \frac{M_p^2 V'^2}{2 V^2} \right) (1 - \epsilon)^{-1} = 2\eta - 6\epsilon, \quad (4.16)$$

using the standard definitions of the slow-roll parameters. In appendix C, the Hamiltonian definitions were used, but the results are the same, compare equation (C.154).

We can now also calculate the running of the spectral index, by calculating the next derivative with respect to $\ln k$. To second order in the slow-roll parameters, the running is given by

$$\frac{d n}{d\ln k} = 16 \left[ \frac{M_p^4 V'^2 V''}{V^3} \right] - 24 \left[ \frac{M_p^4}{4} \left( \frac{V'}{V} \right)^4 \right] - 2 \left[ M_p^4 \frac{V' V'''}{V^2} \right], \quad (4.17)$$

$$= 16\epsilon - 24\epsilon^2 - 2\xi^2 \quad (4.18)$$

where

$$\xi^2 = M_p^4 \frac{V' V'''}{V^2} \quad (4.19)$$

is the next order slow-roll parameter. The superscript in $\xi^2$ means that the parameter is second order. $\xi^2$ is not a square and it might not even be positive. The slow-roll hierarchy can be continued, introducing ever higher order slow-roll parameters [23]. Because the spectral index depends linearly on the slow-roll parameters, it must necessarily be close to unity: slow-roll inflation automatically leads to a nearly scale-invariant spectrum. This may be understood as follows: the multipole range that lends itself for detection is about $O(1) \cdots O(1000)$, this implies a range of $\ln k$ of about 10. Since inflation typically leads to a huge increase of the scale factor in a very short time, conditions will not have changed much in this small range of $k$-values. The spectrum will be nearly scale-invariant therefore.

Using the formulation of the spectrum in terms of the inflaton potential, we can determine the normalisation of the potential. From the CMBR data we have an expression for the amplitude of the spectrum at the scale $k = 7.5a_0 H_0$, derived in [74]:

$$\left| \delta_H(k = \bar{k}) \right| = 1.91 \times 10^{-5} \quad (4.20)$$

$$\bar{k} = 7.5a_0 H_0, \quad (4.21)$$

and thus

$$\left| \frac{V^{3/2}}{M_p^3 V} \right|_{k = \bar{k}} = 5.2 \times 10^{-4}. \quad (4.22)$$

With this expression, we determine the normalisation of the potential. This leads to the choice of $\lambda = 10^{-14}$ in the $\sigma^4$-model for inflation. This constraint was obtained from an analysis of the COBE-data. Of course, the WMAP-data allow for a more sophisticated calculation of this bound, but most of the figures in this thesis already had been compiled using the bound above. Since the improvement of the figures would be marginal, the WMAP-data have not been used in recompiling the figures.

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4.2.2 The scalar and the tensor spectrum

Because the scalar spectrum and the tensor spectrum are both functions of the slow-roll parameters, they can be related. The spectral index for the two spectra is almost the same, but the amplitudes differ. From equations (C.150) and (C.151), we infer that the amplitudes are related by

\[
\frac{A_{h_{ij}}}{A_{R}} = 4\epsilon. \tag{4.23}
\]

If the observational intensities of the spectra are compared, we find for the expansion coefficients at low values for \( l \)

\[
r \equiv \frac{C_{l,h_{ij}}}{C_{l,R}} \approx 10\epsilon. \tag{4.24}
\]

This figure differs from the amplitude relation just above it because of the influence of the journey through space that the radiation and gravitational waves must make in order to reach us, see [11]. The ratio \( r \) can be determined experimentally (at least, an upper bound of it). We will use this bound in section 4.4.

4.3 Polarisation

The WMAP satellite has not only measured the temperature fluctuations of the CMBR, but also its polarisation. This is of great importance, since the theory of inflation makes a prediction about the correlation function of the polarisation from the CMBR which cannot be explained using any other known theory. The WMAP data favours the inflationary scenario. For some time already, inflation has widely been believed to be responsible for the solution to the HBB problems posed in the beginning of this thesis, but there has always been the possibility that some other theory might be responsible for the effects ascribed to inflation. This possibility has become more unlikely.

Let us take a look at the prediction that inflation makes regarding the polarisation. It goes beyond the scope of this work to give a detailed derivation of the results presented in this section, but since polarisation is an important test for inflation, it is a good idea to spend some lines on it. More details can be found in [91, 92, 93, 94, 90, 95].

The polarisation of the CMBR is caused by the fact that the surface of last scattering has a non-zero thickness; it took some time for the universe to become completely transparent to photons. During the process of decoupling, the photons were still frequently scattered, which caused the imprint of the energy density fluctuations on the photons we see in the CMBR today. However, the energy density of the fluid during the decoupling process was not constant. The velocity of the fluid caused the polarisation of the photons.

First, we will make a theoretical estimate of the amount of polarisation and afterwards we will compare the result with the data from WMAP. The main prediction, a negative correlation peak between the density fluctuations and the polarisation at a scale of \( l \approx 150 \), is indeed checked by the data.

The polarisation is defined by

\[
\delta E = \frac{\langle E_x^2 - E_y^2 \rangle}{\langle E_x^2 + E_y^2 \rangle}, \tag{4.25}
\]

for light coming from the \( z \)-direction. \( E_{x/y}^2 \) denotes the polarisation of the light in the \( x/y \)-direction.
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The temperature perturbation and the polarisation perturbation are given by (no derivation given here, see the references)

\[
\delta T = -\frac{1}{3} \Psi(\eta_{\text{dec}}) \cos(kc_s \eta_{\text{dec}}) \tag{4.26}
\]

\[
\delta E = +0.17(1 - \mu^2) \Delta \eta_{\text{dec}} kc_s \Psi(\eta_{\text{dec}}) \sin(kc_s \eta_{\text{dec}}). \tag{4.27}
\]

The quantity \( \Psi \) is the gauge invariant potential (C.59), which controls the behaviour of the photons on their way toward us. We see that the polarisation perturbation is related to the derivative of the temperature perturbation, since the polarisation is caused by the velocity of the matter density and the temperature perturbation by the matter density itself.

With these quantities, we can calculate the correlation function for the temperature and the polarisation fluctuations:

\[
\langle \delta T \delta E \rangle = -0.03(1 - \mu^2)(kc_s \Delta \eta_{\text{dec}}) \langle \Psi^2(\eta_{\text{dec}}) \rangle \sin(2kc_s \eta_{\text{dec}}). \tag{4.28}
\]

There should be a negative peak in this function for

\[
2kc_s \eta_{\text{dec}} = \frac{3\pi}{2}, \tag{4.29}
\]

Where \( \eta \) is defined to be zero for matter radiation equality. Using expression (4.1) for the value of \( k \) at decoupling, we obtain

\[ l \approx 150. \tag{4.30} \]

Of course, a peak for \( 2kc_s \eta_{\text{dec}} = \frac{\pi}{2} \) should also be expected. This peak is not very pronounced, because in the spectrum of the correlation function an extra factor \( k^3 \) is present (compare 4.3), which flattens the peak. A graph of the cross correlation between polarisation and temperature fluctuations is given in figure 4.2. From the figure, we see that the peak at \( l = 150 \), a result from the interference between the cos and sin terms in the density fluctuations and its gradient, is obviously present. Since there is no alternative theory known that is able to explain this anti-correlation, this figure serves as a strong advocate in favour of inflation.

4.4 A comparison of models

The observations that were discussed in the last sections were quite general for inflation models. We will now take a look at some specific models to learn what distinguishes different types of models [19]. A general classification will be made, based on the slow-roll parameters.

4.4.1 The models

We will consider several potentials to get an impression of the possible observational differences between different inflaton potentials. First we will define the models and make some analytical calculation where possible. In figure 4.4, computational results are shown.
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Figure 4.2: The correlation spectrum for the polarisation and the temperature fluctuations. The negative peak at \( l = 150 \) is compelling evidence in favour of inflation. Figure courtesy of the NASA/WMAP Science Team, http://map.gsfc.nasa.gov

**Monomial**

One of the simplest choices for the inflaton potential is the monomial:

\[
V = V_0 \left( \frac{\sigma}{M_P} \right)^\alpha .
\]

(4.31)

in chapter 3, this potential was already used. The slow-roll parameters for this model are given by

\[
\epsilon = \frac{\alpha^2}{2} \left( \frac{\sigma}{M_P} \right)^{-2} ,
\]

(4.32)

\[
\eta = \alpha (\alpha - 1) \left( \frac{\sigma}{M_P} \right)^{-2} ,
\]

(4.33)

\[
\zeta^2 = \alpha^2 (\alpha - 1) (\alpha - 2) \left( \frac{\sigma}{M_P} \right)^{-4} .
\]

(4.34)

For this model, we can make an approximate calculation of the amount of inflation, related to the value of the field. This makes it possible to express the spectrum in terms of the number of e-folds \( N \). To do so, we start with the expression for the amount of inflation (2.94):

\[
N = \frac{1}{M_P} \int_{\sigma_e}^\sigma \frac{V}{V'} d\sigma .
\]

(4.35)

The end of inflation can be estimated by setting \( \epsilon = 1 \), which leads to

\[
\sigma_e = \frac{\alpha M_P}{\sqrt{2}} .
\]

(4.36)

The integral is then given by

\[
N = \frac{\sigma^2}{2\alpha M_P} - \frac{\alpha}{4} .
\]

(4.37)
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and the relation between the number of e-folds and the field value is

$$\sigma \approx M_P \sqrt{2\alpha N}. \quad (4.38)$$

The spectrum can now be determined with help of relation (4.8):

$$P_R(k) = \frac{1}{24\pi^2} \frac{V}{(\epsilon M_P^2)}$$

$$= \frac{1}{12\pi^2\alpha^2 \frac{M_P}{M_P^2}} \left( \frac{\sigma}{M_P} \right)^{2+\alpha} \quad (4.39)$$

$$= \frac{1}{12\pi^2\alpha^2 \frac{M_P}{M_P^2}} (2\alpha N)^{1+\alpha/2}. \quad (4.40)$$

The spectral index is (with (4.16))

$$n(k) - 1 = 2\eta - 6\epsilon = -\frac{2+\alpha}{2N} \quad (4.42)$$

and the running of the spectral index is given by (4.17):

$$\frac{d n}{d \ln k} = 16\epsilon \eta - 24\epsilon^2 - 2\xi^2 = -\frac{2+\alpha}{2N^2}. \quad (4.43)$$

If we choose a typical value $N = 60$, we see that the spectral index will be close to unity if $\alpha$ is not very large. The running of the spectral index is almost negligible, which means that the spectrum will be constant over an extended region.

**Exponential**

The next simplest model is the exponential potential, defined by

$$V = V_0 e^{-\lambda (\sigma / M_P)}. \quad (4.44)$$

The slow-roll parameters are constant.

$$\epsilon = \frac{\lambda^2}{2} \quad , \quad \eta = 2\epsilon \quad , \quad \xi^2 = \lambda^4. \quad (4.45)$$

An exact solution exists for this potential in empty space. It is called **power law inflation**. It is given by [21]

$$a = a_0 (tM_P)^{2/\lambda^2} \quad (4.46)$$

$$\sigma = \sigma_0 + \frac{2M_P}{\lambda} \ln(M_P t) \quad (4.47)$$

$$\sigma_0 = \frac{2M_P}{\lambda} \ln \left( \frac{V_0 \lambda^2}{2M_P^2 (6 - \lambda^2)} \right) \quad (4.48)$$

$$\rho_\sigma = \rho_{\sigma,0} (tM_P)^{-2} \quad (4.49)$$

$$w_\sigma = \frac{\lambda^2}{3} - 1. \quad (4.50)$$

This solution is valid for $0 < \lambda < 6$, corresponding to $-1 < w_\sigma < 1$. Inflation occurs for $\lambda < \sqrt{2}$.

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4. Inflation and the background radiation

Since the slow-roll parameters are constant, inflation will not end. There are ways around this problem. One possibility is to introduce a second field, which interacts in just the right way with the first field to end inflation at some point in time. This multiple field concept has e.g. been reviewed in [103]. We will not concern us with this subject. An other possibility is to slightly change the potential, such that it looks like an exponential for small times, but deviates from this shape if the field rolls further.

Suppose now that inflation does end, through one or another mechanism. We may then calculate the spectrum of the background radiation. Since the slow-roll parameters are constant, this is very easy. The results are

\[ n(k) - 1 = -\lambda^2 \]  
\[ \frac{dn}{d\ln k} = 0. \]  

Exponential inflation leads to a constant spectral index.

Small Field

There is a class of models which need very little motion of the field during inflation. In such small field models, the field typically starts out close to the origin and rolls towards a minimum which is situated at not more than a couple of Planck masses away. An example of such a potential is given by

\[ V = V_0 - \left( \frac{\sigma}{M_P} \right)^4. \]  

This potential looks like the example from figure 2.2. The minimum is situated at \( \sigma = M_P \).

The field starts rolling close to the origin. The slow-roll parameters are then both very small. Closer to the minimum of the potential, both parameters grow very large, to become infinite in the minimum. Inflation ends for \( \epsilon \approx 1 \), approximately at \( \sigma = 0.55 M_P \). This is quite a contrast with the monomial case, where the field value changes easily by tens of Planck masses during inflation.

Hybrid

Hybrid models are models for which \( \epsilon < 1 \) at all times. To stop inflation, an other field is necessary which halts the inflaton through the interaction between the fields. A simple example of a hybrid potential is given by

\[ V = V_0 + \frac{1}{2} m^2 \sigma^2. \]  

where we must have

\[ m^2 \ll \frac{V_0}{M_P^2} \]  

for \( \eta \) to be small. The slow-roll parameters for this potential are

\[ \epsilon = \frac{1}{2} \sigma^2 / M_P^2 \]  
\[ \frac{(V_0/m^2M_P^2 + \frac{1}{2} \sigma^2/M_P^2)^2}{(V_0/m^2M_P^2 + \frac{1}{2} \sigma^2/M_P^2)^2} \]  
\[ \eta = \frac{1}{V_0/m^2M_P^2 + \frac{1}{2} \sigma^2/M_P^2} \]  
\[ \xi^2 = 0. \]  

\[ 53 \]
The amount of inflation can be calculated exactly from the potential:

\[ N = \frac{1}{4} \left( \frac{\sigma - \sigma_c}{M_P} + \frac{V_0}{m^2 M_P^2} \ln \frac{\sigma}{\sigma_c} \right) \]  

(4.59)

\[ \approx \frac{V_0}{4m^2 M_P^2} \ln \frac{\sigma}{\sigma_c}. \]  

(4.60)

The value of \( \sigma_c \), at which inflation ends, depends on the auxiliary field. We will not go into details here, but we will assume that inflation ends at some point, which we will specify later on.

Assuming 60 e-folds inflation, we find

\[ \sigma = \sigma_c e^{240m^2 M_P^2 / V_0}. \]  

(4.61)

The slow-roll parameters at Hubble crossing of the present Hubble scale are then

\[ \epsilon = \frac{1}{2} \frac{m^2 \sigma_c^2 e^{480m^2 M_P^2 / V_0}}{V_0} \]  

(4.62)

\[ \eta = \frac{m^2 M_P^2}{V_0}. \]  

(4.63)

The spectral index can now be calculated to give

\[ n = 1 + \frac{m^2 M_P^2}{V_0} \left[ 2 - 3 \left( \frac{\sigma}{\sigma_c} \right)^2 e^{480m^2 M_P^2 / V_0} \right]. \]  

(4.64)

The running of the index can be neglected, just as the intensity of the gravitational waves. The spectral index can be larger than unity or smaller, depending on the value of \( \sigma_c / M_P \). For small values of \( \sigma_c \), and thus small values for \( \sigma \) during inflation, the index is larger than unity. Let us for definiteness take

\[ \frac{V_0}{M_P^2 m^2} = 100 \]  

(4.65)

and

\[ \sigma_c = 0.01 M_P. \]  

(4.66)

This leads to a spectral index of \( n = 1.020 \). Just as the small field model discussed above, the field travels only a very small distance through the potential during inflation.

**Mixed exponential**

The last model we will discuss is an example of the vast amount of ‘exotic’ models that have been proposed. The potential in this model is given by

\[ V = \frac{e^{-\lambda_1 \sigma / M_P}}{1 + e^{\lambda_2 \sigma / M_P}}, \quad \lambda_1 \ll 1, \quad \lambda_2 = \mathcal{O}(1). \]  

(4.67)

(4.68)

A graph of this function is given in figure 4.3. The flat plateau on the left of the origin causes inflation, which ends when the steep drop at the origin is encountered. We must note that this potential has no absolute minimum and reheating during oscillations in
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Figure 4.3: Graph of the mixed exponential potential (4.67). The edge at $\sigma = 0$ ends inflation.

the minimum is not an option. Like the model from chapter 3, reheating should take place through gravitational particle production.

There exist many more models in the literature, but is not necessary for us to explore many models. The few models we have seen now serve as a good illustration of the concepts we encounter in inflation theory. In the next section, we will finally compare the observational predictions of the models.

4.4.2 A picture of the model zoo

Instead of comparing the specific predictions of the models discussed above, we will try to sketch a somewhat more general picture of the theoretical predictions. We saw that two important parameters are predicted by the inflation models: the spectral index, $n$ and the ratio between the scalar and the tensor perturbations $r$. The running of the spectral index can also be calculated, but in general, this is a very small number. We will therefore not concern ourselves with this running.

The analysis of the WMAP-data by the WMAP-team suggests that indeed the running of the spectral index is very small, but this is by no means a settled issue, see e.g. [98], where a significant running is found.

Let us recapitulate:

$$n = 1 + 2\eta - 6\epsilon$$  \hspace{1cm} (4.69)

$$r = 10\epsilon.$$  \hspace{1cm} (4.70)

We can now make a diagram of $n$ and $r$ in which we can place all inflation models, according to their predictions for $n$ and $r$ at some particular length scale, here chosen to be the present Hubble scale.

We introduce the ratio of the slow-roll parameters $\kappa$, given by

$$\kappa = \frac{\eta}{\epsilon}.$$  \hspace{1cm} (4.71)
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Using this definition, we may find the relation between \( n \) and \( r \) as a function of \( \kappa \):

\[
r = \frac{5}{\kappa - 3} (n - 1).
\]  

(4.72)

In figure 4.4, the \( n - r \)-diagram is displayed. The plane is divided into four regions by the lines \( \kappa = 0, 2, 3 \). The (somewhat arbitrarily defined) regions correspond to different classes of models. The lower region, \( A \), is the domain of the small-field models. Since \( \eta \) and \( \epsilon \) differ in sign in this region, their effects on the rolling of the field are opposite. This will prohibit the field from running over a large distance. The rightmost region, \( D \), has \( n > 1 \), which is not very common for inflation models.

The monomial models lie on the dotted line. Specific values for \( \alpha \) are denoted by \( m_2, m_4, m_6 \) and \( m_8 \). The exponential models lie on the line \( \kappa = 2 \). Four cases are given: \( \lambda = 0.1, 0.2, 0.3, 0.4 \), denoted by \( e0.1 \) etc. The computational results for the mixed exponential, the small-field model and the hybrid potential are indicated by \( me \), \( sf \) and \( H \), respectively. In compiling this figure, the amount of inflation was chosen to be \( N = 60 \).

All the models that we have discussed have a spectral index close to one, as was already argued earlier. Present observations from WMAP constrain the spectral index to be

\[
n = 0.97 \pm 0.03.
\]  

(4.73)
4. Inflation and the background radiation

Gravitational waves have not yet been detected, but it is estimated that [77]

$$r < 0.5.$$  \hfill (4.74)

The observations thus constrain the models to lie in the lower-right corner of the diagram, with $n \lesssim 1$. Unfortunately, the constrains are not tight enough to pin down the right model. However, classes of models that lie outside the lower-right corner, can be dispensed with. For example, it seems unlikely that the D-region contains many viable models.

4.4.3 Reconstructing the potential

Inspired by diagrams like the one we just discussed, several authors (e.g. [88, 89]) have tried to track back from the WMAP-data. The zoo-diagram is taken as a starting point and the evolution equations for the field were solved, using only the slow-roll parameters. As noted earlier in this chapter, higher order slow-roll parameters can be defined. These parameters can be used to approximate the shape of the potential, similar to a Taylor expansion. Starting with a random set of slow-roll parameters at the end of inflation up to some large enough order, the evolution can be traced back, leading to a value for the parameters throughout the evolution. If this has been done, the shape of the potential can be reconstructed. This procedure leads to some remarkable results. Indeed, it seems possible to put constraints on the ratio between the two first order slow-roll parameters $\kappa$. In [88], a value $\kappa > 2$ is suggested.

These methods, however, should be treated with some care. No matter how contrived some potential models may seem, they are usually motivated from some high energy concept, often string theory. The backtracking method described here leads to a potential that is not even defined analytically, but merely reconstructed from the slow-roll evolution. There is thus no theoretical motivation at all for the shape of the potential. Why, then, should the potential be reconstructed at all? One could as well treat the slow-roll parameters as fundamental and ignore the potential altogether.

A perhaps more important objection to this method is the fact that nature might not pick out the slow-roll constants at random. The fact that most randomly picked slow-roll parameters lead to a potential for which $\kappa > 2$, does not constrain the choice. Nature might just have picked that one potential which does not satisfy the $\kappa$-bound.

In this chapter, we have seen some general features of inflation models. The models are seen all to give rise to $n \approx 1$. Although the observations are not yet precise enough to pin down the possible shape of the potential (if the potential formulation is ‘true’), they do constrain the region of possible models. In the next chapter, we will look at the properties of quintessence models.
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5. A closer look at quintessence

In this chapter, we will take a closer look at the possibilities to solve the quintessence problem. First, we will treat several approaches to the solution of the problem. Thereafter, we will focus on field models. An important concept are tracker solutions to the equation for motion of the field. We will consider the tracker concept in some detail in section 5.2. Reviews on the dark energy and quintessence are given in [16, 14, 15, 17, 18].

5.1 Different approaches to the quintessence problem

The quintessence problem basically consists of two parts. The first question to ask is: what is the nature of the dark energy? The second question is: why is the energy density comparable to the energy density in matter at precisely this moment? There are many different answers to these questions. Below, we will treat a few, without claiming completeness in any sense. Three classes of solutions were already presented in section 2.4.

5.1.1 Cosmological constant

By far the simplest solution to the first question, about the nature of the dark energy, is the introduction of a cosmological constant. We simply set $\Lambda \neq 0$ in equation (2.1). Such a cosmological constant can be generated after inflation by slightly displacing the inflaton potential such that its minimum is not exactly at zero. This offset introduces a vacuum energy density which acts as a cosmological constant. This can be seen from the Einstein-Hilbert action for gravitation and matter

$$ S_{EH} = \int d^4x \left( \frac{1}{2\kappa^2} \sqrt{-g} (R - 2\Lambda) + L_m \right). $$

The cosmological constant $\Lambda$ is equal to a vacuum energy contribution to the matter Lagrangian $L_m$.

The energy density associated with $\Lambda$ is constant (see table 2.1) and the value of $\Lambda$ is simply the value we measure today. As already noted in section 1.6, one problem with this approach lies in the fine-tuning. The energy density of the cosmological constant is comparable to that of the matter content, which means that just at the end of inflation, the energy density of the cosmological constant has been approximately one hundred orders of magnitude smaller than the energy density of the radiation. How could this possibly be the case? Could there be a mechanism responsible for this?

An answer to that question may be given by the anthropic principle. This principle states that the cosmological constant has such a small value just because we measure it. It could be possible that the patch of the universe we live in is just one of many possible realisations of the universe. If in such a realisation the constant would have had an appreciably larger value, it would have started dominating much earlier and
structure formation would not have taken place. Obviously, humanity would not be there to contemplate the value of the constant. If the constant would have been much smaller, its effect would not yet be noticeable at this moment. In short, if there is a cosmological constant to be measured, it has to have the value it has, simply because we would not know from its existence if it had not. This is a valid argumentation and perhaps the only one that is ‘right’, but adopting the anthropic principle really means that we stop asking questions about the nature of physics. We simply notice phenomena and ignore the whys and hows. Science then becomes, to use Rutherford’s words, stamp collecting.

One should be very careful in debating such matters, since it is very difficult to argue whether the small value of $\Lambda$ is remarkable. The reason for this is that we do not know how the values of the fundamental parameters at the beginning of our universe are selected. In other words, we do not know the probability measure for the initial parameter space. An interesting discussion of this is given in [30, 31, 32].

Possibly, the value of the ground state energy depends on the probing scale. In that case, it is necessary to find the renormalization group equations describing this behaviour. At present, such a renormalization has not been carried out to general approval. We will further not be concerned with this problem, but it should be kept in mind. References were given in section 2.4

### 5.1.2 Changing gravity

Changing the equations of gravity may of course modify the way in which the energy content of the universe interacts with itself. For example, the gravitational attraction could be larger on large length scales, leaving an observer with the impression that it is lessening in the evolution of the universe, or, that the universe is inflating [57, 58, 59, 60].

A natural possibility to change the equations of gravitation is to include higher order curvature terms. These terms do not come in out of the blue. If we make a quantum mechanical loop expansion of the graviton field, higher order loops correspond to increasing powers of the graviton momenta $p_{\text{grav}}$, which in turn correspond to derivatives of the metric. These derivatives can be rewritten in terms of the curvature tensor and its contractions, see e.g. [5].

A second possibility is provided by brane-world models. In these models, the existence of a space-time with more than four dimensions is supposed. The fifth and higher dimensions are curled up and we effectively live in a four dimensional world. The small extra dimensions may however have some impact on the way gravity behaves. See e.g. [61, 62]

### 5.1.3 Field models

There are a lot of different models that all use a scalar field to act as the source of quintessence [18]. Some models will be reviewed here very briefly.

#### Standard approach

With the ‘standard’ use of fields, the simplest possible use of the field concept is meant. In the standard approach, a scalar field ($Q$, for quintessence field) is introduced, which is moving in some potential [33, 35, 36, 37, 38, 34]. The quintessence field is in general not the same field as the inflaton field, although in chapter 3 it happened to be that
way. The evolution of the quintessence field proceeds according to equation (2.64),

\[ \dot{Q} + 3H\dot{Q} + V' = 0. \quad (5.2) \]

At the start of the evolution, the field does in general not dominate the energy density and the Hubble rate is determined by the evolution of the background matter: radiation and matter. Dependent on the shape of the potential and perhaps the initial conditions for the field, the potential energy becomes dominant at some later time and the quintessence field starts dominating the energy density in the universe. In general, fine-tuning of the potential or the initial conditions or both is necessary to obtain quintessence dominance at present, but in the class of tracker models, this problem is partly solved. The tracker class of models is so important within the field approach to quintessence that they will be reviewed extensively in section 5.2.

**k-essence**

In k-essence models, the assumption is made that there exists a non-linear kinetic term (hence the name) in the field Lagrangian. The equation of motion for the field is then changed, of course. The dynamics of the field should be such as to naturally lead to cosmological constant behaviour at present. An example is worked out in [50, 51]. In the model presented in this reference, the field mimics the equation of state of the background during radiation dominance and starts dominating slowly as matter dominance starts. Since it typically takes some time to reach full dominance for the field, the field will act as a cosmological constant just now.

**Non-minimal coupling**

For a free field, in a curved background, the equation of motion is given by

\[ (-\Box + m^2 + \xi R)Q = 0, \]

with \( \Box = D_\mu D^\mu \). The parameter \( \xi \) parameterises the coupling to gravitational curvature term. \( \xi = 1/6 \) corresponds to conformal coupling, which means that the equation of motion reduces to the the Minkowskian form after a suitable conformal transformation has been carried out. The effects of the curvature and the covariant derivative just cancel for \( \xi = 1/6 \). Throughout this thesis, \( \xi \) is set to zero. A nonzero coupling may lead to intricate interactions between the field and the gravitational background and in various scenarios the coupling may lead to a cosmological constant like behaviour. In general, the mechanisms leading to this behaviour are complicated and we will not elaborate this possibility here. Some models can be found in [52, 53, 54, 55].

A possibility that is a bit like non-minimal coupling, is introducing a direct coupling to the ‘normal’ radiation and matter components in the universe. A conformal transformation relates matter-coupled models to non-minimal models, and they might be considered mathematically equivalent, although the physics surely is different. See e.g. [56].

After this overkill of quintessence models, we will look in some detail at a specific class of models. In the next section, we will treat tracker models, which form a sub-class of the ‘standard’ approach.
5. A closer look at quintessence

5.2 Tracker solutions

Trying to solve the quintessence problem by proposing a quantum field in some specific potential may not seem a very clever idea. Why should the problem of tuning the present value of the dark energy not be transferred to the normalisation of the potential and the initial conditions for the field? The answer to this question, and the reason that the quintessence concept works reasonably well, lies in the existence of tracker solutions. These are solutions to the equation of motion for the field similar to the attractor solution for inflation as found in section 2.7. In due time, to some extent independent of initial conditions, all solutions for the field will ‘tag on’ to the tracker solution, erasing the sensitivity on initial conditions. The dynamics of this process are quite different from the dynamics for the inflation field, since there is matter dominance during an extended time in the history of the universe. In this section, we will evaluate the dynamics leading to tracker solutions. See further [39, 40, 41].

5.2.1 The nature of the tracker solution

Three characteristics determine if a solution falls in the class of tracker solutions:

- the solution for the field leads to a slowly varying equation of state for the field;
- the energy density of the field slowly becomes dominant, which implies that the equation of state of the field should be somewhat smaller than that of the background:
  \[ w_Q \leq w_B \];
- the equation of state of the field mimics the equation of state of the background:
  the change from radiation to matter dominance is reflected in the equation of state of the field.

The tracker solutions can be studied using the evolution equations for the field that we have seen many times already, but it is practical to derive an equivalent set of equations conveying the same information [41]. If we calculate the relative density of the field and its equation of state, the state of the universe is completely determined. We thus need two equations for these two quantities. First the relative density, we define

\[ y \equiv \frac{\rho_Q}{\rho_B}, \quad \Omega_Q = \frac{y}{1+y}. \]  

The background energy density \( \rho_B \) can be supplied by either radiation or matter. We see that \( y \) is directly related to \( \Omega_Q \). Taking the derivative of the definition of \( y \) with respect to the scale factor, we have

\[ \frac{dy}{dN} = \frac{1}{H} \left( \frac{\dot{\rho}_Q}{\rho_B} - \frac{\rho_Q \dot{\rho}_B}{\rho_B^2} \right) \]  

\[ = \frac{y}{H} \left( \frac{\dot{\rho}_Q}{\rho_Q} - \frac{\dot{\rho}_B}{\rho_B} \right). \]  

Using the evolution from the continuity equation (2.40), we find

\[ \frac{dy}{dN} = 3y(w_B - w_Q). \]

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The second equation governs the behaviour of the equation of state:

\[
\frac{dw_Q}{dN} = \frac{1}{H} w_Q
\]
\[
= \frac{1}{H} \frac{\rho_Q}{p_Q} \left( \frac{\dot{\rho}_Q}{\rho_Q} - \frac{\dot{p}_Q}{p_Q} \right). 
\] (5.9)

The quantity \( \frac{\dot{\rho}_Q}{\rho_Q} \) is just the squared sound velocity of the field:

\[
v_Q^2 = \frac{\dot{\rho}_Q}{\rho_Q} = \frac{\dot{Q}Q - V'Q}{QQ + V'Q} 
\]
\[
= 1 + \frac{2}{3} \frac{V'}{HQ}. 
\] (5.10)

\( \dot{Q} \) is given by

\[
\dot{Q} = \sqrt{2V\frac{1 + w_Q}{1 - w_Q}}. 
\] (5.11)

and the Hubble rate is

\[
H = \sqrt{\frac{1}{3M_p^4 \Omega_Q (V + \frac{1}{2} \dot{Q}^2)}} 
\]
\[
= \sqrt{\frac{2V}{3M_p^4 \Omega_Q (1 - w_Q)}}. 
\] (5.12)

We have thus

\[
v_Q^2 = 1 + \frac{V' M_p}{V} (1 - w_Q) \sqrt{\frac{\Omega_Q}{3(1 + w_Q)}} 
\] (5.13)

and the equation for \( w_Q \) becomes

\[
\frac{dw_Q}{dN} = 3(1 + w_Q)(w_Q - v_Q^2). 
\] (5.14)

We can also write this equation as

\[
\frac{dw_Q}{dN} = 3(1 + w_Q)(1 - w_Q)(\Delta - 1), 
\] (5.15)

with

\[
\Delta = \frac{V' M_p}{V} \sqrt{\frac{\Omega_Q}{3(1 + w_Q)}}. 
\] (5.16)

If the quintessence field rolls down the potential to the right side of the vertical axis, \( \Delta \) will be positive. This last form of the equation for \( w_Q \) is especially useful in discussing the ‘settling into’ the tracker solution.

From the set of equations (5.8, 5.18), we note some interesting properties of the tracker solution. Firstly, if the tracker solution has been joined, the field equation of state changes very slowly and we must have

\[
\Delta \approx 1. 
\] (5.17)
5. A closer look at quintessence

Secondly, if the equation of state of the field is nearly constant, the evolution for the relative density of the field is given by

\[ y \propto e^{3(w_B - w_Q)N}. \]  

(5.21)

The energy density of the field overtakes the energy density of the background. The smaller the difference between \( w_Q \) and \( w_B \), the slower this overtaking takes place.

The equations (5.8, 5.18) cannot be solved analytically in general, but it is not difficult to feed them into a computer. Some example calculations will be done in section 5.2.4.

5.2.2 Approaching the tracker solution

Now that we have derived some basic properties of the tracker solutions, it is interesting to see how the tracker solution is approached, to large extent independent of the initial conditions of the field. There are two cases: the energy density of the field is initially larger than the tracker value, or the energy density is initially smaller than the tracker value. We will first consider the case that the initial density is too large, the overshoot case.

If we are given an initial value for the field, the value of \( \frac{V_0}{V} \) is fixed. Overshoot then implies that \( \Delta \) will be greater than unity, since for the tracker solution \( \Delta \approx 1 \). From (5.18), we see that \( w_Q \) will grow. If the overshoot energy density is large enough, the equation of state will tend to \( w_Q = 1 \). Approaching this value, the equation of state will stop growing, because of the factor \((1 - w_Q)\), present in the equation. This was to be expected, since the equation of state for a quantum field can never exceed \( w_Q = 1 \).

An equation of state \( w_Q = 1 \) means that we have kination and the field energy density will scale as \( \frac{\rho}{a^6} \), faster than the background energy density. After some time, the relative density \( \Omega_Q \) will drop so low, that we have \( \Delta < 1 \). This implies that \( w_Q \) starts running again, this time in the direction \( w_Q \to -1 \). As soon as the field approaches \( w_Q = -1 \), the running stops and the field ‘freezes in’ at some particular value. The scaling of the energy density is now \( \rho_Q = \text{const.} \), slower than the background energy density. \( \Omega_Q \) will now increase and eventually we will again have \( \Delta > 1 \), leading to an increase of \( w_Q \) and so on.

The potentials that are suited for tracker behaviour have the property that the ratio \( \frac{V'}{V} \) decreases as the field rolls down the potential. This implies that the oscillations just described will be dampened. The result will be, that the evolution of the field settles down in some smooth, slowly varying function, which we call the tracker solution.

In the case that the initial field energy density is much smaller than the tracker value, the undershoot case, the evolution to the tracker solution more or less proceeds in the same way. Starting with undershoot, we will have \( \Delta < 1 \), and the field settles down in a value \( w_Q \gtrsim -1 \). This will cause the relative density of the field to grow, until \( \Delta > 1 \), leading to \( w_Q \to 1 \) et cetera.

The value of the equation of state of the tracker solution will always be smaller than that of the background, since else the relative density of the field will grow until \( \Delta < 1 \), decreasing the equation of state of the field until indeed \( w_Q < w_B \). This fact implies that the quintessence field will eventually start dominating the energy density of the universe. If this happens, the value of \( \Delta \) will drop below unity, and the equation of state will decrease. This means that if the field becomes dominant, it automatically acquires an equation of state \( w_Q = -1 \). This is a very important observation, since it explains why there is a dark energy component in the universe. The field dynamics force the field into \( w_Q = -1 \).
5. A closer look at quintessence

We have seen that tracker behaviour will occur if the potential satisfies the constraint
\[
\frac{\partial (V'/V)}{\partial V} > 0,
\] (5.22)
which ensures that the ratio \(\frac{V'/M_P}{V}\) decreases as the field rolls down the potential. This dampens the oscillations of the equation of state. This constraint can be written equivalently as
\[
\Gamma \equiv \frac{VV''}{V'} > 1,
\] (5.23)
defining \(\Gamma\), the tracker condition.

5.2.3 Constraints and initial conditions

As we have seen in the last section, evolution to a slowly varying tracker solution is possible if \(\Gamma > 1\). We might wonder if the tracker solution is uniquely determined. It could be possible that the field evolves to a tracker solution with different characteristics, depending on the initial value of the field and the field velocity. To guarantee that the solution is indeed unique, the evolution of the ratio \(\frac{V'/M_P}{V}\) should be roughly equal for the range of plausible initial conditions. This constraint can be formalised by demanding
\[
\frac{\Delta \Gamma}{\Gamma} \ll 1,
\] (5.24)
where \(\Delta \Gamma\) is the difference between the maximum and the minimum value of \(\Gamma\) over the range of possible initial conditions.

Some potentials that satisfy the tracker condition (5.23) are the inverse monomial potential
\[
V = V_0 \left( \frac{Q}{M_P} \right)^{-\alpha}, \quad \Gamma = 1 + \frac{1}{\alpha}, \quad \alpha > 0,
\] (5.25)
and the inverse exponential
\[
V = V_0 e^{\frac{\lambda M_P}{Q}}, \quad \Gamma = 1 + \frac{2}{\lambda M_P} Q.
\] (5.26)

Apart from the constraints on the potential, the initial conditions for the field should satisfy some very weak constraints. The velocity of the field is not much constrained, since \(\Delta \geq 1\) directly determines whether \(w_Q\) tends to +1 or –1, fixing the speed of the field. The initial value of the field is of some more importance, since it determines the initial energy density of the field.

We have seen that overshoot and undershoot both lead to tracker behaviour, but the difference between the initial energy density of the field and the tracker value may not be too large. If the initial energy density of the field is very large, \(\Omega_Q\) will stay large for a prolonged period. The running of the field will last for a long time and the freezing value of the quintessence field will be far down the potential. Therefore, it will take very long for the field to become dominant again. If this is the case, the field equation of state may not have completed its first oscillation. Effectively, the field is frozen in until the present time and it behaves as a pure cosmological constant, which means that the height of the potential should be tuned exactly to give the right value for the energy density in the field at present. Clearly, we want to avoid this situation with the tracker concept. We can make this argument exact. Using the exact solution
5. A closer look at quintessence

Figure 5.1: Graph of \((Q_f - Q_i)/M_P\), from (5.28). Dominance of the field makes the freezing value so large, that tracking will not yet occur at present.

For the transition from a kination dominated universe to a radiation dominated universe (equation (3.50)), we find

\[
Q_f = Q_i + \sqrt{6}M_P\tanh \left[ \left( 1 + \frac{\rho_{r,i}}{\rho_{Q,i}} \right)^{-1/2} \right] \tag{5.27}
\]

\[
= Q_i + \sqrt{6}M_P\tanh \sqrt{\Omega_Q}, \tag{5.28}
\]

If \(\Omega_Q\) starts out very large, the field starts running until the value of \(\sqrt{\frac{\rho_f}{M_P}}\) has dropped so low that \(\Delta < 1\) again. This effect is illustrated in figure 5.1. We encountered this situation in chapter 3, which is the reason that the field did not have an oscillating equation of state, although the potential did satisfy the tracker condition.

On the other hand, there is hardly a limit on the smallness of \(\rho_{Q,i}\). The one hard constraint is that the energy density should be at least equal to the present energy density, since the field energy density can never grow. In the case that the initial energy density of the field indeed is very close to its present value, the field will start rolling only at this moment, making it essentially equivalent to a cosmological constant, which again is not what we want. But even if we do not push things to the edge, the field energy density at the onset of the quintessence evolution may easily be chosen 60 orders of magnitude smaller than the radiation energy density. Indeed a very impressive span of initial conditions.

The range of possible initial conditions includes the reasonable scenario of equipartition. In this scenario, the quintessence field starts out with just as much energy as all the other degrees of freedom after inflation. The quintessence field then has an initial energy density of

\[
\Omega_{Q,\text{equipart.}} \sim 10^{-2} - 10^{-3}, \tag{5.29}
\]

if the number of degrees of freedom is approximately given by \(g_* = 10^2 - 10^3\). One might argue that this equipartition scenario is indeed the most natural possibility.
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5.2.4 Some specific examples

Inverse monomial potential

To get a feeling for the meaning of the aforesaid, we will now look at a specific potential. We take the inverse monomial potential [41]:

\[ V = V_0 \left( \frac{Q}{M_P} \right)^{-\alpha}. \]  (5.30)

We find

\[ \frac{V' M_P}{V} = -\alpha \frac{M_P}{Q} \]  (5.31)

\[ = -\alpha \frac{3 M_P^2 H^2 \Omega_Q - \frac{V}{2 Q^2 + V}}{V_0 M_P^2} \]  (5.32)

\[ = -\alpha \left( \frac{3 M_P^2 H^2 \Omega_Q (1 - w_Q)}{2 V_0} \right)^{1/\alpha} \]  (5.33)

and

\[ v_Q^2 = 1 - \alpha (1 - w_Q) \sqrt{\frac{\Omega_Q}{3(1 + w_Q)}} \left( \frac{3 M_P^2 H^2 \Omega_Q (1 - w_Q)}{2 V_0} \right)^{1/\alpha}. \]  (5.34)

The equations (5.8, 5.17, 5.34) can be integrated using a computer. Two initial conditions and two potential parameters must be chosen: \( w_{Q,\text{init}}, y_{\text{init}}, V_0 \) and \( \alpha \). These parameters determine the complete evolution of the scale factor if supplied by observational knowledge on the energy density of matter and radiation. The four parameters are not independent. We must choose them in such a way as to yield \( \Omega_{Q,0} = 0.7 \). This means that we have to determine \( V_0 \) as a function of \( \alpha \), or the other way around. The tracking behaviour ensures that varying the initial values of \( w_Q \) and \( y \) will not influence this result.

In figure 5.2, the results of the computation are displayed for \( \alpha = 6 \). \( N \) denotes the number of e-folds after inflation ends. Three cases are shown. They differ with respect to the initial value for \( y \). The upper line is an ‘overshoot’ case, in which \( y \) initially is much larger than the tracking value. The equation of state directly heads for \( w_Q = +1 \), until the energy has dropped to the tracker value. The middle line corresponds to an initial energy density just about equal to the tracking value. The lower line corresponds to ‘undershoot’. The value of \( w_Q \) drops to \(-1\) and when the relative density is large enough, it joins the tracker solution. When matter dominance sets in, at \( N \approx 46 \), the value of the equation of state makes a slight drop. Just after that, the value of \( w_Q \) drops even further as the field starts to dominate the energy density. This is not a general feature of the model. The moment at which the field starts dominating is determined by the normalisation of the potential. In this case, a value of \( V_0^{1/4} = 10^{10} \text{ GeV} \) is needed. This is about a hundred million times more than the electroweak unification scale.

We see that the fine-tuning problem has not been totally solved by this quintessence scenario. We had to fit the potential to obtain the right time for dominance of the field. The energy density of the field does not need tuning and that is a big advantage. The tracker scenario automatically tunes the energy density of the field to be of the same order of the background energy density. Thus, part of the dark energy problem has been solved.
5. A closer look at quintessence

Figure 5.2: The equation of state (left) and the relative density (right) of the quintessence field for the inverse monomial potential (5.30). On the horizontal axis the number of e-folds after inflation ends. Three different cases are shown, with differing initial value of $y = \rho_Q/\rho_r$. These cases correspond to an initial energy density higher, lower and almost equal to the tracker energy density. The change from radiation to matter dominance at $N \approx 46$ is clearly visible. To achieve $\Omega_Q = 0.7$ today, a value $V_0 = 10^{40}$ GeV$^4$ is needed.

The borderline case: exponential potential

We will now turn to a very interesting potential: the exponential potential (discussed in [35, 36, 37]), given by

$$V = V_0 e^{-\lambda Q/M_P}.$$  \hfill (5.35)

The tracker condition for this potential reads

$$\Gamma = 1.$$ \hfill (5.36)

Strictly spoken, the potential is not suitable for tracker solutions, since $\Gamma \neq 1$, but the fact that the exponential potential lies just at the borderline of the tracker class leads to some very interesting properties.

If we calculate the velocity of sound in the exponential potential, we see

$$v_Q^2 = 1 - \lambda(1 - w_Q)\sqrt{\frac{\Omega_Q}{3(1 + w_Q)}}. \hfill (5.37)$$

If we want the equation of state to be nearly constant, this implies (using equation (5.17)) that $\Omega_Q = \text{const}$, because the factor $\frac{V/V_{MP}}{1 + \lambda}$ is constant for this model. If $\Omega_Q$ is constant, the equation of state should equal that of the background, (from equation (5.8))

$$w_Q = w_B.$$ \hfill (5.38)

Once the tracker is reached, the value of $\Omega_Q$ is locked in and the field scales identical with the dominant fluid component. If $\Omega_Q$ is constant and $w_Q = w_B$, we see from (5.17) and (5.37) that

$$\Omega_Q = \frac{3(1 + w_B)}{\lambda^2}. \hfill (5.39)$$

Apparently, we can calculate the tracker solution exactly in this case. For radiation dominance, the relative density is

$$\Omega_Q = \frac{4}{\lambda^2}, \quad \text{radiation dom.} \hfill (5.40)$$
5. A closer look at quintessence

Figure 5.3: The equation of state (left) and the relative density of the quintessence field (right) for the exponential potential (5.35). In this case, $\lambda = 3$, which leads to $\Omega_Q = 4/9$ for radiation dominance and $\Omega_Q = 3/9$ for matter dominance. On the horizontal axis the number of e-folds after inflation ends. Again, three cases are shown, undershoot, overshoot and tracking. The constant $V_0$ is not relevant for the calculations.

and for matter dominance, we have

$$\Omega_Q = \frac{3}{\lambda^2}, \quad \text{matter dom.} \quad (5.41)$$

In figure 5.3, this behaviour is shown. The steepness of the potential is $\lambda = 3$ in this example. Just as in the inverse monomial case, the transition from radiation to matter dominance is clearly recognisable. Since the value of $V_0$ does not enter the equations, the height of the potential is of no importance. To arrive at an energy density in the field comparable to that in matter at present, the value of $\lambda$ should be of order one. There is hardly any fine-tuning needed at all! This is a big advantage, but there is a very big problem also. Since the tracking value of the relative density of the field is constant, we must have during nucleosynthesis

$$\Omega_Q = 0.93, \quad \text{nucleosynthesis.} \quad (5.42)$$

Needless to say, this figure would exert quite some influence on the rate of synthesis. Therefore, a tracking scenario with an exponential potential is untenable. It is of course possible to tune the field such that tracking has not yet begun [36, 37], but this reintroduces the fine-tuning problem which we were seeking to avoid.

The conclusion is, that the one tracker potential that hardly needs fine-tuning is not a feasible scenario. The potentials that are viable physical models do need fine-tuning to some extent.

5.2.5 The $\Omega_Q$-$w_Q$ relation

One of the most important results of the tracker model is the prediction of a relation between the energy density of the field and the equation of state. We have already seen that the equation of state tends to $w_Q = -1$ if the field becomes dominant, but we did not make this relation exact. With the data gathered in the last section, it is easy to present the relation between $\Omega_Q$ and $w_Q$. Computational results for the inverse monomial with $\alpha = 6$ and $\alpha = 1$ are displayed in figure 5.4. In the exponential case, the relation is trivial, $\Omega_Q$ and $w_Q$ being fixed during tracking.
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The tendency for $w_Q$ to drop as $\Omega_Q$ increases is clear. For a smaller power $\alpha$, the equation of state comes closer to $w_Q = -1$. An even lower value than shown in the figure can be obtained by allowing for non-integer values of $\alpha$, smaller than unity. The relation between the energy density and the equation of state of the field is subject to observational testing and provides the most direct bound with observations for the tracker model. The maximum value of the equation of state, as found by the WMAP team [76] is given by

$$w_Q \leq -0.78.$$  \hspace{1cm} (5.43)

Obviously, the $Q^{-6}$-model does not fit in this picture. Fortunately, it is easy to construct models that do obey this constraint. Since all inverse monomials (possibly with non-integer power) obey $\Gamma > 1$, any sum of inverse monomials with positive coefficients will also have $\Gamma > 1$, as we see from (5.22). There is thus a very broad class of potentials to choose from in modelling tracker behaviour.

5.3 The link with observations

In the end, the question should always be: does the theory fit the data? Unfortunately, it is very difficult to compare predictions of various quintessence models with data. Of course, the two trivial constraints, $\Omega_Q \approx 0.7$ and recent dominance of the dark energy are satisfied by every model. Other models are not even considered. But then again, how to distinguish between all those models that fit the two main observational constraints? Some references that go into this problem are [80, 81, 82, 84, 85, 83, 65].

The big problem lies in the fact that the background radiation, which forms the most massive and most accurate body of data available in cosmology was formed during recombination, in the early universe. The mayor characteristic of quintessence model is that they predict effects at present, but their effects in the very early universe, at
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decoupling are typically very small, since the tracker field is not yet dominant at that time.

On its way to earth, the background radiation passes all stages of the evolution of the universe and in principle it should be possible to detect the late-time effects. However, the impact of the quintessence model on the path that the light takes through spacetime is pretty obscure. Only through computer simulations are we able to estimate the effects that quintessence models have on the appearance of the background radiation.

Certain models may predict new phenomena, such as the existence of light scalars which couple to matter or changes in the laws of gravity at scales that are small enough to probe directly, such as the solar system. However, such phenomena are typically weak, since else they would have been detected already.

The best opportunity to compare data and theory is provided by the tracker potentials. Because the $\Omega_Q-w_Q$ relation is predicted, measurements of the value of the equation of state today may be an indication of the type of potential.
5. A closer look at quintessence
6. Conclusions and outlook

At the end of this thesis, we will look back at the results and comment on our findings. A brief sketch of what the future will hopefully bring (according to the author) will also be given.

6.1 Inflation

As we have seen already in the first chapters, inflation is an elegant way to solve the problems of the old Hot Big Bang model. Rather than replacing this model, inflation serves as an add-on concept, taking place before the start of what everyone used to call the Big Bang. The predictions of inflation, notably the precise nature of the temperature and polarisation perturbations in the CMBR, have been tested by the WMAP mission.

One direct consequence of the inflationary concept is that the start of the universe is not well defined anymore. The energy density of the inflaton field stays approximately constant during inflation and inflation could have taken place for an indefinitely long time before the cosmological scales that are relevant at present left the horizon. The figure of 60 e-folds inflation applies to the Hubble scale exit of our present Hubble length: what happened long before this event cannot be traced back. In other words, observable inflation started somewhat more than 60 e-folds before the end of inflation.\(^1\)

An interesting question that remains open is, where inflation comes from or, in other words, "Who ordered the inflaton?" We have seen that inflation can be described conveniently in terms of a self-interacting quantum field, but what could be the origin of such a field? Most inflation models found in literature are inspired by GUT theories or even string theory, but at present, consensus about which models deserve our attention and which models are just nonsense is far to be found. A review of possible particle physics motivated models for inflation can be found in [19].

Although it is possible to rephrase every inflationary model in terms of a quantum field model, there is always the possibility that the field concept is just not the right way to think about inflation. Renormalization group running of the ground state energy or a refinement of the Einstein equations could perhaps be more suitable ways of thinking about inflation. To come to a deeper understanding of inflation, it is necessary to further develop a high energy physics theory capable of explaining the presence (or maybe absence) of an inflation field. In the long run, string theory might be a good candidate for such a theory, but the answer may lie pretty close: near future experiments, such as HERA [101] and the Tevatron [102], will probe higher energy scales than has been possible before. New discoveries are perhaps waiting just around the corner.

It is also important to refine the observations of the CMBR to be able to distinguish with higher precision between the possible inflation models. The Planck satellite, to

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\(^1\)It is possible to have far less than 60 e-folds inflation, which then takes place at a much lower energy scale, but most models still assume about 60 e-folds.
be launched by the European Space Agency in 2007 (more details in [99]) will greatly improve on the WMAP observations.

As a word of caution, I would like to pay just a little attention to an argument by Hollands and Wald in [30, 31, 32]. As pointed out, it might not be necessary to invoke the inflation concept at all. The main reason to assume inflation in the early universe is the fact that the ‘initial conditions’ of our universe are not well understood. An initially flat and homogeneous universe would solve the problems of the standard HBB model, but many authors feel uncomfortable ‘blaming’ everything on the initial conditions. Inflation helps by erasing the dependence on initial conditions. But, as the authors argue, there is no theory of how the initial conditions are ‘selected’. Is a perfectly flat and homogeneous universe really strange or coincidental? I feel that the merit of inflation is that it makes the dependence on initial conditions for the universe less strong, thereby making the question whether they are natural in some sense or not redundant, but it is important to keep in mind that there may be other solutions to the flatness and horizon problem. In any case, inflation does make predictions concerning the CMBR which have been found to be true, so some credits must at least be given to the inflation model.

6.2 Quintessence

The quintessence problem, as we have seen, still leaves room for many disputes. First of all, there is not even consensus on what the nature of the dark energy is. A short survey of some possible solutions was given in section 5.1. Secondly, the fine-tuning problem has still not been solved in a compelling way in any of the models. The tracker solutions, for example, do not provide a solution. In most models, the height of the quintessence potential must be tuned accurately. In the one model where hardly any tuning is necessary (the exponential potential), the energy density of the field does not come out right.

Observations of the CMBR do somewhat constrain the possibilities for building quintessence models, but the constraints are certainly not strong enough to rule out classes of models.

In my view, the biggest obstacle to be taken is the solution to the renormalization of the ground state energy in the Standard Model of particle physics. As argued before, the ground state energy could be dependent on the energy scale. This would mean that we should expect some contribution to the dark energy from the ground state energy. As long as the renormalization procedure has not been carried out to satisfaction, it really is impossible even to precisely state the quintessence problem. Perhaps the renormalization will solve the quintessence problem entirely, but there is also a possibility that we will merely discover that the dark energy is in part provided for by the ground state energy and that we should look for an other account of the remaining part.

Whatever the nature of the solution to the quintessence problem may be, it seems not very likely that a unified model, combining inflation and quintessence into one concept, will be found. Since inflation takes place at high energies and the quintessence problem comes into play at very small energy densities, a combination of these concepts is hard to model without the two concepts being effectively independent, as we have seen in chapter 3.

The conclusion may be drawn that there still remains much to be discovered in relation to inflation and quintessence. Theoretical progress in high energy physics and possibly the renormalization group equations for the ground state energy will no doubt take
place and cosmological observations will be refined. And when inflation and quintessence will have been understood, further questions will be waiting around the corner…
6. Conclusions and outlook
A. Conventions, numerical values and conversion tables

In this appendix, the most important conventions are listed. Also, a list of fundamental constants and conversion tables is given.

A.1 Conventions

A.1.1 Metric, connections and curvature

A metric with positive signature is used

\[ \eta_{\mu\nu} = \text{diag} (-1, 1, 1, 1). \]  

(A.1)

The affine connection is

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\nu\mu,\sigma}), \]  

(A.2)

and the curvature tensor is given by

\[ R^\sigma_{\lambda\mu\nu} = \Gamma^\lambda_{\nu\sigma,\mu} + \Gamma^\lambda_{\mu\nu,\sigma} - \Gamma^\lambda_{\mu\sigma,\nu} \]  

(A.3)

\[ R_{\mu\nu} = R^\sigma_{\mu\sigma\nu}, \]  

(A.4)

\[ R = g^{\mu\nu} R_{\mu\nu}. \]  

(A.5)

A.1.2 Differentiation

Partial differentiation is sometimes denoted by a comma (as in the definition of \( \Gamma \) above)

\[ f_{,\mu} = \partial_{\mu} f = \frac{\partial f}{\partial x^{\mu}}. \]  

(A.6)

The covariant derivative is denoted by a capital \( D \) or a semicolon. It is defined to be

\[ D_{\mu} f^{\alpha_1 \ldots \alpha_n,\gamma_1 \ldots \gamma_n}_{\gamma_1 \ldots \gamma_n} = f^{\alpha_1 \ldots \alpha_n,\gamma_1 \ldots \gamma_n}_{\gamma_1 \ldots \gamma_n} \]  

(A.7)

An apostrophe is used to indicate derivatives with respect to a non-space-time argument, such as a quantum field.

\[ V' (\sigma) = \frac{\partial V (\sigma)}{\partial \sigma}. \]  

(A.8)

An apostrophe will also denote differentiation with respect to conformal time \( \eta \). An overdot denotes differentiation with respect to coordinate time.

\[ \dot{a}(t) = \frac{da(t)}{dt}, \quad a' (\eta) = \frac{da(\eta)}{d\eta}. \]  

(A.9)
A. Conventions, numerical values and conversion tables

A.2 Fundamental constants and units

Throughout this thesis, Planck Units are most often employed. This means that \( c = h = 1 \), and all quantities have dimension \([M^\alpha]\), with \( \alpha \) some integer, positive or negative. Below, some values and definitions are given.

\[
\begin{align*}
 h &= 1.054571596 \times 10^{-34} \text{ J s} = \cdots \text{ kg m}^2 \text{ s}^{-1} \quad (A.10) \\
 c &= 2.99792458 \times 10^8 \text{ m s}^{-1} \quad (A.11) \\
 G_N &= 6.673 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \quad (A.12) \\
 k_B &= 1.3806503 \times 10^{-23} \text{ J K}^{-1} = \cdots \text{ kg m}^2 \text{ s}^{-2} \text{ K}^{-1} \quad (A.13) \\
 E_P &= \sqrt{\frac{\hbar c}{G_N}} = 1.956 \times 10^9 \text{ J} = 1.221 \times 10^{19} \text{ GeV} \quad (A.14) \\
 m_P &= \sqrt{\frac{\hbar c}{G_N}} = 2.177 \times 10^{-8} \text{ kg} \quad (A.15) \\
 M_P &= \frac{m_P}{\sqrt{8\pi}} = 2.436 \times 10^{18} \text{ GeV} \quad (A.16) \\
 l_P &= \sqrt{\frac{\hbar G_N}{c^3}} = 1.616 \times 10^{-35} \text{ m} \quad (A.17) \\
 t_P &= \sqrt{\frac{\hbar G_N}{c^5}} = 5.391 \times 10^{-44} \text{ s} \quad (A.18)
\end{align*}
\]

A.2.1 Conversion tables

Astronomers tend to use their own units, like lightyears and Megaparsecs. To facilitate conversion between various units, some conversion tables are given below.

Energy

\[
\begin{align*}
1 \text{ GeV} &\doteq 1.602 \times 10^{-10} \text{ J} \quad (A.19) \\
&\doteq 1.783 \times 10^{-27} \text{ kg} \quad (A.20) \\
&\doteq 5.337 \times 10^{-19} \text{ kg m s}^{-1} \quad (A.21) \\
&\doteq 1.160 \times 10^{13} \text{ K} \quad (A.22) \\
1 \text{ J} &\doteq 6.242 \times 10^9 \text{ GeV} \quad (A.23) \\
1 \text{ kg} &\doteq 5.609 \times 10^{26} \text{ GeV} \quad (A.24) \\
1 \text{ kg m s}^{-1} &\doteq 1.874 \times 10^{18} \text{ GeV} \quad (A.25) \\
1 \text{ K} &\doteq 8.621 \times 10^{-14} \text{ GeV} \quad (A.26) \\
1 \text{ GeV}^{-1} &\doteq 6.582 \times 10^{-25} \text{ s} \quad (A.27) \\
&\doteq 2.087 \times 10^{-32} \text{ yr} \quad (A.28) \\
&\doteq 1.973 \times 10^{-16} \text{ m} \quad (A.29) \\
1 \text{ s} &\doteq 1.519 \times 10^{24} \text{ GeV}^{-1} \quad (A.30) \\
1 \text{ yr} &\doteq 4.792 \times 10^{31} \text{ GeV}^{-1} \quad (A.31) \\
1 \text{ m} &\doteq 5.068 \times 10^{15} \text{ GeV}^{-1} \quad (A.32)
\end{align*}
\]
A. Conventions, numerical values and conversion tables

Time and length

\begin{align*}
1 \text{ yr} & \doteq 3.154 \times 10^7 \text{ s} & (A.33) \\
1 \text{ s} & \doteq 3.171 \times 10^{-8} \text{ yr} & (A.34) \\
1 \text{ s} & \doteq 2.998 \times 10^8 \text{ m} & (A.35) \\
1 \text{ m} & \doteq 3.336 \times 10^{-9} \text{ s} & (A.36) \\
1 \text{ m} & \doteq 1.058 \times 10^{-16} \text{ ly} & (A.37) \\
1 \text{ m} & \doteq 3.254 \times 10^{-23} \text{ Mpc} & (A.38) \\
1 \text{ Mpc} & \doteq 3.073 \times 10^{22} \text{ m} & (A.39) \\
1 \text{ Mpc} & \doteq 3.26 \times 10^6 \text{ ly} & (A.40) \\
1 \text{ ly} & \doteq 9.454 \times 10^{15} \text{ m} & (A.41) \\
1 \text{ ly} & \doteq 3.07 \times 10^{-7} \text{ Mpc} & (A.42)
\end{align*}
A. Conventions, numerical values and conversion tables
Gravitational particle production

During transitions from one type of dominant component in the universe to an other type of dominant component, the evolution of the scale factor changes. This change in time dependence of the scale factor leads to spontaneous particle creation [25]. This curious phenomenon is caused by the fact that the vacuum state of space-time depends on the type of metric; what looks like a vacuum in one space-time, may not look like a vacuum in an other space-time. Below, this argument will be made precise, and a derivation of the amount of particle production will be given for a transition from De Sitter space-time to a radiation dominated space-time. For a detailed treatment, consult [5] and [26].

The result of the calculation that we will perform, is that the particle production is proportional to the fourth power of the Hubble rate at the end of inflation

\[ \rho_{\text{part}} \propto H_e^4. \]  

(B.1)

We start the derivation with a very short summary of some results from quantum field theory (QFT).

B.1 Quantum Field Theory in Minkowski space-time

For a free scalar field, we have the Klein-Gordon equation

\[ (-\Box + m^2)\sigma = 0, \]  

(B.2)

With \( \Box \) the d’Alembertian, \( \Box = \partial_\mu \partial^\mu \). A complete set of solutions is given by \( \{ u_k, u_k^* \} \),

\[ u_k = (2\pi)^{-3/2} \frac{1}{\sqrt{2\omega}} e^{ikx - i\omega t}, \quad \omega^2 = k^2 + m^2. \]  

(B.3)

We define the Minkowski scalar product:

\[ \langle \sigma_1, \sigma_2 \rangle = i \int d^3 x \sigma_1^\dagger(x) \overrightarrow{\partial_x} \sigma_2(x) \]  

(B.4)

\[ = i \int d^3 x \{ \sigma_1^\dagger(x) \partial_i [\sigma_2(x)] - \partial_i [\sigma_1^\dagger(x)] \sigma_2(x) \}, \]  

(B.5)

which gives

\[ \langle u_k, u_{k'} \rangle = \delta(k - k') \]  

(B.6)

\[ \langle u_k, u_k^* \rangle = -\delta(k - k') \]  

(B.7)

\[ \langle u_k, u_{k'} \rangle = 0. \]  

(B.8)
B. Gravitational particle production

Note that this product is not positive definite and it cannot serve as a norm in Hilbert space. Using this scalar product, the field can be written:

$$\sigma = \int d^3k \left( a_k u_k + a_k^\dagger u_k^* \right),$$  

(B.9)

with

$$a_k = \langle \sigma, u_k \rangle$$  \hspace{1cm} (B.10)
$$a_k^\dagger = \langle \sigma, u_k^* \rangle.$$  \hspace{1cm} (B.11)

To quantise the field, we pose the canonical commutation relations

$$[\hat{a}_k, \hat{a}_{k'}] = [\hat{a}_k^\dagger, \hat{a}_{k'}^\dagger] = 0$$  \hspace{1cm} (B.12)
$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta(k - k').$$  \hspace{1cm} (B.13)

The $a_k^{(1)}$ can now be viewed as creation and annihilation operators. Ignoring the vacuum energy, the Hamiltonian is

$$\hat{H} = \int d^3k \ a_k^\dagger a_k \omega_k = \int d^3k \ \hat{N}_k \omega_k.$$  \hspace{1cm} (B.14)

$\hat{N}_k$ is the number operator: $\hat{N}_k dk$ counts the number of particles with momentum in the interval $dk$. We now have a vacuum state, defined by

$$\hat{a}_k |0\rangle = 0, \quad \forall k.$$  \hspace{1cm} (B.15)

From the expression for the Hamiltonian, we see directly that this is indeed the state of lowest energy. We can make multi-particle states by applying the creation operator to the vacuum.

B.2 QFT in curved space-time

In a general space-time, with arbitrary coordinates, the line-element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu,$$  \hspace{1cm} (B.16)

and we must replace some quantities in our theory by their covariant counterparts.

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu}$$  \hspace{1cm} (B.17)
$$d^4x \rightarrow \sqrt{-g} d^4x = \sqrt{-\det g_{\mu\nu}} d^4x$$  \hspace{1cm} (B.18)
$$\Box f \rightarrow D_\mu \partial^\mu f = \sqrt{-g}^{-1} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu f).$$  \hspace{1cm} (B.19)

We will shortly consider the transition from a De Sitter universe in the far past to a radiation dominated universe in the far future, both of which have a Minkowski metric in conformal coordinates. It is not clear, however, how the field behaves for intermediate times. We will treat the effect of the transition as a disturbance, or external force in the Klein-Gordon equation. In the future, when space-time is Minkowski again, we have a solution basis which we will call $u_k$. This basis is not equal to the early-time basis $u_k$ in general.
In future times we may go through the same procedure as we did in the first paragraph. In particular, we have a field decomposition and a vacuum.

\[ \sigma = \int d^3k \ (\tilde{a}_k \tilde{u}_k + \tilde{a}_k^\dagger \tilde{u}_k^\dagger) \]  
(B.20)

\[ \tilde{a}_k \ket{0} = 0, \quad \forall k. \]  
(B.21)

Since both \{\( u_k, u_k^* \)\} and \{\( \tilde{u}_k, \tilde{u}_k^* \)\} constitute a complete set, we may express the bases in terms of one another:

\[ \tilde{u}_k = \int d^3q (\alpha_{kq} u_q + \beta_{kq} u_q^*) \]  
(B.22)

\[ u_q = \int d^3k (\alpha_{kq}^* \tilde{u}_k - \beta_{kq}^* \tilde{u}_k) \]  
(B.23)

\[ \alpha_{kq} = \langle \tilde{u}_k, u_q \rangle \]  
(B.24)

\[ \beta_{kq} = -\langle \tilde{u}_k, u_q^* \rangle. \]  
(B.25)

\( \alpha \) and \( \beta \) are called Bogoliubov coefficients.

Equating the two field-decompositions (B.9, B.20), we see that

\[ a_k = \int d^3q (\alpha_{kq} \tilde{a}_q + \beta_{kq} \tilde{a}_q^*) \]  
(B.26)

\[ \tilde{a}_q = \int d^3k (\alpha_{kq}^* a_k - \beta_{kq}^* a_k^*). \]  
(B.27)

Evaluating the dot product \( \langle \tilde{u}_k, \tilde{u}_q \rangle \), we find the following relation for the Bogoliubov coefficients

\[ \int d^3p (\alpha_{kp} \alpha_{qp}^* - \beta_{kp} \beta_{qp}^*) = \delta(k - q). \]  
(B.28)

We are now in the position to make the difference between the two vacuum states explicit. Calculating the number of particles at late time in the early-time vacuum state \( \ket{0} \), we have

\[ \langle 0 \vert \bar{N}_k \vert 0 \rangle = \langle 0 \vert \tilde{a}_k \tilde{a}_k^\dagger \vert 0 \rangle \]  
(B.29)

\[ = \int d^3p d^3q \langle 0 \vert -\alpha_p \beta_{kp}(-\beta_{kq} \alpha_q^*) \vert 0 \rangle \]  
(B.30)

\[ = \int d^3p d^3q \langle 1_p \vert 1_q \rangle \beta_{kp} \beta_{kq}^* \]  
(B.31)

\[ = \int d^3q |\beta_{kq}|^2. \]  
(B.32)

From the above it is clear that ‘spontaneous’ particle production will occur whenever

\[ \beta_{kq} \neq 0. \]  
(B.33)

The value of \( \beta_{kq} \) depends on the nature of the transition state. We should notice that we cannot set \( \beta_{kq} = 0 \) in general (e.g. by redefining the coordinates). Even if we choose a free falling coordinate system, particle creation may occur.
B. Gravitational particle production

B.3 The Robertson-Walker universe

We will now calculate the Bogoliubov coefficient $\beta_{qk}$ in a Robertson-Walker background. We will assume that the field is non-minimally coupled, that is, there is a direct interaction between the field and the curvature, parameterised by $\xi$. This leads to the modified Klein-Gordon equation

$$(-\Box + m^2 + \xi R)\sigma = 0, \quad (B.34)$$

We will now consider the RW-metric

$$ds^2 = -dt^2 + a(t)^2 d\mathbf{x}^2 = C(\eta)(-d\eta^2 + d\mathbf{x}^2), \quad (B.35)$$

where $\eta$ is the conformal time. Note that $C(\eta)$ is the squared scale factor. We must take care to take all time-differentiations with respect to $\eta$, not coordinate time.

For the curvature, we find

$$R = 3 \left( \frac{C''}{C^2} - \frac{1}{2} \frac{C'^2}{C^3} \right). \quad (B.36)$$

The d’Alembertian becomes

$$\Box \sigma = \frac{1}{C} \left( \Delta \sigma + \sigma'' + \frac{C' \sigma'}{C} \right). \quad (B.37)$$

We write the solution to the KG-equation as a plane wave (which is the general solution in Minkowski space-time) times a factor which depends on time, to account for the expansion of the metric.

$$u_k = (2\pi)^{-3/2} C(\eta)^{-1/2} e^{ik\mathbf{x}} \chi_k(\eta) \quad (B.38)$$

Substituting this form for $u_k$, we obtain

$$\chi_k''(\eta) + \left[ \omega^2(\eta) + (\xi - 1/6)C(\eta)R(\eta) \right] \chi_k(\eta) = 0, \quad \omega^2(\eta) = m^2 C(\eta) + k^2. \quad (B.39)$$

For our purposes, it is sufficient to consider the case $m = 0$. We rewrite the equation for $\chi_k$ to get

$$\chi_k''(\eta) + k^2 \chi_k(\eta) = -(\xi - 1/6)C(\eta)R(\eta) \chi_k(\eta) \equiv J(\eta). \quad (B.40)$$

We may write down the integral equation for this problem in terms of the retarded Green function.

$$\chi_k(\eta) = \chi_k^{\text{in}}(\eta) + \int_{-\infty}^{\eta} d\tilde{\eta} \, G_k^R(\eta - \tilde{\eta}) J(\tilde{\eta}), \quad (B.41)$$

where $G_k^R$ obeys

$$G_k^{\alpha R}(\eta - \tilde{\eta}) + k^2 G_k^R(\eta - \tilde{\eta}) = \delta(\eta - \tilde{\eta}) \quad (B.42)$$

$$G_k^R(\eta - \tilde{\eta}) = 0 \quad , \quad \eta < \tilde{\eta.} \quad (B.43)$$

We must take note that $J(\eta)$ still contains the sought-for function $\chi_k(\eta)$. Equation (B.41) is not an explicit solution for $\chi_k(\eta)$. The function $\chi_k^{\text{in}}$ is the incoming solution, found by solving the KG-equation in the limit $\eta \to -\infty$. For the integral (B.41) to converge, $J(\eta)$ must go to zero sufficiently fast for $\eta \to \pm \infty$. The in-solution is easily found to be

$$\chi_k^{\text{in}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}. \quad (B.44)$$
B. Gravitational particle production

The Green function for \( \eta > \tilde{\eta} \) is (consult any book on quantum field theory)

\[
G^R_{\mathbf{k}}(\eta - \tilde{\eta}) = -\frac{i}{2\hbar} \left[ e^{ik(\eta - \tilde{\eta})} - e^{-ik(\eta - \tilde{\eta})} \right] = i \left[ \chi^\text{in}_k(\eta)\chi^\text{in*}_k(\tilde{\eta}) - \chi^\text{in*}_k(\eta)\chi^\text{in}_k(\tilde{\eta}) \right]. \quad (B.45)
\]

For large \( \eta \), we assume that the integral solution does not contribute significantly to \( \chi_k(\eta) \), and we may safely let \( \eta \to \infty \). The outgoing (large \( \eta \)) solution is thus

\[
\chi^\text{out}_k(\eta) = \chi^\text{in}_k(\eta) + i \int_{-\infty}^{\infty} d\tilde{\eta} \left[ \chi^\text{in}_k(\eta)\chi^\text{in*}_k(\tilde{\eta}) - \chi^\text{in*}_k(\eta)\chi^\text{in}_k(\tilde{\eta}) \right] J(\tilde{\eta}) \quad (B.46)
\]

In terms of Bogoliubov coefficients, we also have

\[
\chi^\text{out}_k(\eta) = \alpha_k \chi^\text{in}_k(\eta) + \beta_k \chi^\text{in*}_k(\eta), \quad (B.47)
\]

where

\[
\alpha_k = \alpha_{kk} \quad (B.48) \\
\beta_k = \beta_{k-k} \quad (B.49)
\]

and we may identify the coefficients by

\[
\alpha_k = 1 + i \int_{-\infty}^{\infty} d\eta \ J_k \chi^\text{in*}_k(\eta) \quad (B.50) \\
\beta_k = -i \int_{-\infty}^{\infty} d\eta \ J_k \chi^\text{in}_k(\eta). \quad (B.51)
\]

Unfortunately, these expressions cannot be evaluated exactly in all but the simplest cases. We may, however, simplify the expression for \( \beta_k \) if the parameter \( (\xi - 1/6) \) is small. We are then allowed to replace \( \chi_k(\eta) \) with \( \chi^\text{in}_k(\eta) \) to first order in \( J(\eta) \), and \( \beta_k \) is

\[
\beta_k = \frac{i}{2\hbar} \int_{-\infty}^{\infty} d\eta \ e^{-2ik\eta} V(\eta), \quad (B.52)
\]

defining

\[
V(\eta) = (\xi - 1/6)C(\eta)R(\eta). \quad (B.53)
\]

This expression suffices to calculate the particle production in going from one RW-universe to another. We have the standard definitions

\[
n = (2\pi^2a^3)^{-1} \int_{0}^{\infty} dk \ k^2 |\beta_k|^2 \quad (B.54) \\
\rho = (2\pi^2a^4)^{-1} \int_{0}^{\infty} dk \ k^3 |\beta_k|^2, \quad (B.55)
\]

which may be expressed in coordinate-space integrals:

\[
n = (16\pi^2a^3)^{-1} \int_{-\infty}^{\infty} d\eta \ V(\eta)^2 \quad (B.56) \\
\rho = -(32\pi^2a^4)^{-1} \int_{-\infty}^{\infty} d\eta \ d\tilde{\eta} \ ln(\mu|\eta - \tilde{\eta}|) V'(\eta)V'(\tilde{\eta}). \quad (B.57)
\]

\( \mu \) is an arbitrary mass scale. If we demand that

\[
\int_{-\infty}^{\infty} d\eta \ V'(\eta) = 0, \quad (B.58)
\]

\( \rho \) will be independent of \( \mu \). One should always check to see if (B.58) holds.
B. Gravitational particle production

Proof of rel. B.57

\[ \rho = (8\pi^2 a^4)^{-1} \int_0^\infty dk \int_{-\infty}^\infty d\eta d\bar{\eta} \ k e^{-2ik(\eta - \bar{\eta})} V(\eta) V(\bar{\eta}) \]  
(B.59)

\[ = (8\pi^2 a^4)^{-1} \int_0^\infty dk \int_{-\infty}^\infty d\eta d\bar{\eta} \, \partial_\eta \left( \frac{i}{2} e^{-2ik(\eta - \bar{\eta} - i\epsilon)} \right) V(\eta) V(\bar{\eta}) \]  
(B.60)

\[ = (32\pi^2 a^4)^{-1} \int d\eta d\bar{\eta} \, \partial_\eta \left( \frac{1}{\eta - \bar{\eta} - i\epsilon} \right) V(\eta) V(\bar{\eta}) \]  
(B.61)

\[ = (32\pi^2 a^4)^{-1} \int d\eta d\bar{\eta} \, \partial_\eta \left( P \frac{1}{\eta - \bar{\eta} - i\epsilon} - i\pi \delta(\eta - \bar{\eta}) \right) V(\eta) V(\bar{\eta}). \]  
(B.62)

\( P \) is the principal value. The last term evaluates to

\[ \int_{-\infty}^{\infty} d\eta d\bar{\eta} \, V(\eta) V(\bar{\eta}) \partial_\eta \delta(\eta - \bar{\eta}) = - \int_{-\infty}^{\infty} d\eta \, V'(\eta) V(\bar{\eta}) \delta(\eta - \bar{\eta}) \]  
(B.63)

\[ = - \int_{-\infty}^{\infty} d\eta \, V'(\eta) V(\eta) \]  
(B.64)

\[ = - \frac{1}{2} V(\eta)^2 \bigg|_{\eta = \infty} = 0. \]  
(B.65)

To evaluate the first term, we note that the principle value may also be written as the derivative of a logarithm

\[ P \frac{1}{\eta - \bar{\eta}} = -\partial_\eta \ln |\eta - \bar{\eta}|, \]  
(B.66)

as can be checked by applying this identity to a test function. We finally have

\[ \rho = -(32\pi^2 a^4)^{-1} \int_{-\infty}^{\infty} d\eta d\bar{\eta} \, V(\eta) V(\bar{\eta}) \partial_\eta \ln |\eta - \bar{\eta}| \]  
(B.67)

\[ = -(32\pi^2 a^4)^{-1} \int_{-\infty}^{\infty} d\eta d\bar{\eta} \, V'(\eta) V'(\bar{\eta}) \ln |\eta - \bar{\eta}| \]  
(B.68)

and (B.57) has been proven.

B.4 Transition from a De Sitter Universe to a radiation dominated universe

We will now look at the transition from a De Sitter (cosmological constant dominated) universe to a radiation dominated universe in some detail, following [26]. First we investigate some properties of both universes.

**De Sitter universe**

A vacuum dominated universe evolves according to

\[ a(t) = a_0 e^{Ht} \]  
(B.69)

in comoving coordinates, where \( H \) is constant. Integrating (2.24) explicitly, we have

\[ \eta = \eta^\infty - \frac{e^{-Ht}}{a_0 H}, \]  
(B.70)

\[ C(\eta) = \frac{1}{[H(\eta - \eta^\infty)]^2}. \]  
(B.71)
B. Gravitational particle production

The curvature is then given by

$$R(\eta) = 3 \left( \frac{C''(\eta)}{C(\eta)^2} - \frac{1}{2} \frac{C''(\eta)^2}{C'(\eta)^3} \right) = 12H^2. \quad (B.72)$$

In this case, \( J(\eta) \) falls off as \( \eta^{-2} \), just enough to guarantee convergence in the past. Convergence is not achieved in late-time De Sitter space. Fortunately, \( J(\eta) = 0 \) for the radiation dominated era.

If we want to use the formulæ (B.56, B.57), we must have an unambiguous definition of the vacuum state. Such a definition cannot be given in general, but in [28] it is shown that the energy density of a system of massless particles with positive conformal coupling \( \xi \) evolves to a limiting value. This so-called Bunch-Davies-attractor is just the in-solution (B.44). The energy-momentum tensor is given by [27]

$$\langle T_{\mu\nu} \rangle_{BD} = - \frac{g_{\mu\nu}}{4608\pi^2} R^2 \left( (1 - 6\xi)^2 - \frac{1}{30} \right). \quad (B.73)$$

For the energy density in a comoving volume, we have

$$\rho_{BD,\text{com.}} = \langle T^{00} \rangle_{BD,\text{com.}} = \frac{H^4}{32\pi^2} \left[ \frac{1}{30} - (1 - 6\xi)^2 \right], \quad (B.74)$$

and the physical energy density indeed goes to zero as the scale factor grows. The state \( \chi^\text{in} \) can thus be considered the vacuum state.

Radiation dominated universe

For an RD universe, the scale factor is given by

$$a(t) = a_0 \sqrt{t - t_0} \quad (B.75)$$

and we have thus

$$\eta = \frac{2}{a_0} \sqrt{t - t_0} + \eta^0, \quad (B.76)$$

$$C(\eta) = \left[ \frac{a_0^2}{2} (\eta - \eta^0) \right]^2. \quad (B.77)$$

The curvature is zero

$$R(\eta) = 0, \quad (B.78)$$

$$J(\eta) = 0. \quad (B.79)$$

Since the radiation era is Minkowskian in the far future, we may use the Minkowskian vacuum.

The transition

We could now substitute the values we found for \( J(\eta) \) into (B.57), but this gives an infinite result. This is caused by the discontinuity in \( J(\eta) \) (see [26]). We therefore must consider a smooth function \( f \) that makes a transition between the two scale factors. We define

$$C(\eta) = f(H\eta), \quad (B.80)$$
B. Gravitational particle production

with

$$f(x) = \begin{cases} 
  x^{-2}, & x < -1 \\
  \cdots, & -1 < x < x_0 - 1 \\
  c_1(x + c_2)^2, & x > x_0 - 1
\end{cases} \quad (B.81)$$

The value of $\eta^\infty$ has been chosen zero and the value of $\eta^0$ has been absorbed into the constant $c_2$. This function defines a transition from a De Sitter to an RD universe on a timescale

$$\Delta \eta = x_0 / H. \quad (B.82)$$

For $H\eta < -1$, the dS regime applies, for $H\eta > x_0 - 1$, the RD regime applies. The exact nature of the intermediate function for $-1 < x < x_0 - 1$ is not important. If we substitute this scale factor in (B.57), this leads to

$$\frac{1}{\rho} = \frac{(1 - 6\xi)^2}{2\pi^2 a^4} F(x_0), \quad (B.83)$$

where

$$F(x_0) = -\int_{-\infty}^{x_0-1} dx dx' \ln |x - x'| \tilde{V}'(x) \tilde{V}'(x'), \quad (B.84)$$

$$\tilde{V}(x) = f^{-2}(f'' f - 1/2 f'^2). \quad (B.85)$$

The integral is truncated for $H\eta > x_0 - 1$, since $V'(\eta) = 0$ from there on anyway. For $x < -1$, we have $\tilde{V}(x) = 4x^{-2}$ and for $x > x_0 - 1$, we have $\tilde{V}(x) = 0$. We approximate $\tilde{V}(x)$ by a straight line with slope $-4/x_0$ in the region $-1 < x < x_0 - 1$

$$\tilde{V}(x)' = \begin{cases} 
  -\frac{8}{x^2}, & x < -1 \\
  -\frac{8}{x_0}, & -1 < x < x_0 - 1 \\
  0, & x > x_0 - 1
\end{cases} \quad (B.86)$$

The value of the integral is

$$F(x_0) = 4 + 16 \frac{x_0 + 1}{x_0 - 1} \ln x_0. \quad (B.87)$$

Assuming that the transition takes place on a time-scale that is comparable to the Hubble scale during inflation ($\Delta \eta \approx H$), we approximate

$$F(x_0) = 36, \quad x_0 \approx 1. \quad (B.88)$$

The particle density then becomes

$$\rho = \frac{(1 - 6\xi)^2 H^4}{2\pi^2 a^4}. \quad (B.89)$$

If we denote the Hubble rate at the end of inflation by $H_e$, this can be written as

$$\rho = c H_e^4 \left( \frac{a}{a_e} \right)^{-4}, \quad (B.90)$$

assuming that the produced matter is highly relativistic. The constant $c$ may be estimated to be approximately $c = 0.05$. However, in [26], a value of $c = 0.01$ is argued for, assuming that $x_0 \ll 1$. But such a quick transition from inflation to radiation
dominance seems a little bit contrived. In [29], a value of $c = 0.05$ is also found, but in [24] a much smaller value is proposed, $c = 0.0015$. These different estimates depend on the specifications of the transition and it is not easy to determine which of the values is the 'best'. For definiteness, we will stick to the value for $c$ that is also used by Peebles and Vilenkin in describing their $\sigma^\pm$ model [33].

The result we have found is valid for one scalar field. Of course, the particle production will not be confined to just one field. To give an estimate for the energy density in particles just after inflation, we treat each degree of freedom as one scalar field. The total energy density is then given by

$$\rho = 0.01 g_* H_i^4 \left( \frac{a}{a_e} \right)^{-4},$$  

where $g_*$ denotes the effective number of degrees of freedom.
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In this appendix the mechanism for generating structure in the universe after inflation will be discussed. The formation of structure is ultimately caused by quantum fluctuations of the inflaton field. During inflation, all length scales that are of cosmological interest will exit the Hubble scale. The length scale then grows so large that the quantum fluctuations on that scale ‘freeze in’. As the universe starts decelerating, all length scales eventually re-enter the Hubble scale. At that moment, the fluctuations come into causal contact again. Zones with a somewhat higher density than that of the background will attract more matter, because of the gravitational attraction. This process takes place before decoupling of radiation and matter, while the universe is still radiation dominated. Because relativistic matter resists compression, oscillations will start in the hot plasma.

When the universe cools and the matter content becomes non-relativistic, the oscillations stop. Places where the density is high will exhibit enhanced clustering, while in places with lower density, the density will diminish. The lumps of clustered matter will in time develop to the structures that we observe today.

The fluctuations of the matter density are left as a ‘fingerprint’ in the cosmic microwave background radiation (CMBR). At decoupling, more light is emitted from places where the density is higher. We can therefore observe the size of the fluctuations on every length scale at the time of decoupling by measuring the correlation function of the perturbations of the CMBR at that particular length scale.

The task is now of course to relate the perturbation spectrum of the CMBR to the dynamics of the inflaton field.

Although the principle is not difficult to understand, the calculations will become very involved. In this appendix, a short overview of the calculations is given, without the aim to present a full derivation. A more elaborate derivation can be found in e.g. [63, 64, 67, 70, 71, 11, 8].

C.1 Some statistical concepts

Since the perturbations in the inflaton field are stochastic, we should treat the CMBR perturbations by means of statistical methods. This implies that all measurements will have a minimal level of uncertainty, caused by the fact that our universe is not infinite, but rather a finite realisation of a stochastic process. In this section, the statistical concepts that we will need will be explained.

The most important assumption we make about the perturbations of the inflaton field is that they are small. At least, we assume that we are in the linear regime, which means that all equations that we will find are linear. We expand a general perturbed
C. Perturbations to the background radiation

quantity $g$ in its Fourier modes:

$$g(x, t) = \int \frac{dk}{(2\pi)^3} g_k(t) e^{ikx}. \quad (C.1)$$

Because the perturbations are small, the different $k$-modes will not mix, and we may treat each $g_k$ separately.

**Transfer function**

The fundamental quantity which causes all perturbations in the universe is the perturbation of the inflaton field $\delta \phi$. We may relate every perturbation to the perturbation of the field. When the field fluctuations are formed, they are clearly in the quantum regime, and should be treated as such. When a fluctuation scale crosses the Hubble scale, the fluctuations on that scale freeze in and become classical quantities. This happens a few e-folds after the scale has left the Hubble scale, at time $t = t_*$. Some time after inflation, the inflaton field has transferred all its energy to ordinary matter and we cannot longer relate perturbations to $\delta \phi$. Therefore, we must choose an other quantity to relate the perturbations to. As we will see later, the curvature perturbation $R$ is just $\delta \phi$ for this purpose, since it will take on a constant value after the field fluctuations freezes in until the scale reentries the Hubble scale. We write

$$g_k(t) = T_g(t, k) R_k. \quad (C.2)$$

$T_g$ is called the transfer function. The curvature perturbation is dependent on the scale $k$, but the transfer function only depends on the magnitude of the wave vector, since the evolution equations which govern the behaviour of the perturbations are rotationally invariant.

**Power spectrum**

The power spectrum of a perturbation is defined by

$$P_g(k) = \frac{k^3}{2\pi^2} \langle |g_k|^2 \rangle. \quad (C.3)$$

By $\langle f \rangle$, we denote the expectation value of the quantity $f$. We assume that the expectation value of the perturbations of the field does not depend on the direction of $k$, but only on its magnitude:

$$P_g(k) = \mathcal{P}_g(k). \quad (C.4)$$

With this property, the definition for $\mathcal{P}$ is particularly useful, since

$$\int \frac{dk}{k} \mathcal{P}_g(k) = \int \frac{dk}{2\pi^2} \langle |g_k|^2 \rangle = \langle |g(x)|^2 \rangle$$

(C.5)

gives the expectation value for the perturbation in coordinate space. This expression would not hold if $\langle |g_k|^2 \rangle$ depended on direction. To summarise: the perturbations are not homogeneous, but their expectation value is.

**Correlation function and spherical harmonics**

The basis quantities of physical interest that can be measured from the CMBR are correlation functions. Since the evolution equations for the perturbations are rotationally invariant, the expectation value for the perturbations should be independent of the
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direction. This does not mean that the correlation between perturbations at different length scales will be zero, merely that this correlation is not dependent on the patch of sky that is observed. The temperature fluctuations in the background are defined by

$$\frac{\delta T}{T}(\theta, \phi) = \frac{T(\theta, \phi) - T_b}{T_b},$$  \hspace{1cm} (C.6)

where $T_b \approx 2.725$ is the background temperature. The correlation function for the perturbations depends only on the angular distance under consideration and not on the direction. It is defined by

$$C(\alpha) = \left\langle \frac{\delta T}{T}(\hat{e}) \frac{\delta T}{T}(\hat{e}') \right\rangle.$$ \hspace{1cm} (C.7)

This function compares the perturbations in two directions, $\hat{e}$ and $\hat{e}'$. The angle $\alpha$ is given by

$$\cos \alpha = \hat{e} \cdot \hat{e}'.$$ \hspace{1cm} (C.8)

Since the background is isotropic, it is natural to perform all calculations in spherical harmonics. The fluctuations are then expanded as

$$\frac{\delta T}{T}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi).$$ \hspace{1cm} (C.9)

The spherical harmonics are defined by

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P^m_l(\cos \theta) e^{im\phi},$$ \hspace{1cm} (C.10)

where $P^m_l(x)$ is an associated Legendre function. More information can be found in [7].

Substituting this form in the correlation function leads to

$$C(\alpha) = \sum_{l,m} \sum_{l',m'} \langle a_{lm} a_{l'm'}^* \rangle Y_{lm}(\hat{e}) Y_{l'm'}^*(\hat{e}').$$ \hspace{1cm} (C.11)

Because of statistical isotropy, we have

$$\langle a_{lm} a_{l'm'}^* \rangle = C_l \delta_{l'l} \delta_{mm'},$$ \hspace{1cm} (C.12)

and

$$C(\alpha) = \sum_{l,m} C_l Y_{lm}(\hat{e}) Y_{l'm'}^*(\hat{e}').$$ \hspace{1cm} (C.13)

The addition theorem for spherical harmonics yields

$$C(\alpha) = \sum_l \frac{2l+1}{4\pi} C_l P_l(\cos \alpha),$$ \hspace{1cm} (C.14)

where $P_l(x)$ is the $l$th Legendre function.

Without derivation (consult [8], section 5.2) we state that $C_l$ is given by

$$C_l \approx \frac{2\pi}{l(l+1)} \frac{1}{4} \delta_H^2(k), \hspace{1cm} k = \frac{lH_0}{2},$$ \hspace{1cm} (C.15)
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provided that the spectral index $n$ is close to one, or if $l \gg |n|$ and $l \gg 1$.

$$\delta_H(k) = \frac{4}{25} P_R(k)$$

(C.16)

is roughly the value of the density contrast $\delta$ at Hubble scale entry (see (C.129)). This expression holds for a weak dependence of $\delta_H$ on $k$. This form of $C_l$ holds at Hubble scale entry of the scale under consideration. Once the Hubble scale is entered, the spectrum starts to evolve, thus causing a more complex appearance of the spectrum. This is explained in section 4.1.

C.2 The spectrum of the field perturbation

We will now calculate the spectrum of the field perturbations. Throughout this section, we assume that $H$ is constant, since the calculations apply to the inflationary era. We start the calculation by examining the equation of motion for the field, assuming the field is dependent on space as well as time. The equation of motion is then given by

$$\ddot{\sigma} + 3H\dot{\sigma} - \nabla^2\sigma + V' = 0.$$  

(C.17)

Actually, this equation treats a perturbed field, within an unperturbed metric. The metrical perturbation, induced by the field perturbations will be ignored since this is a second order effect. Introducing the perturbation

$$\sigma = \sigma_b(t) + \delta\sigma(x,t)$$

and linearising, we have

$$\ddot{\delta\sigma} + 3H\dot{\delta\sigma} - \nabla^2\delta\sigma + V''\delta\sigma = 0.$$  

(C.19)

In momentum space, this linearised equation becomes

$$\ddot{\delta\sigma_k} + 3H\dot{\delta\sigma_k} + \left(\frac{k}{a}\right)^2 \delta\sigma_k + V''\delta\sigma_k = 0.$$  

(C.20)

At Hubble scale exit of the length scale $k^{-1}$, we have $k = aH$ and thus

$$\left(\frac{k}{a}\right)^2 + V'' = H^2 + V'' = \frac{V}{3M_p^2} + V''$$

(C.21)

and the slow-roll condition ($\eta \ll 1 \Leftrightarrow V'' \ll V/M_p^2$) implies that we may ignore the last term.

$$\ddot{\delta\sigma_k} + 3H\dot{\delta\sigma_k} + \left(\frac{k}{a}\right)^2 \delta\sigma_k = 0 \quad \text{(Hubble scale exit)}$$

(C.22)

Long after Hubble scale exit, this equation will not hold since the scale factor grows exponentially, decreasing $k/a$. At times well before Hubble scale exit, the approximation holds all the better.

We expand the field perturbation as

$$\delta\sigma_k = w_k a_k(k) + w_k^a a_l(k).$$

(C.23)
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This leads to
\[ \langle |\delta \sigma_k|^2 \rangle = \langle 0 | w_k a(k) w_k^* a^\dagger(k) | 0 \rangle = |w_k|^2 \langle 0 | (1 + a^\dagger(k) a(k)) | 0 \rangle = |w_k|^2. \] (C.24)

The \( w_k \)'s must satisfy (C.22). The solution is given by
\[ w_k = C_1 (i + x) e^{ix} + C_2 (i - x) e^{-ix}, \quad x = \frac{k}{aH}, \] (C.25)
with \( C_1, C_2 \) arbitrary complex constants. This solution must be matched to the flat-space solution for early times, which results in
\[ w_k = \frac{H}{\sqrt{2k^3}} \left( i + \frac{k}{aH} \right) e^{-i \frac{k}{aH}}. \] (C.26)

We can now evaluate the spectrum:
\[ P_\sigma = \frac{k^3}{2\pi^2} \langle |\delta \sigma_k|^2 \rangle = \frac{k^3}{2\pi^2} |w_k|^2 = \left( \frac{H}{2\pi} \right)^2 \left( 1 + \frac{k^2}{a^2 H^2} \right) e^{-t_*} \frac{H}{2\pi}. \] (C.27)

The quantity is evaluated at a time \( t_* \), a couple of e-folds after Hubble scale exit, when the field perturbations have frozen in. For \( t = t_* \), \( k/aH \approx 0 \).

C.3 The curvature perturbation

As mentioned in the first section, the curvature perturbation \( R \) is constant on larger lengths than Hubble scale. In this section, we will define the curvature perturbation and we will relate it to the perturbation of the field, see [8]. In doing so, we will have a means of linking the observations with inflationary theory, provided that we are able to link the curvature perturbation to the spectrum of the CMBR. This we will undertake in the next section.

Note! During this section, we will work in conformal time.

C.3.1 Perturbations of the metric

To calculate the curvature perturbation, we need to know how to work with perturbations of the metric, induced by the field perturbations. We will start by defining the perturbed metric:
\[ g_{\mu\nu} = g^{RW}_{\mu\nu} + \delta g_{\mu\nu} \] (C.28)
with
\[ \delta g_{\mu\nu} = a^2 \left\{ -2\phi \delta^0_\mu \delta^0_\nu - B_i (\delta^i_\mu \delta^0_\nu + \delta^0_\mu \delta^i_\nu) + (2\psi \delta_{ij} + 2E_{ij}) \delta^i_\mu \delta^j_\nu \right\}. \] (C.29)

The tensor \( E_{ij} \) should be traceless, to make this notation unambiguous.

Any rank 2 tensor \( L_{\mu\nu} \) can be composed into three parts, a scalar, a vector and a tensor part, as follows
\[ L_{\mu\nu} = L^S_{\mu\nu} + L^V_{\mu\nu} + L^T_{\mu\nu}, \] (C.30)
where we have, in \( N \) dimensions,
\[ L^S_{\mu\nu} = \frac{1}{N} \delta_{\mu\nu} + D_\mu D_\nu S \] (C.31)
\[ L^V_{\mu\nu} = \frac{1}{2} (D_\mu V_\nu - D_\nu V_\mu) \] (C.32)
\[ L^T_{\mu\nu} = L_{\mu\nu} - L^S_{\mu\nu} - L^V_{\mu\nu}. \] (C.33)
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In these expressions, \( S \) is a scalar field, \( V_\mu \) is a vector field and \( L_{\mu\nu}^T \) is a tensor. \( D_\mu \) is the covariant derivative. If we want the decomposition to be unique, we must demand

\[
D^\mu V_\mu = D^\mu L_{\mu\nu}^T = 0 \quad (C.34)
\]

and

\[
\text{Tr} L_{\mu\nu}^T = 0. \quad (C.35)
\]

A vector \( W_\mu \) can be decomposed in two parts:

\[
W_\mu = V_\mu + D_\mu S, \quad D^\mu V_\mu = 0. \quad (C.36)
\]

We will now rewrite the metric, using

\[
B_i = B_i^S + B_i^V \quad (C.37)
\]

\[
E_{ij} = (E_{ij}^S + \frac{1}{3}\delta_{ij} E^S) + \frac{1}{2}(E_{ij}^V + E_{ji}^V) + \frac{1}{2}h_{ij}. \quad (C.38)
\]

This leads to

\[
\delta g_{\mu\nu}^S = a^2 \left[ -2\phi_{\mu}^{\nu} \phi^{\mu}_\nu - B_{ij}^S (\delta_{\mu_i}^\nu \delta_{\nu_j}^\mu + \delta_{\mu_j}^\nu \delta_{\nu_i}^\mu) \\
+ (2\psi_{\mu}^{\nu} + 2(E_{ij}^S + 1/3\delta_{ij} E^S))\delta_{\mu_i}^\nu \delta_{\nu_j}^\mu \right] \quad (C.39)
\]

\[
\delta g_{\mu\nu}^V = a^2 \left[ -B^V (\delta_{\mu_i}^\nu + \delta_{\mu_j}^\nu) + (E_{ij}^V + E_{ji}^V)\delta_{\mu_i}^\nu \delta_{\nu_j}^\mu \right] \quad (C.40)
\]

\[
\delta g_{\mu\nu}^T = a^2 [2h_{ij}\delta_{\mu_i}^\nu \delta_{\nu_j}^\mu]. \quad (C.41)
\]

The Einstein equations split in three parts, which are independent of one another, a scalar, a vector and a tensor part, which just correspond to the decomposition above. Vector perturbations are not generated by inflation (see [103, section 4.5]), tensor perturbations are gravitational waves and together with the scalar perturbations, they are generated by the perturbation of the inflaton field \( \sigma \).

We will now see what happens if we transform the metric to another gauge, that is, if we change the space-time coordinates according to

\[
x \rightarrow \tilde{x} = x + \xi(x), \quad (C.42)
\]

with

\[
\xi = (\xi^0, \xi) \quad , \quad \xi = \xi^S + \xi^V. \quad (C.43)
\]

For a general tensor, we have the transformation law

\[
\tilde{L}_{\mu\nu} = \frac{\partial x^\alpha}{\partial \tilde{x}^\alpha} \frac{\partial x^\beta}{\partial \tilde{x}^\beta} L_{\alpha\beta}(x - \xi) \quad (C.44)
\]

\[
= L_{\mu\nu}(x) - L_{\mu\beta}(x)\partial_{\xi^\beta}(x) - L_{\alpha\nu}(x)\partial_{\xi^\alpha}(x) - \xi^\lambda(x)\partial_{\lambda}L_{\mu\nu}(x). \quad (C.45)
\]

If we use this transformation to find the metric in new coordinates, we obtain to first order in the perturbations

\[
\tilde{g}_{\mu\nu}^S = -a^2(1 + 2\psi - 2\partial_\mu \xi^0 - 2\xi^0 H) \quad (C.46)
\]

\[
\tilde{g}_{\mu i}^S = -a^2(B_i^S + \xi^0 - \partial_\mu \xi^S)_i \quad (C.47)
\]

\[
\tilde{g}_{ij}^S = a^2 \left[ (1 + 2\psi + 2/3E^S - 2\xi^0 H)\delta_{ij} + 2E_{ij}^S - 2\xi_{ij}^S \right] \quad (C.48)
\]

\[
\tilde{g}_{ij}^T = 2a^2 h_{ij}. \quad (C.49)
\]
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On the other hand, we have by definition
\[
\tilde{g}^0_0 = -a^2(1 + 2\tilde{\phi}) \quad (C.50)
\]
\[
\tilde{g}^0_i = -a^2\tilde{B}^S_i \quad (C.51)
\]
\[
\tilde{g}^S_j = a^2[(1 + 2\tilde{\psi} + 2/3\tilde{E}^S_S)\delta_{ij} + 2\tilde{E}^S_{ij}] \quad (C.52)
\]
\[
\tilde{g}^T_{ij} = 2a^2\tilde{h}_{ij}. \quad (C.53)
\]

Comparing the two expressions yields eventually
\[
\tilde{\phi} = \phi - \xi^0 - \xi^0\dot{\mathcal{H}} \quad (C.54)
\]
\[
\tilde{B}^S = B^S + \xi^0 - \xi^0\dot{S} \quad (C.55)
\]
\[
\tilde{\psi} = \psi + \frac{1}{3}\xi^S - \xi^0\dot{\mathcal{H}} \quad (C.56)
\]
\[
\tilde{E}^S = E^S - \xi^S \quad (C.57)
\]
\[
\tilde{h}_{ij} = h_{ij}. \quad (C.58)
\]

The last result should be expected, since gravitational waves are gauge invariant to the order we are working in.

For later reference, we derive one final result in this section. Combining the expressions in (C.54), we can make gauge invariant quantities. Two such quantities, which are often used, are given by
\[
\Psi = \psi + \frac{1}{3}E^S + \mathcal{H}(B^S - E^S) \quad (C.59)
\]
\[
\Phi = \phi + \frac{1}{a}[(B^S - E^S)a]' \quad (C.60)
\]

This concludes our survey of the perturbations to the metric.

C.3.2 Definition of the curvature perturbation

Finally, we are ready to define the curvature perturbation. It is given by
\[
\mathcal{R} = \psi + \frac{1}{3}E^S. \quad (C.61)
\]

We thus find the transformation behaviour
\[
\tilde{\mathcal{R}} = \mathcal{R} - \xi^0\dot{\mathcal{H}}. \quad (C.62)
\]

We can link the curvature perturbation directly to the spatial curvature of the metric. In matrix form, the spatial part of the metric is given by
\[
g_{ij} = a^2 \begin{pmatrix}
1 + 2\psi + 2E_{xx} & 2E_{xy} & 2E_{xz} \\
2E_{yx} & 1 + 2\psi + 2E_{yy} & 2E_{yz} \\
2E_{zx} & 2E_{zy} & 1 + 2\psi + 2E_{zz}
\end{pmatrix} \quad (C.63)
\]
and \(E_{xx} + E_{yy} + E_{zz} = 0\). We suppressed the index ‘S’, which denotes the scalar part. The vector and tensor part are ignored, since the Einstein equations decouple. The inverse of the metric is (to first order)
\[
g^{ij} = \frac{1}{a^2(1 + 6\psi)} \begin{pmatrix}
1 + 4\psi - 2E_{xx} & -2E_{xy} & -2E_{xz} \\
-2E_{yx} & 1 + 4\psi + 2E_{yy} & -2E_{yz} \\
-2E_{zx} & -2E_{zy} & 1 + 4\psi - 2E_{zz}
\end{pmatrix}. \quad (C.64) \]
The connection is
\begin{equation}
\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{jk,l}) \tag{C.65}
\end{equation}
\begin{equation}
= \delta_{ij} \psi_{,k} + \delta_{ik} \psi_{,j} - \delta_{jk} \psi_{,i} + E_{ij,k} + E_{ik,j} - E_{jk,i}. \tag{C.66}
\end{equation}

For the Riemann curvature tensor we find, still working to first order,
\begin{equation}
R^i_{jk} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} \tag{C.67}
\end{equation}
\begin{equation}
= \delta^i_{jl} \psi_{,k} - \delta^i_{jk} \psi_{,l} + \delta_{jk} \psi_{,l} - \delta_{jl} \psi_{,k} + E^i_{l,j,k} + E^i_{k,j,l} - E^i_{j,k,l}. \tag{C.68}
\end{equation}
\begin{equation}
R_{ij} = R^k_{ikj} = -\psi_{,i,j} - \delta_{ij} \psi_{,k} + E^k_{j,i,k} + E^k_{i,j,k} - E^k_{ij,k} - E^k_{k,i,j}. \tag{C.69}
\end{equation}

Finally, the spatial curvature is
\begin{equation}
3^R = g^{ij} R_{ij} = a^{-2} (-4 \psi_{,l} + 2 E^l_{m,l,m} - 2 E^l_{l,m,m}). \tag{C.70}
\end{equation}

We may evaluate the derivatives, using the fact that $E_{ij}$ is traceless:
\begin{equation}
2 E^l_{m,l,m} = \frac{4}{3} k^2 E, \quad 2 E^l_{l,m,m} = 0 \tag{C.71}
\end{equation}
and thus, in coordinate space,
\begin{equation}
3^R = - \frac{4}{a^2} \nabla^2 \left( \psi + \frac{1}{3} E \right) = - \frac{4}{a^2} \nabla^2 R. \tag{C.72}
\end{equation}

The name curvature perturbation is now clear.

### C.3.3 The curvature perturbation and the inflaton perturbation

In this subsection we will, finally, link the curvature perturbation to the inflaton perturbation. If we choose a flat coordinate system, $\psi = E^{S} = 0$, we have for a transformation from a flat to a comoving coordinate system, see (C.62)
\begin{equation}
R_{\text{com}} = - \xi^{0} \mathcal{H}. \tag{C.73}
\end{equation}

The time shift $\xi^{0}$ can be evaluated in terms of the field perturbation $\delta \phi$.

**Form of $\xi^{0}$**

We consider a general perturbed quantity $\rho$, which we write as
\begin{equation}
\rho(x, \eta) = \rho_{0}(\eta) + \delta \rho(x, \eta). \tag{C.74}
\end{equation}
$\rho_{0}(\eta)$ is the homogeneous background value of the quantity. We now change the time-coordinate
\begin{equation}
\eta \rightarrow \tilde{\eta} = \eta + \xi^{0}(x, \eta), \quad \frac{\xi^{0}}{\eta} \ll 1. \tag{C.75}
\end{equation}
We will take the spatial mean of the perturbations to be zero. We now define the quantity in new coordinates by
\begin{equation}
\tilde{\rho}(x, \tilde{\eta}) = \tilde{\rho}_{0}(\tilde{\eta}) + \delta \tilde{\rho}(x, \tilde{\eta}). \tag{C.76}
\end{equation}
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while on the other hand, we have to first order
\[ \rho(x, \eta) = \rho_\eta(\eta) + \delta \rho(\eta) + \delta \rho(x, \eta). \]  
(C.77)

These two quantities should be equal. Subtracting the latter from the former, we have
\[ 0 = \tilde{\rho}_\eta(\eta) - \rho_\eta(\eta) + \tilde{\delta \rho}(x, \eta) - \delta \rho(x, \eta) - \xi^0 \rho'_b(\eta). \]  
(C.78)

At some fixed moment in time, the background value of the quantity \( \rho \) should also be equal in both coordinate systems, and we then have
\[ \tilde{\delta \rho}(x, \eta) = \delta \rho(x, \eta) + \xi^0 \rho'_b(\eta). \]  
(C.79)

We now look at the field perturbation. In comoving coordinates, this perturbation is zero, by definition. If we go from flat to comoving coordinates, we have
\[ \xi^0 = \delta \eta = \frac{\delta \sigma(x, \eta)}{\sigma'_b(\eta)}, \]  
(C.80)

where the RHS is to be evaluated in flat coordinates. Combining all results we have
\[ \mathcal{R} = \mathcal{H} \frac{\delta \sigma}{\sigma'} = H \frac{\delta \sigma}{\sigma}. \]  
(C.81)

We have now eventually found the correspondence between \( \delta \sigma \) and \( \mathcal{R} \). The power spectra can now also be related:
\[ P_R = P_{\sigma} \left( \frac{H}{\sigma} \right)^2 = \left( \frac{H^2}{2\pi \sigma} \right)^2 t. \]  
(C.82)

C.4 The link between \( \mathcal{R} \) and the CMBR

In this section, we will show that the curvature perturbation \( \mathcal{R} \) is constant just after Hubble scale exit until re-entry. This fact will simplify the link between observations of the CMBR and the inflationary perturbations considerably, since the curvature perturbation can be determined directly from the CMBR. In doing so, the value of the curvature perturbation just after Hubble scale exit is then automatically known also. We thus have a means of probing the inflation dynamics directly. The proof of the statement that \( \mathcal{R} \) is constant is involved and only a very short summary will be presented in the next section. A much more detailed derivation is given in [11], but beware, some (minor) errors are made in this reference and it should be treated with care.

C.4.1 Proof that \( \mathcal{R} \) is constant

To prove that the curvature perturbation is constant, we must start with the first order perturbation of the Einstein equations. We can solve the behaviour of the perturbations and it is possible to link the solutions to \( \mathcal{R} \). This is not an easy task and only a brief outline is presented here, without full derivations.

The field perturbation gives rise to a perturbation of the energy momentum tensor:
\[ T_{\mu\nu} \Rightarrow T_{\mu\nu} + \delta T_{\mu\nu}. \]  
(C.83)
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Also, a perturbation of the metric is induced, which leads to a perturbation of the Einstein tensor

\[ G_{\mu\nu} \Rightarrow G_{\mu\nu} + \delta G_{\mu\nu}. \]  

(C.84)

The non-perturbed Einstein equation is satisfied (determining the evolution of the RW-metric as a function of the dominant background energy density in the universe), and the first order perturbation to the Einstein equation is

\[ \delta G_{\mu\nu} = \frac{1}{M_P^2} \delta T_{\mu\nu}. \]  

(C.85)

We will now choose a specific gauge, and evaluate this perturbed equation. For this purpose, we choose the conformal Newtonian gauge (cng). It is defined by setting

\[ E_{ij} = B_i = 0 \]  

(C.86)

\[ \psi \rightarrow -\psi_{cng} \]  

(C.87)

The gauge invariant quantities \( \Psi \) and \( \Phi \) are then given by

\[ \Psi = -\psi_{cng}, \quad \Phi = \phi_{cng}. \]  

(C.88)

The metrical perturbation is

\[ \delta g_{\mu\nu}^{cng} = a^2 \begin{pmatrix} 0 & -2\psi_{cng} & 0 \\ -2\psi_{cng} & 2\psi_{cng}\delta_{ij} \end{pmatrix}. \]  

(C.89)

With this form of the metric, the perturbation to the Einstein tensor becomes

\[ \delta G_0^0 = -\frac{2}{a^2} \left[ -3H(\dot{\Phi} - \psi') + \nabla^2(-\psi) \right] \]  

(C.90)

\[ \delta G_i^0 = -\frac{2}{a^2} [\dot{\dot{\Phi}} - \psi']_i \]  

(C.91)

\[ \delta G_j^i = \frac{2}{a^2} \left\{ \delta_j^i \left[ \Phi(2\dot{\dot{\Phi}} + \ddot{\Phi}^2) + \dot{\Phi}\dot{\Psi}' - \Psi'' - 2\dot{\Psi}^2 \Phi + \frac{1}{2} \nabla^2(\Phi + \Psi) \right] \right. \]  

\[ \left. -\frac{1}{2}(\Phi + \Psi) \right\} \]  

(C.92)

The perturbation to the energy momentum tensor can be found by substituting

\[ \sigma = \sigma_0(t) + \delta \sigma(x, t) \]  

(C.93)

and the metric perturbation in

\[ T_{\mu}^\nu = \partial^\mu \sigma \partial_\nu \sigma - g_{\mu\nu} \left[ \frac{1}{2} \partial^0 \sigma \partial_0 \sigma + V(\sigma) \right]. \]  

(C.94)

This leads to

\[ \delta T_0^0 = \frac{1}{a^2} \left( \Phi \sigma_b'^2 - \sigma'_b \delta \sigma' - a^2 \frac{\partial V}{\partial \sigma} \delta \sigma \right) \]  

(C.95)

\[ \delta T_i^0 = -\frac{1}{a^2} \sigma'_b \delta \sigma, \]  

(C.96)

\[ \delta T_j^i = \frac{\delta_j^i}{a^2} \left( \Phi \sigma_b'^2 + \sigma'_b \delta \sigma' - a^2 \frac{\partial V}{\partial \sigma} \delta \sigma \right). \]  

(C.97)
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Note the subtle difference between $\delta T_{0i}$ and $\delta T_{ij}$.

We are now ready to solve the perturbed Einstein equations. We first note a very convenient simplification. Since $\delta T_{ij}$ only has diagonal components, $\delta G_{ij}$ must also vanish for $i \neq j$. This implies that

$$(\Phi + \Psi)^{,i} = 0, \quad i \neq j.$$  \hfill (C.98)

The solution to this equation is

$$(\Phi + \Psi) = \alpha + \beta_i x^i$$  \hfill (C.99)

and regularity at infinity and a spatial mean of zero demand that $\alpha = \beta = 0$. We thus must have $\Phi = -\Psi$ for the cng. With this simplification, the perturbed Einstein equations read

$$\nabla^2 \Phi - 3H \Phi' - \Phi(2H^2 + H') = -\frac{1}{2} (\sigma' \delta \sigma' + a^2 \frac{\partial V}{\partial \sigma} \delta \sigma) \quad \text{00-component (C.100)}$$

$$H \Phi + \Phi' = \frac{\sigma' \delta \sigma}{2} \quad \text{0i-component (C.101)}$$

$$3(H \Phi)' + \Phi'' = \frac{1}{2} \left( \sigma' \delta \sigma' - a^2 \frac{\partial V}{\partial \sigma} \right) \quad \text{ij-component (C.102)}$$

In deriving this form of the equations, the first order Einstein equations are also used. To solve these three equations, we introduce new variables:

$$z \equiv \frac{a \sigma'}{H},$$  \hfill (C.103)

$$u \equiv a \sigma + z \Phi.$$  \hfill (C.104)

This transformation is highly non-trivial, but it can be shown (see [11]) that it leads to (in momentum space)

$$-k^2 \Phi = \frac{1}{2M_p^2} \frac{\mathcal{H}}{a^2} (zu' - z'u)$$  \hfill (C.105)

$$\left( \frac{a^2 \Phi}{H} \right)' = \frac{1}{2M_p} uz$$  \hfill (C.106)

$$u'' = \left( \frac{z''}{z} - k^2 \right) u.$$  \hfill (C.107)

The last equation is the most interesting. Using the Hamilton-Jacobi form of the slow-roll parameters (2.81, 2.82), we find

$$\frac{z''}{z} = \mathcal{H} \left[ \epsilon_H - \eta_H + \mathcal{H}(2 - \eta_H)(1 + \epsilon_H - \eta_H) \right] \approx 2 \mathcal{H}^2.$$  \hfill (C.108)

We can now consider two regimes in which we solve (C.107):

• $k \gg \mathcal{H}$ If this is the case, the scale $k$ lies within the Hubble scale. We may ignore the Hubble rate and (C.107) becomes

$$u'' = -k^2 u,$$  \hfill (C.110)
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with positive frequency solution

\[ u = \frac{1}{\sqrt{2k}} e^{-i k \eta} , \quad \text{within Hubble scale.} \quad (C.111) \]

- \( k \ll \mathcal{H} \)  
  If this is the case, the scale is well outside the Hubble scale and (C.107) becomes

\[ u'' = \frac{z''}{z} u. \quad (C.112) \]

The solution to this equation is

\[ u = \zeta z , \quad \text{outside Hubble scale,} \quad (C.113) \]

with \( \zeta \) a constant. The value of \( \zeta \) is given by

\[ \zeta = \Phi - \frac{\mathcal{H}}{\mathcal{H}' - \mathcal{H}^2} (\Phi' + \mathcal{H} \Phi). \quad (C.114) \]

From the definition of \( \mathcal{R} \), we see

\[ \mathcal{R}_{\text{eng}} = -\Phi \quad (C.115) \]

Using equation (2.34) and equation (C.80), we see that

\[ \zeta = -\mathcal{R}_{\text{eng}} - \mathcal{H} \delta \eta_{\text{eng}} - \mathcal{R}_{\text{com}} \]
\[ = -\mathcal{R}_{\text{com}} - \mathcal{H} \delta \eta_{\text{eng}} - \mathcal{R}_{\text{com}} - \delta \eta_{\text{com}} - \mathcal{R}_{\text{com}} \]
\[ = -\mathcal{R}_{\text{com}}. \quad (C.118) \]

We thus see that \( \zeta \) is just the negative comoving curvature perturbation. Since \( \zeta \) is constant on larger scales than the Hubble scale, so is the curvature perturbation.

C.4.2 The measurement of \( \mathcal{R} \) from the CMBR

The next step to take is to relate the curvature perturbation to the CMBR. The perturbation \( \Phi \) is just the Newtonian potential that acts on the density perturbations and it can be determined directly from the CMBR. Our goal is thus to find the relation between \( \mathcal{R} \) and \( \Phi \). This relation can be found using equation (C.106):

\[ \left( \frac{a^2 \Phi}{\mathcal{H}} \right)' = \frac{1}{2 M_p^2} \zeta^2 \]
\[ = \zeta a^2 \left( 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \right) , \quad (C.119) \]

where we used equation (2.34). Integration yields

\[ \Phi = \zeta \frac{\mathcal{H}}{a^2} \int_0^n d\eta \ a^2 \left( 1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \]
\[ = \zeta \frac{\mathcal{H}}{a^2} \int_0^n d\eta \ a^2 \left[ 1 + \left( \frac{1}{\mathcal{H}} \right)' \right] \]
\[ = \zeta \left( 1 - \frac{\mathcal{H}}{a^2} \int_0^n d\eta \ a^2 \right) . \quad (C.123) \]
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The integration constant can be set to zero, since it gives a decaying solution. The integral can be evaluated exactly, using the form of the scale factor:

\[ a \propto \eta^{3w} \].

(C.124)

Substituting this form leads to

\[ \Phi = -\mathcal{R} \left( \frac{3 + 3w}{5 + 3w} \right) \].

(C.125)

From the CMBR, the density contrast \( \delta \) is determined. It is defined by

\[ \delta \equiv \frac{\delta \rho}{\rho} \].

(C.126)

With this definition, we can write for the Newtonian (weak field) approximation to the Einstein equations

\[ \nabla^2 \Phi = \frac{\rho}{2M_P^2} \delta \].

(C.127)

Switching to momentum space, we find

\[ \Phi = -\frac{a^2}{k^2} \frac{\rho}{2M_P^2} \delta \]

(C.128)

and for the density contrast, we have

\[ \delta = \frac{k^2}{a^2} \frac{2M_P^2}{\rho} \left( \frac{3 + 3w}{5 + 3w} \right) \mathcal{R} \].

(C.129)

This equation is the direct link between the perturbations that we observe in the CMBR (\( \delta \)) and the inflation dynamics.

C.5 Tensor perturbations

Sofar, we have only been considered with scalar perturbations. Since the stress-energy tensor does not contain a pure tensor part, the equations governing the tensor perturbation are in a way simpler. This is, however, compensated for by the mass of indices arising when varying the metric. We will now calculate the tensor perturbations, which can be viewed as gravitational waves [72].

We start with the metric from (C.28). We only consider the tensor part of the perturbation, which leads to

\[ g_{\mu\nu} = g_{\mu\nu}^{RW} + a^2 h_{ij} \delta^i_{\mu} \delta^j_{\nu} \],

(C.130)

with \( h_{ij} \) the tensor perturbation.

To calculate the equation of motion for the tensor perturbations, we need to calculate the action up to second order in \( h_{ij} \). We start with

\[ \delta S = \int d^4x \sqrt{-g} \frac{2M_P^2}{\delta g_{\mu\nu}} G^\mu_{\nu} \delta g_{\mu\nu} \].

(C.131)
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where $G^{\mu\nu}$ is the Einstein tensor. We can now calculate this expression, using

\[ g_{\mu\nu} = a^2 (\eta_{\mu\nu} + h_{\mu\nu}) \]  
\[ g^{\mu\nu} = a^{-2} (\eta^{\mu\nu} - h^{\mu\nu}) \]  
\[ h_{\mu\nu} = 0 \]  
\[ h_{\mu\nu,\lambda} = 0. \]

The scale factor may be ignored because raising the indices introduces just a factor of $a^{-2}$, which cancels the factor $a^2$. This will be a great simplification. A long calculation [2, chapter 20] yields the result

\[ \delta^2 S = \int d^3x d\eta \frac{a^2}{4M_P^2} (h''_{\mu\nu} h_{\mu\nu} - h^{\mu\nu} \lambda h_{\mu\nu}). \]

The primes denote differentiation with respect to conformal time.

The action leads to the equation of motion (in momentum space)

\[ h''_{ij} + 2\mathcal{H} h'_{ij} + k^2 h_{ij} = 0. \]

We now factor the perturbations into its two polarisations:

\[ h_{ij} = \sum_{\lambda=+,\times} e^{(\lambda)}_{ij} h^{(\lambda)}. \]

Introducing the new variables

\[ v^{+,\times} = a h^{+,\times} \]  
\[ z = a, \]

we find for each polarisation the equation of motion

\[ v'' = \left( \frac{z''}{z} - k^2 \right) v, \]

exactly equal to the equation for the scalar perturbations (C.141)! The solutions are the same as in the scalar case: the perturbations are constant at larger scales than the Hubble scale. We are now in the position to compare the spectrum of the scalar perturbations and the gravitational perturbations. The scalar spectrum is given by

\[ P_R = \frac{k^3}{2\pi^2} \left| u_k \right|^2 . \]

This expression assumes that we may treat the perturbation $\mathcal{R} = \zeta$ from equation C.114 as a quantum field, with expansion coefficients $u_k$. For the gravitational perturbations we have a similar expression:

\[ P_{h_{ij}} = \frac{k^3}{2\pi^2} \left| v_k \right|^2 . \]

For a full comparison between these spectra, we need the exact solution to (C.107) and (C.141). It is given by a Haenkel function of the first kind (see [11] for details)

\[ u_k(\eta) = \frac{\sqrt{\pi}}{2} e^{i(n + \frac{1}{2})} \zeta \sqrt{-\eta} H_n^{(1)}(-k\eta) \]  
\[ v_k^{(\lambda)}(\eta) = \frac{\sqrt{\pi}}{2} e^{i(n + \frac{1}{2})} \zeta \sqrt{-\eta} H_n^{(1)}(-k\eta). \]
The indices of the Haenkel functions are not the same, because the definition for $z$ is not exactly the same in the scalar and the tensor case. We have

$$\nu = \frac{1 + \epsilon_H - \eta_H}{1 - \epsilon_H} + \frac{1}{2} \approx \frac{1}{2} (3 + 4\epsilon_H - 2\eta_H) \quad (C.146)$$

$$\nu_g = \frac{1}{1 - \epsilon_H} + \frac{1}{2} \approx \frac{1}{2} (3 + 2\epsilon_H). \quad (C.147)$$

The large scale behaviour, for $k \ll \mathcal{H}, -k\eta \to 0$, is given by

$$|u_k|^2 = \frac{C(\nu)}{\sqrt{2k}} \left( \frac{k}{aH} \right)^{\frac{3}{2} - \nu} \quad (C.148)$$

$$|v_k^{(\lambda)}|^2 = \frac{C(\nu_g)}{\sqrt{2k}} \left( \frac{k}{aH} \right)^{\frac{3}{2} - \nu_g}, \quad (C.149)$$

where the constants are approximately unity if the slow-roll regime applies. This is enough information to work out the power spectra. The final result is

$$\mathcal{P}_\mathcal{R} = \frac{1}{2M_p^2} \epsilon \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{aH} \right)^{3 - 2\nu} \quad (C.150)$$

$$\mathcal{P}_{hi} = \frac{2}{M_p^2} \left( \frac{H}{2\pi} \right)^2 \left( \frac{k}{aH} \right)^{3 - 2\nu_g}. \quad (C.151)$$

It is customary to write the power spectra in terms of a power of the scale $k$. We then have

$$\mathcal{P}_\mathcal{R} \propto k^{n-1} \quad (C.152)$$

$$\mathcal{P}_{hi} \propto k^{n_T}. \quad (C.153)$$

The numbers $(n - 1)$ and $n_T$ are called the spectral indices.\footnote{The different way of defining the spectral index is due to historical reasons. There is no physical reason for doing so.} Comparing these expressions with the ones above it gives for the spectral indices

$$n - 1 \approx 2\eta_H - 4\epsilon_H \quad (C.154)$$

$$n_T \approx -2\epsilon_H. \quad (C.155)$$

A direct calculation of $(n - 1)$ using the inflation potential is also possible. This will be done in chapter 4.
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Notes to the bibliography

To help the reader find its way in the references, they have been grouped according to subject, which should make it more easy to find useful material.

Background material

General relativity: [1, 2]
Quantum field theory: [3, 4, 5]
Galactic dynamics: [6]
Mathematics reference: [7]

Reviews and overviews

Cosmology in general: [8, 9, 10, 11]
Inflation: [12, 13]
Quintessence: [14, 15, 16, 17, 18]

Inflation

Inflation models: [19, 20, 21, 33, 34]
Attractor nature of inflation: [22, 23]
Gravitational reheating: [24, 25, 26, 27, 28, 29]
Initial conditions: [30, 31, 32]

Quintessence

Simple field models: [33, 34, 35, 36, 37, 38]
Tracking: [39, 40, 41]
Renormalization group running: [42, 43, 44, 45, 46, 47, 48, 49]
K-essence: [50, 51]
Non-minimal coupling: [52, 53, 54, 55, 56]
Changing the equations of gravity: [57, 58, 59, 60, 61, 62]

Observations

Theory of structure formation: [63, 64, 65, 66, 67, 68, 69, 70, 71, 72]
COBE: [73, 74]
WMAP: [75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89]
Polarisation of the CMBR: [77, 90, 91, 92, 93, 94, 95]
Supernovae: [96, 97]
Miscellaneous: [98, 99, 100]

Miscellaneous

High energy physics experiments: [101, 102]
Multiple field inflation: [103]
Historic papers: [104]


BIBLIOGRAPHY


