1450–1550.

- **The Collapse of the Eastern Roman Empire** 1453.
- **The Discovery of the New World** 1492.
- **The Reformation** 1517.

The Fall of Constantinople.

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The Council of Ferrara-Florence, "the Union Council" (1438-1445).

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- Johannes Argyropoulos (1415-1487).

May 29th, 1453. Constantinople falls to the Ottomans; Constantine XI Palaeologos dies.

The discovery of the Americas.

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- **October 31, 1517.** Luther posts the 95 Theses on the church door.

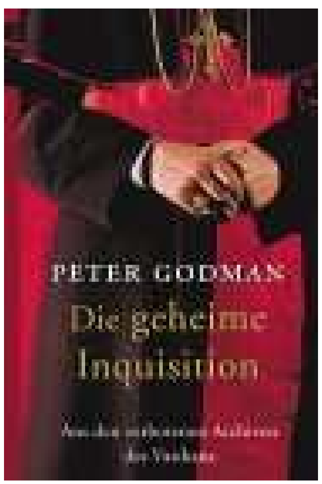
The Reformation (2).

- **1520**. Luther is excommunicated.
- **1527.** Philipp I the Magnanimous, *Landgraf* of Hesse (1504-1567) founds the first protestant university: **University of Marburg**.
- **Jun 25, 1530.** *Confessio Augustana* (written by Melanchthon).
- **1531.** Formation of the Schmalkaldic League.
- **1534.** The Act of Supremacy: Henry VIII becomes the head of the Church of England.
- **1542.** Pope Paul III founds the Roman Inquisition (= *Officium Sacrum*).

The Reformation (3).

- **1546-1547.** The Schmalkaldic war.
- **1555.** The Peace of Augsburg.
- **1571.** Pope Pius V founds the Congregation of the Index.

Peter **Godman**, Die geheime Inquisition, 2002
`amazon.de`: € 9.95



Pierre de la Ramée.

Pierre de la Ramée (Petrus Ramus; 1515-1572)

- *Animadversiones in Dialecticam Aristotelis* (1543).
- Professor at the *Collège de France*.
- Ramistic Logic. *ars disserendi*. Logic of natural discourse.
- Protestant. Died in the Massacre of St. Bartholomew (August 26th, 1572).

Port Royal.

- Cornelius Jansen (1585-1638), bishop of Ypres; *Augustinus* (1640), doctrine of strict predestination.
- Abbey of Port Royal, since 1638 centre of Jansenism.
- Pierre Nicole (1625-1695); Antoine Arnauld (1612-1694)

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- Pierre Nicole (1625-1695); Antoine Arnauld (1612-1694)
- **1662.** *La logique, ou l'art de penser*. Opposing scholasticism, “epistemological turn”.
- *Comprehension* vs *Extension*.
- Letters between Arnauld and Leibniz: 1687-1690.

Comprehension vs Extension.

- *Comprehension of X.* The set of properties that x has to have in order to be an X.
- *Extension of X.* The set of all X.
- *An example.*
  - Universe of Discourse: $U = \{a, A, A, B, b\}$
  - Properties: $\mathbf{Consonant}$, $\mathbf{Capital}$, $\mathbf{Blue}$.
  - Extensions:
    - $\mathbf{Consonant} \rightsquigarrow \{B, b\}$
    - $\mathbf{Capital} \rightsquigarrow \{A, A, B\}$
    - $\mathbf{Blue} \rightsquigarrow \{a, B, b\}$
  - The Comprehension of $\mathbf{Consonant}$ in this universe of discourse includes the property $\mathbf{blue}$.

Leibniz (1).

Gottfried Wilhelm von Leibniz (1646-1716)

- Work on philosophy, mathematics, law (Doctorate in Law from the University of Altdorf (1667), alchemy, theology, physics, engineering, geology, history.
- Diplomatic tasks (1672).
- Attempts to build a calculating machine (1672).

Leibniz (2).

- **1673-1677**: Invented calculus independently of Sir Isaac Newton (1643-1727).
- **1679**: Binary numbers.
- **1684**: Determinant theory.
- Research politics; foundation of Academies: Brandenburg, Dresden, Vienna, and St Petersburg.
- **1710**: *Théodicée*. “The best of all possible worlds”.

Leibniz (3).

Properties.

- *Identity of Indiscernibles:* If $\{\Phi\,;\,\Phi(x)\} = \{\Phi\,;\,\Phi(y)\}$, then $x = y$.
- Primary substances ("Plato", "Socrates") can be expressed in terms of properties: a uniform language of predication.
- Connected to Leibniz' monadology (1714).

Relations.

- Call for an analysis of relations.
- Attempt to reduce relations to unary predicates:

"Plato is taller than Socrates"	$\mathbf{Taller}(\mathrm{Pla}, \mathrm{Soc})$
"Plato is tall *in as much as* Socrates is short"	$\mathbf{Tall}(\mathrm{Pla}) \oplus \mathbf{Short}(\mathrm{Soc})$

Calculemus!

“quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: ***calculemus***.*”*

$\rightsquigarrow$ Arithmetization of Language and Automatization of Reasoning

Arithmetization of Language (1).

- *characteristica universalis*: general notation system for everything, based on the unanalyzable basics.
- *calculus ratiocinator*: formal system with a mechanizable deduction system.
- "*calculus de continentibus et contentis est species quaedam calculi de combinationibus*"
- The properties correspond to the natural numbers $n > 1$. The unanalyzable properties correspond to the prime numbers.
- **Example.** If *animal* corresponds to 2, and *rationalis* corresponds to 3, then *homo* would correspond to 6. If *philosophicus* corresponds to 5, then *philosophus* = *homo philosophicus* would be 30.

Arithmetization of Language (2).

animal $\rightsquigarrow 2$, *rationalis* $\rightsquigarrow 3$, *homo* $\rightsquigarrow 6$, *philosophicus* $\rightsquigarrow 5$, *philosophus* $\rightsquigarrow 30$.

- All individuals are determined by their properties, so *Socrates* is represented by a number n. Since Socrates is a philosopher, $30|n$.
- In general, "the individual represented by n has the property represented by m" is rendered as $m|n$.
- Now we can formalize $A\mathrm{a}B$ and $A\mathrm{i}B$. Let n_A and n_B be the numbers representing A and B, respectively.
  - $A\mathrm{a}B$: $n_A|n_B$.

    "Every human is an animal": $2|6$.
  - $A\mathrm{i}B$: $\exists k(n_A|k \cdot n_B)$.

    "Some human is a philosopher": $30|5 \cdot 6$.

Arithmetization of Language (3).

$A\mathrm{a}B$: $n_A|n_B$; $A\mathrm{i}B$: $\exists k(n_A|k \cdot n_B)$.

- **Barbara** becomes: "If $n|m$ and $m|k$, then $n|k$."
  So, the laws of arithmetic prove **Barbara**.
- **Darii** becomes: "If $n|m$ and there is some w such that $m|w \cdot k$, then there is some w^* such that $n|w^* \cdot k$."
  (Let $w^* := w$.)
- **But:** $A\mathrm{i}B$ is always true, as $n|n \cdot m$ for all n and m.
- If n represents *homo* and m represents *asinus*, then $n \cdot m$ would be a "man with the added property of being a donkey".
- This simple calculus is not able to deal with negative propositions.

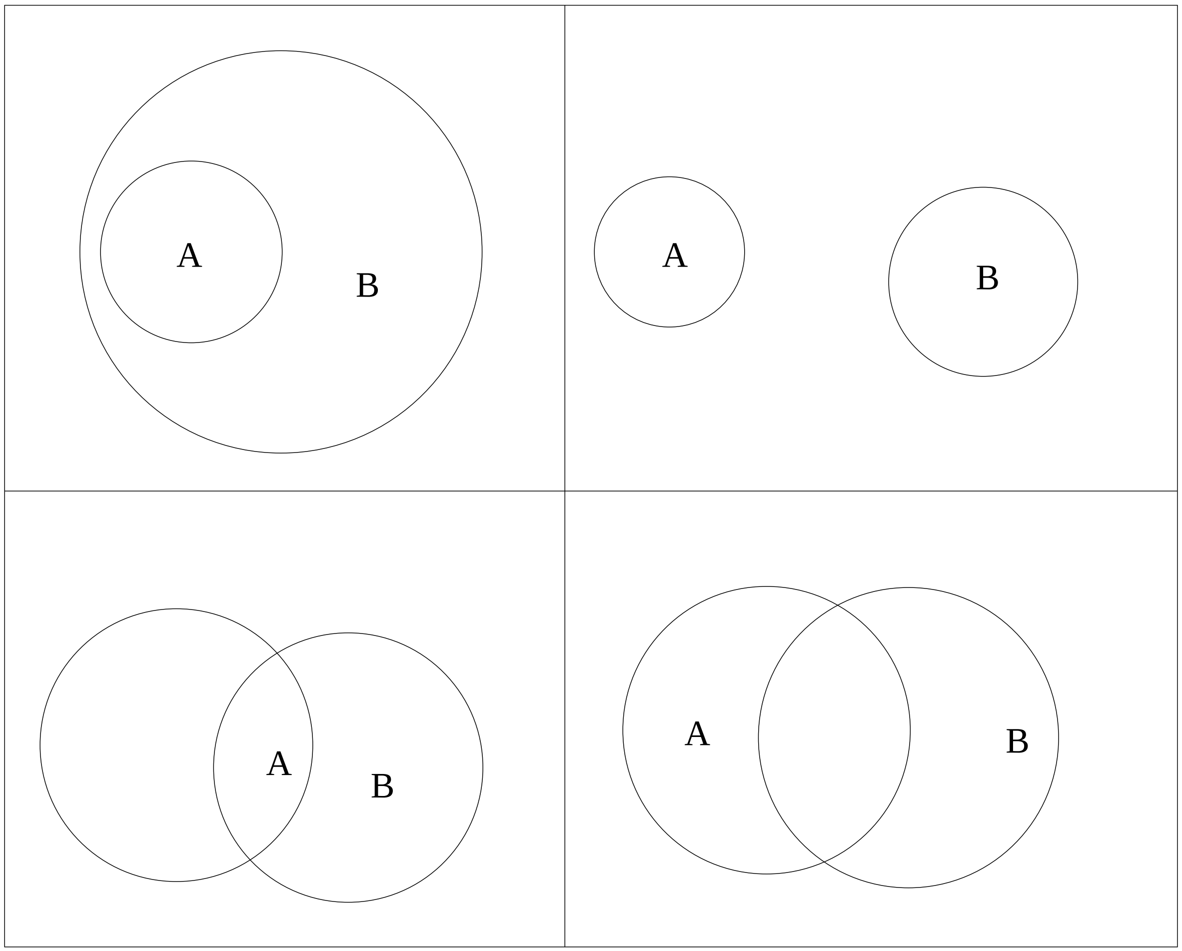
Euler.

Leonhard Euler (1707-1783)

- Successor of Nicolaus Bernoulli in St. Petersburg (1726-1727).
- Member of the newly founded St. Petersburg Academy of Sciences (1727).
- 1741-1766: Director of Mathematics, later inofficial head of the Berlin Academy.

Euler diagrams.

Lettres à une Princesse d'Allemagne (1768-72).



“Every A is B.”

“No A is B.”

“Some (but only some) A is B.”

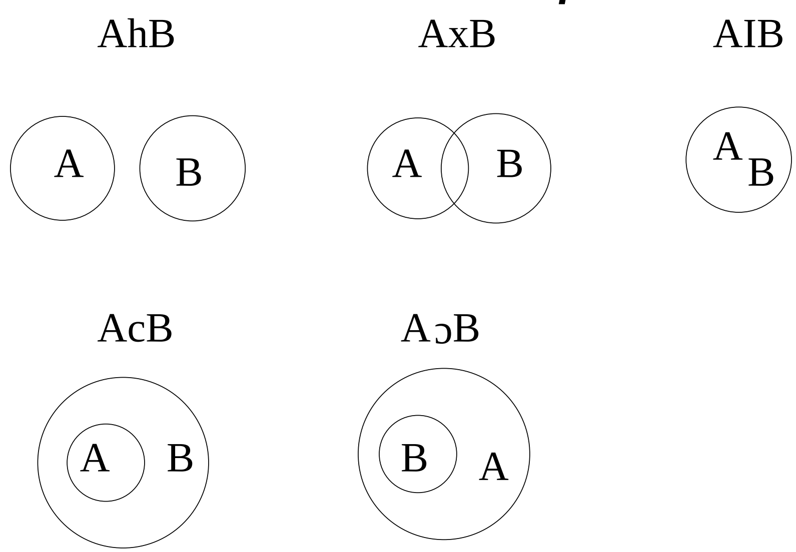
“Some (but only some) A is not B.”

(Diagrams with Existential import!)

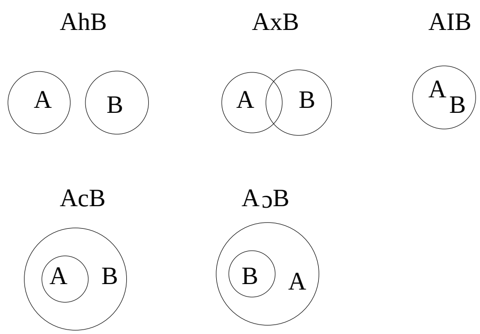
Gergonne (1).

Joseph Diaz Gergonne (1771-1859).

- Very active in the wars after the French revolution.
- Discoverer of the duality principle in geometry.
- *Essais de dialectique rationnelle* (1816-1817):



Gergonne (2).



Any two non-empty extensions ("sets") A and B are in exactly one of Gergonne's five relations:

h *est hors de*

x *s'entre-croise avec*

I *est identique à*

c *est contenue dans*

ɔ *contiens*

Gergonne (3).

Syllogisms of the first figure: $A \bullet_0 B, B \bullet_1 C : A \bullet_2 C$.

	h	x	I	c	ɔ
h		¬I,¬ɔ	h	¬I,¬ɔ	h
x	¬I,¬c		x	¬h,¬I,¬ɔ	¬I,¬c
I	h	x	I	c	ɔ
c	h	¬I,¬ɔ	c	c	
ɔ	¬I,¬c	¬h,¬I,¬c	ɔ	¬h	ɔ

If $A\mathrm{x}B$ and $B\mathrm{c}C$, then $\neg A\mathrm{I}C$ and $\neg A$ɔC.

De Morgan.

Augustus de Morgan (1806-1871).

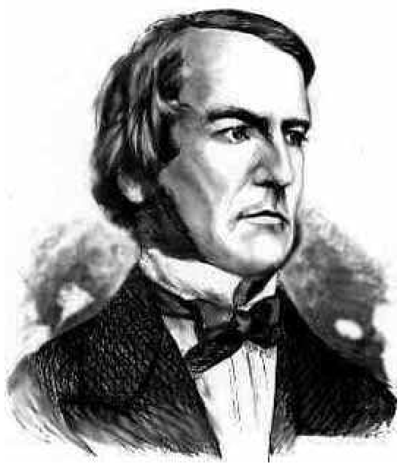
- Professor of Mathematics at UCL (1828).
- Corresponded with Charles Babbage (1791-1871) and William Rowan Hamilton (1805-1865).
- **1866.** First president of the London Mathematical Society.
- $x = 43$, $x^2 = 1849$. $y = 45$, $y^2 = 2025$.

De Morgan rules.

$$\neg(\Phi \wedge \Psi) \equiv \neg\Phi \vee \neg\Psi$$
$$\neg(\Phi \vee \Psi) \equiv \neg\Phi \wedge \neg\Psi$$

Boole (1).

George Boole (1815-1864).



- School teacher in Doncaster, Liverpool, Waddington (1831-1849).
- Correspondence with de Morgan.
- Professor of Mathematics at Cork (1849).

- Developed an algebra of logic based on the idea of taking the extensions of predicates as objects of the algebra.
- 1 is the “universe of discourse”, 0 is the empty extension.

Boole (2).

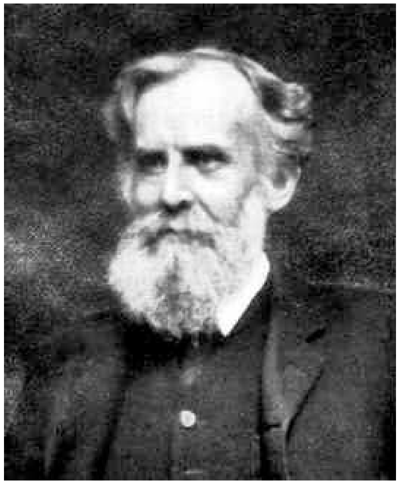
"All B are A"	$b(\mathbf{1}-a)=\mathbf{0}$.
"No B is an A"	$ba=\mathbf{0}$.
"Some B is an A"	$ba\neq\mathbf{0}$.
"Some B is not an A"	$b(\mathbf{1}-a)\neq\mathbf{0}$.

Celarent.

- *We assume:* $ba=\mathbf{0}$ and $c(\mathbf{1}-b)=\mathbf{0}$.
- *We have to show:* $ca=\mathbf{0}$.
- $ba=\mathbf{0}$ implies that $cba=c\mathbf{0}=\mathbf{0}$.
- $ca=ca-\mathbf{0}=ca-cba=a(c-bc)=a(c(\mathbf{1}-b))=ac\mathbf{0}=\mathbf{0}$.

Venn.

John Venn (1834-1923).



- Lecturer in Moral Science at Cambridge (1862).
- Area of interest: logic and probability theory.
- *Symbolic Logic* (1881).
- *The Principles of Empirical Logic* (1889).
- *Alumni Cantabrigienses*.

Venn diagrams.

Boolean Algebras (1).

A structure $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ is a Boolean algebra if

- B is a set with $0, 1 \in B$.
- $+$ and $\cdot$ are binary operations on B satisfying the commutative and associative laws.
- $-$ is a unary operation on B.
- $+$ distributes over $\cdot$ and *vice versa*: $x + (y \cdot z) = (x + y) \cdot (x + z)$ and $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
- $x \cdot x = x + x = x$ (idempotence), $- - x = x$.
- $-(x \cdot y) = (-x) + (-y)$, $-(x + y) = (-x) \cdot (-y)$ (de Morgan's laws).
- $x \cdot (-x) = 0$, $x + (-x) = 1$, $x \cdot 1 = x$, $x + 0 = x$, $x \cdot 0 = 0$, $x + 1 = 1$.
- $-1 = 0$, $-0 = 1$.

Example. $B = \{0, 1\}$.

$\cdot$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	1

Boolean Algebras (2).

$X := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}, \text{Themistokles}\}$

$\mathbf{Phil} := \{\text{Platon}, \text{Aristotle}, \text{Speusippus}\}$

$\mathbf{Rhet} := \{\text{Themistokles}\}$

$\mathbf{Acad} := \{\text{Platon}, \text{Speusippus}\}$

$\mathbf{Peri} := \{\text{Aristotle}\}$

$B := \{\varnothing, X, \mathbf{Phil}, \mathbf{Rhet}, \mathbf{Acad}, \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Peri}, \mathbf{Rhet} + \mathbf{Acad}\}.$

$$\begin{tikzcd}
& X \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbf{Phil} \arrow[dr] \arrow[drr] & \mathbf{Rhet} + \mathbf{Peri} \arrow[dl] \arrow[dr] & \mathbf{Rhet} + \mathbf{Acad} \arrow[dll] \arrow[dl] \\
\mathbf{Rhet} \arrow[dr] & \mathbf{Acad} \arrow[d] & \mathbf{Peri} \arrow[dl] \\
& \varnothing &
\end{tikzcd}$$

Boolean Algebras (3).

If X is a set, let $\wp(X)$ be the power set of X, *i.e.*, the set of all subsets of X.
For $A, B \in \wp(X)$, we can define

- $A \cdot B := A \cap B$,
- $A + B := A \cup B$,
- $0 := \varnothing$,
- $1 := X$,
- $-A := X \backslash A$.

Then $\langle \wp(X), 0, 1, +, \cdot, - \rangle$ is a Boolean algebra, denoted by $\mathbf{Pow}(X)$.

Boolean Algebras (4).

Define the notion of isomorphism of Boolean algebras: Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ and $\mathbf{C} = \langle C, \perp, \top, \oplus, \otimes, \ominus \rangle$ be Boolean algebras. A function $f : B \to C$ is a Boolean isomorphism if

- f is a bijection,
- for all $x, y \in B$, we have $f(x + y) = f(x) \oplus f(y)$, $f(x \cdot y) = f(x) \otimes f(y)$, $f(-x) = \ominus f(x)$, $f(0) = \perp$, $f(1) = \top$.

Stone Representation Theorem. If $\mathbf{B}$ is a Boolean algebra, then there is some set X such that $\mathbf{B}$ is isomorphic to a subalgebra of $\mathbf{Pow}(X)$.

Circuits.

- $+$ corresponds to having two switches in parallel: if either (or both) of the switches are **ON**, then the current can flow.
- $\cdot$ corresponds to having two switches in series: if either (or both) of the switches are **OFF**, then the current is blocked.

Partial Orders (1).

- Let X be a set. A binary relation R on X is just a subset of $X \times X$, *i.e.*, a set of ordered pairs $\langle x_0, x_1 \rangle$ with $x_0, x_1 \in X$. We write xRy for $\langle x, y \rangle \in R$.
- A binary relation R is called reflexive if for all $x \in X$, we have xRx.
- A binary relation R is called transitive if for all $x, y, z \in X$, we have that if xRy and yRz, then xRz.
- A binary relation R is called antisymmetic if for all $x, y \in X$, we have that if xRy and yRx, then $x = y$.
- A structure $\langle X, R \rangle$ is called a partial order if R is a reflexive, transitive, antisymmetric relation on X. Typically, the relation in a partial order is written as $\leq$.

Partial Orders (2).

Example. If $X = \wp(A)$, then $\subseteq$ is a reflexive, transitive and antisymmetric relation on $\wp(A)$, so $\langle X, \subseteq \rangle$ is a partial order.

Let $\mathbf{B} = \langle B, 0, 1, +, \cdot, - \rangle$ be a Boolean algebra. We define a binary relation $\leq$ on B as follows:

$$x \leq y :\equiv x \cdot y = x.$$

Note. $A \subseteq B$ if and only if $A \cap B = A$.

Theorem. If $\mathbf{B}$ is a Boolean algebra, then $\langle B, \leq \rangle$ is a partial order.

Proof. Clearly, $x \cdot x = x$, so $x \leq x$. If $x \cdot y = x$ and $y \cdot z = y$, then $x \cdot z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot y = x$, so $\leq$ is transitive. If $x \cdot y = x$ and $y \cdot x = y$, then $x = y$, so $\leq$ is antisymmetric. q.e.d.

Lattices (1).

- If $\langle X, \leq \rangle$ is a partial order and $x, y \in X$, we call z an upper bound of x and y if $x \leq z$ and $y \leq z$. We call z the least upper bound of x and y if z is an upper bound and for all upper bounds z^*, we have $z \leq z^*$.
- If $\langle X, \leq \rangle$ is a partial order and $x, y \in X$, we call z an lower bound of x and y if $z \leq x$ and $z \leq y$. We call z the greatest lower of x and y if z is a lower bound and for all lower bounds z^*, we have $z^* \leq z$.
- We call a partial order $\langle X, \leq \rangle$ a lattice if and only if for all $x, y \in X$ the least upper bound and the greatest lower bound of x and y exist.

Lattices (2).

Theorem. If B is a Boolean algebra, then $\langle B, \leq \rangle$ is a lattice.

Proof. We claim that for x and y, $x \cdot y$ is the greatest lower bound and $x + y$ is the least upper bound.

"$x + y$ is an upper bound": $x \cdot (x + y) = (x \cdot x) + (x \cdot y) = x + (x \cdot y) = (x \cdot 1) + (x \cdot y) = x \cdot (1 + y) = x \cdot 1 = x$, so $x \leq x + y$. Similarly, $y \leq x + y$.

"$x \cdot y$ is a lower bound": $x \cdot (x \cdot y) = (x \cdot x) \cdot y = x \cdot y$, so $x \cdot y \leq x$. Similarly, $x \cdot y \leq y$.

Suppose $x \leq z$ and $y \leq z$. Then $(x + y) \cdot z = (x \cdot z) + (y \cdot z) = x + y$, so $x + y \leq z$.

Suppose $z \leq x$ and $z \leq y$. Then $z \cdot (x \cdot y) = (z \cdot x) \cdot y = z \cdot y = z$, so $z \leq x \cdot y$. q.e.d.

Lattices (3).

A structure $\mathbf{L} = \langle L, +, \cdot \rangle$ is a lattice* if

- L is a set.
- $+$ and $\cdot$ are binary operations on L satisfying the commutative and associative laws.
- $x \cdot x = x + x = x$ (idempotence),
- $x + (x \cdot y) = x = x \cdot (x + y)$ (absorption).

If $\mathbf{L}$ is a lattice*, we can define a binary relation $\leq$ by

$$x \leq y :\equiv x \cdot y = x.$$

Theorem. A structure $\mathbf{L}$ is a lattice* if and only if $\langle L, \leq \rangle$ is a lattice.