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Intuitionism

Brouwer, Heyting and Dummett

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Motivation

The long belief in the principle of the excluded third in mathematics is considered by Intuitionism as a phenomenon of history of civilisation.

Intuitionism tries to explain the long persistence of this dogma by the practical validity of classical logic for an extensive group of simple everyday phenomena.

(excerpt from Brouwer 1948:94,96)

TND (tertium non datur) traced to RIO and/or RITV.

Metaphysics!

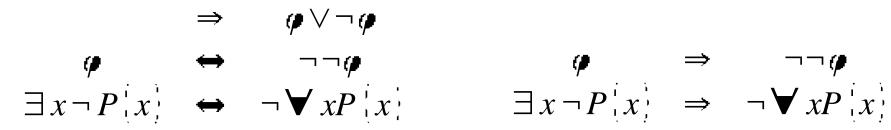
General comparison

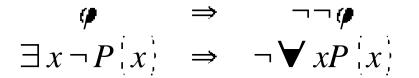
Classical mathematics

Classical logic

Intuitionistic mathematics

Intuitionistic logic





Brouwer's Mathematics

- Mathematics is synthetic.
- Truth is a priori.
- Mathematical knowledge is mind dependent.
 - Anti Realist in Ontology
 - Anti Realist in Truth Value
 - But: non-empiricist
- Found mathematics on Kantian view of time.
 - Sequence of moments gives natural numbers
 - Infinite divisibility of moments gives linear continuum
- Found geometry on reals using Cartesian techniques.

Brouwer's Mathematical Method

- Mathematics is about idealized mental constructions.
- Existential proofs require a construction procedure.
- No unknowable truths.
 - Knowledge is provability. Truths are provable statements.
- Impredicative definitions are circular.
- Impossible to consider infinite collections of mathematical objects as completed totalities.
 - Construction of the elements can never be completed.
 - Example of a meaningless sentence: The set of continuous univariate functions has property P.

Example: reals

Classical reals

- Any decimal expansion
- Digit n is fixed from the beginning for each n

Intuitionistic reals

- Early Brouwer
 - rule-governed decimal expansions (Cauchy sequences)
- Later Brouwer
 - Free-choice sequences.
 - Digit are iterated on demand by a *creative subject*.
- Potential infinity

Actual infinity

Properties of reals

- Theorems of real numbers must follow from finite amounts of information
 - The rule
 - An initial segment of the free-choice sequence
- Method of weak counterexamples
 - An object is not well-defined if it implies TND.
- Any function from reals to reals is continuous.

Heyting

Logic

- Codification of the rules of communicating mathematics via language
- Dependent on mathematics. New methods of reasoning require extensions of logical system
- Not truth conditions, proof conditions.

Heyting Calculus

A proof of $e_{\mathcal{T}}$ consists of a proof of $e_{\mathcal{T}}$ and a proof of \square

- A proof of $e_{\mathcal{T}}$ consists of a proof of $e_{\mathcal{T}}$ or a proof of \bigtriangleup
- A proof of $\mathcal{CT} \to \bigtriangleup$ consists of a method of transforming any proof of \mathcal{CT} into a proof of \bigtriangleup
- A proof of $\triangleright e_{\mathcal{T}}$ consists of a method of transforming any proof of $e_{\mathcal{T}}$ into an absurdity

A proof of $\Im x e_{\mathcal{T}}(x)$ consists of a procedure that, given *n*, produces a proof of $e_{\mathcal{T}}(n)$

A proof of $\operatorname{er}(x)$ consists of a construction of an item *n* and a proof of $\operatorname{er}(n)$

Problem and conclusion

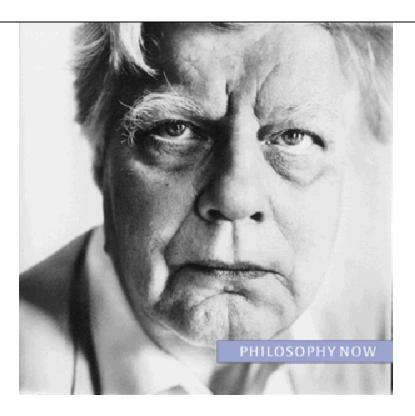
- Discontinuous functions are essential to physics
 - Brouwer: Physics holds less "mathematical truth". Divorce.

Which mathematics to pick?

- Brouwer: Intiutionistic mathematics.
- Heyting: Both, but Intiutionistic mathematics preferred.
- Bishop: Common core of both

Sir Michael A.E. Dummett

1925 -



Biography

Born 1925

1943-1947 Military Service

1944 Received into Roman Catholic Church

1947 Studies at Christ Church College, Oxford

- 1950 Graduates with First Class Honours in Philosophy, Politics and Economics, fellowship at All Souls College
- 1962-1974 Reader Philosophy Of Math. at Oxford
- 1979-1992 Professorship of Logic at Oxford

1999 Received Knighthood

Publications

- 1973 Frege: Philosophy of Language
- 1977 Elements of Intuition
- 1981 The Interpretation of Frege's Philosophy
- 1991 The Logical Basis of Metaphysics, Frege: Philosophy of Mathematics
- Collected papers:

Truth and Other Enigmas

Frege and Other Philosophers

Origins of Analytical Philosophy

Philosophy of Language

- "Why prefer Intuitionism?"
- Correctness of logic turns on questions of *meaning*.
- "Meaning consists solely in its role as instrument of communication" (connection to language).
- Meaning = Use (Manifestation Requirement).
- What is use?

Humpty Dumpty



"There's glory for you!" `I don't know what you mean by "glory,"' A lice said. Humpty Dumpty smiled contemptuously. `Of course you don't--till I tell you.I meant "there's a nice knock-down argument for you!"' `But "glory" doesn't mean "a nice knock-down argument,"' A lice objected. `When I use a word,' Humpty Dumpty said in rather a scornful tone, `it means just what I choose it to mean --neither more nor less.'

Holism

- Web of Belief.
- Reject holism and insist that each statement must have determinate individual content.
- Alternative theory: *Molecular semantics*.

Molecular Semantics

- "…individual sentences carry a content which belongs to them in accordance with the way they are compounded out of their own constituents, independent of other sentences of the language not involving those constituents.."
- Logic can be understood independently.
- LEM doesn't follow from \forall and \geq separately.
- Hence: classical mathematics is wrong!

Truth / Verifiability

Classical mathematics:

- Central notion is *truth*,
- Truth cannot be determined in general,
- Problem with Manifestation Requirement,
- Hence: realism is wrong!

Dummett:

- Replace truth by *verifiability*.
- In mathematics: verifiability is proof.
- Heyting semantics.

Reply to Dummett

- All truths are knowable, or aren't they?
- Intuitionism needs idealizations (> x > (x) > (x) > (x) > (n) > (n)).
- But not too much idealization.
- Accept Dummett?
- Reject Dummett?

Dummett's Reply

Dummett tries to particularize the theory to mathematics:

- Hard-headed finitism
- Indefinite extensibility

An intuitionist:

- Rejects holism?
- Idealizes, but not too much?
- Is in denial?