Homework Set #4

Capita Selecta: Set Theory
2016/17: 1st Semester; block a
Universiteit van Amsterdam

Homework. There will be six homework sheets; handed in by each student individually. Homework has to contain the name and student ID of the student. If your homework is handwritten, make sure that it is legible. You submit your solutions either by e-mail to `h(dot)nobrega(at)uva(dot)nl` or in person before the Tuesday lecture or by placing them in Hugo's mailbox at the ILLC at Science Park 107.

Deadline. This homework set is due on **Tuesday, 4 October 2016** before the lecture.

In all of the exercises, work in a sufficiently strong metatheory. Throughout this set of exercises, $\langle \mathbb{P}, \leq \rangle$ is a partial order and M is a countable transitive model of ZFC, unless stated otherwise.

1. Suppose $\mathbf{V} = \mathbf{L}$. Show that for all ordinals $\alpha > \omega$, the following are equivalent:
   (a) $\mathrm{L}_\alpha = \mathrm{V}_\alpha$;
   (b) $\alpha = \beth_\alpha$.

2. Let $\Phi_{\aleph_1}(x)$ be a formula that is true of x if and only if x is $\aleph_1$, let $\Phi_{\mathbb{R}}(x)$ be a formula that is true of x if and only if x is the set of all functions from ω to $\{0,1\}$, and let $\Phi_\sim(x, y)$ be a formula that is true of x and y if and only if there is a bijection between x and y. Then CH is expressible as

$$\exists x \exists y \Phi_{\aleph_1}(x) \wedge \Phi_{\mathbb{R}}(y) \wedge \Phi_\sim(x, y).$$

   Let $M_0 \subseteq M_1 \subseteq M_2$ be three transitive models of ZFC and $x, y \in M_1$ such that

$$M_1 \models \Phi_{\aleph_1}(x) \wedge \Phi_{\mathbb{R}}(y) \wedge \Phi_\sim(x, y).$$

   Show:
   (a) If $M_2 \models \neg\Phi_{\aleph_1}(x)$, then there is some ordinal $\beta > x$ such that $M_2 \models \Phi_{\aleph_1}(\beta)$.
   (b) If $M_2 \models \neg\Phi_{\aleph_1}(x)$, then $M_2 \models \neg\Phi_{\mathbb{R}}(y)$.
   (c) If $M_0 \models \neg\Phi_{\aleph_1}(x)$, then $y \notin M_0$.

3. Let $\langle \mathbb{P}, \leq \rangle$ be a partial order. Show that there is a complete Boolean algebra $\mathcal{B}$ and a map $i : \mathbb{P} \to \mathcal{B} \setminus \{0\}$ such that
   (i) the image of $\mathbb{P}$ under i is dense in $\mathcal{B} \setminus \{0\}$,
   (ii) for all $p, q \in \mathbb{P}$, we have that $p \leq q$ implies $i(p) \leq i(q)$, and
   (iii) for all $p, q \in \mathbb{P}$, we have that $p \perp q$ implies $i(p) \wedge i(q) = 0$.

4. A partial order $\langle \mathbb{P}, \leq \rangle$ is called *separative* iff whenever $p \not\leq q$, there is an r such that $r \leq p$ and $r \perp q$. Show that $\langle \mathbb{P}, \leq \rangle$ is separative iff the function i that exists by Exercise 3. is injective and satisfies

$$\forall p, q \in \mathbb{P}\,(p \leq q \;\leftrightarrow\; i(p) \leq i(q)).$$

5. Let M be a countable transitive model of ZFC. Assume that $\langle \mathbb{P}, \leq \rangle \in M$ and that $\mathbb{P}$ is infinite. Show that there is an $H \subseteq \mathbb{P}$ such that $M[H]$ is not a model of $\mathsf{ZF}-$. [*Hint.* Fix $f \in M$ such that f maps $\omega \times \omega$ injectively into $\mathbb{P}$. Choose H so that $f^{-1}(H)$ is a well-order of ω of type strictly larger than $o(M)$.]