Homework Set #6

Capita Selecta: Set Theory
2016/17: 1st Semester; block a
Universiteit van Amsterdam

Homework. Homework has to contain the name and student ID of the student. If your homework is handwritten, make sure that it is legible. You submit your solutions either by e-mail to `h(dot)nobrega(at)uva(dot)nl` or in person before the Tuesday lecture or by placing them in Hugo's mailbox at the ILLC at Science Park 107.

Deadline. This homework set is due on **Tuesday, 18 October 2016** before the lecture.

In all of the exercises, work in a sufficiently strong metatheory and assume that M is a countable transitive model of ZFC.

1. In class, we proved that if $M \models \mathsf{ZFC}$ then $M[G] \models \mathrm{AC}$ by proving that for every set x, there is a surjection from an ordinal onto x. Show $M[G] \models \mathrm{AC}$ directly by providing for every set x consisting of non-empty sets a $\mathbb{P}$-name τ for a choice function $f : x \to \bigcup x$ such that $f(z) \in z$ for all $z \in x$.

2. If X is any set, we define the *constructible hierarchy over* X as follows: let $\mathbf{L}_0(X)$ be the transitive closure of $\{X\}$; let $\mathbf{L}_{\alpha+1}(X) := \mathcal{D}(\mathbf{L}_\alpha(X))$; let $\mathbf{L}_\lambda(X) := \bigcup\{\mathbf{L}_\alpha(X)\,;\, \alpha < \lambda\}$ for limit ordinals λ; and, finally, let $\mathbf{L}(X) := \bigcup\{\mathbf{L}_\alpha\,;\, \alpha \in \mathrm{Ord}\}$. Using the same proof as for $\mathbf{L}$, one can show that $\mathbf{L}(X) \models \mathsf{ZF}$ (you may assume this).

   (a) Give a precise formulation of the following informal statement and prove it: "$\mathbf{L}(X)$ is the smallest transitive model of ZF containing X".

   (b) Assume that for some ξ, we have that $\mathbf{L}_\xi \models \mathsf{ZFC}$. Let $\mathbb{P} \in \mathbf{L}_\xi$ and assume that G is $\mathbb{P}$-generic over $\mathbf{L}_\xi$. Show that $\mathbf{L}_\xi[G] = \mathbf{L}_\xi(G)$.

3. Let $\langle \mathbb{P}_0, \leq_0, \mathbf{1}_0 \rangle$ and $\langle \mathbb{P}_1, \leq_1, \mathbf{1}_1 \rangle$ be forcing partial orders. A map $i : \mathbb{P}_0 \to \mathbb{P}_1$ is called a *complete embedding* if

   (a) for all $p, q \in \mathbb{P}_0$ we have that $p \leq_0 q$ implies $i(p) \leq_1 i(q)$,

   (b) for all $p, q \in \mathbb{P}_0$ we have that $p \perp_0 q$ if and only if $i(p) \perp_1 i(q)$, and

   (c) for all $q \in \mathbb{P}_1$ there is a $p \in \mathbb{P}_0$ such that for all $p' \leq_0 p$, we have that $i(p')$ and q are compatible.

   Show that if there is a complete embedding i from $\langle \mathbb{P}_0, \leq_0, \mathbf{1}_0 \rangle$ to $\langle \mathbb{P}_1, \leq_1, \mathbf{1}_1 \rangle$ and G is $\mathbb{P}_1$-generic over M, then $i^{-1}(G) := \{p \in \mathbb{P}_0\,;\, i(p) \in G\}$ is $\mathbb{P}_0$-generic over M.

4. Let $\mathbb{P}$ be the partial order of partial functions p from $\aleph_\omega^M$ to 2 with $|\mathrm{dom}(p)| < \aleph_\omega^M$, ordered by reverse inclusion. Suppose that G is $\mathbb{P}$-generic over M. Show that in $M[G]$, the ordinal $\aleph_\omega^M$ is countable.