CS:ST : On quantifier exchanges

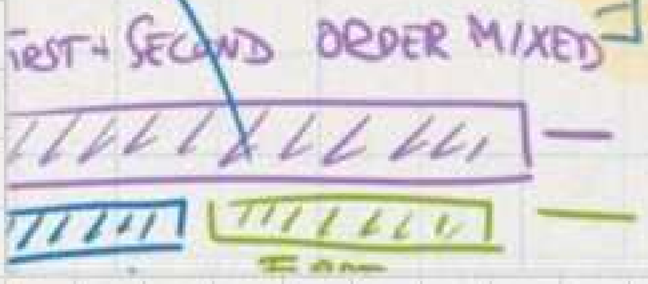
At the end of lecture IX, Paul Maurice asked about the following quantifier exchange rule listed on page 9 of the lecture notes for Lecture VIII:

Observe

$$\exists x^{(1)} \exists y^{(2)} \longleftrightarrow \exists z^{(2)}$$

$$\forall x^{(1)} \exists y^{(2)} \longleftrightarrow \exists z^{(2)}$$

$$\exists x^{(1)} \forall y^{(2)} \xrightarrow[\not\leftarrow]{} \forall y^{(2)} \exists x^{(1)}$$



$$\exists x^{(2)} \exists y^{(2)} \longleftrightarrow \exists z^{(2)}$$

Indeed, this is not correct!

It should have been:

$$\exists x^{(2)} \forall y^{(1)} \longleftrightarrow \forall y^{(1)} \exists x^{(2)}$$

As mentioned, the direction from LEFT to RIGHT is in general true, independently of the type of qfs:

$$\text{if } \mathfrak{A} \models \exists x \forall y\, \varphi(x,y), \text{ then}$$

$$\mathfrak{A} \models \forall y \exists x\, \varphi(x,y).$$

The direction from RIGHT to LEFT is not in general true, but is true in this particular case.

Suppose

$$\mathcal{A}^{(2)} \models \forall y^{(1)} \exists x^{(2)} \varphi(y^{(1)}, x^{(2)})$$

Then $x^{(2)}$ stands for an element $x \in \omega^\omega$ and $y^{(1)}$ for a natural number $n \in \omega$.

So the formula says:

for all n there is an $x_n \in \omega^\omega$ s.t.

$$\varphi(n, x_n)$$

We define $\hat{x}(\langle k, \ell \rangle) := x_k(\ell)$.

$\hat{x}$ is a single element of ω^ω that encodes all of the x_n; in particular,

$(\hat{x})_n(k) = x_n(k)$

[cf. Andretta p. 72 for the definition of $(f)_n$.]

Define $\hat{\varphi}(v^{(1)}, v^{(2)}) :\Longleftrightarrow \varphi(v^{(1)}, (v^{(2)})_{v^{(1)}})$

Then

$\mathcal{A}^{(2)} \models \forall y^{(1)} \hat{\varphi}(y^{(1)}, \hat{x})$, and thus

$\mathcal{A}^{(2)} \models \exists x^{(2)} \forall y^{(1)} \hat{\varphi}(y^{(1)}, x^{(2)})$.