LECTURE 4 CS:ST 2020

- FRAGMENTS OF AC.
- $AC_I(X)$
- $AD \Rightarrow \neg AC$.

PATHOLOGICAL MODELS OF ZF

IF ZF IS CONSISTENT SO ARE:

① ZF + "$\mathbb{R}$ IS COUNTABLE UNION OF COUNTABLE SETS"

② ZF + "CONTINUITY IS NOT THE SAME AS SEQUENTIAL CONTINUITY"

DEF WE CALL AXIOM OF COUNTABLE CHOICE AC_ω THE STATEMENT

$$\forall X \; AC_\omega(X)$$

MOST OF THE TIMES $AC_\omega(\mathbb{R})$ IS ENOUGH.

DEF LET X BE A SET. WE CALL AXIOM OF DEPENDENT CHOICE ON X $DC(X)$ THE SENTENCE:

$$\forall R \subseteq X \times X \left[\underbrace{\forall x \in X \, \exists y \in X \text{ st } x R y}_{} \Rightarrow \exists f \in X^{\omega} \, \forall n \in \omega \; f(n) \, R \, f(n+1) \right]$$

WE CALL AXIOM OF DEPENDENT CHOICE DC THE STATEMENT

$$\forall X \, (\underbrace{X \neq \emptyset}_{\text{BECAUSE WE WANT } AC \Rightarrow DC.} \Rightarrow DC(X))$$

<u>PROPOSITION</u> $AC \Rightarrow DC$.

<u>PROPOSITION</u> $DC \Rightarrow AC_\omega$

<u>PROOF</u>

I will prove

$$\forall X \quad DC(\omega \times X) \Rightarrow AC_\omega(X)$$

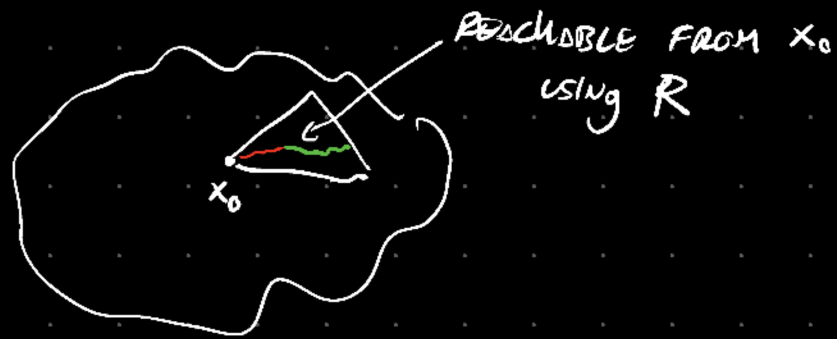
<u>FACT 1</u> IF $f: X \twoheadrightarrow Y$ AND $Y \neq \emptyset$ THEN $DC(X) \Rightarrow DC(Y)$

$$[\![\; x R' y \Leftrightarrow f(x) R f(y) \;]\!]$$

$$R' \subseteq X \times X \qquad R \subseteq Y \times Y$$

<u>FACT 2</u> $DC(X)$ IS EQUIVALENT TO

$$\forall R \subseteq X \times X \; [\forall x \in X \; \exists y \in X \;\; xRy \Rightarrow$$
$$\forall x_0 \in X \; \exists f \in X^\omega \;\; f(0) = x_0 \wedge \forall n \in \omega \; f(n) R f(n+1)]$$



ASSUME $DC(\omega \times X)$ WTS $AC_\omega(X)$.

LET F BE A COUNTABLE FAMILY OF NON-EMPTY SUBSETS OF X.

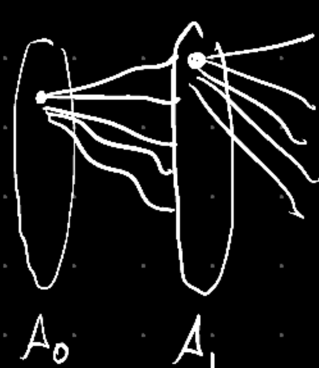
$$F = \{A_n \mid n \in \omega\}$$

DEFINE F' AS FOLLOWS:

$$F' = \{\{n\} \times A_n \mid n \in \omega\}$$

DEFINE R AS FOLLOWS

$$x R y \text{ IFF } \exists n \quad x \in \{n\} \times A_n \text{ AND } y \in \{n+1\} \times A_{n+1}$$



LET $Y = \bigcup F' \quad \emptyset \neq Y \subseteq X$.

LET $x \in Y$ SO BY DEF THERE IS $n \in \omega$ st.
$x \in \{n\} \times A_n$ SINCE $A_{n+1} \neq \emptyset$ THERE IS
$y \in \{n+1\} \times A_{n+1}$ BUT THEN $x R y$.

BY FACT 1 I CAN USE $DC(Y)$. LET $x_0 \in \{0\} \times A_0$
NOW BY FACT 2 THERE IS f st

$$f(0) = x_0 \wedge \forall n \in \omega \; f(n) \, R \, f(n+1).$$

LET $g(n) = \pi_2(f(n))$ AN INDUCTION SHOWS THAT
THIS IS A CHOICE FUNCTION FOR F $\blacksquare$

PROPOSITION (JENSEN) $AC_\omega \not\Rightarrow DC$.

<u>COROLLARY</u> $DC(\mathbb{R}) \Rightarrow AC_\omega(\mathbb{R})$

<u>PROOF</u> NOTE THAT $|\omega \times \mathbb{R}| = |\mathbb{R}|$ SO
USE FACT 1 AND THE PREVIOUS PROOF. $\blacksquare$

TREES

Definition 1.25. Let X be a non-empty set. A **tree on** X is a $T \subseteq {}^{<\omega}X$ closed under initial segments, that is

$$\forall t \in T \, \forall s \subseteq t \, (s \in T).$$

The elements of T are called **nodes**. If $s \subset t$ and $s, t \in T$, then t is an **extension** of s, and if $\mathrm{lh}(t) = \mathrm{lh}(s) + 1$ then t is an **immediate extension** of s. An $s \in T$ is a **terminal node** if it has no extensions, and the set of all terminal nodes is denoted by $\mathbf{tn}(T)$. A tree T is **pruned** if it has no terminal nodes, i.e., $\mathbf{tn}(T) = \emptyset$. A **branch** of a tree T on X is a sequence $f \in {}^{\omega}X$ such that

$$\forall n \in \omega \, (f \restriction n \in T).$$

The **body** of T is the set of all of its branches

$$[T] = \{f \in {}^{\omega}X \mid \forall n \, (f \restriction n \in T)\}.$$

A **sub-tree** of T is an $S \subseteq T$ which is closed under initial segments.

THEOREM (1.26) FOR EVERY NON-EMPTY SET X TFAE:

① $DC(X^{<\omega})$.

② EVERY NON-EMPTY PRUNED TREE ON X HAS A BRANCH.

PROOF

1 ⇒ 2 LET $T \subseteq X^{<\omega}$ NON-EMPTY AND PRUNED.

DEFINE $R \subseteq T \times T$ AS FOLLOWS

$$x R y \Leftrightarrow \exists z \in X \quad y = x^\frown \langle z \rangle$$

LET $x \in T$ SINCE T IS PRUNED THEN THERE IS $y \in T$ st $x \subset y$ BUT THEN $x R \underbrace{y \restriction lh(x)}_{\Uparrow\, T}$.

SO BY FACT 1 $DC(X^{<\omega}) \Rightarrow DC(T)$ AND BY FACT 2 THERE IS $f \in (X^{<\omega})^{<\omega}$

$$f(0) = \emptyset \wedge \forall n \; f(n) R f(n+1)$$

<u>CLAIM</u> $\bigcup f \in [T]$.

<u>PROOF</u>

BY INDUCTION $\forall n$

$$\left.\begin{array}{l} f(n) \subset f(n+1) \text{ AND } \mathrm{dom}(f(n)) = n \\ \mathrm{RANGE}(f(n)) \subseteq X \end{array}\right\} \bigcup f \in X^\omega$$

ALSO $(\bigcup f) \restriction n = f(n) \in T$ SO $\bigcup f \in [T]$.

$2 \Rightarrow 1$

FACT 3 IF $X \neq \emptyset$ THEN $|X^{<\omega} \setminus \{\emptyset\}| = |X^{<\omega}|$.

$[\![$ FIX $a \in X$ SEND $s \longmapsto \langle a \rangle^\frown s$ THIS IS AN INJECTION $X^{<\omega} \rightarrowtail X^{<\omega} \setminus \{\emptyset\}]\!]$

SO WE "ONLY" NEED TO SHOW

$$DC(X^{<\omega} \setminus \{\emptyset\})$$

LET $R \subseteq X^{<\omega} \setminus \{\emptyset\} \times X^{<\omega} \setminus \{\emptyset\}$ st

$$\forall x \in X^{<\omega} \setminus \{\emptyset\} \; \exists y \in X^{<\omega} \setminus \{\emptyset\} \; xRy.$$

DEFINE $\tilde{R} \subseteq R$

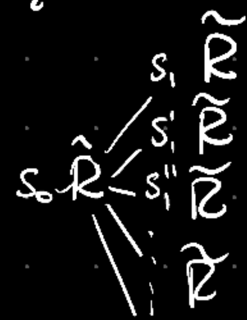
$$\tilde{R} := \{ (s,t) \in R \mid \forall u \in X^{<\omega} \setminus \{\emptyset\} \; sRu \Rightarrow lh(u) \geq lh(t) \}$$

IF I BUILD A CHAIN ON $\tilde{R}$ I AM DONE!

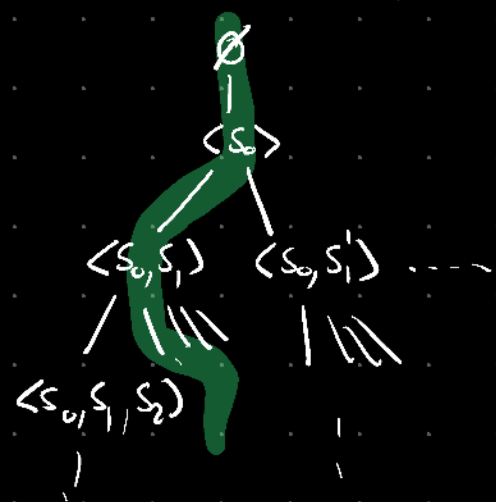
ALSO $\forall x \in \text{Field}(\hat{R})\ \exists y \in \text{Field}(\hat{R})$ st $x \tilde{R} y$.

$$s_0 \tilde{R} s_1 \tilde{R} s_2 \ldots$$

Fix $s_0 \in X^{<\omega} \setminus \{\emptyset\}$



$\tilde{T}$



DEFINE $\hat{T}$ AS FOLLOWS:

$$\hat{T} = \{ \tilde{u} \in (X^{<\omega} \setminus \{\emptyset\})^{<\omega} \mid \tilde{u} = \emptyset \vee \tilde{u}(0) = s_0 \wedge \forall n+1 < lh(\tilde{u})\ \tilde{u}(n) \hat{R} \tilde{u}(n+1) \}$$

IF $[\hat{T}] \neq \emptyset$ WE ARE DONE!

- $\tilde{T}$ is not-empty $\emptyset \in \tilde{T}$
- $\tilde{T}$ is pruned

if $\tilde{u} \in \tilde{T}$ $\tilde{u} = \langle s_0 \ldots s_n \rangle$ then there is s_{n+1} st $s_n \tilde{R} s_{n+1}$ so then $\tilde{v} = \tilde{u}^\frown \langle s_{n+1} \rangle$ is in $\tilde{T}$ and $\tilde{u} \subset \tilde{v}$.

APPLY THE ASSUMPTION TO $\tilde{T}$! ⚡

ASSUMPTION WAS THAT EVERY NON-EMPTY PRUNED TREE ON X HAS A BRANCH.

BUT $\tilde{T}$ IS A TREE ON $X^{<\omega} \setminus \{\emptyset\}$

LEMMA (1.22) IF $X \neq \emptyset$ THE THERE IS AN INJECTION $f : (X^{<\omega} \setminus \{0\})^{<\omega} \longrightarrow X^{<\omega}$

st:

① $f(\emptyset) = \emptyset$

② $s \subset t \iff f(s) \subset f(t)$

③ $lh(F(\langle s_0 \dots s_{i+1} \rangle)) = lh(f(\langle s_0 \dots s_i \rangle)) + g(lh(s_{i+1}))$

where $g : \omega \to \omega \setminus \{\emptyset\}$.

Proof DONE VIA CODING.

IF $|X| = 1$ SO $|X^{<\omega}| = |\omega|$ [[CANTOR MAP!]]

IF $|X| > 1$ LET $0, 1 \in X$ with $0 \neq 1$

For each $s = x_0 \dots x_n \in X^{<\omega} \setminus \{\emptyset\}$.

DEFINE

$$\tilde{s} = 0 x_0 0 x_2 0 x_3 \dots, 0 x_n 0 1 1$$

$$F(\langle s_0, s_1 \dots s_n \rangle) = \tilde{s}_0 ^\frown \tilde{s}_1 ^\frown \tilde{s}_2 \dots \tilde{s}_n$$

NOW WE USE f TO DEFINE $T \subseteq X^{<\omega}$

LET T BE THE TREE INDUCED BY $\tilde{T}$ VIA f.

$$T = \{ u \in X^{<\omega} \mid \exists \tilde{v} \in \tilde{T} \ \ u \subseteq f(\tilde{v}) \}$$

① T IS NON-EMPTY $\emptyset \in T$.

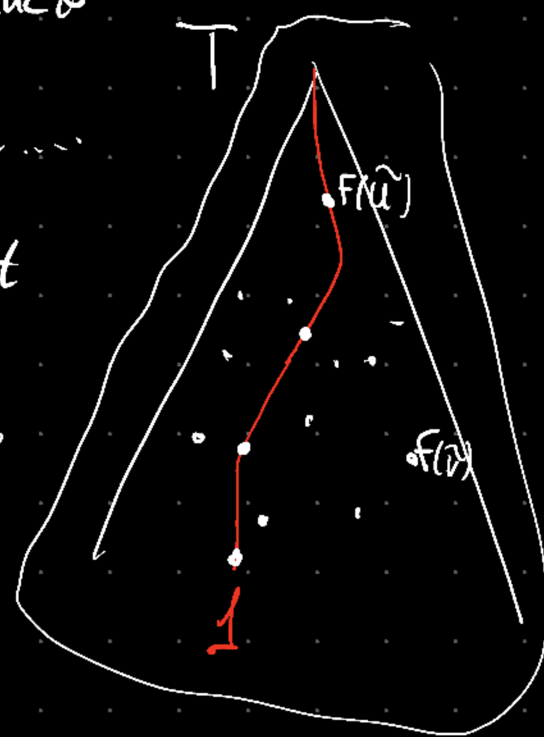
② T IS PRUNED:

IF s IS TERMINAL IN T THE SO $s = f(\tilde{v})$ FOR SOME $\tilde{v} \in \tilde{T}$. BUT $\tilde{T}$ IS PRUNED SO THERE IS $\tilde{u} \in \tilde{T}$ st $\tilde{v} \subset \tilde{u}$ BY LEMMA 1.22 $f(\tilde{v}) \subset f(\tilde{u})$ SO s IS NOT TERMINAL! ↯ s''

③ $[T] \neq \emptyset$ BY OUR ASSUMPTIONS.

LET $f \in [T]$ WE WANT A BRANCH ON $\tilde{T}$.

WE WILL DEFINE A SEQUENCE



$$\emptyset = \tilde{u}_0 \subset \tilde{u}_1 \subset \dots$$

ST $\forall n \ \tilde{u}_n \in \tilde{T}$ AND ST

$$\forall n \ F(\tilde{u}_n) \subset f$$

By RECURSION:

- $\tilde{u}_0 = \emptyset$
- ASSUME $\tilde{u}_n$ IS DEFINED AND SUCH THAT

$$f(\tilde{u}_n) \subset f$$

BY DEFINITION OF $\tilde{R}$ AND $\tilde{T}$ IF

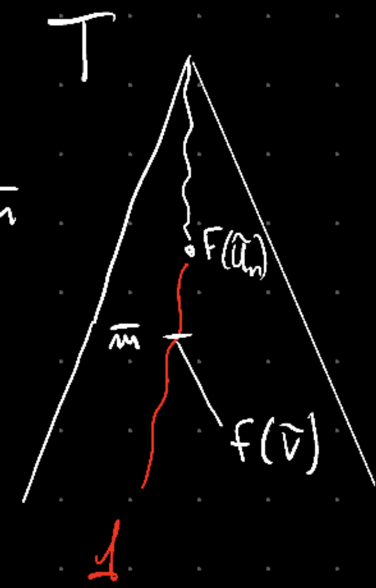
$$\left.\begin{array}{l} \tilde{u}_n {}^\frown \langle t_1 \rangle \\ \tilde{u}_n {}^\frown \langle t_2 \rangle \end{array}\right. \in \tilde{T}$$

Then $lh(t_1) = lh(t_2)$ and by Lemma 1.22

$$lh(f(\tilde{u}_n{}^\frown\langle t_1\rangle)) = lh(f(\tilde{u}_n{}^\frown\langle t_2\rangle))$$

so there is $\bar{m}$ st

$$\forall t \quad lh(f(\tilde{u}_n{}^\frown\langle t\rangle)) = \bar{m}$$



since $x \in [T]$ then $x \restriction \bar{m} \in T$.

Let $\tilde{v} \in \tilde{T}$ of minimal length
st

$$x \restriction m \subseteq f(\tilde{v})$$

(*)

Let $\tilde{v}^* = \tilde{v} \restriction lh(\tilde{v})-1$ we know that
$\underline{f(\tilde{v}^*) \subset x \restriction \bar{m}}$.

If $\tilde{v}^* = \tilde{u}_n$ we are done

Indeed $f(\tilde{v}) = x \restriction \bar{m}$ because $lh(f(\tilde{v})) = \bar{m}$
and $x \restriction \bar{m} \subseteq f(\tilde{v})$ then $x \restriction \bar{m} = f(\tilde{v})$.

$f(\tilde{u}_n), f(\tilde{V}^*) \subset f \restriction \bar{m}$ so they

are compatible. By Lemma 1.2

$$\tilde{u}_n \subseteq \tilde{V}^* \quad \text{or} \quad \tilde{V}^* \subseteq \tilde{u}_n$$

- If $\tilde{u}_n \subset \tilde{V}^*$: $\exists k \quad \tilde{V}^* \restriction k = \tilde{u}_n$

so $lh(f(\underbrace{\tilde{V}^* \restriction k+1}_{\tilde{u}_n {}^\frown \langle t \rangle})) = \bar{m}$ but $f(\tilde{V}^*) \subset f \restriction \bar{m}$ $\lightning$

- If $\tilde{V}^* \subset \tilde{u}_n$: then $\tilde{V} \subseteq \tilde{u}_n$

$$\bar{m} \leq lh(f(\tilde{V})) \leq lh(f(\tilde{u}_n)) < \bar{m} \ \lightning$$

so $\tilde{V}^* = \tilde{u}_n$ then $f(\tilde{V}) = f \restriction m \subset f$

SO LET $\tilde{U}_{n+1} = \breve{V}$.

$\blacksquare$

COROLLARY THE FOLLOWING ARE EQUIVALENT

1. DC
2. EVERY NON-EMPTY PRUNED TREE HAS A BRANCH.