CS:ST Lecture VII

23 September 2020

3.B The Borel Hierarchy

The collection of all Borel sets can be stratified in a hierarchy.

Definition 3.5. Let Z be a topological space. The classes

$$\boldsymbol{\Sigma}^0_\alpha \restriction Z, \boldsymbol{\Pi}^0_\alpha \restriction Z, \boldsymbol{\Delta}^0_\alpha \restriction Z \subseteq \mathscr{P}(Z)$$

are defined by induction on $\alpha \geq 1$:

- $\boldsymbol{\Sigma}^0_1 \restriction Z = \{V \subseteq Z \mid V \text{ is open}\}$;
- $\boldsymbol{\Sigma}^0_\alpha \restriction Z = \{\bigcup_n X_n \mid \exists \langle \beta_n \mid n \in \omega \rangle \, (1 \leq \beta_n < \alpha \wedge X_n \in \boldsymbol{\Pi}^0_{\beta_n} \restriction Z)\}$, for $\alpha > 1$;
- $\boldsymbol{\Pi}^0_\alpha \restriction Z = \{Z \setminus X \mid X \in \boldsymbol{\Sigma}^0_\alpha \restriction Z\}$, for $\alpha \geq 1$;
- $\boldsymbol{\Delta}^0_\alpha \restriction Z = \boldsymbol{\Sigma}^0_\alpha \restriction Z \cap \boldsymbol{\Pi}^0_\alpha \restriction Z$, for $\alpha \geq 1$.

AMBIGUOUS

Definition 3.16. A set $U \subseteq {}^\omega\omega \times {}^\omega\omega$ is **universal for a pointclass** Γ iff

$$\forall A \subseteq {}^\omega\omega \, (A \in \Gamma \Rightarrow \exists a \in {}^\omega\omega \, (A = U_{(a)})) .$$

Proposition 3.17. *If Γ is closed under trivial substitutions and it is self-dual, then it does not have a universal set.*

Corollary 3.19. *Assume $\mathsf{AC}_\omega(\mathbb{R})$. For each $0 < \alpha < \omega_1$, $\boldsymbol{\Sigma}^0_\alpha$ and $\boldsymbol{\Pi}^0_\alpha$ have a universal set. Therefore the Borel hierarchy is proper, i.e.,*

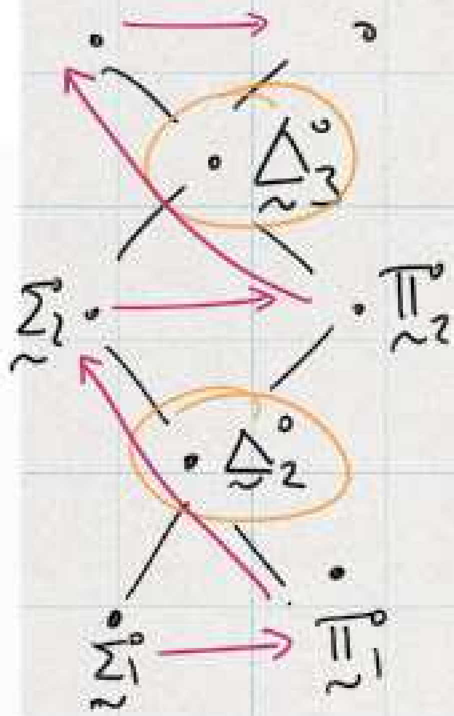
- $\boldsymbol{\Sigma}^0_\alpha \neq \boldsymbol{\Pi}^0_\alpha$,
- $\boldsymbol{\Delta}^0_\alpha \subset \boldsymbol{\Sigma}^0_\alpha$ *and* $\boldsymbol{\Delta}^0_\alpha \subset \boldsymbol{\Pi}^0_\alpha$,
- $\boldsymbol{\Sigma}^0_\alpha \subset \boldsymbol{\Delta}^0_\beta$ *and* $\boldsymbol{\Pi}^0_\alpha \subset \boldsymbol{\Delta}^0_\beta$ *for* $\alpha < \beta$.

$\boldsymbol{\Sigma}^0_{\omega_1} = \boldsymbol{\Pi}^0_{\omega_1} = \boldsymbol{\Delta}^0_{\omega_1}$

Key point of C 3.19

If Γ has a universal set, then $\check{\Gamma}$ has a universal set.

If Γ has a universal set then $U(\omega;\Gamma)$ has a universal set.



"collapse"

$\downarrow$

ω_1

$\mathsf{AC}_\omega(\mathbb{R}) \Longrightarrow$ regularity of ω_1

$\boxed{\text{DEFINABLE}}$

Section 4 of ANDRETTA

Definability in "SECOND ORDER ARITHMETIC"

$(\mathbb{N}, +, \cdot, <, 0)$

FIRST ORDER ARITHMETIC

Just first-order logic over this structure

"Second order" usually means":

$$\mathbb{S} = (S, f, R, c)$$

$$\rightsquigarrow \quad (\underset{\text{OBJECTS}}{\underline{S}}, \underset{\text{SUBSETS OF } S}{\underline{\mathcal{P}(S)}}, f, R, c, \in)$$

$\mathcal{P}(\mathbb{N}) \sim 2^{\omega}$

Here: Instead of subsets, we want $\mathbb{N}$ and $\mathbb{N}^{\mathbb{N}} = \omega^{\omega}$.

Subsection 4A, p. 70

Two-sorted language, variables of two sorts:

"first order-variables" $v_0^{(1)}, v_1^{(1)}, v_2^{(1)}, \dots, x^{(1)}, y^{(1)}, z^{(1)}$

"second order-variables" $v_0^{(2)}, v_1^{(2)}, v_2^{(2)}, \dots, x^{(2)}, y^{(2)}, z^{(2)}$

$\mathbb{N}$

$\mathbb{N}^{\mathbb{N}}$

Non-logical symbols:

$\underbrace{<, S, +,}_{\text{arithmetical}} \cdot, \underbrace{\langle \cdot ; \cdot \rangle}_{\text{coding}}, (\cdot)_{\mathrm{I}}, (\cdot)_{\mathrm{II}}, \ell, \oplus, \;\; , \;\; ,$ $\leftarrow \frown$

Val, Code, 0

bij: $\langle \cdot ; \cdot \rangle : \mathbb{N}^2 \longrightarrow \mathbb{N}$

A term is either a first order term or a second order term. The first order terms denote natural numbers and are defined as follows:

- the first order variables are first order terms,
- the constant 0 is a first order term,
- if t and u are first order terms, then so are

$$S(t), \quad t+u, \quad t\cdot u, \quad \langle t,u\rangle, \quad t_{\mathbf{I}}, \quad t_{\mathbf{II}}, \quad \ell(t), \quad \Pr(t,u).$$

- if u is a first order term and t is a second order term, then

$$\mathrm{Val}(t,u), \quad \mathrm{Code}(t,u),$$

  are first order terms

First Order Terms

The second order terms denote elements of the Baire space and are defined as follows:

- a second order variable is a second order term,
- if t and u are second order terms, then so are

$$S(t), \quad t^{\leftarrow}, \quad t\oplus u, \quad t_{\mathbf{I}}, \quad t_{\mathbf{II}},$$

- if u is a first order term and t is a second order term, then

$$u^\frown t, \quad (t)_u$$

  are second order terms.

Second Order Terms

$$x_{I}(n) := x(2n)$$

$$x_{II}(n) := x(2n+1)$$

$$\boxed{x\oplus y(n)} := \begin{cases} x(k) & 2k=n \\ y(k) & 2k+1=n \end{cases}$$

TURING JOIN

This is called TURING JOIN since it is the least upper bound operation in the Turing degrees.

Syntax & semantics of this two-sorted language are completely standard.

Note $\exists v^{(1)}$ quantifies over first order objects

$\exists v^{(2)}$ quantifies over second order objects.

$\exists x^{(1)} (\varphi \wedge x^{(1)} < y^{(1)})$

Definition 4.5. A formula φ of $A^{(2)}$ is:

(i) Δ^0_0 if it is obtained from atomic formulæ using the boolean connectives and bounded quantification over first order variables, i.e. quantification of the form

$$\boxed{\exists x^{(1)} \left(x^{(1)} < y^{(1)} \wedge \varphi\right)} \quad \text{and} \quad \boxed{\forall x^{(1)} \left(x^{(1)} < y^{(1)} \Rightarrow \varphi\right).}$$

BOUNDED QUANTIFIER

Δ^0_0

For notational simplicity, we will write $\exists x^{(1)} < y^{(1)}\, \varphi$ and $\forall x^{(1)} < y^{(1)}\, \varphi$. $\underline{\Delta^0_0}$ formulæ are also called $\underline{\Sigma^0_0}$ formulæ or $\underline{\Pi^0_0}$ formulæ;

(ii) Σ^0_{n+1} iff it is of the form

$$\underline{\exists x^{(1)}\, \psi}$$

with ψ a Π^0_n formula;

$$\underbrace{\exists y^{(1)} \exists x^{(1)} (x^{(1)} < y^{(1)} \wedge y^{(1)} = y^{(1)})}_{\Sigma^0_1}$$

(iii) Π^0_{n+1} iff it is of the form

$$\underline{\forall x^{(1)}\, \psi}$$

with ψ a Σ^0_n formula. $\boxed{\text{Therefore } \varphi \text{ is } \Pi^0_{n+1} \text{ iff it is equivalent to the negation of a } \Sigma^0_{n+1} \text{ formula,}}$

NOT CORRECT

$\neg \exists x^{(1)} \neg \varphi \iff \forall x^{(1)} \varphi$

$\neq$

Instead of the syntactic notion of $\Delta_0^0, \Sigma_n^0, \Pi_n^0$, we interested in a "semi-semantic" notion of the class of formulas **EQUIVALENT TO** $\Delta_0^0, \Sigma_n^0, \Pi_n^0$ formulas.

Then, we can define

φ is Δ_n^0 iff it's eq. to a Σ_n^0 fmla and a Π_n^0 fmla.

So far, we talked about LOGICAL EQUIVALENCE, but if interpreted in the concrete structure $\mathcal{A}^{(2)}$, there could be fmlas eq. over $\mathcal{A}^{(2)}$, but not logically equivalent.

$$\mathcal{A}^{(2)} \models \exists x^{(1)} \exists y^{(1)} \varphi(x^{(1)}, y^{(1)})$$

$$\Updownarrow$$

$$\mathcal{A}^{(2)} \models \exists z^{(1)} \varphi\left(z^{(1)}_{\mathrm{I}}, z^{(1)}_{\mathrm{II}}\right)$$

$\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$

inverses

$\cdot_{\mathrm{I}}, \cdot_{\mathrm{II}}$

Δ^0_0

$\downarrow$

φ

$$\underbrace{\exists x^{(1)} \exists y^{(1)} \exists z^{(1)} \exists a^{(1)}}_{\exists d^{(1)}} \forall b^{(1)} \exists c^{(1)} \varphi$$

So this is Σ^0_3

REMEMBER Prenex Normal Form

Summary If φ is a formula that only has first order qf, I can use prenex normal form to make it equivalent to $Q_0 x_0^{(1)} \dots Q_N x_N^{(1)} \psi$ ← no quantifiers

and then use the previous $\mathcal{A}^{(2)}$-eq. to reduce seq. of the same quantifier to one.

Σ^0_N — $\exists x_0^{(1)} \forall x_1^{(1)} \exists x_2^{(1)} \dots Q x_N^{(1)} \psi$

OR

Π^0_N — $\forall x_0^{(1)} \exists x_1^{(1)} \exists x_2^{(1)} \dots Q x_N^{(1)} \psi$

Q depends on the parity of N

Lemma 4.21. *Suppose $A \subseteq \mathcal{N}_{l,m}$ is in $\Delta^0_0(X)$. Then A is clopen in $\mathcal{N}_{l,m}$.*

Lemma 4.22. *Let $C \subseteq \mathcal{N}_{l,m}$. If $C \in \Pi^0_1(X)$ then C is closed.*

Theorem 4.23. *Let $A \subseteq \mathcal{N}_{l,m}$ and $X \subseteq \mathbb{R}$. For every $n \geq 1$:*

(a) *If $A \in \Sigma^0_n(X)$ or $A \in \Pi^0_n(X)$ or $A \in \Delta^0_n$, then $A \in \mathbf{\Sigma}^0_n$ or $A \in \mathbf{\Pi}^0_n$ or $A \in \mathbf{\Delta}^0_n$. Conversely*

(b) *Assume $\mathsf{AC}_\omega(\mathbb{R})$. If $A \in \mathbf{\Sigma}^0_n$ or $A \in \mathbf{\Pi}^0_n$ or $A \in \mathbf{\Delta}^0_n$, then there is a $p \in \mathbb{R}$ such that $A \in \Sigma^0_n(p)$ or $A \in \Pi^0_n(p)$, or $A \in \Delta^0_n(p)$.*

$\underset{\sim}{\Sigma}^0_2 = F_\sigma$

$\underset{\sim}{\Pi}^0_2 = G_\delta$

$\underset{\sim}{\Sigma}^0_3 = G_{\delta\sigma}$

$\underset{\sim}{\Pi}^0_3 = F_{\sigma\delta}$

$\underset{\sim}{\Sigma}^0_4 = F_{\sigma\delta\sigma}$

$\underset{\sim}{\Pi}^0_4 = G_{\delta\sigma\delta}$

ADDISON'S THEOREM

If φ is a formula we say φ defines A with parameters p $\Longleftrightarrow$

$$x \in A \iff \mathcal{A}^{(2)} \models \varphi(x, p)$$

Some things about the $\Sigma^0_n \longmapsto \underset{\sim}{\Sigma}^0_n$ direction.

In practice, we often look at the defining formula of a set and observe their Borel level.

E.g.: "A is Borel because I can define it with a formula that only quantifies over nat. numbers".

If $A \in \underset{\sim}{\Sigma}^0_{n+1}$, then

$$A = \bigcup_{n \in \mathbb{N}} C_n \qquad C_n \in \underset{\sim}{\Pi}^0_n.$$

$$x \in A \longleftrightarrow \exists n \in \mathbb{N} \ \ x \in C_n$$

use the universal set for $\underset{\sim}{\Pi}^0_n$

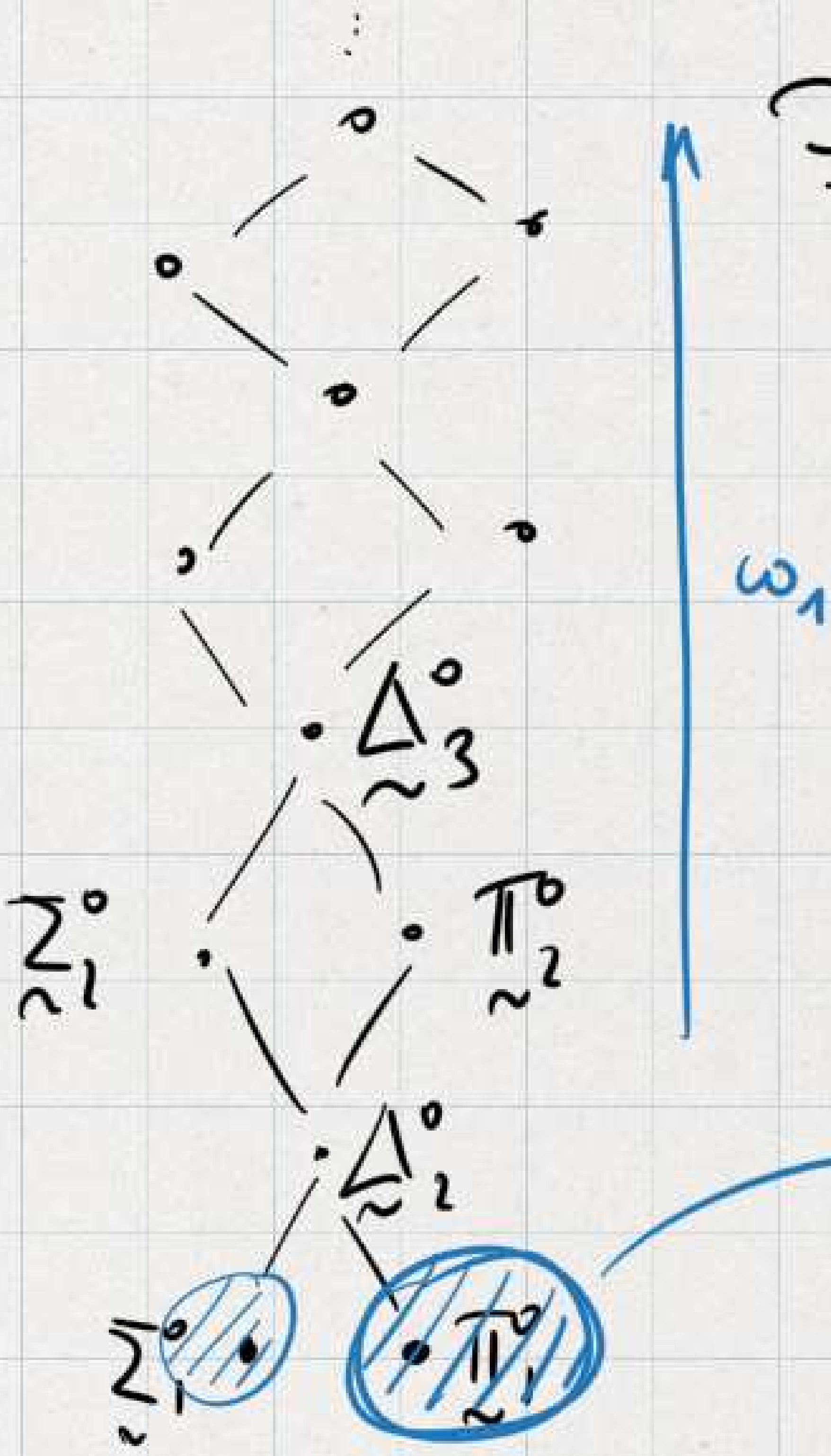
universal for $\underset{\sim}{\Pi}^0_n$

$$(x, n) \in U$$

find φ Π^0_n s.t. φ describes U

$$x \in A \longleftrightarrow \underbrace{\exists n^{(1)} \underbrace{\varphi_{(1)}(x, n^{(1)})}_{}}_{} \quad \Sigma_{n+1}.$$

DETERMINACY



Q: How far up in this hierarchy does determinacy go?

$$\mathcal{D} := \{A \subseteq \omega^\omega ;\ G(A) \text{ is determined}\}$$

$$\boldsymbol{\Sigma}^0_1 \cup \boldsymbol{\Pi}^0_1 \subseteq \mathcal{D} \quad [\text{ZF}]$$

$$\mathcal{P}(\omega^\omega) \neq \mathcal{D} \quad [\text{ZFC}]$$

GALE-STEWART proves that each set in $\boldsymbol{\Sigma}^0_1 \cup \boldsymbol{\Pi}^0_1$ is determined.

PREVIEW The Borel hierarchy is not the right hierarchy to measure the extent of $\mathcal{D}$?

Pacific Journal of Mathematics

PJM 5:5 (1955), 841-847

THE STRICT DETERMINATENESS OF CERTAIN INFINITE GAMES

PHILIP WOLFE

1. **Introduction.** Gale and Stewart [1] have discussed an infinite two-person game in extensive form which is the generalization of a game as defined by Kuhn [3] obtained by deleting the requirement of finiteness of the game tree and regarding as plays all unicursal paths of

Thm (Wolfe, 1955)

In the theory $ZC^- + \Sigma_1$-Repl.
[Zermelo without Powerset + Choice + Repl. for Σ_1-Fmls]

$$\boldsymbol{\Sigma}^0_2 \subseteq \mathcal{D}$$

[Proof on HW set #4]

ADVANCES
IN GAME THEORY

R. J. AUMANN	J. MYCIELSKI
L. D. BERKOVITZ	E. D. NERING
M. DAVIS	H. NIKAIDO
W. H. FLEMING	G. OWEN
A. R. GALMARINO	B. PELEG
O. GROSS	H. RADSTRÖM
J. C. HARSANYI	R. A. RESTREPO
M. HEBERT	J. L. ROSENFELD
J. R. ISBELL	C. RYLL-NARDZEWSKI
G. JENTZSCH	R. SELTEN
S. M. JOHNSON	L. S. SHAPLEY
M. MASCHLER	R. E. STEARNS
K. MIYASAWA	L. E. ZACHRISSON

EDITED BY
M. Dresher, L. S. Shapley, A. W. Tucker

PRINCETON, NEW JERSEY
PRINCETON UNIVERSITY PRESS
1964

INFINITE GAMES OF PERFECT INFORMATION

Morton Davis

§ 1. INTRODUCTION

It is well-known that finite, two-person, zero-sum games with perfect information are strictly determined [1]. There have been attempts to remove from this result each of these restrictions. It is the first of these with which we will be concerned, i.e., we consider infinite games.

In a paper by Gale and Stewart [2], zero-sum, two-person, infinite games with perfect information are defined. Familiarity with this paper will be assumed.

The notation of this paper will lean heavily on the above paper, but some additions and modifications will be made. In referring to the game Γ we will mean the $(x_0, X_I, X_{II}, X, f, S, S_I, S_{II})$ of Gale and Stewart, where each element is understood as given in the game. We will also write $\Gamma(S_I = A)$ to stand for the game $(x_0, X_I, X_{II}, X_n, f, S, A, A^c)$ where $S = A \cup A^c$. We use here and elsewhere in the paper the superscript c to denote complement. We assume $f^{-1}(x)$ is always a finite set.

In this paper we extend the results of Gale and Stewart [2] and Wolfe [3], answer Questions 1 and 2 of [2] (assuming the Continuum Hypothesis), and finally, characterize the winning sets of a game suggested to me by

Thm (Morton Davis 1964)

$ZC^- + \Sigma_1$-Repl

proves

$\utilde{\Sigma}^0_3 \subseteq \mathcal{D}$.

ANNALS OF MATHEMATICAL LOGIC – Volume 2, No. 3 (1971) pp. 325–357

HIGHER SET THEORY AND MATHEMATICAL PRACTICE *

Harvey M.FRIEDMAN
Stanford University

Received 17 April 1970

Introduction

When we examine the classical set-theoretic foundations of mathematics, we see that the only sets that play a role are sets of restricted type; at the risk of understatement, only sets of rank $< \omega + \omega$. Further examination reveals four fundamental principles about sets used: the existence of an infinite set; the existence of the power set of any set; every property determines a subset of any set; and the axiom of choice.

You cannot prove

$$\underset{\sim}{\Sigma}^0_4 \ \cancel{\underset{\sim}{\Sigma}^0_5} \subseteq \mathcal{D}$$

without using Power set axiom!

$$ZFC^- \nvdash \cancel{\underset{\sim}{\Sigma}^0_5} \subseteq \mathcal{D}.$$

$$\underset{\sim}{\Sigma}^0_4$$

↑
ZFC minus power set

[$\underset{\sim}{\Sigma}^0_5 \longrightarrow \underset{\sim}{\Sigma}^0_4$ improvement due to MARTIN.]

THE JOURNAL OF SYMBOLIC LOGIC
Volume 37, Number 4, Dec. 1972

ZF ⊢ Σ^0_4 DETERMINATENESS

J. B. PARIS

Introduction. In this paper we show that in Zermelo-Fraenkel set theory (ZF) Σ^0_4 sets of reals are determinate.

Before proceeding to the proof it will be helpful to consider some previous work in this area. The first major result was obtained by Gale and Stewart [3] who showed that in ZF open games are determinate. This was then successively improved by Wolfe [4] to Π^0_2 (and so of course Σ^0_2) and then by Morton Davis [1] to Π^0_3. The results of Morton Davis further showed that countable unions of sufficiently 'simple' determinate sets are also determinate. At this time, however, Π^0_3 sets did not appear sufficiently simple for this method to be applied in order to get Σ^0_4 determinacy.

Annals of Mathematics, **102** (1975), 363-371

Borel determinacy

By DONALD A. MARTIN

Introduction

Let Y be a set of finite sequences such that every initial segment (including the empty one) of an element of Y belongs to Y and such that every element of Y is a proper initial segment of an element of Y. Let $\mathcal{F}(Y)$ be the collection of all infinite sequences $\langle y_0, y_1, \cdots \rangle$ all of whose finite initial segments belong to Y. For each $A \subseteq \mathcal{F}(Y)$ we define a two person game of perfect information $\mathcal{G}(A, Y)$. Two players, I and II, take turns moving: I picks y_0, with $\langle y_0 \rangle \in Y$, II picks y_1 with $\langle y_0, y_1 \rangle \in Y$, I picks y_2 with $\langle y_0, y_1, y_2 \rangle \in Y$, etc. I *wins* just in case $\langle y_i : i \in \omega \rangle \in A$. ($\omega$ = the set of all natural numbers.) A *strategy for* I is a function s with domain the set of all elements

Paris (1972)

ZFC $\quad \underset{\sim}{\Sigma}^0_4 \subseteq \mathcal{D}$

Martin (1975)

ZFC $\quad$ BOREL $\subseteq \mathcal{D}$

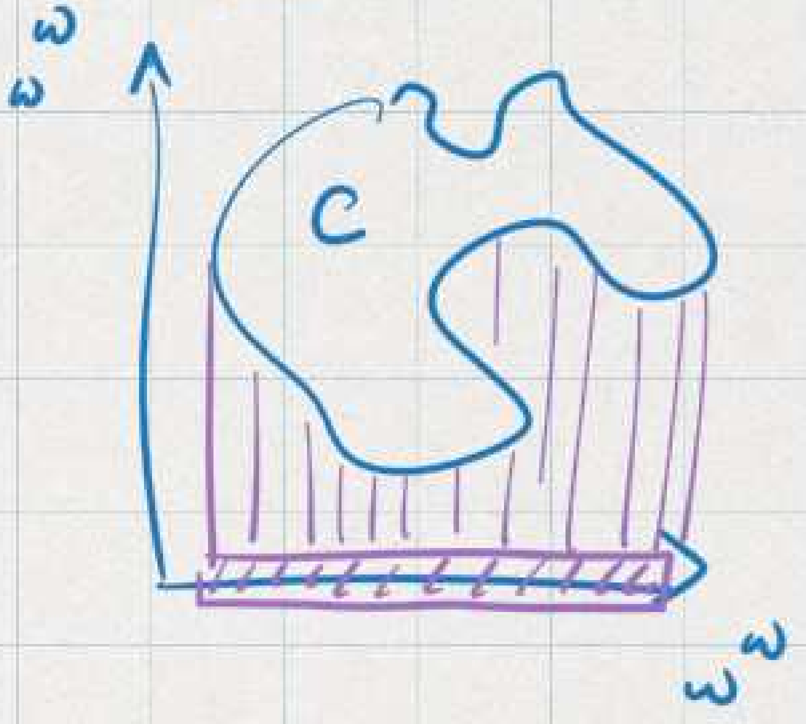
Summary The Borel hierarchy is not the right measure of complexity for determining "the extent of $\mathcal{D}$".

What is the right hierarchy?

THE FAMOUS ERROR OF LEBESGUE:

Lebesgue claimed:

$C \subseteq \omega^\omega \times \omega^\omega$



J. Math. Pure Appl. 1 (1905) 139-216 (quote pp. 191-192)

Sur les fonctions représentables analytiquement;

PAR M. H. LEBESGUE.

Cela est évident si E est un intervalle, car alors *e* en est un aussi. Or tout ensemble mesurable B se déduit d'intervalles par l'application répétée des opérations I et II', lesquelles se conservent en projection (1); la proposition est établie.

LEBESGUE'S FALSE CLAIM:

C is Borel, then pC is Borel !!

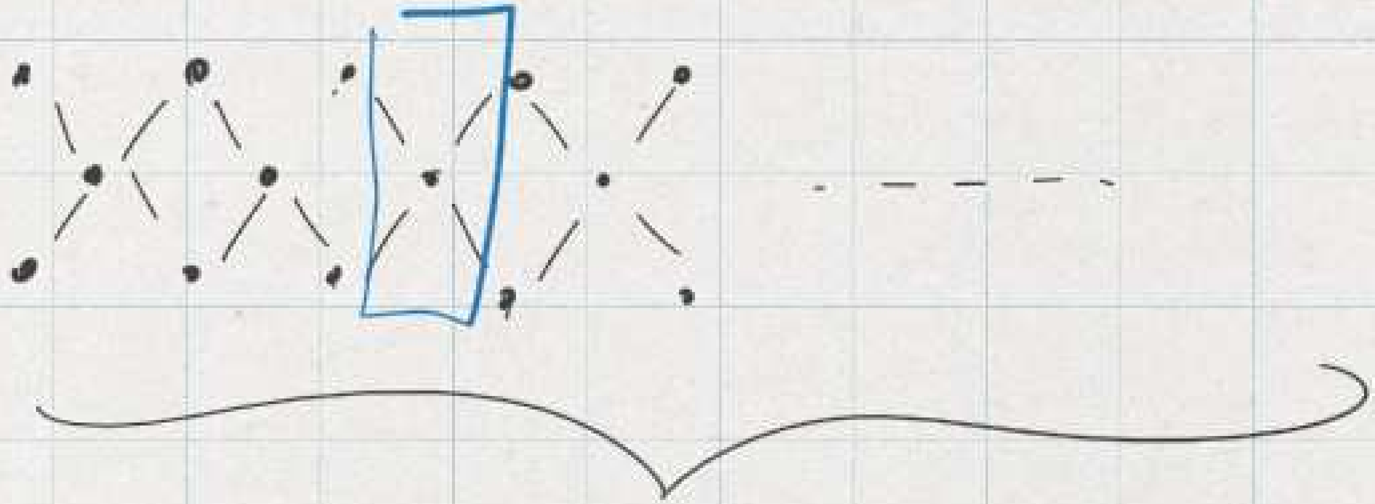
$pC = \{x ; \exists y\ (x,y) \in C\}$

PROJECTION

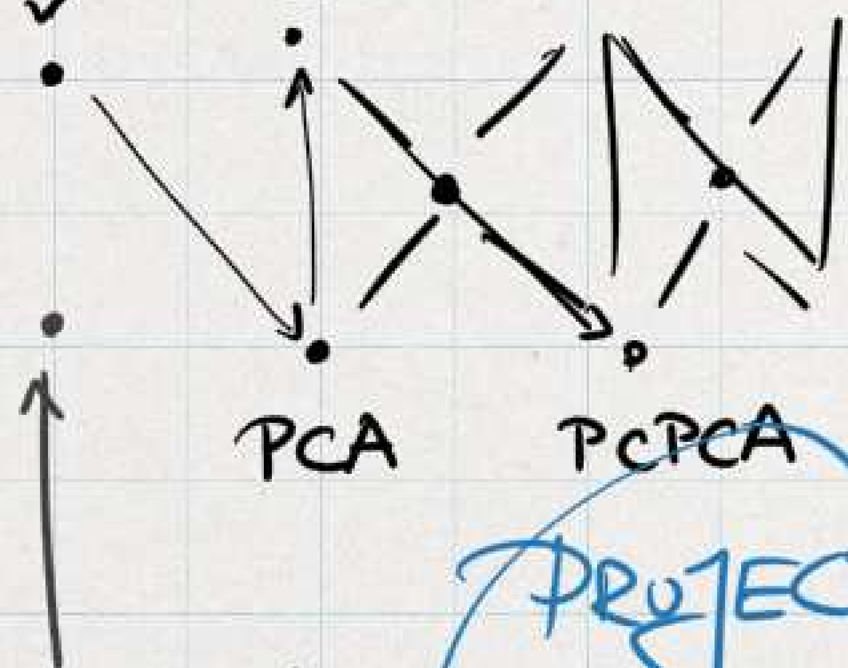
Suslin (1917) proved that this is wrong.

CA co-analytic

CPCA



ω_1



PCA PcPCA

PROJECTIVE HIERARCHY

closure of Borel under projections

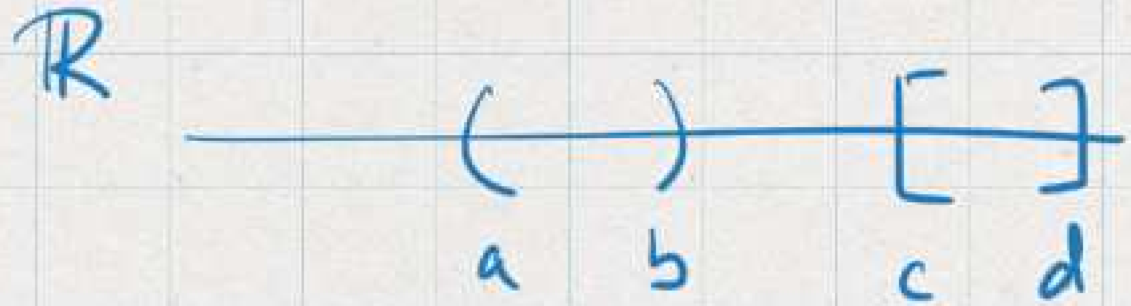
A $\longrightarrow$ ANALYTIC

$\mathfrak{A}, \underset{\sim}{\Sigma}^1_1$

$\Delta^0_{\alpha+1}$

$\mathrm{diff}(\beta, \Sigma^0_\alpha)$ β-Σ^0_α

Σ^0_α



(a,b) $[c,d]$

$(a,b] = \{x ; a < x \leq b\}$

$\underbrace{(a,b+1)}_{\varphi} \setminus \underbrace{(b,b+1)}_{\psi} = (a,b]$

$\varphi \wedge \neg \psi$

SOME SCRIBBLINGS ABOUT THE DIFFERENCE HIERARCHY

(feel free to ignore!)