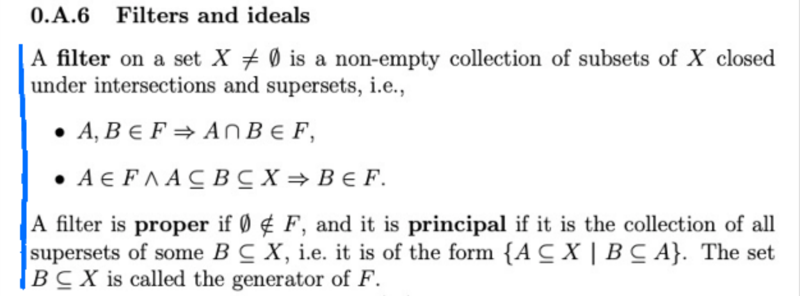
LECTURE XIII CS:ST 2020

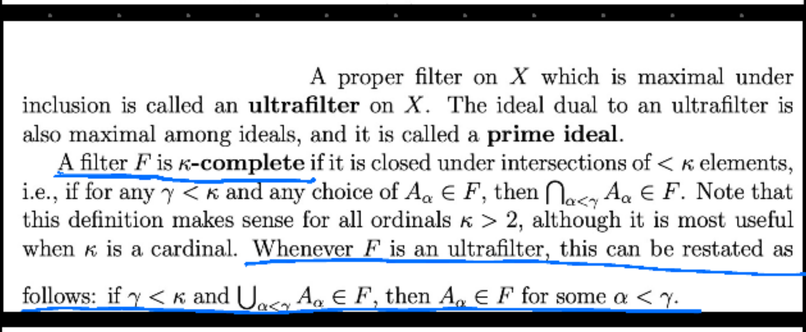


0.A.6 Filters and ideals

A **filter** on a set $X \neq \emptyset$ is a non-empty collection of subsets of X closed under intersections and supersets, i.e.,

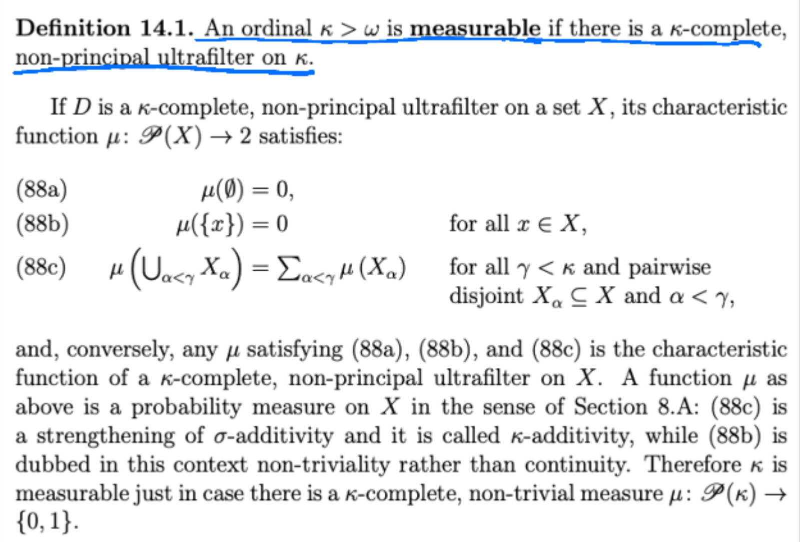
- $A, B \in F \Rightarrow A \cap B \in F$,
- $A \in F \wedge A \subseteq B \subseteq X \Rightarrow B \in F$.

A filter is **proper** if $\emptyset \notin F$, and it is **principal** if it is the collection of all supersets of some $B \subseteq X$, i.e. it is of the form $\{A \subseteq X \mid B \subseteq A\}$. The set $B \subseteq X$ is called the generator of F.



A proper filter on X which is maximal under inclusion is called an **ultrafilter** on X. The ideal dual to an ultrafilter is also maximal among ideals, and it is called a **prime ideal**.

A filter F is **κ-complete** if it is closed under intersections of $< \kappa$ elements, i.e., if for any $\gamma < \kappa$ and any choice of $A_\alpha \in F$, then $\bigcap_{\alpha<\gamma} A_\alpha \in F$. Note that this definition makes sense for all ordinals $\kappa > 2$, although it is most useful when κ is a cardinal. Whenever F is an ultrafilter, this can be restated as follows: if $\gamma < \kappa$ and $\bigcup_{\alpha<\gamma} A_\alpha \in F$, then $A_\alpha \in F$ for some $\alpha < \gamma$.



Definition 14.1. An ordinal $\kappa > \omega$ is **measurable** if there is a κ-complete, non-principal ultrafilter on κ.

If D is a κ-complete, non-principal ultrafilter on a set X, its characteristic function $\mu \colon \mathscr{P}(X) \to 2$ satisfies:

(88a) $\mu(\emptyset) = 0$,

(88b) $\mu(\{x\}) = 0$ for all $x \in X$,

(88c) $\mu\left(\bigcup_{\alpha<\gamma} X_\alpha\right) = \sum_{\alpha<\gamma} \mu(X_\alpha)$ for all $\gamma < \kappa$ and pairwise disjoint $X_\alpha \subseteq X$ and $\alpha < \gamma$,

and, conversely, any μ satisfying (88a), (88b), and (88c) is the characteristic function of a κ-complete, non-principal ultrafilter on X. A function μ as above is a probability measure on X in the sense of Section 8.A: (88c) is a strengthening of σ-additivity and it is called κ-additivity, while (88b) is dubbed in this context non-triviality rather than continuity. Therefore κ is measurable just in case there is a κ-complete, non-trivial measure $\mu \colon \mathscr{P}(\kappa) \to \{0, 1\}$.

THEO (HW Q20) ZFC $\vdash$ "$\aleph_1$ is not measurable."

AD $\Rightarrow$ $\aleph_1$ is measurable.

PROP (T19.16) ASSUME AD. THEN THERE ARE NO NON-PRINCIPAL ULTRAFILTERS ON ω. IN PARTICULAR EVERY ULTRAFILTER IS ω_1-COMPLETE.

PROOF

FOR THE SECOND PART:
LET $\mathcal{U}$ BE NON-PRINCIPAL ULTRAFILTER ON S. ASSUME U IS NOT ω_1-COMPLETE. THEN THERE ARE $A_n \notin \mathcal{U}$ BUT $\bigcup_{n\in\omega} A_n \in \mathcal{U}$. WLOG WE CAN ASSUME THAT THE A_n ARE DISJOINT. $\left(A_n' = A_n \setminus \bigcup_{i<n} A_n \right)$. DEFINE THE ULTRAFILTER

$$\mathcal{U}' = \left\{ A \subseteq \omega \mid \bigcup_{n\in A} A_n \in \mathcal{U} \right\}$$

IT IS EASY TO SEE THAT $\mathcal{U}'$ IS A NON-PRINCIPAL ULTRAFILTER ON ω.

FOR THE FIRST PART: LET $\mathcal{U}$ BE A NON-PRINCIPAL ULTRAFILTER OVER ω. WTS $\neg AD$.

CONSIDER THE FOLLOWING GAME:

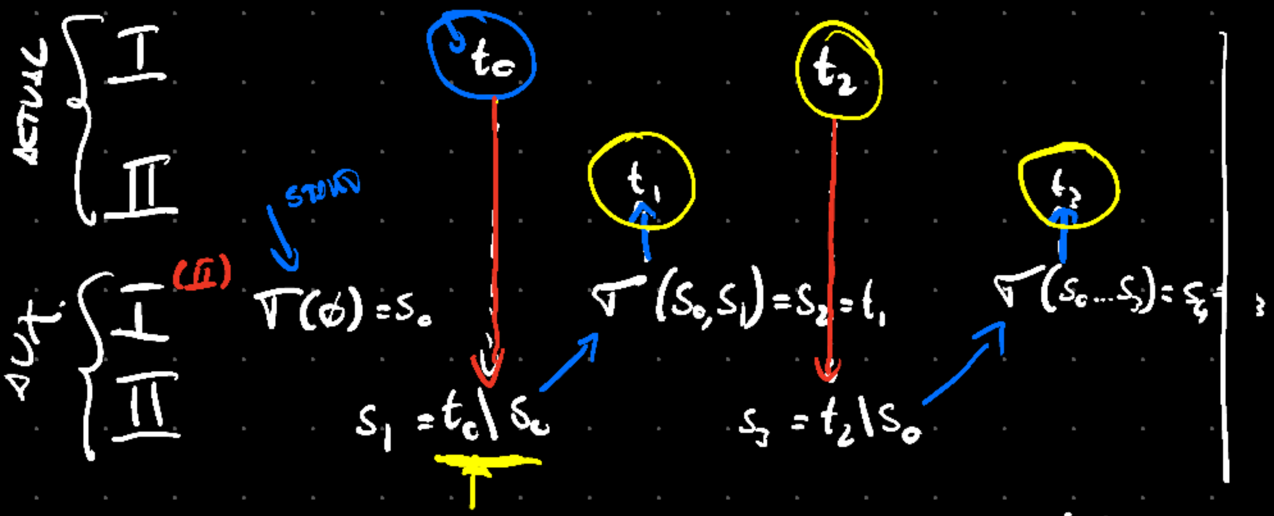
$\in [\omega]^{<\omega}$:= FINITE SUBSETS OF ω.

I $\quad s_0 \quad\quad s_2 \quad \ldots$

II $\quad\quad s_1 \quad\quad \ldots$

In each step s_n is disjoint from $\bigcup_{i<n} s_i$.
I will win if $\bigcup_{new} s_{2n} \in \mathcal{U}$.

- Assume I has a w.s. σ:



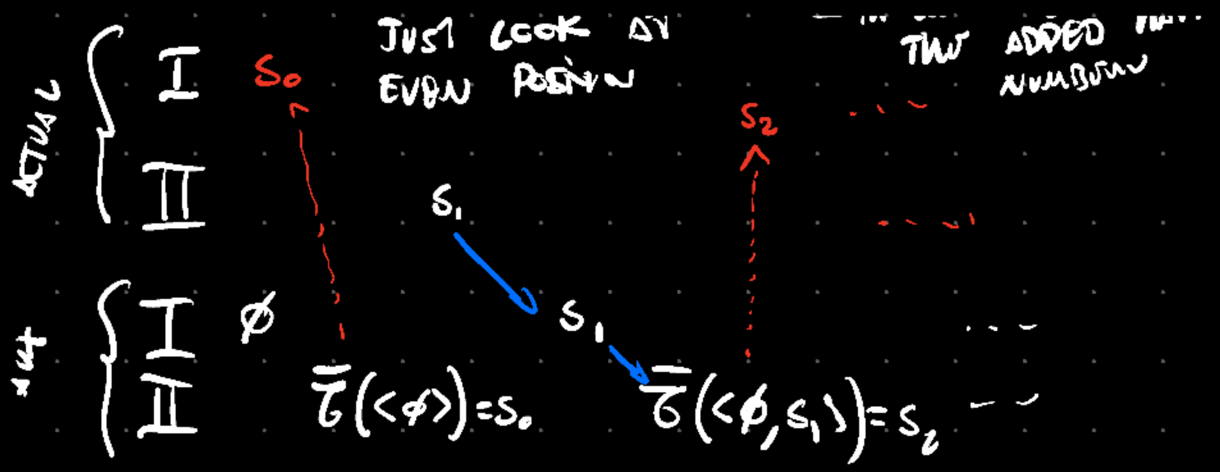
So in the aux game I wins. But $\bigcup_{new} t_{2n+1} = \bigcup_{new} s_{2n} \setminus s_0$ since $\bigcup_{new} s_{2n} \in \mathcal{U}$ and $\mathcal{U}$ is non-principal and s_0 is finite then $\bigcup_{new} t_{2n+1} \in \mathcal{U}$. So the new strategy is winning for II ↯

- If II has a w.s. τ: extend τ to

SEE MY COMMENT BELOW!

$$\bar{\tau}(\underbrace{\langle s_0,\dots s_{2n}\rangle}_{\uparrow}) = \begin{cases} \tau(\underbrace{\langle s_0\dots s_{2n}\rangle}) \cup \{n_k\} & \text{if } n_k \notin \bigcup_{i\le 2n} s_i \\ \tau(\underbrace{\langle s_0\dots s_{2n}\rangle}_{\uparrow}) & \text{ow.} \end{cases}$$

in which you remove



$\bar{\sigma}$ is winning for II so $\bigcup_{n\in\omega} s_{2n} \in \mathcal{U}$

so I actually wins the actual game. ↯ ∎

DEGREES OF DEFINABILITY

Let $x,y\in\omega^\omega$ we say that x is arithmetically definable from y iff there is an arithmetical formula φ which defines x with parameter y.

We write $x \leq_A y$.

It is easy to see that $\leq_A$ is reflexive and transitive and we denote by $\equiv_A$ the corresponding equivalence class.

$$x \equiv_A y \iff x \leq_A y \wedge y \leq_A x.$$

Let $x\in\omega^\omega$ we denote by $[x]_A$ the $\equiv_A$-equivalence class of x and we call it

DEGREE OF x. I DENOTE BY D THE SET OF DEGREES

$$D=\{[x]_A \mid x\in\omega^\omega\}$$

WE CAN NATURALLY EXTEND $\leq_A$ TO $D\times D$

$$[x]_A \leq_A [y]_A \text{ IFF } x\leq_A y$$

LET $d\in D$ DEFINE

$$C(d)=\{d'\in D \mid d\leq_A d'\}$$

WE CALL $C(d)$ THE CONE OF d. WE DEFINE $\mathcal{M}_A$ AS FOLLOWS:

$$X\in\mathcal{M}_A \Leftrightarrow \exists d\in D\left(C(d)\subseteq X\right)$$

THIS IS THE FILTER GENERATED BY CONES.

PROPOSITION $\leq_A$ HAS THE FOLLOWING PROPERTIES:

① $\forall x\in\omega^\omega$ $\{y \mid y\leq_A x\}$ IS COUNTABLE.

② IF $\{x_0,x_1,\dots\}$ IS A COUNTABLE SUBSET OF ω^ω THEN THERE IS $x\in\omega^\omega$ st $x_i\leq_A x$ $\forall i\in\omega$.

SO $\mathcal{M}_\Delta$ IS A FILTER. [DO IT!] IF $\underline{AC_\omega}$ THEN 2 IMPLIES $\mathcal{M}_\Delta$ IS ω_1-COMPLETE.

IF YOU ASSUME AC $\mathcal{M}_\Delta$ IS NOT ULTRA!

<u>PROP</u> (19.21) AD IMPLIES $\mathcal{M}_\Delta$ IS AN ULTRAFILTER.

<u>PROOF</u>

LET $X \subseteq D$ CONSIDER THE GAME $G(\cup X)$ ON ω. BY AD $G(\cup X)$ IS DETERMINED.

WE SHOW THAT IF I WINS $G(\cup X)$ THEN $X \in \mathcal{M}_\Delta$. A SIMILAR PROOF SHOWS II WINS THE GAME IMPLIES $D - X \in \mathcal{M}_\Delta$.

LET τ BE W.S. FOR I. BY USING A DEFINABLE BIJECTION $\omega^{<\omega} \to \omega$ WE CAN IDENTIFY τ WITH AN ELEMENT OF ω^ω.

LOOK AT $[\tau]_\Delta$. FOR $d \in D$ IF $[\tau]_\Delta \leq_\Delta d$ AND IF $y \in d$ THEN $\tau * y \in d$ BECAUSE $\tau * y$ IS DEFINABLE FROM τ AND y AND τ IS DEFINABLE BY y. BUT τ IS WINNING FOR I SO $\tau * y \in \cup X$ $d = [\tau * y]_\Delta \in X$ ∎

SO X CONTAINS $\overline{C([T]_\Delta)}$ AND IS IN $\mathcal{U}$.

THEOREM 19.25 (SOLOVAY) AD IMPLIES THAT ω_1 IS MEASURABLE

PROOF (MARTIN) WE NEED TO:

① TRANSFER $\mathcal{M}_\Delta$ TO ω_1

② PROVE THAT THE RESULTING UF IS ω_1-COMPLETE NON-PRINCIPAL ON ω_1

BY PROP 1.10 (ANDRETTA) THERE IS $g:\omega^\omega \twoheadrightarrow \omega_1$.
DEFINE $f: D \to \omega_1$ AS FOLLOWS:

$$f(d) = \sup \underbrace{\{ g(x) \mid [x]_\Delta \leq_\Delta d \}}$$

SINCE BY PROPERTY 1 OF DEGREES $\{g(x) \mid [x]_\Delta \leq d\}$ IS COUNTABLE AND SINCE $\aleph_1$ IS REGULAR, THEN $f(d) \in \omega_1$ $\forall d \in D$.

DEFINE $\mathcal{U}$ ON ω_1 AS FOLLOWS

$$X \in \mathcal{U} \iff f^{-1}[X] \in \mathcal{M}_\Delta.$$

THE FACT THAT $\mathcal{U}$ IS AN ω_1-COMPLETE ULTRAFILTER

FOLLOWS FROM THE FACT THAT $\mathcal{M}_\Delta$ IS. [[CHECK]]

SO TO PROVE THAT $\mathcal{U}$ IS NON-PRINCIPAL WE SHOW THAT $\forall\alpha<\omega_1$

$$A_\alpha = \{d\in D \mid f(d)\geq\alpha\}\in\mathcal{M}_\Delta$$

THIS IS ENOUGH SINCE $\underset{\mathcal{M}_\Delta \ni\ \uparrow}{\underline{A_{\alpha+1}}} \cap \underset{\mathcal{M}_\Delta \in\ \uparrow}{\underline{f^{-1}[\{\alpha\}]}} = \emptyset$

↳ IF $\mathcal{U}$ IS PRINCIPAL THEN $\{\alpha\}\in\mathcal{U}$ FOR SOME $\alpha<\omega_1$

BUT THIS MEANS THAT $f^{-1}[\{\alpha\}]\in\mathcal{M}_\Delta$ $\overset{\cap}{A_{\alpha+1}}=\emptyset$ ↯

SO $\mathcal{U}$ IS NON-PRINCIPAL.

LET $\alpha<\omega_1$ AND LET $x\in\omega^\omega$ $g(x)=\alpha$.

$$C([x]_\Delta)=\{d \mid [x]_\Delta \leq_\Delta d\}\in\mathcal{M}_\Delta$$

BUT $f([x]_\Delta)\geq g(x)$ BY DEF BUT THEN

$f([x]_\Delta)\geq\alpha$ SO $C([x])\subseteq A_\alpha$

SO $A_\alpha\in\mathcal{M}_\Delta$. ∎

SOLOVAY PROVED THIS USING ENCODINGS OF LONG GAMES (GAMES ON ω_1)

AND AFTER MARTIN GAVE HIS PROOF SOLOVAY PROVED THE FOLLOWING:

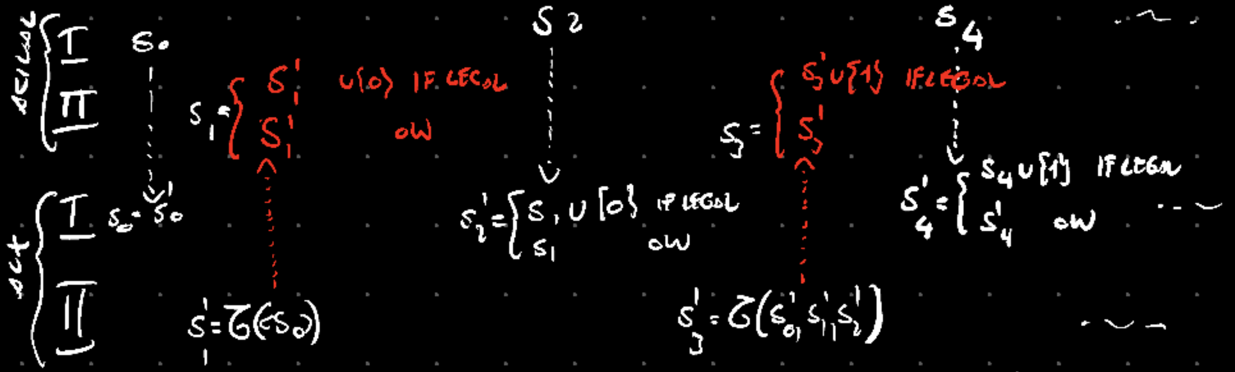
THEO ASSUME AD THEN ω_2 IS MEASURABLE.

AFTER YOU SEE THIS YOU MAY THINK MAYBE

$\forall n$ ω_n IS MEASURABLE UNDER AD.

THEO (MARTIN) ASSUME AD $\forall 3 \leq n < \omega$ ω_n IS SINGULAR WITH COFINALITY ω_2

CONCERNING $\overline{\sigma}$ THE IDEA IS THE FOLLOWING:



NOTE THAT IF II WINS THE AUX GAME SHE WINS THE ACTUAL GAME SINCE $\bigcup s_{2n} \subseteq \bigcup s'_{2n}$. MOREOVER $\bigcup_{n \in \omega} s_{2n+1} \in \mathcal{U}$ IF $\bigcup_{n \in \omega} s_{2n} \notin \mathcal{U}$ SINCE $\bigcup_{n \in \omega} s_{2n+1} \cup \bigcup_{n} s_{2n} = \omega$.