Homework Sheet #6

Capita Selecta: Set Theory
2020/21: 1st Semester; block 1
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Homework. Homework is due on Fridays. Please submit your work as a single pdf file via Canvas. You will receive one point for each question that you attempt, independent of your performance on the question. The homework will not be formally marked, but the lecturers may give individual feedback in the case of problems.

Deadline. This homework set is due on **Friday, 16 October 2020, 1pm**.

18. Let T be a tree on $\omega \times \omega$. Let P be the set of partial functions from $\omega^{<\omega}$ into $\aleph_1$. If $s \in \omega^{<\omega}$ and $u \in P^{<\omega}$ such that $\mathrm{lh}(u) \leq \mathrm{lh}(s)$, we say that u is *coherent with* s if

    (a) for all $i < \mathrm{lh}(u)$, we have that $\mathrm{dom}(u_i) = T_{s\restriction i}$,

    (b) for all $i < \mathrm{lh}(u)$, $u(i)$ is an order preserving map from $(T_{s\restriction i}, \supsetneq)$ into $(\kappa, <)$, and

    (c) for $i \leq j$, we have that $u_i \subseteq u_j$.

    Define a version of the Shoenfield tree on $\omega \times P$ and prove the corresponding version of Shoenfield's theorem. For which κ can you show κ-Suslinness with this proof?

19. Let X be a set, F be a filter on X, and κ be a cardinal. We say that F is *free* if $\bigcap F \neq \varnothing$.

    (a) Show that F is an ultrafilter in the sense of Andretta's § 0.A.6 if and only if for all $A \in X$, we have $A \in F$ or $X \backslash A \in F$.

    (b) Show that F is free if and only if F is not principal (in the sense of Andretta's § 0.A.6).

    (c) Show that if F is an ultrafilter then F is principal if and only if F contains a singleton.

    (d) Show that if F is an ultrafilter than F is κ-complete (in the sense of § 0.A.6) if and only if for every partition $\{P_\alpha\,;\,\alpha < \lambda\}$ of X into $\lambda < \kappa$ many pieces, there is precisely one $\alpha < \lambda$ such that $P_\alpha \in F$.

20. Work in ZFC and consider a subset $A \subseteq \omega^\omega$ of cardinality $\aleph_1$ with a bijection $b : A \to \aleph_1$. Consider the lexicographic order $<_{\mathrm{lex}}$ on ω^ω (check that this is a linear order on ω^ω). Define $f : [A]^2 \to 2$ by

$$f(\{x, y\}) = \begin{cases} 0 & \text{if and only if } x <_{\mathrm{lex}} y \iff b(x) < b(y) \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

    Show that there cannot be an uncountable homogeneous set for f. Conclude that $\aleph_1$ is not measurable.

    [For the last conclusion, you may use Rowbottom's theorem without proof.]

21. We write AD_X for the Axiom of Determinacy for games with moves in X.

    (a) Show that $\mathsf{AC}_{\aleph_1}(\mathbb{R})$ implies $\neg\mathsf{PSP}$.

    (b) Show that for every X, AD_X implies $\mathsf{AC}_X(X^\omega)$.

    (c) Show that $\mathsf{AD}_{\aleph_1}$ is inconsistent.