An introduction to

COMPLEX NUMBERS

Jan van de Craats

Last update: April 25, 2022

Illustrations and LaTeX typesetting: Jan van de Craats

Prof. dr. J. van de Craats is professor emeritus in mathematics
at the University of Amsterdam

This is an English translation of chapters 1, 2 and 3 of
Jan van de Craats: *Complexe getallen voor wiskunde D*

Translated by the author.

How to use this book

This is an exercise book. Each chapter starts with exercises, printed on the left-hand pages. Once you have finished an exercise, you can check your answer at the end of the book. On the right-hand pages, the theory behind the exercises is explained in a clear and concise manner. Use this information if and when required.

Background knowledge may be obtained from:

Jan van de Craats en Rob Bosch: *Basisboek wiskunde. Tweede editie*
Pearson, Amsterdam, 2009, ISBN 978-90-430-1673-5 (in Dutch)

or from its English translation:

Jan van de Craats and Rob Bosch: *All you need in maths!*
Pearson, Amsterdam, 2014, ISBN 978-90-430-3285-8.

The Greek Alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ϵ	E	epsilon	ν	N	nu	φ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	o	O	omicron	ψ	Ψ	psi
ϑ	Θ	theta	π	Π	pi	ω	Ω	omega

Contents

1 Calculating with complex numbers

In this chapter you learn how to calculate with complex numbers. They constitute a number system which is an extension of the well-known real number system. You also learn how to represent complex numbers as points in the plane. But for complex numbers we do not use the ordinary planar coordinates (x, y) but a new notation instead: $z = x + \mathrm{i}\,y$. Adding, subtracting, multiplying and dividing complex numbers then becomes a straightforward task in this notation.

1 Calculating with complex numbers

1.1 Calculate:

a. $(3\,\mathrm{i}\,)^2$

b. $(-3\,\mathrm{i}\,)^2$

c. $-(4\,\mathrm{i}\,)^2$

d. $(-\,\mathrm{i}\,)^3$

e. $\mathrm{i}^{\,4}$

1.2 Calculate:

a. $(\frac{1}{2}\sqrt{2}\,\mathrm{i}\,)^2$

b. $(-\frac{1}{3}\sqrt{6}\,\mathrm{i}\,)^2$

c. $(\frac{1}{2}\sqrt{4}\,\mathrm{i}\,)^2$

d. $(\frac{2}{3}\sqrt{3}\,\mathrm{i}\,)^2$

e. $(-\frac{1}{2}\sqrt{3}\,\mathrm{i}\,)^2$

Write the following roots in the form $r\,\mathrm{i}$ where r is a positive real number. Example: $\sqrt{-5} = \sqrt{5}\,\mathrm{i}$. Give exact answers and simplify the roots as much as possible (for instance, write $3\sqrt{3}$ instead of $\sqrt{27}$).

1.3

a. $\sqrt{-3}$

b. $\sqrt{-9}$

c. $\sqrt{-8}$

d. $\sqrt{-25}$

e. $\sqrt{-15}$

1.4

a. $\sqrt{-33}$

b. $\sqrt{-49}$

c. $\sqrt{-48}$

d. $\sqrt{-45}$

e. $\sqrt{-75}$

Solve the following quadratic equations. Give eaxct answers and simplify roots if possible.

1.5

a. $x^2 - 2x + 2 = 0$

b. $x^2 + 4x + 5 = 0$

c. $x^2 + 2x + 10 = 0$

d. $x^2 - 6x + 10 = 0$

e. $x^2 - 4x + 8 = 0$

1.6

a. $x^2 - 12x + 40 = 0$

b. $x^2 - 4x + 6 = 0$

c. $x^2 + 2x + 4 = 0$

d. $x^2 - 6x + 12 = 0$

e. $x^2 + 8x + 20 = 0$

The following exercise is a real challenge! If you get stuck, you may look at the solution in the answer section at the end of the book. But first try to solve this exercise yourself!

1.7 In calculating with roots from negative numbers you have to be very careful, as will be apparent from the following paradoxical 'derivation':

$$-1 = (\sqrt{-1})^2 = \sqrt{(-1)^2} = \sqrt{1} = 1.$$

Try to find the error(s)! In other words, which of the four equality signs is (or are) unjustified, and why?

Square roots of negative numbers

In school, you learned that there doesn't exist a number x for which $x^2 = -1$. Indeed, squares are never negative. But what if we *imagine* that there does exist such a number? A number—we call it "i" (from **i***maginary*)—for which

$$\mathrm{i}^2 = -1$$

holds true? One could call such a number a *square root of* -1, so $\mathrm{i} = \sqrt{-1}$. Then also for other negative numbers a square root can be found if we apply the ordinary rules of calculation. For example, $6\mathrm{i}$ is a square root of -36 since $(6\mathrm{i})^2 = 6\mathrm{i} \times 6\mathrm{i} = 36 \times \mathrm{i}^2 = 36 \times (-1) = -36$. In a similar way, it can be shown that $\sqrt{-13} = \sqrt{13}\,\mathrm{i}$, or that $\sqrt{-12} = \sqrt{12}\,\mathrm{i} = 2\sqrt{3}\,\mathrm{i}$ (note that $\sqrt{12} = \sqrt{4 \cdot 3} = 2\sqrt{3}$).

What we have done, is finding a solution of the equation $x^2 = -a$, where a is a positive number. We have discovered that $\sqrt{a}\,\mathrm{i}$ is a solution, but, of course, also $-\sqrt{a}\,\mathrm{i}$ is a solution: $(-\sqrt{a}\,\mathrm{i})^2 = (-1)^2(\sqrt{a})^2\mathrm{i}^2 = 1 \cdot a \cdot (-1) = -a$. The complete soltion of the equation $x^2 = -a$ therefore is $x = \pm\sqrt{a}\,\mathrm{i}$.

The *abc*-formula

If we have a number i for which $\mathrm{i}^2 = -1$ holds, then we can solve any quadratic equation, even if its discriminant is negative. For instance, take $x^2 + 2x + 5 = 0$. Indeed:

$$\begin{aligned} x^2 + 2x + 5 &= 0 \\ (x+1)^2 + 4 &= 0 \\ (x+1)^2 &= -4 \end{aligned}$$

yielding $x + 1 = \pm 2\mathrm{i}$ so $x = -1 + 2\mathrm{i}$ or $x = -1 - 2\mathrm{i}$.

This boils down to applying the well-known *abc*-formula for solving quadratic equations. The solutions of the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If the discriminant $b^2 - 4ac$ is negative, then $4ac - b^2$ is positive, so $\sqrt{b^2 - 4ac} = \sqrt{(4ac - b^2)(-1)} = \sqrt{4ac - b^2}\,\mathrm{i}$. In the example above we had $a = 1$, $b = 2$, $c = 5$ and $b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot 5 = -16$, so, indeed, $x_{1,2} = \dfrac{-2 \pm 4\mathrm{i}}{2} = -1 \pm 2\mathrm{i}$.

Calculate the following complex numbers, draw them in the complex plane and calculate their absolute value.

1.8

a. $(1-2\mathrm{i})+(3-4\mathrm{i})$
b. $2\mathrm{i}-(4-2\mathrm{i})$
c. $(2-2\mathrm{i})+(-1+2\mathrm{i})$
d. $(4-6\mathrm{i})-(1-3\mathrm{i})$
e. $(2-\mathrm{i})+(3-2\mathrm{i})$

1.9

a. $(1-2\mathrm{i})(3-4\mathrm{i})$
b. $2\mathrm{i}(4-2\mathrm{i})$
c. $(2-2\mathrm{i})(2+2\mathrm{i})$
d. $(1-3\mathrm{i})^2$
e. $(2-\mathrm{i})^2$

1.10

a. i^3
b. i^4
c. i^5
d. i^{10}
e. i^{2006}

1.11

a. $(-\mathrm{i})^5$
b. $(2\mathrm{i})^3$
c. $(-2\mathrm{i})^7$
d. $(1+\mathrm{i})^3$
e. $(1-\mathrm{i})^3$

The complex numbers z for which $\mathrm{Re}(z)=5$ holds, constitute the vertical line $x=5$ in the complex plane. Draw the following lines in the complex plane.

1.12

a. $\mathrm{Re}(z)=4$
b. $\mathrm{Re}(z)=-3$
c. $\mathrm{Im}(z)=2$
d. $\mathrm{Im}(z)=-2$
e. $\mathrm{Im}(z)=\mathrm{Re}(z)$

1.13

a. $\mathrm{Re}(z)+\mathrm{Im}(z)=1$
b. $\mathrm{Re}(z)=2\,\mathrm{Im}(z)$
c. $\mathrm{Re}(z)-2\,\mathrm{Im}(z)=1$
d. $\mathrm{Re}(z)+\mathrm{Im}(z)=5$
e. $\mathrm{Re}(z)+\mathrm{Im}(z)=\mathrm{Re}(z)-\mathrm{Im}(z)$

The complex numbers z for which $|z|=5$ holds, constitute the circle with radius 5 and center 0. Verify that the complex numbers z for which $|z-1|=5$ holds, constitute the circle with radius 5 and center 1. Draw the following circles in the complex plane and for each circle give its center and its radius.

1.14

a. $|z|=4$
b. $|z-1|=3$
c. $|z-2|=2$
d. $|z-3|=1$
e. $|z+1|=5$

1.15

a. $|z+3|=4$
b. $|z-\mathrm{i}|=5$
c. $|z+2\mathrm{i}|=1$
d. $|z-1-\mathrm{i}|=3$
e. $|z+3-\mathrm{i}|=2$

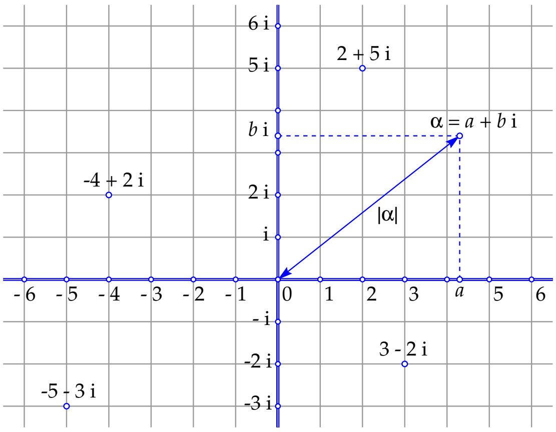
The complex plane

In solving quadratic equations we have encountered numbers of the form $a + b\,\mathrm{i}$. They are called *complex numbers*. For instance $-1 + 2\,\mathrm{i}$ or $3 - 5\,\mathrm{i}$. Such numbers can be *added* as follows: $(-1 + 2\,\mathrm{i}) + (3 - 5\,\mathrm{i}) = 2 - 3\,\mathrm{i}$. Or *subtracted* as follows: $(-1 + 2\,\mathrm{i}) - (3 - 5\,\mathrm{i}) = -4 + 7\,\mathrm{i}$. Or even *multiplied*:

$$(-1 + 2\,\mathrm{i})(3 - 5\,\mathrm{i}) = -3 + 5\,\mathrm{i} + 6\,\mathrm{i} - 10\,\mathrm{i}^2 = -3 + 11\,\mathrm{i} + 10 = 7 + 11\,\mathrm{i}.$$

Just expand brackets and use that $\mathrm{i}^2 = -1$.

A complex number $a + b\,\mathrm{i}$ is completely determined by the two real numbers a and b. Real numbers may be thought of as points on a line, the *real number line*. In a similar way, the complex numbers may be thought of as points in a plane, the *complex plane*. In this plane first a coordinate system has to be chosen. The complex number $a + b\,\mathrm{i}$ then corresponds to the point with coordinates (a, b):



For the points on the x-axis we have $b = 0$. Instead of $a + 0\,\mathrm{i}$ we then simply write a. And for the points on the y-axis we have $a = 0$. Then we do not write $0 + b\,\mathrm{i}$ but simply $b\,\mathrm{i}$. And instead of $1\,\mathrm{i}$, of course, we simply write i.

The x-axis from now on will be called the *real axis* and the numbers on it the *real numbers*. The y-axis is called the *imaginary axis* and the numbers on it are called the *imaginary numbers*. Complex numbers often are denoted by the letter z or by Greek letters like α (alpha). We then write $z = x + y\,\mathrm{i}$ or $\alpha = a + b\,\mathrm{i}$.

If $\alpha = a + b\,\mathrm{i}$ is a complex number, then a is called its *real part*, notation $a = \mathrm{Re}(\alpha)$, and b is called its *imaginary part*, notation $b = \mathrm{Im}(\alpha)$. The imaginary part, therefore, is a *real* number! The real number $\sqrt{a^2 + b^2}$ is called the *absolute value* of α, notation $|\alpha|$. Instead of absolute value one also often uses the word *modulus*. The absolute value of α is the distance from α to the origin (by the Pythagorean theorem). If α happens to be a real number, then $|\alpha|$ equals the ordinary absolute value of α.

Calculate the following quotients of complex numbers, i.e., write each quotient in the form $a + b\,\mathrm{i}$ with a and b real.

1.16

a. $\dfrac{1}{3-4\,\mathrm{i}}$

b. $\dfrac{3}{4-2\,\mathrm{i}}$

c. $\dfrac{2-2\,\mathrm{i}}{-1+2\,\mathrm{i}}$

d. $\dfrac{4-6\,\mathrm{i}}{1-3\,\mathrm{i}}$

e. $\dfrac{2-\mathrm{i}}{3-2\,\mathrm{i}}$

1.17

a. $\dfrac{1-2\,\mathrm{i}}{3+4\,\mathrm{i}}$

b. $\dfrac{2\,\mathrm{i}}{1-2\,\mathrm{i}}$

c. $\dfrac{1}{\mathrm{i}}$

d. $\dfrac{1-3\,\mathrm{i}}{\mathrm{i}}$

e. $\dfrac{1+\mathrm{i}}{1-\mathrm{i}}$

1.18

a. $\dfrac{3\,\mathrm{i}}{4+3\,\mathrm{i}}$

b. $\dfrac{3+\mathrm{i}}{1-2\,\mathrm{i}}$

c. $\dfrac{2-\mathrm{i}}{-1+2\,\mathrm{i}}$

d. $\dfrac{2-\mathrm{i}}{1+2\,\mathrm{i}}$

e. $\dfrac{1+2\,\mathrm{i}}{2-\mathrm{i}}$

1.19

a. $\dfrac{1-2\,\mathrm{i}}{4\,\mathrm{i}}$

b. $\dfrac{2-\mathrm{i}}{3+2\,\mathrm{i}}$

c. $\dfrac{1+\mathrm{i}}{4\,\mathrm{i}}$

d. $\dfrac{2-\mathrm{i}}{-\mathrm{i}}$

e. $\dfrac{1+3\,\mathrm{i}}{3-\mathrm{i}}$

Multiplication and division

Multiplying complex numbers is done by expanding brackets and using that $\mathrm{i}^2 = -1$. In the previous section you already have seen an example of multiplying two complex numbers. It always goes like this:

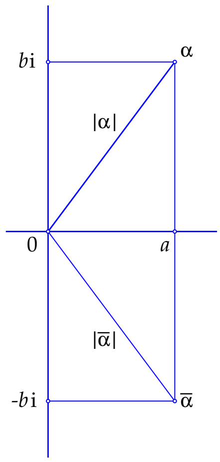
$$\begin{aligned}(a_1 + b_1\,\mathrm{i})(a_2 + b_2\,\mathrm{i}) &= a_1a_2 + a_1b_2\,\mathrm{i} + a_2b_1\,\mathrm{i} + b_1b_2\,\mathrm{i}^2 \\ &= (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)\,\mathrm{i}\,.\end{aligned}$$

Division is the inverse operation of multiplication. We will teach you a trick to calculate the quotient of two complex numbers in a fast and simple manner. First we give an example:

$$\frac{1-2\mathrm{i}}{2+3\mathrm{i}} = \frac{(1-2\mathrm{i})(2-3\mathrm{i})}{(2+3\mathrm{i})(2-3\mathrm{i})} = \frac{-4-7\mathrm{i}}{4-6\mathrm{i}+6\mathrm{i}-9\mathrm{i}^2} = \frac{-4-7\mathrm{i}}{13} = -\frac{4}{13} - \frac{7}{13}\,\mathrm{i}\,.$$

In the third step we obtained in the denominator the *real* number 13, from which we subsequently could easily calculate the quotient in the desired form $a + b\,\mathrm{i}$.

The trick apparently consists of multiplying numerator and denominator with *the same* complex number (which doesn't change the quotient). This number is the so-called *conjugated complex number* of the denominator. The conjugated complex number of $\alpha = a + b\,\mathrm{i}$ is the complex number $a - b\,\mathrm{i}$, notation $\overline{\alpha}$. One gets $\overline{\alpha}$ by flipping the sign of the imaginary part of α.
The trick above works because it produces in the denominator a number of the form



$$\alpha\overline{\alpha} = (a + b\,\mathrm{i})(a - b\,\mathrm{i}) = a^2 - ab\,\mathrm{i} + ab\,\mathrm{i} - b^2\,\mathrm{i}^2 = a^2 + b^2.$$

This is always a *positive real number* (except if $a = b = 0$, but then we have $\alpha = 0$, and also for complex numbers it is impossible to divide by 0).

In the previous section we defined the *absolute value* $|\alpha|$ of α as $|\alpha| = \sqrt{a^2 + b^2}$. Consequently, we have $\alpha\overline{\alpha} = |\alpha|^2$ and also $|\overline{\alpha}| = |\alpha|$.

The only rules you should memorize on multiplication and division are:

> *To multiply complex numbers one should expand brackets using* $\mathrm{i}^2 = -1$.
> *To divide complex numbers one should multiply numerator and denominator by the conjugated complex number of the denominator and expand brackets.*

Summary

Complex numbers are numbers of the form $\alpha = a + b\,\mathrm{i}$, where a and b are *real* numbers. Complex numbers can be represented as points in a plane in which a coordinate system is chosen. The complex number $\alpha = a + b\,\mathrm{i}$ then is the point with coordinates (a, b).

Terminology and notations:

If $\alpha = a + b\,\mathrm{i}$ then a is called the *real part* and b the *imaginary part* of α.
If $\alpha = a + b\,\mathrm{i}$ then $\overline{\alpha} = a - b\,\mathrm{i}$ is called the *conjugated complex number* of α.
If $\alpha = a + b\,\mathrm{i}$ then $|\alpha| = \sqrt{a^2 + b^2}$ is called the *absolute value* or *modulus* of α. This is a non-negative real number. It is the distance from the point α to the origin.

Instead of $\alpha = a + b\,\mathrm{i}$ one sometimes writes $\alpha = a + \mathrm{i}\,b$. The imaginary part then is written *after* the i instead of before the i.

Calculation rules:

Addition and subtraction (by coordinates):

$$\begin{aligned}
\alpha_1 + \alpha_2 &= (a_1 + b_1\,\mathrm{i}) + (a_2 + b_2\,\mathrm{i}) &&= (a_1 + a_2) + (b_1 + b_2)\,\mathrm{i} \\
\alpha_1 - \alpha_2 &= (a_1 + b_1\,\mathrm{i}) - (a_2 + b_2\,\mathrm{i}) &&= (a_1 - a_2) + (b_1 - b_2)\,\mathrm{i}\,.
\end{aligned}$$

Multiplication:

$$\alpha_1\alpha_2 = (a_1 + b_1\,\mathrm{i})(a_2 + b_2\,\mathrm{i}) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)\,\mathrm{i}\,.$$

This is easy to memorize: expand brackets and use that $\mathrm{i}^2 = -1$.
In particular: $\alpha\overline{\alpha} = (a + b\,\mathrm{i})(a - b\,\mathrm{i}) = a^2 + b^2$, so $\alpha\overline{\alpha} = |\alpha|^2$.

Division:

$$\begin{aligned}
\frac{\alpha_1}{\alpha_2} &= \frac{\alpha_1\,\overline{\alpha_2}}{\alpha_2\,\overline{\alpha_2}} = \frac{(a_1 + b_1\,\mathrm{i})(a_2 - b_2\,\mathrm{i})}{(a_2 + b_2\,\mathrm{i})(a_2 - b_2\,\mathrm{i})} \\
&= \frac{(a_1a_2 + b_1b_2) + (-a_1b_2 + a_2b_1)\,\mathrm{i}}{{a_2}^2 + {b_2}^2} = \frac{a_1a_2 + b_1b_2}{{a_2}^2 + {b_2}^2} + \frac{-a_1b_2 + a_2b_1}{{a_2}^2 + {b_2}^2}\,\mathrm{i}\,.
\end{aligned}$$

This also is easy to memorize: multiply numerator and denominator by the conjugated complex number of the denominator and expand brackets.

For *real* numbers (i.e., complex numbers $a + b\,\mathrm{i}$ with $b = 0$) the calculation rules are in agreement with the ordinary calculation rules for addition, subtraction, multiplication and division of real numbers. Consequently, the complex number system is an *extension* of the real number system. The real numbers are located on the horizontal axis, which is also called the *real axis*. The vertical axis is called the *imaginary axis*; the numbers on this axis are called *imaginary numbers*.

2 The geometry of complex calculations

In the first section of this chapter you will learn how to view complex numbers as vectors. The sum and the difference of two complex numbers then are easily represented in a geometric setting. It is also easy to describe circles in this way. Next you will learn a new notation for complex numbers, the (r, φ)-notation, which is related to polar coordinates. With this notation, the calculation rules for multiplication and division of complex numbers will get a geometric interpretation in which a famous formula of Leonhard Euler plays an important role. We close this chapter by introducing the complex exponential function e^z, the complex sine function $\sin z$ and the complex cosine function $\cos z$.

2 The geometry of complex calculations

2.1 In each of the following exercises two complex numbers α and β are given. Calculate the complex number that corresponds to the vector with initial point α and endpoint β. Always check your answer with a drawing.

a. $\alpha = \mathrm{i}, \beta = -2\mathrm{i}$
b. $\alpha = 1 - \mathrm{i}, \beta = -2$
c. $\alpha = -2 + 3\mathrm{i}, \beta = 1 - 2\mathrm{i}$
d. $\alpha = 4, \beta = -4$
e. $\alpha = 8\mathrm{i}, \beta = 8\mathrm{i}$

2.2 For each of the previous exercises draw the vector corresponding to the complex number $\alpha + \beta$. Take the origin as initial point.

2.3 Determine the equation of the following circles and write each in the form

$$z\overline{z} - \overline{\alpha}z - \alpha\overline{z} + \alpha\overline{\alpha} - r^2 = 0$$

Example: The equation of the circle with center $1 + \mathrm{i}$ and radius 2 is $(z - (1 + \mathrm{i}))(\overline{z} - (1 - \mathrm{i})) = 4$ which upon expanding brackets yields

$$z\overline{z} - (1 - \mathrm{i})z - (1 + \mathrm{i})\overline{z} - 2 = 0.$$

a. The circle with center i and radius 3
b. The circle with center $1 - \mathrm{i}$ and radius $\sqrt{2}$
c. The circle with center 1 and radius 1
d. The circle with center $-2 + \mathrm{i}$ and radius 2
e. The circle with center $1 - 2\mathrm{i}$ and radius 1

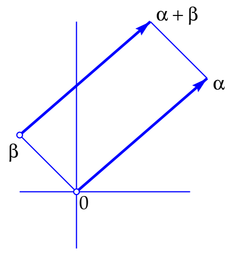
Not every equation which at first sight looks like a circle equation represents a circle. For instance, take $z\overline{z} = -1$. This does not represent a circle, since the left-hand side is greater than or equal to 0 for each complex number z. Therefore, there are no complex numbers z satisfying this equation.

2.4 Investigate which of the following equations represents a circle. If so, then determine its center and radius.

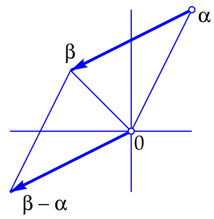
a. $z\overline{z} - \mathrm{i}z + \mathrm{i}\overline{z} = 0$
b. $z\overline{z} + (1 + \mathrm{i})z + (1 - \mathrm{i})\overline{z} = 2$
c. $z\overline{z} + 2\mathrm{i}z - 2\mathrm{i}\overline{z} + 4 = 0$
d. $z\overline{z} + 2z + 2\overline{z} + 5 = 0$
e. $z\overline{z} + (2 - \mathrm{i})z + (2 + \mathrm{i})\overline{z} - 1 = 0$

Complex numbers as vectors

A vector in the plane can be represented by an arrow pointing from an *initial point* to an *end point*. Parallel arrows with the same direction and the same size represent the same vector.



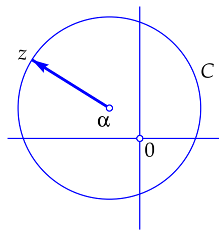
In the complex plane, to each complex number α a vector may be associated by drawing an arrow from the origin to the point α. This vector then also can be represented by an arrow pointing from an arbitrary point β to the point $\alpha + \beta$, since the points $\alpha + \beta$, β, 0, and α form a parallelogram (parallelogram construction of $\alpha + \beta$).



The vector representation is convenient to picture the *difference* $\beta - \alpha$ of two complex numbers α and β:

$\beta - \alpha$ is the vector (arrow) pointing from α to β.

Note that to find the complex number $\beta - \alpha$ as a point in the complex plane, you have to start the arrow at the origin. Example: $\alpha = 1 + 2\,\mathrm{i}$, $\beta = -1 + \mathrm{i}$ so $\beta - \alpha = -2 - \mathrm{i}$.



The vector representation is also convenient in working with circles. If C is a circle with center α and radius r then for each point z on C we have

$$|z - \alpha| = r.$$

Thus, the absolute value of $z - \alpha$ equals r, in other words, the arrow pointing from α to z has length r.

In the figure above we have taken $\alpha = -1 + \mathrm{i}$ and $r = 3$. Therefore, that circle is given by $|z - (-1 + \mathrm{i})| = 3$ so $|z + 1 - \mathrm{i}| = 3$.

Sometimes it is also convenient not to work with the absolute value, but to use $|w|^2 = w\overline{w}$ (see page 7) with $w = z - \alpha$. Then the equation of the circle C with center α and radius r can be written as

$$(z - \alpha)(\overline{z} - \overline{\alpha}) = r^2.$$

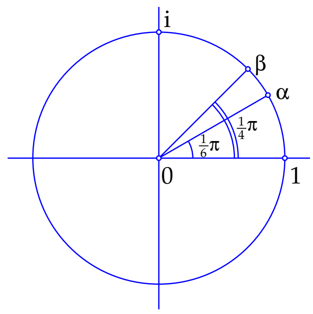
The circle above therefore is given by $(z + 1 - \mathrm{i})(\overline{z} + 1 + \mathrm{i}) = 9$, or, upon expanding brackets,

$$z\overline{z} + (1 - \mathrm{i})\overline{z} + (1 + \mathrm{i})z - 7 = 0.$$

The *unit circle*, the circle with center 0 and radius 1, is given by

$$z\overline{z} = 1.$$

For some angles (given in radians) the sine and the cosine have a well-known exact value. For example, $\cos\frac{1}{6}\pi = \frac{1}{2}\sqrt{3}$, $\sin\frac{1}{6}\pi = \frac{1}{2}$ *and* $\cos\frac{1}{4}\pi = \sin\frac{1}{4}\pi = \frac{1}{2}\sqrt{2}$. *This means that the points* α *and* β *in the drawing to the right have nice exact rectangular coordinates, namely* $\alpha = (\frac{1}{2}\sqrt{3}, \frac{1}{2})$ *and* $\beta = (\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$. *Writing* α *and* β *as complex numbers, we therefore get*



$$\alpha = \frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i} \quad \textit{and} \quad \beta = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,\mathrm{i}\,.$$

In the next exercises all given points are located on the unit circle. Draw these points and give their argument (in radians) in the form $\varphi + 2k\pi$ (where k is an arbitrary integer). *Example:* $\arg \mathrm{i} = \frac{1}{2}\pi + 2k\pi$.

2.5

a. $-\mathrm{i}$

b. -1

c. 1

d. $-\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,\mathrm{i}$

e. $\frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}$

2.6

a. $\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}$

b. $\frac{1}{2}\sqrt{3} - \frac{1}{2}\,\mathrm{i}$

c. $-\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}$

d. $-\frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i}$

e. $\frac{1}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i}$

Multiplication or division of complex numbers on the unit circle is done by adding or subtracting their arguments (see the explanation on the next page). Use this in the next exercises. Therefore, find solutions in a *geometric* way; not by expanding brackets!

2.7

a. $(\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,\mathrm{i}\,)^2$

b. $(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}\,)^3$

c. $(-\frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i}\,)^3$

d. $(-\frac{1}{2}\sqrt{3} - \frac{1}{2}\,\mathrm{i}\,)^5$

e. $(\frac{1}{2}\sqrt{3} - \frac{1}{2}\,\mathrm{i}\,)(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}\,)^2$

2.8

a. $(-\mathrm{i}\,)/(\frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i}\,)$

b. $(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}\,)^2/(\frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i}\,)$

c. $(-\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}\,)/(\frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i}\,)^3$

d. $(\frac{1}{2}\sqrt{3} - \frac{1}{2}\,\mathrm{i}\,)^6/(\frac{1}{2}\sqrt{3} + \frac{1}{2}\,\mathrm{i}\,)$

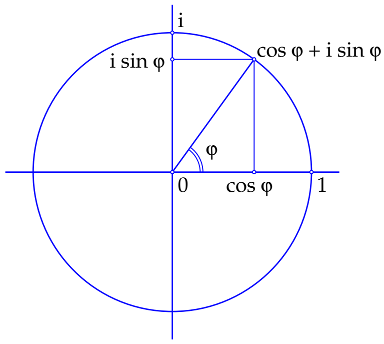
e. $(\frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}\,)^3/(\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,\mathrm{i}\,)^3$

Complex numbers on the unit circle

Each point on the unit circle (the circle with radius 1 and the origin as its center) has rectangular coordinates of the form $(\cos\varphi, \sin\varphi)$. Here, φ is the angle from the positive x-axis to the radius vector, the vector pointing from the origin to the given point (φ is de Greekse letter 'phi'). We always measure angles counterclockwise in radians (180° equals π radians). Angles are determined up to integer multiples of 2π.

Written as a complex number, a point on the unit circle therefore takes the form

$$z = \cos\varphi + \mathrm{i}\,\sin\varphi.$$



Then, indeed, we have

$$|\cos\varphi + \mathrm{i}\,\sin\varphi| = \sqrt{\cos^2\varphi + \sin^2\varphi} = \sqrt{1} = 1.$$

The angle φ is called the *argument* of z, notation $\varphi = \arg(z)$. The argument is determined up to integer multiples of 2π.

What happens if two such numbers, for example $z_1 = \cos\varphi_1 + \mathrm{i}\,\sin\varphi_1$ and $z_2 = \cos\varphi_2 + \mathrm{i}\,\sin\varphi_2$ are multiplied? Then

$$\begin{aligned} z_1 z_2 &= (\cos\varphi_1 + \mathrm{i}\,\sin\varphi_1)(\cos\varphi_2 + \mathrm{i}\,\sin\varphi_2) \\ &= (\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + \mathrm{i}\,(\cos\varphi_1\sin\varphi_2 + \sin\varphi_1\cos\varphi_2) \end{aligned}$$

But according to well-known trigonometric rules, we have

$$\begin{aligned} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 &= \cos(\varphi_1+\varphi_2) \quad \text{and} \\ \cos\varphi_1\sin\varphi_2 + \sin\varphi_1\cos\varphi_2 &= \sin(\varphi_1+\varphi_2) \end{aligned}$$

so

$$z_1 z_2 = \cos(\varphi_1+\varphi_2) + \mathrm{i}\,\sin(\varphi_1+\varphi_2).$$

Therefore, this again is a a number *on the unit circle*, with as its argument the sum $\varphi_1 + \varphi_2$ of the arguments of z_1 and z_2, so we have:

> *The product z_1z_2 of two complex numbers on the unit circle again is a number on the unit circle. Its argument is the sum $\arg(z_1) + \arg(z_2)$ of the arguments of z_1 and z_2.*

Similarly, for the quotient of two such complex numbers, we have:

> *The quotient $\dfrac{z_1}{z_2}$ of two complex numbers on the unit circle again is a number on the unit circle. Its argument is the difference $\arg(z_1) - \arg(z_2)$ of the arguments of z_1 and z_2.*

By substituting $\varphi = \pi$ in Euler's formula on the next page, we get:

$$e^{\pi i} = \cos\pi + i\sin\pi = -1 + i\,0 = -1$$

so

$$e^{\pi i} + 1 = 0.$$

This also is a famous formula by Euler. It combines the five most important constants in mathematics, e, π, i, 1 and 0.

Calculate:

2.9

a. $e^{-\pi i}$
b. $e^{2\pi i}$
c. $e^{\frac{1}{2}\pi i}$
d. $e^{3\pi i}$
e. $e^{4\pi i}$

2.10

a. $e^{-\frac{3}{2}\pi i}$
b. $e^{\frac{2}{3}\pi i}$
c. $e^{\frac{5}{2}\pi i}$
d. $e^{-\frac{13}{6}\pi i}$
e. $e^{2006\pi i}$

2.11

a. $e^{-\pi i}e^{\frac{2}{3}\pi i}$
b. $e^{3\pi i}e^{-2\pi i}$
c. $e^{\frac{1}{3}\pi i}e^{-\pi i}$
d. $\dfrac{e^{\frac{1}{2}\pi i}}{e^{\frac{3}{2}\pi i}}$
e. $\dfrac{e^{-\frac{1}{4}\pi i}}{e^{\frac{3}{4}\pi i}}$

2.12

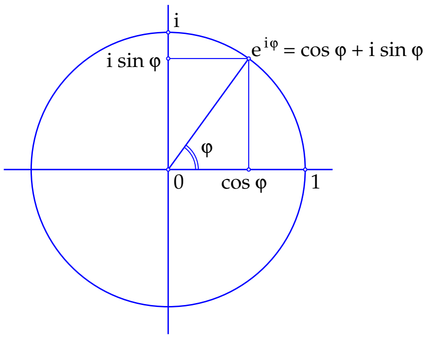
a. $\dfrac{e^{-\frac{3}{4}\pi i}}{e^{\frac{3}{4}\pi i}}$
b. $\dfrac{e^{\frac{2}{3}\pi i}}{e^{\frac{1}{6}\pi i}}$
c. $e^{\frac{5}{2}\pi i}e^{3\pi i}$
d. $\dfrac{e^{\frac{7}{6}\pi i}}{e^{\frac{2}{3}\pi i}}$
e. $\dfrac{e^{\pi i}}{e^{4\pi i}}$

Formulas of Euler

In the eighteenth century the great mathematician Leonhard Euler proved the formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

We will not explain Euler's proof here, but rather present his formula as a *definition*, or, if you like, an *abridged notation*. Instead of $\cos\varphi + i\sin\varphi$, from now on we will write $e^{i\varphi}$ (or $e^{\varphi i}$). Note, however, that this doesn't involve the well-known real exponential function, since the exponent $i\varphi$ is not a real number but an *imaginary* number. And, of course, there is more to it: later we will define the function e^z for arbitrary complex numbers z (see page 19).



In the previous section we showed that

$$(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2) = \cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2).$$

In the new notation, this looks much more familiar:

$$e^{i\varphi_1}e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}.$$

Just as for real exponential functions, we have in this case: *in multiplying imaginary powers of* e*, the exponents must be added.* And, of course, also:

$$\frac{e^{i\varphi_1}}{e^{i\varphi_2}} = e^{i(\varphi_1-\varphi_2)}.$$

In dividing imaginary powers of e*, the exponents must be subtracted.*

Substituting $-\varphi$ for φ in Euler's formula above, yields

$$e^{-i\varphi} = \cos(-\varphi) + i\sin(-\varphi) = \cos\varphi - i\sin\varphi.$$

Adding the two formulas, we get $e^{i\varphi} + e^{-i\varphi} = 2\cos\varphi$, i.e.

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}.$$

Subtracting the two formulas, we get $e^{i\varphi} - e^{-i\varphi} = 2i\sin\varphi$, i.e.

$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

Also these two famous formulas were found by Euler.

Write the following complex numbers in (r, φ) notation. Use the arctangent function (inverse tangent function) on a calculator. Use radians and give your answers rounded to 4 decimals.

2.13

a. $1+2\,\mathrm{i}$

b. $4-2\,\mathrm{i}$

c. $2-3\,\mathrm{i}$

d. $-2-3\,\mathrm{i}$

e. -3

2.14

a. $2+\mathrm{i}$

b. $2-\mathrm{i}$

c. $-\mathrm{i}$

d. $-5+\mathrm{i}$

e. $-3\,\mathrm{i}$

2.15 Write the next complex numbers in $x+\mathrm{i}y$ notation. Use a calculator and give your answers rounded to 4 decimals.

a. $2\mathrm{e}^{2\,\mathrm{i}}$

b. $3\mathrm{e}^{-\mathrm{i}}$

c. $0.2\mathrm{e}^{0.3\,\mathrm{i}}$

d. $1.2\mathrm{e}^{2.5\,\mathrm{i}}$

e. $\mathrm{e}^{3.1415\,\mathrm{i}}$

2.16 Take $z = 3\mathrm{e}^{-2\,\mathrm{i}}$. Write the following numbers in (r, φ) notation:

a. $\overline{z}$

b. z^2

c. $(\overline{z})^5$

d. $\dfrac{1}{z}$

e. z^n (integer n)

2.17 Let be given the complex numbers $z_1 = 2\mathrm{e}^{5\,\mathrm{i}}$ and $z_2 = 3\mathrm{e}^{-2\,\mathrm{i}}$. Show, using the (r, φ) notation, that

a. $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$

b. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

2.18 Using the (r, φ) notation, show that in general for all complex numbers $z_1 = r_1\,\mathrm{e}^{\,\mathrm{i}\varphi_1}$ and $z_2 = r_2\,\mathrm{e}^{\,\mathrm{i}\varphi_2}$ we have

a. $\overline{z_1 z_2} = \overline{z_1}\;\overline{z_2}$

b. $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

Hint: use the same method as in the previous exercise, but this time with letters instead of numbers.

The (r, φ) notation for complex numbers

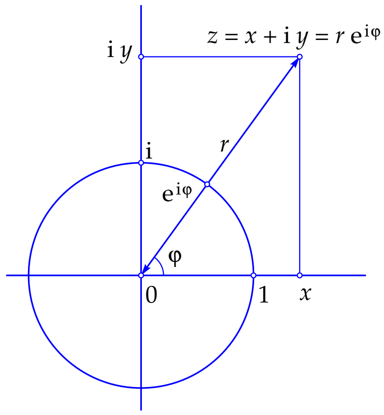
Each complex number $z = x + \mathrm{i}\, y$ can be written in the form

$$z = r(\cos\varphi + \mathrm{i}\,\sin\varphi)$$

where $r = |z| = \sqrt{x^2 + y^2}$ is the absolute value of z, and $\varphi = \arg(z)$ is the argument of z, i.e., the angle from the positive x-axis to the radius vector (the vector from the origin to the point z). Then $x = r\,\cos\varphi$ and $y = r\,\sin\varphi$.

The abridged notation from the previous section yields

$$z = r\,\mathrm{e}^{\,\mathrm{i}\,\varphi}.$$



This is called the (r, φ) *notation* or *polar notation* (since it is related to polar coordinates).
The (r, φ) notation is very convenient for multiplication and division:

$$\begin{aligned} z_1 z_2 &= r_1\,\mathrm{e}^{\,\mathrm{i}\,\varphi_1}\,r_2\,\mathrm{e}^{\,\mathrm{i}\,\varphi_2} = r_1 r_2\,\mathrm{e}^{\,\mathrm{i}\,(\varphi_1+\varphi_2)} \\ \frac{z_1}{z_2} &= \frac{r_1\,\mathrm{e}^{\,\mathrm{i}\,\varphi_1}}{r_2\,\mathrm{e}^{\,\mathrm{i}\,\varphi_2}} = \frac{r_1}{r_2}\,\mathrm{e}^{\,\mathrm{i}\,(\varphi_1-\varphi_2)}. \end{aligned}$$

For multiplying complex numbers the absolute values are multiplied and the arguments are added. For dividing complex numbers the absolute values are divided and the arguments are subtracted.

The relations between x, y, r and φ are

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

$$r = \sqrt{x^2 + y^2}, \quad \tan\varphi = \frac{y}{x}.$$

Using these relations, the notation $z = x + \mathrm{i}\,y$ of a complex number z can be converted to the (r, φ) notation $z = r\,\mathrm{e}^{\,\mathrm{i}\,\varphi}$ and vice versa. There are, however, some subtle intricacies in determining φ if x and y are given. In the first place, φ is not defined if $x = y = 0$. Furthermore, if $x = 0$, $y > 0$ then $\varphi = \frac{\pi}{2}$, and if $x = 0$, $y < 0$ then $\varphi = -\frac{\pi}{2}$. In all other cases φ can be calculated using the arctangent function (the inverse tangent function). Note, however that the arctangent function always produces a value between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. For complex numbers $z = x + \mathrm{i}\,y$ in the left half plane (i.e., with $x < 0$) you have to add π:

$$\varphi = \arctan\frac{y}{x} + 2k\pi \ \text{ if } x > 0 \qquad \text{and} \qquad \varphi = \arctan\frac{y}{x} + \pi + 2k\pi \ \text{ if } x < 0.$$

Recall that the argument is determined up to integer multiples of 2π.

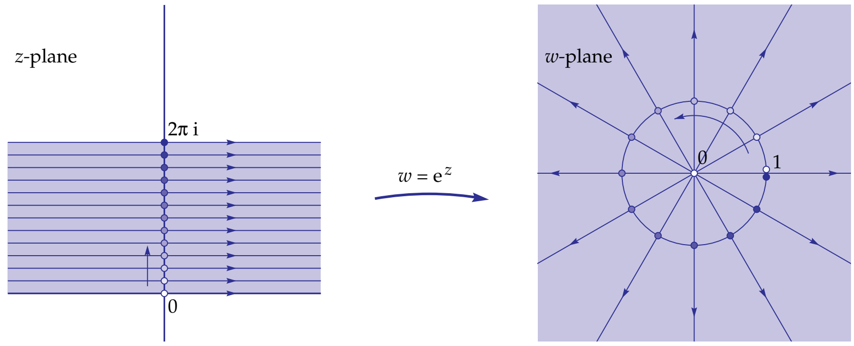
Write the following complex numbers in the notation $x + \mathrm{i}\,y$. Use a calculator and give your answers rounded to 4 decimals.

2.19
a. $\mathrm{e}^{2+2\mathrm{i}}$
b. $\mathrm{e}^{3-\mathrm{i}}$
c. $\mathrm{e}^{0.2+0.3\mathrm{i}}$
d. $\mathrm{e}^{1.2+2.5\mathrm{i}}$
e. $\mathrm{e}^{-0.5+3.14\mathrm{i}}$

2.20
a. $\mathrm{e}^{-3+\mathrm{i}}$
b. $\mathrm{e}^{3-0.5\mathrm{i}}$
c. $\mathrm{e}^{-2-3\mathrm{i}}$
d. $\mathrm{e}^{-1+5\mathrm{i}}$
e. $\mathrm{e}^{0.8+3\mathrm{i}}$

2.21
a. $\cos(\mathrm{i})$
b. $\cos(\pi + \mathrm{i})$
c. $\sin(-2\mathrm{i})$
d. $\sin(-2 - 3\mathrm{i})$
e. $\sin(4\pi - 2 - 3\mathrm{i})$

In the next exercise you will be exploring the complex function $w = e^z$ using the figure below. It is not easy to 'graph' complex functions, since both z and w are complex numbers, situated in a complex plane. To picture a 'graph' of such a function, a four-dimensional space would be needed. Instead, we draw two complex planes, one z-plane and one w-plane, and indicate corresponding z-values and w-values in some way. Below, we already have given it a start.



2.22
a. Verify that the vertical segment $0 \leq y \leq 2\pi$ of the y-axis in the z-plane is mapped onto the unit circle in the w-plane in the indicated way.
b. The images of $z = 0$ and $z = 2\pi\,\mathrm{i}$ in the w-plane are not drawn completely correct. What is wrong, and why are they drawn in this way?
c. In the blue strip in the z-plane a number of horizontal lines have been drawn. Find their images in the w-plane.
d. Does there exist a point z that is mapped onto the origin in the w-plane? If so, find it, if not, explain why such a z doesn't exist.
e. Determine the images in the w-plane of the vertical segments $x = 1$, $0 \leq y \leq 2\pi$ and $x = -1$, $0 \leq y \leq 2\pi$ in the z-plane.
f. What is the image of the full y-axis?
g. Calculate all points z that are mapped onto $w = \mathrm{i}$ in the w-plane.

The complex functions e^z, $\cos z$ and $\sin z$

The real exponential function e^x is well-known, and we have learned in former sections that
$$e^{iy} = \cos y + i \sin y.$$
This means that we know the function e^z for all z values that are purely real and all z values that are purely imaginary. For arbitrary complex numbers $z = x + iy$ we now define
$$e^z = e^x e^{iy}.$$
With this definition we have $e^{z_1+z_2} = e^{z_1} e^{z_2}$ for all complex numbers z_1 and z_2 since
$$\begin{aligned} e^{z_1+z_2} &= e^{(x_1+x_2)+i(y_1+y_2)} = e^{x_1+x_2} e^{i(y_1+y_2)} = e^{x_1} e^{x_2} e^{iy_1} e^{iy_2} \\ &= e^{x_1} e^{iy_1} e^{x_2} e^{iy_2} = e^{x_1+iy_1} e^{x_2+iy_2} = e^{z_1} e^{z_2}. \end{aligned}$$
For all integer real numbers k we have $e^{2k\pi i} = 1$, so
$$e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z.$$
In other words: *the function* e^z *is periodic with period* $2\pi i$. Furthermore, we have $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$ since $e^x e^{iy}$ is no else than the polar notation for e^z with $r = e^x$.

Using Euler's formulas we define the cosine function and the sine function for arbitrary complex numbers z as follows:
$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$
From the fact that e^z is periodic with period $2\pi i$ it follows that the cosine and the sine functions are *periodic with period* 2π, just as expected:
$$\cos(z + 2k\pi) = \cos z \quad \text{and} \quad \sin(z + 2k\pi) = \sin z \quad \text{for all integers } k.$$
Without proof or further explanations we mention that the functions e^z, $\cos z$ and $\sin z$ are also *differentiable* and that the well-known differentiation rules also hold in the complex case. For example, we have
$$\frac{d}{dz}(e^z) = e^z$$
in other words: *the function e^z equals its own derivative*. Also the following well-known formulas keep valid for the complex sine and cosine functions:
$$\frac{d}{dz}(\cos z) = -\sin z \quad \text{and} \quad \frac{d}{dz}(\sin z) = \cos z.$$

Summary

Complex numbers may be viewed as *vectors*: the complex number α corresponds to the vector (arrow) pointing from the origin to the point α in the complex plane. Arrows with the same size and direction represent the same vector. Addition of complex numbers corresponds to *vector addition* (parallelogram construction).

The complex number $\beta - \alpha$ corresponds to the vector pointing from α to β.

The circle with center α and radius r is given by the equation

$$(z - \alpha)(\overline{z} - \overline{\alpha}) = r^2.$$

Euler's formulas:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}.$$

The (r, φ) *notation (polar notation)* of a complex number $z = x + iy$ is $z = r\,e^{i\varphi}$, where $r = |z| = \sqrt{x^2 + y^2}$ and $x = r\cos\varphi,\ y = r\sin\varphi$.

The number r is called the *absolute value* or *modulus* of z, and φ is called the *argument* of z, notation $\varphi = \arg(z)$. The argument is measured in radians. It is determined up to integer multiples of 2π.

If $x \neq 0$ then $\tan\varphi = y/x$ and φ can be calculated as follows from x and y:

$$\begin{array}{llll} \text{if} & x > 0 & \text{then} & \varphi = \arctan\dfrac{y}{x} + 2k\pi \\ \text{if} & x < 0 & \text{then} & \varphi = \arctan\dfrac{y}{x} + \pi + 2k\pi. \end{array}$$

In multiplying complex numbers the absolute values are multiplied and the arguments are added. In dividing complex numbers the absolute values are divided and the arguments are subtracted.

The *complex exponential function* e^z:

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

Then $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$. The complex exponential function is periodic with period $2\pi i$.

The *complex cosine function* and the *complex sine function*:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

The complex cosine and sine functions are periodic with period 2π.

3 Roots and polynomials

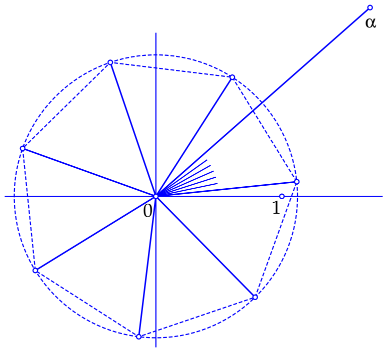
In this chapter you will meet complex roots and complex polynomials. You will learn that each complex number has exactly n complex n^{th} roots, and, moreover, that these roots in the complex plane are the vertices of a regular n-gon with the origin as its center. Next, you will learn about complex polynomials and complex polynomial equations. According to the *Fundamental Theorem of Algebra* each polynomial equation of degree n has exactly n solutions, provided they are counted with appropriate multiplicity. Finally you will learn that each polynomial with real numbers as coefficients can be written as the product of real linear factors and real quadratic factors with a negative discriminant.

3 Roots and polynomials

Write all n^{th} roots in the following exercises in the (r, φ) notation (for each n^{th} root there are n possibilities). Give exact answers or answers rounded to four decimals. In each case show in a drawing that the n^{th} roots form the vertices of a regular n-gon with the origin as its center.

As an *example* we have drawn the seven 7^{th} roots $\sqrt[7]{\alpha}$ where $\alpha = 1.7 + 1.5\,\mathrm{i}$. Then $|\alpha| \approx 2.2672$ and $\arg(\alpha) \approx 0.7230 + 2k\pi$ so $|\sqrt[7]{\alpha}| = \sqrt[7]{|\alpha|} \approx 1.1240$ and $\arg(\sqrt[7]{\alpha}) = \frac{1}{7}\arg(\alpha) \approx 0.1033 + \frac{2k\pi}{7}$, hence

$$\sqrt[7]{\alpha} \approx 1.1240\,\mathrm{e}^{\,(0.1033+\frac{2k\pi}{7})\,\mathrm{i}} \quad \text{for} \quad k = 0, 1, \ldots, 6$$



3.1

a. $\sqrt[3]{\mathrm{i}}$

b. $\sqrt[3]{-\mathrm{i}}$

c. $\sqrt[3]{1}$

d. $\sqrt[3]{8}$

e. $\sqrt[3]{8\,\mathrm{i}}$

3.2

a. $\sqrt[3]{1+\mathrm{i}}$

b. $\sqrt[3]{-27}$

c. $\sqrt[3]{-27\,\mathrm{i}}$

d. $\sqrt[3]{\frac{1}{2} - \frac{1}{2}\sqrt{3}\,\mathrm{i}}$

e. $\sqrt[2]{4}$

3.3

a. $\sqrt[4]{-1}$

b. $\sqrt[4]{-\mathrm{i}}$

c. $\sqrt[5]{1}$

d. $\sqrt[4]{3-4\,\mathrm{i}}$

e. $\sqrt[6]{6\,\mathrm{i}}$

3.4

a. $\sqrt[4]{1-\mathrm{i}}$

b. $\sqrt[5]{-32}$

c. $\sqrt[4]{81\,\mathrm{i}}$

d. $\sqrt[7]{2\,\mathrm{i}}$

e. $\sqrt[3]{3+3\,\mathrm{i}}$

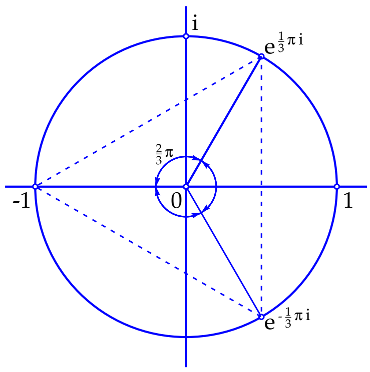
What is a complex n^{th} root?

We already know that $\sqrt{-1} = \mathrm{i}$, since $\mathrm{i}^2 = -1$. Or rather we should say that $\sqrt{-1} = \pm\mathrm{i}$, since also $(-\mathrm{i})^2 = -1$. But what about $\sqrt[3]{-1}$? It must be a complex number z satisfying $z^3 = -1$. Do we know such a number? Sure: $z = -1$ is a solution, since $(-1)^3 = -1$. But also $z = \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}$ is a solution, since

$$z^3 = \left(\mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}\right)^3 = \mathrm{e}^{3(\frac{1}{3}\pi\mathrm{i})} = \mathrm{e}^{\pi\mathrm{i}} = -1.$$

And, of course, also $z = \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}$ is a solution, since $z^3 = \mathrm{e}^{-\pi\mathrm{i}} = -1$.

Thus, we have found *three* complex numbers z for which $z^3 = -1$ holds. All three may claim to be $\sqrt[3]{-1}$. But unlike for real square roots, which by convention are always taken nonnegative, in working with complex numbers one doesn't make a choice on the 'preferred' value of a complex root. As will be shown later, preferred choices for complex n^{th} roots are not desirable. Therefore, if we write, e.g. $\sqrt[3]{-1}$, then it should be clear from the context which of the three possible values is meant.



The radius vectors of the three cube roots of -1 include angles of $\frac{2}{3}\pi$. The roots themselves are the vertices of an equilateral triangle with the origin as its center. Why this is the case, becomes clear when we look more in detail at the way we have defined $\sqrt[3]{-1}$.

We are looking for complex numbers $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$ for which $z^3 = -1$ holds. But $z^3 = \left(r\,\mathrm{e}^{\mathrm{i}\varphi}\right)^3 = r^3\mathrm{e}^{3\mathrm{i}\varphi}$, which should be equal to -1. Since $|-1| = 1$ and $\arg(-1) = \pi + 2k\pi$ we should have $r^3 = 1$ and $3\varphi = \pi + 2k\pi$. It follows that $r = 1$ (since r is a *nonnegative real* number) and $\varphi = \frac{1}{3}\pi + \frac{2}{3}k\pi$. For $k = 0$, $k = 1$ and $k = 2$ we get

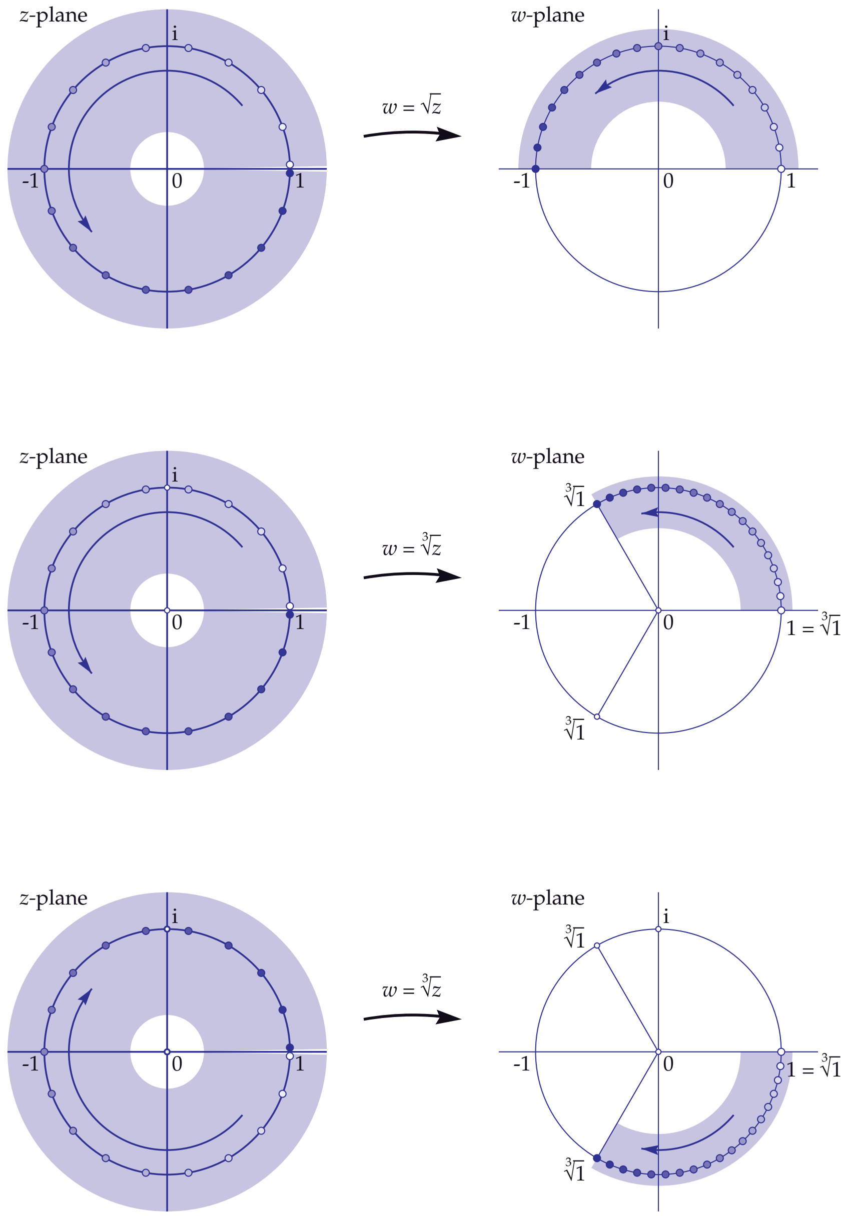
$$z = \mathrm{e}^{\frac{1}{3}\pi\mathrm{i}}, \quad z = \mathrm{e}^{\pi\mathrm{i}} = -1, \quad z = \mathrm{e}^{\frac{5}{3}\pi\mathrm{i}} = \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}}.$$

For any integer value of k we always get one of these three values. Indeed, the argument increases by $\frac{2}{3}\pi$ each time k increases by 1, and after three steps you are back on the initial value. This holds in general for the cube roots of an arbitrary complex number $\alpha \neq 0$: in all cases one finds *three* cube roots, and their radius vectors include angles of $\frac{2}{3}\pi$.

More generally, for any positive integer n:

Each complex number $\alpha = r\,e^{\mathrm{i}\varphi}$ with $r = |\alpha| > 0$ has exactly n n^{th} roots, namely

$$\sqrt[n]{\alpha} = \sqrt[n]{r}\,\mathrm{e}^{(\frac{1}{n}\varphi + \frac{2k\pi}{n})\mathrm{i}} \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$


z-plane
w-plane
i
-1
0
1
w = √z
w = ∛z
1 = ∛1
∛1

Why complex roots are multi-valued

In the previous section we defined the n^{th} root of a complex number α as follows:

$$\sqrt[n]{\alpha} = \sqrt[n]{r}\,\mathrm{e}^{\,(\frac{1}{n}\varphi + \frac{2k\pi}{n})\,\mathrm{i}} \quad \text{for} \quad k = 0, 1, \ldots, n-1$$

so (unless $\alpha = 0$) there are exactly n distinct possibilities for such a root. For the usual square root of a positive real number, for instance $\sqrt{4}$, there are in principle 2 possibilities, namely 2 and -2, but for real numbers we use the *convention* that for $\sqrt{4}$ always the *positive* root is taken, so $\sqrt{4} = 2$. Then the other root is $-\sqrt{4}$.

Why don't we have a similar convention for complex roots? For instance: always take the root with a minimum nonnegative argument? The reason is that such a convention would cause much trouble in working with complex functions in which complex roots are involved.

For instance, take the function $w = \sqrt{z}$. A 'graph' of this function is not easy to draw, since both the z-values and the w-values are located in a complex plane, so a graph should require four dimensions. Instead, we draw two complex planes next to each other: a z-plane and a w-plane as in the top figure on the previous page. Consider the root function $w = \sqrt{z}$ on a ring shaped domain around the origin in the z-plane. Let z follow a path in the z-plane starting at 1 and describing the unit circle in the counterclockwise sense. If we choose $\sqrt{1} = 1$ at the starting point then $\sqrt{w}$ also describes the unit circle, starting at $w = 1$, but with half speed, since

$$w = \sqrt{z} = \sqrt{\mathrm{e}^{\,\mathrm{i}\,\varphi}} = \left(\mathrm{e}^{\,\mathrm{i}\,\varphi}\right)^{1/2} = \mathrm{e}^{\,\frac{1}{2}\,\mathrm{i}\,\varphi}.$$

After one complete turn around the origin in the z-plane, $w = \sqrt{z}$ has described only a half turn in the w-plane, so $w = \sqrt{z}$ has arrived at the *other* root, in other words, we would have $\sqrt{1} = -1$. If there would be a fixed convention for the meaning of $\sqrt{z}$, then somewhere underway we should have jumped from one root to the other, which is much troublesome.

The same occurs for each path in the z-plane that winds once around the origin: you always end at the other 'branch' of the square root. The origin therefore is called a *branching point* of the root function.

For higher roots a similar phenomenon happens. The middle and bottom figures on the previous page illustrate the function $w = \sqrt[3]{z}$. We have indicated how a path along the unit circle in the z-plane that turns once around the origin produces in the w-plane a path from $\sqrt[3]{1} = 1$ to $\sqrt[3]{1} = \mathrm{e}^{\,2\pi\,\mathrm{i}\,/3}$ (for a counterclockwise turn) or to $\sqrt[3]{1} = \mathrm{e}^{\,-2\pi\,\mathrm{i}\,/3}$ (for a clockwise turn). Here, also, a fixed convention on the meaning of $\sqrt[3]{z}$ would cause much trouble.

Therefore, a notation like $\sqrt[n]{z}$ always indicates one of the n possible n^{th} roots of z, but from the context it should be made clear which root is meant.

Determine a quadratic polynomial of the form $p(z) = z^2 + \alpha_1 z + \alpha_0$ which has the numbers z_1 and z_2 as zeroes, where

3.5
- a. $z_1 = 1, z_2 = -1$
- b. $z_1 = 1, z_2 = 5$
- c. $z_1 = 1, z_2 = \mathrm{i}$
- d. $z_1 = \mathrm{i}, z_2 = -2\mathrm{i}$
- e. $z_1 = 1 + \mathrm{i}, z_2 = 1 - \mathrm{i}$

3.6
- a. $z_1 = 0, z_2 = -\mathrm{i}$
- b. $z_1 = 1, z_2 = 2$
- c. $z_1 = 0, z_2 = -2\mathrm{i}$
- d. $z_1 = 1 + 2\mathrm{i}, z_2 = 1 - 2\mathrm{i}$
- e. $z_1 = 1 + \mathrm{i}, z_2 = -1 + \mathrm{i}$

Determine a cubic polynomial of the form $p(z) = z^3 + \alpha_2 z^2 + \alpha_1 z + \alpha_0$ with zeroes z_1, z_2 and z_3, where

3.7
- a. $z_1 = 1, z_2 = -1, z_3 = 0$
- b. $z_1 = \mathrm{i}, z_2 = -\mathrm{i}, z_3 = 0$
- c. $z_1 = \mathrm{i}, z_2 = -\mathrm{i}, z_3 = 1$
- d. $z_1 = \mathrm{i}, z_2 = 2\mathrm{i}, z_3 = 3\mathrm{i}$
- e. $z_1 = 1, z_2 = -1, z_3 = \mathrm{i}$

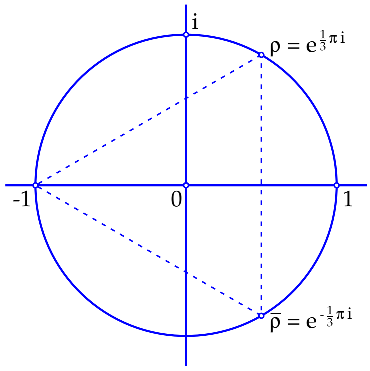
3.8
- a. $z_1 = 1, z_2 = 2, z_3 = 3$
- b. $z_1 = 1 + \mathrm{i}, z_2 = 1 - \mathrm{i}, z_3 = 1$
- c. $z_1 = 1, z_2 = 0, z_3 = 2$
- d. $z_1 = \mathrm{i}, z_2 = 0, z_3 = 1$
- e. $z_1 = \mathrm{i}, z_2 = -\mathrm{i}, z_3 = 2\mathrm{i}$

On n^{th} roots and n^{th}-degree polynomials

We have seen that there are three cube roots of -1, namely -1, $\mathrm{e}^{\frac{1}{3}\pi\,\mathrm{i}}$ and $\mathrm{e}^{-\frac{1}{3}\pi\,\mathrm{i}}$. For simplicity we write $\rho = \mathrm{e}^{\frac{1}{3}\pi\,\mathrm{i}}$ and $\overline{\rho} = \mathrm{e}^{-\frac{1}{3}\pi\,\mathrm{i}}$ (ρ is the Greek letter 'rho'). Therefore, the three roots are -1, ρ and $\overline{\rho}$.

The cube roots of -1 are the complex numbers z for which $z^3 = -1$, in other words, they are the solutions of the *cubic equation*

$$z^3 + 1 = 0.$$



Or, put it still differently, they are the *zeroes* of the *cubic polynomial* $z^3 + 1$. But consider the equation

$$(z - (-1))(z - \rho)(z - \overline{\rho}) = 0$$

It is obvious that its solutions are also equal to -1, ρ and $\overline{\rho}$. Would the left-hand side perhaps be equal to $z^3 + 1$?

Expanding brackets yields:

$$(z - (-1))(z - \rho)(z - \overline{\rho}) = (z + 1)(z^2 - (\rho + \overline{\rho})z + \rho\overline{\rho}).$$

But $\rho = \frac{1}{2} + \frac{1}{2}\sqrt{3}\,\mathrm{i}$ so $\rho + \overline{\rho} = 1$ and $\rho\overline{\rho} = 1$ (verify this!), hence, indeed

$$(z+1)(z^2 - (\rho+\overline{\rho})z + \rho\overline{\rho}) = (z+1)(z^2 - z + 1) = z^3 - z^2 + z + z^2 - z + 1 = z^3 + 1.$$

What we have seen in this particular case, is also true in general:

If an n^{th}-degree polynomial

$$p(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$

has n distinct zeroes $z_1, z_2, \ldots, z_n$, then $p(z)$ can be written as

$$p(z) = (z - z_1)(z - z_2)\cdots(z - z_n).$$

In the given example we had $n = 3$, $p(z) = z^3 + 1$ and $z_1 = -1$, $z_2 = \rho$, $z_3 = \overline{\rho}$.

In the previous section we determined the n^{th} roots of a complex number α. The corresponding polynomial then always has the form $p(z) = z^n - \alpha$. We have seen that, indeed, it always has n zeroes (the n^{th} roots of α), except in the trivial case $\alpha = 0$. What happens in general for n^{th}-degree polynomials, will be treated in the next section.

3.9 In each of the next exercises a polynomial $p(z)$ and a number α are given. Verify in each case that $p(\alpha) = 0$, and determine the polynomial $q(z)$ for which $p(z) = (z-\alpha)q(z)$ holds.

a. $p(z) = z^4 - z^3 - 2z^2, \quad \alpha = -1$
b. $p(z) = z^4 - z^3 + 3z^2 - 3z, \quad \alpha = 1$
c. $p(z) = z^5 - \mathrm{i}\, z^4 - z + \mathrm{i}\,, \quad \alpha = \mathrm{i}$
d. $p(z) = z^4 - 4z^3 + 5z^2 - 4z + 4, \quad \alpha = 2$
e. $p(z) = z^4 - 1, \quad \alpha = -\mathrm{i}$
f. $p(z) = z^4 + 2z^2 + 1, \quad \alpha = \mathrm{i}$

Hint: you could solve the previous exercises by systematically trying, but those who know *long division*, will see that this is also a fast and easy method for the division of polynomials. As an example we present the long division corresponding to the case $p(z) = 3z^4 - 7z^3 + 3z^2 - z - 2$ and $\alpha = 2$. Substitution shows that $\alpha = 2$ is a zero, i.e., $p(2) = 0$. The following long division yields the quotient polynomial $q(z) = 3z^3 - z^2 + z + 1$.

$$
\begin{array}{rcrrrrrcl}
z-2 & / & 3z^4 & -7z^3 & +3z^2 & -z & -2 & \backslash & 3z^3-z^2+z+1 \\
 & & 3z^4 & -6z^3 & & & & & \\ \cline{3-4}
 & & & -z^3 & +3z^2 & & & & \\
 & & & -z^3 & +2z^2 & & & & \\ \cline{4-5}
 & & & & z^2 & -z & & & \\
 & & & & z^2 & -2z & & & \\ \cline{5-6}
 & & & & & z & -2 & & \\
 & & & & & z & -2 & & \\ \cline{6-7}
 & & & & & & 0 & &
\end{array}
$$

3.10 Determine in each of the previous exercises the multiplicity of the zero α and subsequently find the other zeroes of $p(z)$.

The Fundamental Theorem of Algebra

Before continuing, we first give a formal definition of the term polynomial and some related concepts.

> **Definition:** A *polynomial* is a function of the form $p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$. The complex numbers α_n, α_{n-1}, ..., α_1, α_0 are called the *coefficients*, and the complex number z is called the *variable*. We always suppose that $\alpha_n \neq 0$ (since otherwise the term $\alpha_n z^n$ can be omitted). The other coefficients, however, may be zero. The number n is called the *degree* of the polynomial.

For each complex number z the polynomial yields a complex number $p(z)$ as its *function value*. If for a certain z_0 we have $p(z_0) = 0$ then z_0 is called a *zero* of the polynomial. The number z_0 then is a *solution* of the equation

$$\alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0 = 0.$$

Such an equation is called an n^{th}*-degree equation*. Instead of a solution of the equation one also often uses the term a *root* of the equation, even if in the notation of such solutions no root signs are used.

For each n^{th}-degree polynomial $p(z)$ with $n > 1$ the following theorem holds.

> **Factor theorem:** *If z_0 is a zero of $p(z)$, then there exists a polynomial $q(z)$ such that $p(z) = (z - z_0)q(z)$. Therefore it is possible to split off a factor $(z - z_0)$ from $p(z)$.*

A first-degree polynomial is also called a *linear* polynomial; the corresponding equation is also called a *linear* equation. A second-degree equation is also called a *quadratic equation*. Quadratic equations can be solved by means of the *abc*-formula.

Complex quadratic equations always have two complex solutions z_1 and z_2. The corresponding polynomial $p(z) = \alpha_2 z^2 + \alpha_1 z + \alpha_0$ then can be written as $p(z) = \alpha_2(z - z_1)(z - z_2)$. If the discriminant $\alpha_1^2 - 4\alpha_2\alpha_0$ is zero, then z_1 and z_2 coincide and we have $p(z) = \alpha_2(z - z_1)^2$.

In general for complex n^{th}-degree polynomials the following theorem holds. It is known as the *fundamental theorem of algebra* and it was proved for the first time by C.F. Gauss in the beginning of the nineteenth century.

> **Fundamental Theorem of Algebra:** *For each n^{th}-degree polynomial $p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$ with $n \geq 1$ there are n complex numbers $z_1, \ldots, z_n$ such that $p(z) = \alpha_n (z - z_1) \ldots (z - z_n)$.*

The numbers $z_1, \ldots, z_n$ are the zeroes of $p(z)$. They need not be distinct. If a zero occurs k times, it is called a *k-fold* zero and k is called the *multiplicity* of the zero. Each n^{th}-degree polynomial with $n \geq 1$ therefore has exactly n complex zeroes when counted with their multiplicity.

3.11 Split the following real polynomials in real linear factors and real quadratic factors with a negative discriminant.
(Hint: if you don't see such a factorization immediately, then first look for a zero and use the factor theorem.)

a. $z^3 + 1$
b. $z^3 - 1$
c. $z^4 - 1$
d. $z^3 + 27$
e. $z^4 + 2z^2 + 1$
f. $z^4 - 2z^2 + 1$

3.12 Let n be an odd integer and suppose that

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a real n^{th}-degree polynomial. Even without using complex numbers it can be proven that $p(x)$ has at least one real zero, namely by writing $p(x)$ in the form

$$p(x) = x^n \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$$

and compare the behavior of $p(x)$ for big positive and big negative x-values. Give such a proof.

On solving n^{th}-degree equations
For quadratic equations the solutions can be found by using the *abc*-formula. Also for cubic and quartic equations such exact formulas exist, but they are much more complicated; we don't treat them here.
For n^{th}-degree equations with $n \geq 5$, however, no such algebraic methods exist to find all solutions. In such cases one has to turn to numerical approximation methods. The *fundamental theorem of algebra* guarantees that there are always n solutions (counted with multiplicity), but the theorem doesn't provide a general method for finding them!

Real polynomials

If all coefficients of a polynomial are real numbers, it is called a *real polynomial*. Then it is of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

for certain real numbers $a_n, \ldots a_0$. As always, we suppose that $a_n \neq 0$. A real polynomial of degree $n \geq 1$ also has n complex zeroes (counted with multiplicity), but they need not be real. For instance, $p(z) = z^2 + 1$ has no real zeroes. It is clear, however, that a real polynomial always has *at most* n real zeroes. Since real polynomials occur in many applications, we will deduce some special properties.

> **Theorem:** *If $z_0 = x_0 + \mathrm{i}\, y_0$ is a non-real zero of a real polynomial $p(z) = a_n z^n + \cdots + a_0$ (so $y_0 \neq 0$) then also the conjugated complex number $\overline{z_0} = x_0 - \mathrm{i}\, y_0$ is a zero of $p(z)$.*

The proof, which is very simple, relies on three properties which follow immediately from the definition of conjugated complex number:

1. If a is a real number, then $\overline{a} = a$.
2. For each α and β we have $\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$.
3. For each α and β we have $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$.

It follows that, in particular, $\overline{z^k} = (\overline{z})^k$ for each positive integer k.

Proof of the theorem: Let $z_0 = x_0 + \mathrm{i}\, y_0$ be a zero of $p(z)$, so $p(z_0) = 0$. Then

$$\begin{aligned} p(\overline{z_0}) &= a_n (\overline{z_0})^n + a_{n-1} (\overline{z_0})^{n-1} + \cdots + a_1 \overline{z_0} + a_0 \\ &= \overline{a_n}\, \overline{z_0^n} + \overline{a_{n-1}}\, \overline{z_0^{n-1}} + \cdots + \overline{a_1}\, \overline{z_0} + \overline{a_0} \\ &= \overline{a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_1 z_0 + a_0} = \overline{p(z_0)} = \overline{0} = 0. \end{aligned}$$

If in the complex factorization $p(z) = a_n (z - z_1) \ldots (z - z_n)$ a factor $(z - z_k)$ occurs for which $z_k = x_k + \mathrm{i}\, y_k$ is not real, then there must also be a factor $(z - \overline{z_k})$. Combining these factors and expanding brackets yields:

$$(z - z_k)(z - \overline{z_k}) = z^2 - (z_k + \overline{z_k}) z + z_k \overline{z_k} = z^2 - 2 x_k z + x_k^2 + y_k^2.$$

This is a real quadratic polynomial with discriminant $4x_k^2 - 4(x_k^2 + y_k^2) = -4y_k^2$. It is negative, as expected. Splitting off that quadratic factor yields a real polynomial of degree $n - 2$, on which the same recipe can be applied, and so on. Thus, we have also proved the following theorem:

> **Theorem:** *Each real polynomial can be written as a product of real linear factors and real quadratic factors with a negative discriminant.*

As a consequence the degree of any real polynomial without real zeroes must be even. It follows that each real polynomial of odd degree has at least one real zero.

Symmary

The n^{th}*-roots* of a complex number $\alpha = r\,\mathrm{e}^{\,\mathrm{i}\,\varphi}$ are defined as follows

$$\sqrt[n]{\alpha} = \sqrt[n]{r}\,\mathrm{e}^{\,(\frac{1}{n}\varphi + \frac{2k\pi}{n})\,\mathrm{i}} \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$

If $\alpha \neq 0$ then there are exactly n possibilities. The n roots then are the vertices of a regular n-gon with the origin as its center.

A *polynomial* is a function of the form

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0 \qquad (\text{with } \alpha_n \neq 0).$$

The complex numbers α_n, α_{n-1}, ..., α_1, α_0 are called the *coefficients*, and the complex number z is called the *variable*. The number n is called the *degree* of the polynomial. If $p(z_0) = 0$ for a certain complex number z_0, then z_0 is called a *zero* of $p(z)$.

Factor theorem: If a polynomial $p(z)$ of degree $n > 1$ has a zero z_0, then there exists a polynomial $q(z)$ for which $p(z) = (z - z_0)q(z)$. Therefore, it is possible to split off a factor $(z - z_0)$ from $p(z)$.

Fundamental Theorem of Algebra: For each n^{th}-degree polynomial

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0$$

of degree $n \geq 1$ there are n complex numbers z_1, ..., z_n such that

$$p(z) = \alpha_n (z - z_1) \ldots (z - z_n).$$

The numbers z_1, ..., z_n are the *zeroes* of $p(z)$, in other words, they are the solutions of the n^{th}-degree equation $p(z) = 0$.

The zeroes need not be distinct. If a zero occurs k times in the factorization above, then it is called a k*-fold* zero; k is called the *multiplicity* of the zero. Each n^{th}-degree polynomial with $n \geq 1$ therefore has exactly n complex zeroes if counted with their multiplicity.

Theorem: Each real polynomial can be written as the product of real linear polynomials and real quadratic polynomials with a negative discriminant.

As a consequence, we have:
1. The degree of any real polynomial without real zeroes must be even.
2. Each real polynomial with an odd degree has at least one real zero.

Answers to the exercises

1. Calculating with complex numbers

1.1 a. -9 b. -9 c. 16 d. i e. 1

1.2 a. $-\frac{1}{2}$ b. $-\frac{2}{3}$ c. -1 d. $-\frac{4}{3}$ e. $-\frac{3}{4}$

1.3 a. $\sqrt{3}\,\mathrm{i}$ b. $3\,\mathrm{i}$ c. $2\sqrt{2}\,\mathrm{i}$ d. $5\,\mathrm{i}$ e. $\sqrt{15}\,\mathrm{i}$

1.4 a. $\sqrt{33}\,\mathrm{i}$ b. $7\,\mathrm{i}$ c. $4\sqrt{3}\,\mathrm{i}$ d. $3\sqrt{5}\,\mathrm{i}$ e. $5\sqrt{3}\,\mathrm{i}$

1.5 a. $1 \pm \mathrm{i}$ b. $-2 \pm \mathrm{i}$ c. $-1 \pm 3\,\mathrm{i}$ d. $3 \pm \mathrm{i}$ e. $2 \pm 2\,\mathrm{i}$

1.6 a. $6 \pm 2\,\mathrm{i}$ b. $2 \pm \sqrt{2}\,\mathrm{i}$ c. $-1 \pm \sqrt{3}\,\mathrm{i}$ d. $3 \pm \sqrt{3}\,\mathrm{i}$ e. $-4 \pm 2\,\mathrm{i}$

1.7 The first, third and fourth equality signs are correct; the first one by the definition of $\sqrt{-1}$ as a number for which the square equals -1, and the fourth one by the definition of $\sqrt{1}$ as the positive real number for which the square equals 1. The third equality sign is true because $(-1)^2 = 1$. Hence, the second equality sign cannot be true. Apparently it isn't true for negative real numbers a that $\sqrt{a^2} = (\sqrt{a})^2$ (this rule is true if $a \geq 0$).

For the following exercises we only give the answer and its absolute value. Make a drawing for yourself.

1.8 a. $4 - 6\,\mathrm{i}, 2\sqrt{13}$ b. $-4 + 4\,\mathrm{i}, 4\sqrt{2}$ c. $1, 1$ d. $3 - 3\,\mathrm{i}, 3\sqrt{2}$ e. $5 - 3\,\mathrm{i}, \sqrt{34}$

1.9 a. $-5 - 10\,\mathrm{i}, 5\sqrt{5}$ b. $4 + 8\,\mathrm{i}, 4\sqrt{5}$ c. $8, 8$ d. $-8 - 6\,\mathrm{i}, 10$ e. $3 - 4\,\mathrm{i}, 5$

1.10 a. $-\mathrm{i}, 1$ b. $1, 1$ c. $\mathrm{i}, 1$ d. $-1, 1$ e. $-1, 1$

1.11 a. $-\mathrm{i}, 1$ b. $-8\,\mathrm{i}, 8$ c. $128\,\mathrm{i}, 128$ d. $-2 + 2\,\mathrm{i}, 2\sqrt{2}$ e. $-2 - 2\,\mathrm{i}, 2\sqrt{2}$

For the following exercises we only give the lines as an equation in xy-coordinates. Make a drawing for yourself.

1.12 a. $x = 4$ b. $x = -3$ c. $y = 2$ d. $y = -2$ e. $x = y$

1.13 a. $x + y = 1$ b. $x = 2y$ c. $x - 2y = 1$ d. $x + y = 5$ e. $y = 0$

For the following exercises we only give the center and the radius, separated by a comma.

1.14 a. $0, 4$ b. $1, 3$ c. $2, 2$ d. $3, 1$ e. $-1, 5$

1.15 a. $-3, 4$ b. $\mathrm{i}, 5$ c. $-2\,\mathrm{i}, 1$ d. $1 + \mathrm{i}, 3$ e. $-3 + \mathrm{i}, 2$

1.16 a. $\frac{3}{25} + \frac{4}{25}\,\mathrm{i}$ b. $\frac{3}{5} + \frac{3}{10}\,\mathrm{i}$ c. $-\frac{6}{5} - \frac{2}{5}\,\mathrm{i}$ d. $\frac{11}{5} + \frac{3}{5}\,\mathrm{i}$ e. $\frac{8}{13} + \frac{1}{13}\,\mathrm{i}$

1.17 a. $-\frac{1}{5}-\frac{2}{5}\,\mathrm{i}$ b. $-\frac{4}{5}+\frac{2}{5}\,\mathrm{i}$ c. $-\mathrm{i}$ d. $-3-\mathrm{i}$ e. i

1.18 a. $\frac{9}{25}+\frac{12}{25}\,\mathrm{i}$ b. $\frac{1}{5}+\frac{7}{5}\,\mathrm{i}$ c. $-\frac{4}{5}-\frac{3}{5}\,\mathrm{i}$ d. $-\mathrm{i}$ e. i

1.19 a. $-\frac{1}{2}-\frac{1}{4}\,\mathrm{i}$ b. $\frac{4}{13}-\frac{7}{13}\,\mathrm{i}$ c. $\frac{1}{4}-\frac{1}{4}\,\mathrm{i}$ d. $1+2\,\mathrm{i}$ e. i

2. The geometry of complex calculations

2.1 a. $-3\,\mathrm{i}$ b. $-3+\mathrm{i}$ c. $3-5\,\mathrm{i}$ d. -8 e. 0

2.2 Left to the reader.

2.3 a. $\bar{z}z+\mathrm{i}\,z-\mathrm{i}\,\bar{z}-8=0$ b. $\bar{z}z-(1+\mathrm{i})z-(1-\mathrm{i})\bar{z}=0$ c. $\bar{z}z-z-\bar{z}=0$
d. $\bar{z}z+(2+\mathrm{i})z+(2-\mathrm{i})\bar{z}+1=0$ e. $\bar{z}z-(1+2\,\mathrm{i})z-(1-2\,\mathrm{i})\bar{z}+4=0$

2.4 a. $(z+\mathrm{i})(\bar{z}-\mathrm{i})=1$, center $-\mathrm{i}$, radius 1
b. $(z+(1-\mathrm{i}))(\bar{z}+(1+\mathrm{i}))=4$, center $-1+\mathrm{i}$, radius 2
c. $(z-2\,\mathrm{i})(\bar{z}+2\,\mathrm{i})=0$, center $2\,\mathrm{i}$, radius 0
d. $(z+2)(\bar{z}+2)=-1$, no circle
e. $(z+(2+\mathrm{i}))(\bar{z}+(2-\mathrm{i}))=6$, center $-2-\mathrm{i}$, radius $\sqrt{6}$

For the following two exercises we only give the arguments.

2.5 a. $-\frac{\pi}{2}+2k\pi$ b. $\pi+2k\pi$ c. $2k\pi$ d. $\frac{3\pi}{4}+2k\pi$ e. $\frac{\pi}{3}+2k\pi$

2.6 a. $-\frac{\pi}{4}+2k\pi$ b. $-\frac{\pi}{6}+2k\pi$ c. $-\frac{3\pi}{4}+2k\pi$ d. $\frac{5\pi}{6}+2k\pi$ e. $-\frac{\pi}{3}+2k\pi$

2.7 a. i b. $-\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\,\mathrm{i}$ c. i d. $\frac{1}{2}\sqrt{3}-\frac{1}{2}\,\mathrm{i}$ e. $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$

2.8 a. $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$ b. $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$ c. $-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\,\mathrm{i}$ d. $-\frac{1}{2}\sqrt{3}+\frac{1}{2}\,\mathrm{i}$
e. $\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}\,\mathrm{i}$

2.9 a. -1 b. 1 c. i d. -1 e. 1

2.10 a. i b. $-\frac{1}{2}+\frac{1}{2}\sqrt{3}\,\mathrm{i}$ c. i d. $\frac{1}{2}\sqrt{3}-\frac{1}{2}\,\mathrm{i}$ e. 1

2.11 a. $\frac{1}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$ b. -1 c. $-\frac{1}{2}-\frac{1}{2}\sqrt{3}\,\mathrm{i}$ d. -1 e. -1

2.12 a. i b. i c. $-\mathrm{i}$ d. i e. -1

For the following two exercises we have taken the argument in the interval $[-\frac{\pi}{2},\frac{3\pi}{2}\rangle$.

2.13 a. $2.2361\,\mathrm{e}^{1.1072\,\mathrm{i}}$ b. $4.4721\,\mathrm{e}^{-0.4636\,\mathrm{i}}$ c. $3.6056\,\mathrm{e}^{-0.9828\,\mathrm{i}}$ d. $3.6056\,\mathrm{e}^{4.1244\,\mathrm{i}}$
e. $3\,\mathrm{e}^{3.1416\,\mathrm{i}}$

2.14 a. $2.2361\,\mathrm{e}^{.4636\,\mathrm{i}}$ b. $2.2361\,\mathrm{e}^{-0.4636\,\mathrm{i}}$ c. $\mathrm{e}^{-1.5708\,\mathrm{i}}$ d. $5.0990\,\mathrm{e}^{2.9442\,\mathrm{i}}$
e. $3\,\mathrm{e}^{-1.5708\,\mathrm{i}}$

2.15 a. $-0.8323+1.8186\,\mathrm{i}$ b. $1.6209-2.5244\,\mathrm{i}$ c. $0.1911+0.0591\,\mathrm{i}$
d. $-0.9614+0.7182\,\mathrm{i}$ e. $-1.0000+0.0001\,\mathrm{i}$

2.16 a. $3\,\mathrm{e}^{2\,\mathrm{i}}$ b. $9\,\mathrm{e}^{-4\,\mathrm{i}}$ c. $243\,\mathrm{e}^{10\,\mathrm{i}}$ d. $\frac{1}{3}\,\mathrm{e}^{2\,\mathrm{i}}$ e. $3^n\,\mathrm{e}^{-2n\,\mathrm{i}}$

2.17 a. $\overline{z_1z_2}=\overline{2\,\mathrm{e}^{5\,\mathrm{i}}3\,\mathrm{e}^{-2\,\mathrm{i}}}=\overline{6\,\mathrm{e}^{3\,\mathrm{i}}}=6\,\mathrm{e}^{-3\,\mathrm{i}}=2\,\mathrm{e}^{-5\,\mathrm{i}}3\,\mathrm{e}^{2\,\mathrm{i}}=\overline{z_1}\,\overline{z_2}$.
Part (b.) is done in the same way.

2.18 a. $\overline{z_1z_2}=\overline{r_1\,\mathrm{e}^{\,\mathrm{i}\,\varphi_1}r_2\,\mathrm{e}^{\,\mathrm{i}\,\varphi_2}}=\overline{r_1r_2\,\mathrm{e}^{\,\mathrm{i}\,(\varphi_1+\varphi_2)}}=r_1r_2\,\mathrm{e}^{-\mathrm{i}\,(\varphi_1+\varphi_2)}=r_1\,\mathrm{e}^{-\mathrm{i}\,\varphi_1}r_2\,\mathrm{e}^{-\mathrm{i}\,\varphi_2}=$
$\overline{z_1}\,\overline{z_2}$. Part (b.) is done in the same way.

2.19 a. $-3.0749+6.7188\,\mathrm{i}$ b. $10.8523-16.9014\,\mathrm{i}$ c. $1.1669+0.3609\,\mathrm{i}$
d. $-2.6599+1.9870\,\mathrm{i}$ e. $-0.6065+0.0010\,\mathrm{i}$

2.20 a. $0.0269+0.0419\,\mathrm{i}$ b. $17.6267-9.6295\,\mathrm{i}$ c. $-0.1340-0.0191\,\mathrm{i}$
d. $0.1044-0.3528\,\mathrm{i}$ e. $-2.2033+0.3141\,\mathrm{i}$

2.21 a. 1.5431 b. -1.5431 c. $-3.6269\,\mathrm{i}$ d. $-9.1545+4.1689\,\mathrm{i}$ e. $-9.1545+4.1689\,\mathrm{i}$

2.22 a. On the y-axis we have $x=0$, so $\mathrm{e}^{z}=\mathrm{e}^{\mathrm{i}y}$ and if y runs from 0 to 2π, then $\mathrm{e}^{\mathrm{i}y}$ describes the complete unit circle.
b. They actually should coincide, but drawn in this way you more clearly see how this part of the y-axis is mapped.
c. The horizontal lines in the z-plane are the lines $y=\frac{k}{6}\pi$ for $k=0,1,\ldots,12$. They are mapped in the w-plane onto the drawn radii. If x runs from minus infinity to plus infinity, such a radius is described from the origin to infinity.
d. No. If $w=\mathrm{e}^{x+\mathrm{i}y}=0$ would hold, then the absolute value would be 0. However, the absolute value equals e^{x}, which is positive for every x.
e. The circles in the w-plane with center 0 and radius e and $\mathrm{e}^{-1}=\frac{1}{\mathrm{e}}$, respectively.
f. The unit circle, which is described infinitely often if y runs from minus infinity to plus infinity.
g. $z=\frac{1}{2}\pi\,\mathrm{i}$ has $w=\mathrm{i}$ as its image, but since e^{z} is periodic with periode $2\pi\,\mathrm{i}$ all points $z=\frac{1}{2}\pi\,\mathrm{i}+2k\pi\,\mathrm{i}$ (with integer k) are also mapped onto i.

3. Roots and polynomials

3.1 a. $\mathrm{e}^{(\frac{\pi}{6}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$ b. $\mathrm{e}^{(-\frac{\pi}{6}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$ c. $\mathrm{e}^{(\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$
d. $2\,\mathrm{e}^{(\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$ e. $2\,\mathrm{e}^{(\frac{\pi}{6}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$

3.2 a. $\sqrt[6]{2}\,\mathrm{e}^{(\frac{\pi}{12}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$ b. $3\,\mathrm{e}^{(\frac{\pi}{3}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$
c. $3\,\mathrm{e}^{(-\frac{\pi}{6}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$ d. $\mathrm{e}^{(-\frac{\pi}{9}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$ e. $2\,\mathrm{e}^{k\pi\,\mathrm{i}}\,(k=0,1)$ (dit zijn de getallen 2 en -2)

3.3 a. $\mathrm{e}^{(\frac{\pi}{4}+\frac{k\pi}{2})\mathrm{i}}\,(k=0,1,2,3)$ b. $\mathrm{e}^{(-\frac{\pi}{8}+\frac{k\pi}{2})\mathrm{i}}\,(k=0,1,2,3)$ c. $\mathrm{e}^{(\frac{2k\pi}{5})\mathrm{i}}\,(k=0,1,2,3,4)$
d. $\sqrt[4]{5}\,\mathrm{e}^{(-0.2318+\frac{k\pi}{2})\mathrm{i}}\,(k=0,1,2,3)$ e. $\sqrt[6]{6}\,\mathrm{e}^{(\frac{\pi}{12}+\frac{k\pi}{3})\mathrm{i}}\,(k=0,1,2,3,4,5)$

3.4 a. $\sqrt[8]{2}\,\mathrm{e}^{(-\frac{\pi}{16}+\frac{k\pi}{2})\mathrm{i}}\,(k=0,1,2,3)$ b. $2\,\mathrm{e}^{(\frac{\pi}{5}+\frac{2k\pi}{5})\mathrm{i}}\,(k=0,1,2,3,4)$
c. $3\,\mathrm{e}^{(\frac{\pi}{8}+\frac{k\pi}{2})\mathrm{i}}\,(k=0,1,2,3)$ d. $\sqrt[7]{2}\,\mathrm{e}^{(\frac{\pi}{14}+\frac{2k\pi}{7})\mathrm{i}}\,(k=0,\ldots,6)$
e. $\sqrt[6]{18}\,\mathrm{e}^{(\frac{\pi}{12}+\frac{2k\pi}{3})\mathrm{i}}\,(k=0,1,2)$

3.5 a. z^2-1 b. z^2-6z+5 c. $z^2-(1+\mathrm{i})z+\mathrm{i}$ d. $z^2+\mathrm{i}z+2$ e. z^2-2z+2

3.6 a. $z^2+\mathrm{i}z$ b. z^2-3z+2 c. $z^2+2\mathrm{i}z$ d. z^2-2z+5 e. $z^2-2\mathrm{i}z-2$

3.7 a. z^3-z b. z^3+z c. z^3-z^2+z-1 d. $z^3-6\mathrm{i}z^2-11z+6\mathrm{i}$ e. $z^3-\mathrm{i}z^2-z+\mathrm{i}$

3.8 a. $z^3-6z^2+11z-6$ b. z^3-3z^2+4z-2 c. z^3-3z^2+2z d. $z^3-(1+\mathrm{i})z^2+\mathrm{i}z$
e. $z^3-2\mathrm{i}z^2+z-2\mathrm{i}$

3.9 a. z^3-2z^2 b. z^3+3z c. z^4-1 d. z^3-2z^2+z-2 e. $z^3-\mathrm{i}z^2-z+\mathrm{i}$
f. $z^3+\mathrm{i}z^2+z+\mathrm{i}$

3.10 a. simple, $z=0$ (double), $z=2$ b. simple, $z=0$, $z=\sqrt{3}\,\mathrm{i}$, $z=-\sqrt{3}\,\mathrm{i}$
c. double, $z=1$, $z=-1$, $z=-\mathrm{i}$ d. double, $z=\mathrm{i}$, $z=-\mathrm{i}$ e. simple, $z=1$, $z=-1$,

$z = \mathrm{i}$ f. double, $z = -\mathrm{i}$ (also double)

3.11 a. $z^3 + 1 = (z+1)(z^2 - z + 1)$ b. $z^3 - 1 = (z-1)(z^2 + z + 1)$
c. $z^4 - 1 = (z^2+1)(z-1)(z+1)$ d. $z^3 + 27 = (z+3)(z^2 - 3z + 9)$
e. $z^4 + 2z^2 + 1 = (z^2+1)^2$ f. $z^4 - 2z^2 + 1 = (z-1)^2(z+1)^2$

3.12 If you write $p(x)$ in the indicated way, you see that $p(x)$ for big *positive* x-values nearly equals x^n, so it certainly is positive. Also for big *negative* x-values $p(x)$ nearly equals x^n, but then x^n is negative (since n is odd). Therefore, $p(x)$ is negative for big negative x-values and positive for big positive x-values. In between $p(x)$ must therefore at least once become zero. Those who are versatile in working with limits may make this reasoning more precise as follows:

$$\lim_{x\to\infty} \frac{p(x)}{x^n} = \lim_{x\to\infty} \left(1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) = 1$$

and similarly $\lim\limits_{x\to-\infty} \dfrac{p(x)}{x^n} = 1$.

Index