Symmetric spherical and planar patterns

Jan van de Craats*

version August 3, 2011

1 Spherical patterns

1.1 Introduction

Many objects in daily life exhibit various forms of symmetry. Balls, bowls, cylinders or cones have infinitely many reflectional and rotational symmetries, at least if there are no special patterns on their surface. But for other symmetrical objects the number of symmetries is finite. Take, for instance, a rectangular table, a chair, a cupboards or, in a more mathematical language, a cube, a regular prism or a regular pyramid. Or take balls with characteristic symmetrical patterns on their surface, such as a football, a volleyball or a basketball.

For all these objects, their symmetries can be viewed as *isometries* in Euclidean three-space. They form a group, the *symmetry group* of the object. It is interesting to enumerate all possible finite symmetry groups. This has been done by many authors in the past, and in many different ways. It turns out that there are exactly fourteen types of finite symmetry groups in three-space (see, e.g., Fejes Tóth (1965), p. 59 ff. or Coxeter (1969), p. 270 ff.).

In the literature many names and notations for these groups have been proposed, but one of the most illuminating is the *signature notation*, proposed recently by John H. Conway and further developed by Conway and his coauthors Heidi Burgiel and Chaim Goodman-Strauss in their beautiful and lavishly illustrated book *The Symmetries of Things*. The first part of their book is devoted to the symmetries of spherical and planar patterns. In this article, we shall present the signature notation in a slightly adapted form. It has the advantage of reflecting in a very direct and simple way all symmetry features of a pattern.

*University of Amsterdam, e-mail: J.vandeCraats@uva.nl

1.2 Symmetrical objects and spherical patterns

Any finite group $\mathcal{G}$ of isometries must have at least one fixed point, a point that is unchanged under all isometries of $\mathcal{G}$. Indeed, take any point P together with its images under all isometries of $\mathcal{G}$. Together they form a finite point set, the *orbit* of P. Any isometry of $\mathcal{G}$ maps this orbit onto itself by permuting its points, so its *centroid* O must be invariant under all isometries of $\mathcal{G}$. In other words, O is a fixed point of all isometries in $\mathcal{G}$.

Moreover, since $\mathcal{G}$ consists of isometries, i.e., mappings that preserve distance between points, any sphere with center O is mapped onto itself by the elements of $\mathcal{G}$. Therefore, enumerating all finite groups of isometries in Euclidean three-space is the same as enumerating all finite groups of isometries on the surface of a sphere. Loosely spoken, any symmetrical object in three-space can be associated with a symmetrical pattern on a sphere around it by central projection from O. Studying spherical patterns, which are two-dimensional, seems easier than studying three-dimensional objects. Moreover, this approach connects the study of (bounded) symmetrical objects in three-space to the study of planar symmetrical patterns like wallpaper patterns and frieze patterns.

1.3 Examples of spherical patterns

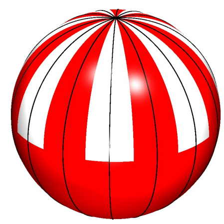
In this subsection we give examples of spherical patterns, one for each type, each with its signature. On this moment, these signatures are just names, but later we shall explain how they are formed and how the signature of any symmetric spherical pattern can be found in a very easy way. We shall also prove that our list is complete: there are no other spherical patterns with a finite symmetry group.

The first seven examples are called *parametric patterns* since they depend on a parameter, in this case an integer $N \geq 1$. Each choice of N yields a different pattern. In our examples we have taken $N = 7$, but it is immediately clear how other choices of N yield similar patterns. The other seven examples are so-called *platonic patterns* since they may be associated with certain symmetry groups of the platonic solids tetrahedron, cube, octahedron, dodecahedron and icosahedron.

In each example reflections (if any) are marked by the great circles, drawn in black, in which the reflection planes intersect the sphere. Rotations are indicated by dots in various colors, marking the points where the rotation axis intersects the sphere. Full explanations will be given later. Note that the coloring of the patterns is essential: symmetries must leave colors unchanged.

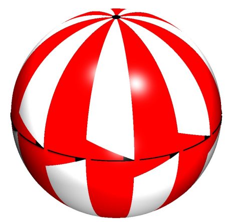
Parametric patterns

[g(7,7)]

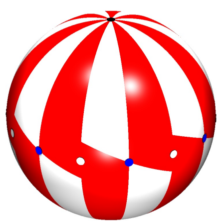


[s(7,7)]

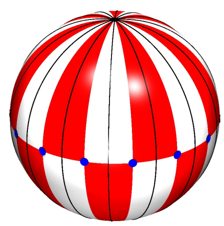
[g(7) s]



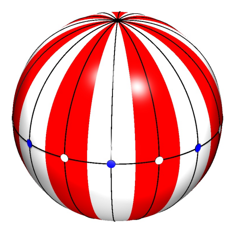
[g(7) x]



[g(7,2,2)]

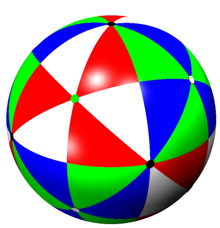


[g(2) s(7)]

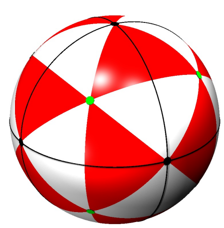


[s(7,2,2)]

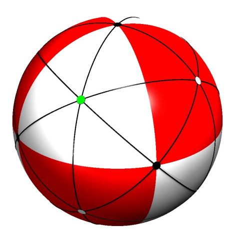
Platonic patterns



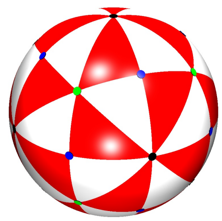
[g(3,3,2)]



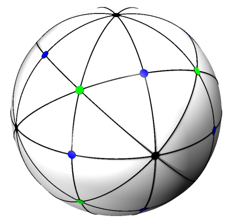
[g(3) s(2)]



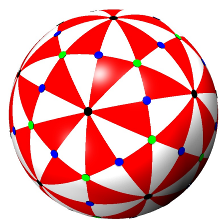
[s(3,3,2)]



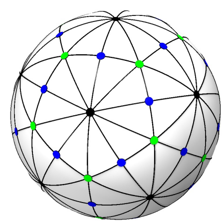
[g(4,3,2)]



[s(4,3,2)]



[g(5,3,2)]



[s(5,3,2)]

1.4 Finite spherical groups of isometries

It is well-known that any isometry on the sphere is either a rotation around an axis through its center, or a reflection in a plane through its center, or a rotatory reflection (a reflection followed by a rotation around an axis perpendicular to the plane of reflection).

Let $\mathcal{G}$ be a finite group of isometries on the sphere. Since the product of two rotations again is a rotation and the product of two (possibly rotatory) reflections also is a rotation, either the group $\mathcal{G}$ consists solely of rotations (including the identity) or the rotations in $\mathcal{G}$ form a normal subgroup $\mathcal{H}$ of index 2. Rotations are *direct* isometries, (rotatory) reflections are *opposite* isometries. Opposite isometries change orientation while direct isometries don't.

The axis of a nontrivial rotation intersects the sphere in two diametrically opposed points, its *poles*. The rotations in $\mathcal{G}$ with a common axis, including the identity, form a cyclic subgroup of $\mathcal{G}$. If p is the order of this subgroup, the two poles are called p-gonal poles, or p-fold centers, or p-poles for short. The smallest positive angle of rotation in this subgroup is $2\pi/p$.

Two poles are called *equivalent* in $\mathcal{G}$ if there is an isometry in $\mathcal{G}$ that transforms one into the other. Note that antipodal poles always have the same order, but need not be equivalent. In the examples in subsection 1.3 equivalent poles are colored the same.

1.5 Chiral patterns

A pattern on the sphere, or a bounded object in three-space, is called *chiral* if its only symmetries are direct isometries. In that case, its mirror image looks different from the original, just as a left hand and a right hand differ in appearance. (The Greek word ”cheir” means hand.) If the pattern or the object permits at least one opposite isometry, it looks exactly the same as its mirror image. Then it is called *achiral*. Chirality of patterns or objects is easily recognized by using a mirror.

In the present subsection, we shall investigate the chiral patterns on the sphere. The symmetry group $\mathcal{G}$ of such a pattern consists entirely of rotations, including the identity. The following derivation of the enumeration of the finite rotation groups is taken from Coxeter (1969), pp. 274-275.

Let n be the order of $\mathcal{G}$ and let P be a p-gonal pole on the sphere. We claim that the number of poles that are equivalent to P (including P itself) equals n/p. To prove this, take a point Q on the sphere near P. The rotations with center P transform Q into a small regular p-gon with center P. The other isometries in $\mathcal{G}$ transform these p-gons into p-gons around the poles that are equivalent to P. In total, the vertices of these p-gons (including Q) form a set of exactly n points on the sphere, divided into congruent regular p-gons. Thus there are n/p of these p-gons, and just as many equivalent p-poles.

Now let us count the $n-1$ non-trivial rotations in $\mathcal{G}$ in a special way, grouping together the nontrivial rotations in classes of equivalent poles. For any p-gonal pole there are $p-1$ nontrivial rotations, so for any set of n/p equivalent p-poles we have $(p-1)n/p$ nontrivial rotations. In total this yields

$$\sum(p-1)n/p$$

nontrivial rotations, where the summation is over al sets of equivalent poles, each with its own value for p. But antipodal poles have the same set of rotations, so each nontrivial rotation occurs twice in this summation. Thus we get the equation

$$2(n-1) = n\sum\frac{p-1}{p}$$

which can also be written as

$$2-\frac{2}{n} = \sum\left(1-\frac{1}{p}\right) \tag{1}$$

If $n=1$ the group $\mathcal{G}$ is trivial: there are no poles and the sum is empty.
For $n \geq 2$ we have

$$1 \leq 2-\frac{2}{n} < 2$$

It follows from $p \geq 2$ that $1/2 \leq 1-1/p < 1$ so there can only be 2 or 3 sets of equivalent poles.

If there are 2 sets, we have

$$2-\frac{2}{n} = 1-\frac{1}{p_1}+1-\frac{1}{p_2}$$

that is

$$\frac{n}{p_1} + \frac{n}{p_2} = 2$$

But both terms on the left are positive integers, so both are equal to 1, and

$$p_1 = p_2 = n$$

Each of the two sets of equivalent poles then consists of one n-gonal pole and $\mathcal{G}$ is the cyclic group with one n-gonal pole at each end of its single axis. The signature of this group is [g(n,n)]. See page 2 for an example with $n = 7$. [g(n,n)]
On page 10 a full explanation is given of this notation. To facilitate reading, we shall always include signatures in square brackets. The letter "g" is from the term "gyration point" used by Conway to denote a rotation center *not* lying in a reflection plane. (The Greek word "gyros" means round.)

If there are 3 sets of equivalent poles, we have

$$2 - \frac{2}{n} = 1 - \frac{1}{p_1} + 1 - \frac{1}{p_2} + 1 - \frac{1}{p_3}$$

whence

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 + \frac{2}{n} > 1$$

It follows that not all p_i can be 3 or more, so at least one of them, say p_3, is 2, and we have

$$\frac{1}{p_1} + \frac{1}{p_2} > \frac{1}{2}$$

This can be rewritten as $2p_1 + 2p_2 > p_1 p_2$, or, equivalently, as

$$(p_1 - 2)(p_2 - 2) < 4$$

Taking $p_1 \geq p_2$ we find as solutions $(p_1, p_2) = (p, 2)$ (for any $p \geq 2$)), $(p_1, p_2) = (3, 3)$, $(p_1, p_2) = (4, 3)$ and $(p_1, p_2) = (5, 3)$.

Summarizing, we have the following possibilities for finite rotation groups on the sphere:

p_1	p_2	p_3	n	signature
p	p	–	p	[g(p,p)]
p	2	2	$2p$	[g(p,2,2)]
3	3	2	12	[g(3,3,2)]
4	3	2	24	[g(4,3,2)]
5	3	2	60	[g(5,3,2)]

Each of the possibilities enumerated above can be realized as the symmetry group of an chiral spherical pattern (see page 2), which proves that there are exactly five types of chiral symmetry patterns on the sphere: two parametric patterns, with signatures [g(p,p)] and [g(p,2,2)] for any $p \geq 2$, and three platonic patterns, with signatures [g(3,3,2)], [g(4,3,2)] and [g(5,3,2)]. The last three are called *platonic* since their symmetry groups are the rotation groups of the platonic solids: the tetrahedron ([g(3,3,2)]), the cube and the octahedron ([g(4,3,2)]) and the dodecahedron and the icosahedron ([g(5,3,2)]).

1.6 Achiral patterns

We now suppose that the spherical pattern is achiral, in other words, that its symmetry group $\mathcal{G}$ contains at least one opposite isometry (reflection or rotatory reflection). Let $n > 1$ be the order of $\mathcal{G}$. Then the rotations in $\mathcal{G}$ form a subgroup $\mathcal{H}$ of order $n/2$. Note that the plane of a reflection intersects the sphere in a great circle. For isometries of the sphere, it is convenient to speak of a reflection in this circle instead of a reflection in the plane of the circle. A reflection circle is sometimes called a *mirror circle* or mirror, for short.

Again, we investigate the nontrivial rotations in $\mathcal{G}$. There are $n/2 - 1$ of them. In $\mathcal{G}$, there can be two different kinds of rotation centers: they may or may not be on a mirror. In the first case, if the order of rotation of such a center Q is q, there must be q mirrors through Q, equally spaced with angles π/q.

For any point T near to Q but not on one of these mirrors, the rotations and reflections through Q transform T into a $2q$-gon. The other isometries in $\mathcal{G}$ transform this $2q$-gon into congruent $2q$-gons around rotation centers that are equivalent to Q. In total, these $2q$-gons have n vertices, so the set of q-poles equivalent to Q (including Q itself) consists of $n/(2q)$ elements. For each q-pole, there are $q - 1$ nontrivial rotations, so in the set of q-poles equivalent to Q we count $(q-1)n/(2q)$ nontrivial rotations.

As in the former section, for a p-pole P not on a mirror the set of p-poles equivalent to P accounts for $(p-1)n/p$ nontrivial rotations. Adding in this way all nontrivial rotations in all sets of q-poles on mirrors and p-poles not on mirrors, we count every rotation twice, since antipodal poles have the same set of rotations. Thus we get the following equation:

$$2\left(\frac{n}{2} - 1\right) = n \sum \frac{q-1}{2q} + n \sum \frac{p-1}{p}$$

where the first summation is over all sets of equivalent poles on mirrors and the second summation is over all sets of equivalent poles not on mirrors.

We write the last equation in the form

$$1 - \frac{2}{n} = \sum \left(\frac{1}{2} - \frac{1}{2q}\right) + \sum \left(1 - \frac{1}{p}\right) \tag{2}$$

If $n = 2$, the left hand side is 0, so both sums on the right must be 0 and there are no nontrivial rotations. The group $\mathcal{G}$ then consists of the identity and either a single reflection or the single rotatory reflection with rotation angle π, which is the *central inversion* in the center of the sphere: the isometry that transforms each point in its antipodal point. In the first case, where $\mathcal{G}$ contains a single reflection, its signature is [s], while in the second case, where $\mathcal{G}$ contains the central inversion, its signature is [x].[1] [s] [x]

Now suppose that $n > 2$. Then

$$\frac{1}{3} \leq 1 - \frac{2}{n} < 1$$

For $p > 1$ we have $1 - 1/p \geq 1/2$, so the second sum contains at most one term.

Case 1: not all rotation centers are on mirrors

We first investigate the case that the second sum is not empty, in other words, that there is at least one p-pole not on a mirror ($p > 1$). The symmetry group then is called a *mixed group*.

As we have seen, the second summation in equation (2) contains only one term, so all p-poles are equivalent and equation (2) can be written as

$$\frac{1}{p} - \frac{2}{n} = \sum \left(\frac{1}{2} - \frac{1}{2q} \right)$$

If $n = 2p$ both sides are zero. Since we supposed that $n > 2$, there are two, necessarily antipodal p-gonal poles, counting for p rotations in $\mathcal{G}$ (including the identity). If there are no mirrors then $\mathcal{G}$ must also contain p rotatory reflections. These rotatory reflections must interchange the two antipodal p-gonal poles. This is the group with signature [g(p) x]. [g(p) x]

If $n = 2p$ and there is a mirror, it must interchange the two poles. Then $\mathcal{G}$ consists of p rotations (including the identity), one reflection and $p - 1$ rotatory reflections. This is the group with signature [g(p) s]. [g(p) s]

Now suppose that $n > 2p$. Then there is at least one q-pole on a mirror circle with $q \geq 2$. Then $1/2 - 1/(2q) \geq 1/4$ and since $p \geq 2$ it follows that

$$\sum \frac{1}{4} \leq \sum \left(\frac{1}{2} - \frac{1}{2q} \right) = \frac{1}{p} - \frac{2}{n} < \frac{1}{p} \leq \frac{1}{2}$$

Therefore there can only be one set of equivalent q-poles and, moreover, p can only be 2 or 3.

If $p = 2$ equation (2) yields

$$\frac{1}{2} - \frac{2}{n} = \frac{1}{2} - \frac{1}{2q}$$

so $n = 4q$. There are two antipodal q-poles lying on q equally spaced mirror circles, bounding $2q$ ‘lunes’ (2-gons) with angles π/q. The centers of these lunes are 2-poles. The signature is [g(2) s(q)]. [g(2) s(q)]

[1] Conway uses a star ($*$) and a multiplication sign ($\times$), but we thought that it is more convenient to use the letters ”s” and ”x”, respectively.

If $p = 3$ then, since $1/2 - 1/(2q) < 1/p = 1/3$, only $q = 2$ is possible. Then equation (2) yields

$$1 - \frac{2}{n} = \frac{1}{4} + \frac{2}{3}$$

so $n = 24$. It follows that there are $n/p = 8$ three-poles P not on mirrors, and $n/(2q) = 6$ two-poles Q on mirrors. These mirrors intersect each other at right angles in the six points Q. This implies that there must be exactly three mirror circles, dividing the sphere into eight octants, which are spherical triangles with three right angles at the points Q. The eight 3-poles P are the centers of these triangles. This pattern has signature [g(3) s(2)]. [g(3) s(2)]

Note that the groups of order 2 with signatures [s] and [x] may be seen as special cases of the groups with signatures [g(p) s] and [g(p) x] for $p = 1$, since a 1-gonal rotation (rotation angle $2\pi/1 = 2\pi$) is the same as the identity.

We summarize our results in the following table:

p	q	n	signature
p	–	$2p$	[g(p) x]
p	1	$2p$	[g(p) s]
2	q	$4q$	[g(2) s(q)]
3	2	24	[g(3) s(2)]

Case 2: all rotation centers are on mirrors

Finally, we consider the case that all rotation centers are on mirrors. The group $\mathcal{G}$ then is called a *reflection group*. Equation (2) then yields

$$1 - \frac{2}{n} = \sum \left(\frac{1}{2} - \frac{1}{2q} \right)$$

But if we put $n' = n/2$ (note that n is even!) this can be written as

$$2 - \frac{2}{n'} = \sum \left(1 - \frac{1}{q} \right)$$

which is exactly the same as equation (1), the equation that describes all possible finite rotation groups. In this case, the equation describes the possibilities for the normal subgroup $\mathcal{H}$ of all rotations in $\mathcal{G}$.

All rotation centers are on mirrors and all intersections of mirrors are rotation centers, with q equally spaced mirrors through each q-fold center. Once again enumerating all possibilities for $\mathcal{H}$ of order $n' = n/2$, we find that for $n' \geq 2$ there are 2 or 3 equivalence classes of q-centers.

For 2 equivalence classes $q_1 = q_2 = n' = n/2$ must hold, so there are two antipodal q-poles ($q = n/2$)) with q equally spaced mirror circles connecting them, forming $2q$ lunes with angles π/q at the poles. The signature is [s(q,q)]. [s(q,q)]

For 3 equivalence classes, the same reasoning as on page 6 leads to the conclusion that there are four more possible types of symmetry groups, and the examples on page (2) ff. show that each possibility can be realized. The following table yields all reflection groups.

q_1	q_2	q_3	n	signature
q	q	–	$2q$	[s(q,q)]
q	2	2	$4q$	[s(q,2,2)]
3	3	2	24	[s(3,3,2)]
4	3	2	48	[s(4,3,2)]
5	3	2	120	[s(5,3,2)]

This completes our enumeration of the finite groups of isometries on the sphere. We have proved that there are exactly 14 types of finite symmetry groups on the sphere. Their signatures are given in the following table, where $p \geq 1$ and $q \geq 1$ are arbitrary parameters. Note that [g(1,1)] is the trivial group consisting of the identity alone, that the signatures [g(1) s] and [s(1,1)] both can be simplified to [s], that [g(1) x] can be simplified to [x] and that [s(1,2,2)] is the same group as [s(2,2)].

	Rotation groups (chiral patterns)	*Mixed groups (achiral patterns)*	*Reflexion groups (achiral patterns)*
Parametric groups	[g(p,p)]	[g(p) s)] [g(p) x)]	[s(q,q)]
	[g(p,2,2)]	[g(2) s(q)]	[s(q,2,2)]
Platonic groups	[g(3,3,2)] [g(4,3,2)] [g(5,3,2)]	[g(3) s(2)]	[s(3,3,2)] [s(4,3,2)] [s(5,3,2)]

1.7 Explaining examples and signatures

For each type of pattern we have given an example in subsection 1.3. For the parametric patterns we have taken $p = 7$ and $q = 7$, respectively. In each pattern the mirror circles are marked in black, while the rotation centers are marked with colored dots. Equivalent centers are colored the same.

In this way, Conway's signature notation becomes almost self-evident. All equivalence classes of gyration points are enumerated between brackets after the symbol ”g”, while all equivalence classes of rotation centers on mirrors are enumerated between brackets after the symbol ”s”. In other words, rotation centers of each color are entered only once in the signature. Finally, the occurence of a rotatory reflection that is *not* the composition of a rotation and a reflection in the group, is marked by the symbol ”x”.

The reader is invited to check all mirror circles, gyration points and rotation centers on mirrors in the given examples, thus verifying the correctness of the given signatures. Also check that, indeed, equivalent poles are colored the same (but not all poles are visible in the drawings). Note that the coloring of the patterns is important: a symmetry of a pattern must leave all colors unchanged.

In fact, it is very easy to find the signature of any spherical pattern with a finite symmetry group. First find out whether it is chiral or achiral, then mark all mirror circles and all poles, giving equivalent poles the same color. The signature then can be read off by inspection, the only subtle point being the possibility that the pattern doesn't possess mirror circles but nevertheless is achiral. Then there must be at least one rotatory reflection, and the signature must be [g(p) x] for some $p \geq 1$.

Note that equations (1) and (2) are satisfied by all chiral and achiral spherical patterns, respectively. The order n of the corresponding finite symmetry group follows from these equations. Also, recall that the number of equivalent p-gonal gyration points equals n/p while the number of equivalent q-gonal rotation centers on mirrors equals $n/(2q)$.

Equations (1) and (2) enabled us to enumerate all 14 types of symmetric patterns in a rather simple manner and to prove that there are no other types. As a matter of fact, equations (1) and (2) are essentially the same as Conway's 'Magic Theorem' for spherical patterns (Conway (2008). p. 53). However, our derivation (for chiral patterns based on Coxeter (1969), pp. 274-275) is more elementary, not making use of any advanced topological results.

2 Planar rosette patterns and frieze patterns

2.1 Rosette patterns

Now we turn to patterns in the Euclidean plane. First we consider patterns with a finite number of symmetries. For reasons that will become clear in a moment, these are called *rosette patterns*.

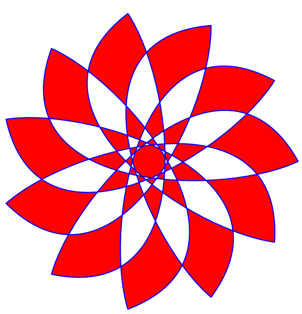
A rosette pattern may have various kinds of symmetry, but there must be at least one fixed point, a point that is unchanged under all elements of its group $\mathcal{G}$ of symmetries. Indeed, since the orbit of any point P is a finite point set, its centroid O is a fixed point of all symmetries in $\mathcal{G}$.

If there are two distinct fixed points O_1 and O_2, the line ℓ through O_1 and O_2 must consist entirely of fixed points, since any symmetry is an isometry. If there is also a fixed point O_3 not on ℓ, any point in the plane is a fixed point under all elements of $\mathcal{G}$, so the group only consists of the identity, and the pattern has no symmetry at all. If only the points of a line ℓ are fixed points of all symmetries, $\mathcal{G}$ has only two elements: the identity and the reflection in ℓ. In common language, the pattern has the reflection in ℓ as its only symmetry.

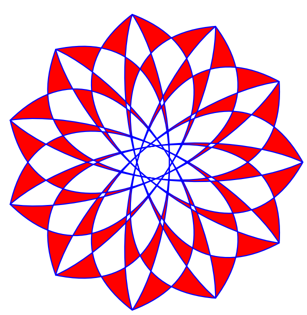
In all other cases, there is exacty one point O that is fixed under all elements of $\mathcal{G}$. The symmetries then can only be rotations with center O or reflections in lines through O.

If the pattern is chiral, its only symmetries are rotations. The group $\mathcal{G}$ then is cyclic. If its order is p, then the smallest positive rotation angle is $2\pi/p$ and all orbits are regular p-gons with center O. We denote its signature by [g(p)].

If the pattern is achiral, there must be a finite number, say q, of mirror lines through O, equally spaced with angles π/q between adjacent lines. The group then is the so-called *dihedral group* of order $2q$, consisting of q reflections and q rotations (including the identity). We denote its signature by [s(q)].



[g(11)]



[s(11)]

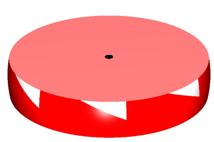
Above, two examples of rosette patterns with their signatures are given. Note that any rosette pattern may be pasted on a big sphere, say at its north pole, thus yielding a parametric spherical pattern with signature [g(p, p)] for the cyclic group or [s(q,q)] for the dihedral group. In this way any rosette pattern may be identified with a spherical pattern, having the 'same' group of symmetries, which in the spherical case are spherical rotations and reflections in circles through the antipodal poles and in the planar case rotations around its center and line reflections in lines through its center.

2.2 Frieze patterns

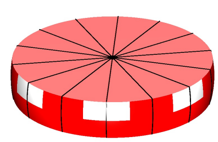
Frieze patterns are planar patterns in which a *motif* is repeated in a row, say, a horizontal row. Although a frieze pattern in real life is always finite, we shall imagine that it is continued infinitely in both directions, so that its symmetries include horizontal *translations*. If T is the translation taking each motif to its neighbour to the right, any other translation in the symmetry group of the pattern can be written as T^n for some integer n. But there may also be other symmetries in the pattern.

It is well-known that there are exactly seven types of symmetry groups for frieze patterns, and in fact we already know them, since they can be seen as limiting cases of the seven types of symmetry groups for *parametric spherical patterns*. As Conway observed: take a finite segment of a frieze pattern consisting of N copies of the motif, and wrap it around the equator of a sphere of a suitably chosen radius. Then a parametric spherical pattern results with one of the seven signatures [g(N,N)], [s(N,N)], [g(N) s], [g(N) x], [g(N,2,2)], [g(2) s(N)] and [s(N,2,2)]. Conversely, any parametric spherical pattern can be unwrapped into the plane, yielding an N-segment of a frieze pattern.

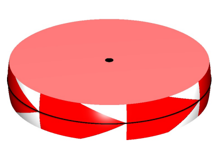
Below, parametric spherical patterns with $N = 7$ are drawn, restricting the sphere to a disk around the equator.



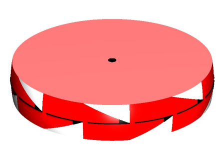
[g(7,7)]



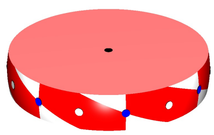
[s(7,7)]



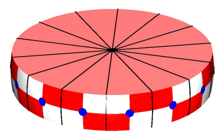
[g(7) s]



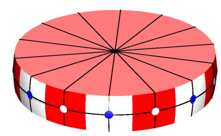
[g(7) x]



[g(7,2,2)]

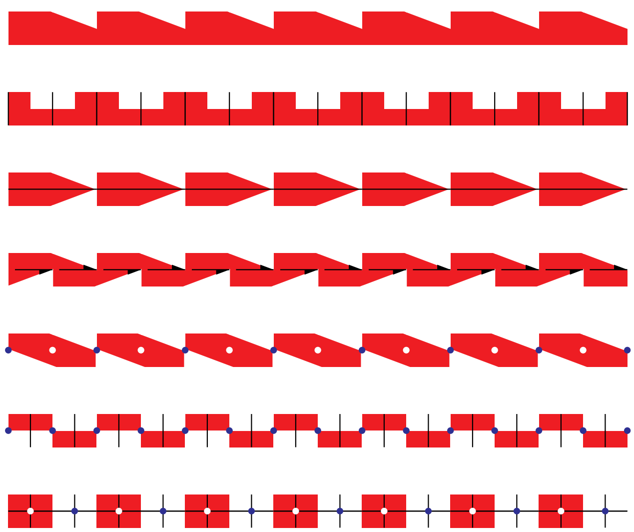


[g(2) s(7)]



[s(7,2,2)]

Unwrapping the patterns into the plane leads to segments of length 7 of the corresponding infinite frieze patterns. It is natural, letling $N \to \infty$, to denote the signatures of these frieze patterns by [g(∞,∞)], [s(∞,∞)], [g(∞) s], [g(∞) x], [g(∞,2,2)], [g(2) s(∞)] and [s(∞,2,2)], respectively.



[g(∞,∞)]

[s(∞,∞)]

[g(∞) s]

[g(∞) x]

[g(∞,2,2)]

[g(2) s(∞)]

[s(∞,2,2)]

Since the symmetry group of any frieze pattern is infinite, equations (1) and (2) on pages 5 and 7, by letting $n \to \infty$. yield

$$2 = \sum \left(1 - \frac{1}{p}\right) \tag{3}$$

for chiral frieze patterns and

$$1 = \sum \left(\frac{1}{2} - \frac{1}{2q}\right) + \sum \left(1 - \frac{1}{p}\right) \tag{4}$$

for achiral frieze patterns, where the p-summations in (3) and (4) are over equivalence classes of gyration points, while the q-summation in (4) is over equivalence classes of rotation centers on mirror lines.

Note that the 'north pole' and the 'south pole' have gone to infinity in the vertical direction, and that the rotations around these poles now have become horizontal translations. These translations may be seen as 'rotations' with 'rotation angle' $2\pi/\infty = 0$ and 'center' at plus or minus infinity in the vertical direction.

For these centers 'at infinity' we must take $p = \infty$ or $q = \infty$, respectively, so $1/p = 1/(2q) = 0$, contributing a term 1 or 1/2 in the above summations. In this way, equations (3) and (4) are easily verified for all frieze patterns.

Rotatory reflections on the sphere become *glide reflections* of frieze patterns in the plane, a glide reflection being the combination of a reflection in a mirror line (its *axis*) and a translation in the direction of the axis. Note that applying a glide reflection twice results in a translation along the axis.

See pattern [g(∞) x] above for an example with a horizontal glide axis that is not a mirror line of the pattern. In our example, the glide axis is indicated by a dashed line with alternating half arrows.

Pattern [g(∞) s] also possesses horizontal glide reflections, but these are composed of a horizontal reflection and a translation already present in the group, so the glide reflections are not mentioned in the signature. Conway's signature notation never contains redundant information. This is also the reason why equivalent poles are only entered once between the brackets.

A horizontal glide reflection also results when combining a vertical reflection and a 2-gonal rotation (a half-turn) with center *not* on the mirror line. Then the axis of the glide reflection is the line through the 2-center perpendicular to the mirror line. Conversely, combining a glide reflection with a half-turn with center on the glide axis results in an ordinary reflexion with mirror axis perpendicular to the glide axis.

Patterns [g(2) s(∞)] and [s(∞,2,2)] yield examples with vertical mirrors and 2-gonal rotation centers on a horizontal line. Since the vertical reflections and the 2-gonal rotations are already present in the group, the glide reflections are not mentioned in the signature.

This completes our treatment of frieze patterns.

3 Wallpaper patterns

3.1 Introduction

While rosette patterns admit no translations, and frieze patterns admit translations in one direction only, in the so-called *wallpaper patterns* translations in more than one direction are present. We shall suppose that the symmetry group $\mathcal{G}$ of such a pattern is *discrete*, meaning that for any point P there is a circle around P which contains no other points of the orbit of P. This implies that there is a bounded *motif* that is repeated infinitely all over the plane by the translations in $\mathcal{G}$. Note that, although we didn't mention it explicitly, we tacitly assumed that also for spherical patterns, rosette patterns and frieze patterns the symmetry groups are discrete. For spherical patterns and rosette patterns this was guaranteed by demanding that the symmetry groups be finite, thus excluding, e.g., cases in which the pattern only consists of one (unmarked) circle, which has infinitely many rotations around its center.

The discreteness of the symmetry group $\mathcal{G}$ of a wallpaper pattern implies that there exists a translation T_1 with minimum positive translation distance d_1 and a second translation T_2, independent of T_1, with a minimum positive translation distance $d_2 \geq d_1$. By reversing if necessary the direction of T_2, we may suppose that the angle α between T_1 and T_2 is non-obtuse. Furthermore, if $\alpha < \pi/3$ would hold, the translation distance of $T_1 - T_2$ would be less than d_2 by the rule of sines, contradicting the choice of T_2. So $\pi/3 \leq \alpha \leq \pi/2$.

It is easy to prove that any translation T in $\mathcal{G}$ can be written as $T = T_1^n T_2^m$ for certain integers n and m, in other words, that the normal subgroup $\mathcal{T}$ of all translations in $\mathcal{G}$ is generated by T_1 and T_2. For any point P, the orbit of P under $\mathcal{T}$ is a *lattice* spanned by vectors representing T_1 and T_2.

But wallpaper patterns may also have symmetries other than translations. In fact, it is well-known, and we shall prove it again in this article, that there are 17 distinct types of wallpaper patterns. Each type has its own Conway signature, which in a very concise way describes the symmetry features of the pattern. As was the case with spherical patterns, rosette patterns and frieze patterns, the signature of wallpaper patterns is easily read off from the pattern as soon as one has identified its chirality and, if present, its mirror lines, its rotation centers and its glide reflexion axes that are not mirror lines.

3.2 Examples of the 17 types of wallpaper patterns

In this subsection for each of the 17 types of wallpaper patterns we shall present a simple example together with its signature. In each instance, mirror lines will be marked in black and rotation centers by colored dots, where equivalent centers get the same color. Glide reflexion axes that are not mirror lines will be marked with dashed lines.

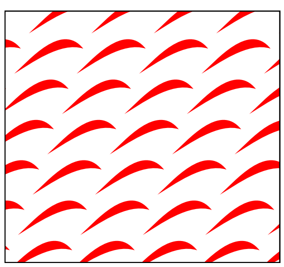
The reader will have no difficulty interpreting and verifying the given signatures. Following Conway, we use the signature [O] for the pattern in which translations are the only symmetries. All other patterns have signatures which should look familiar by now.

In pattern [s s] there are two types of non-equivalent mirror lines, both vertical. In pattern [s x] vertical mirror lines and glide reflexion axes alternate at equal distances. All mirror lines are equivalent, as are all glide reflexion axes. The pattern [x x] has two types of non-equivalent glide reflexion axes, both vertical and both with the same vertical translation distance.

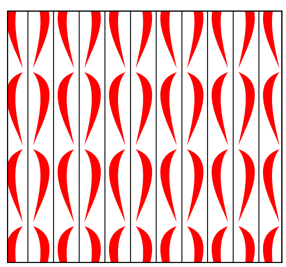
Special mention deserves pattern [g(2,2) x], in which two non-equivalent types of two-fold gyration points occur, colored blue and green, and two types of glide reflexion axes that are no mirror lines. Note that a horizontal glide reflexion, combined with a half turn around a center not on its axis, yields a glide reflection with a vertical axis. This is why in the signature [g(2,2) x] only one symbol "x" occurs.

In the next subsections we shall give more details on the various patterns and we also shall prove that there are no more than 17 types.

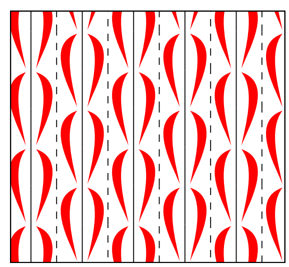
Seventeen wallpaper patterns with their signatures:



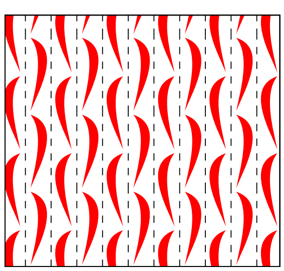
[O]



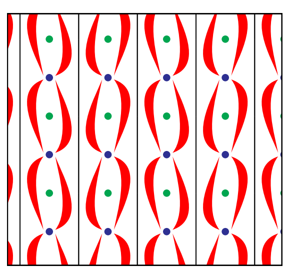
[s s]



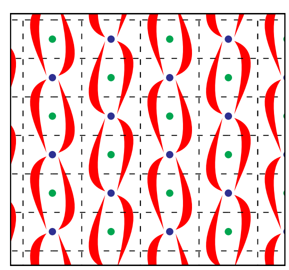
[s x]



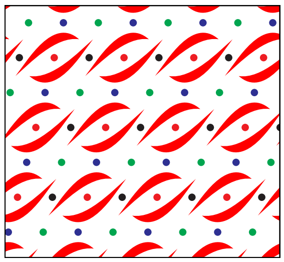
[x x]

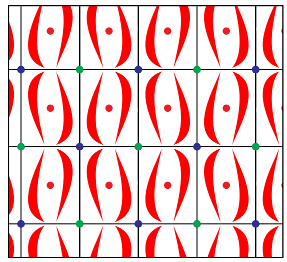


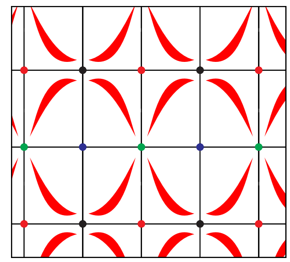
[g(2,2) s]

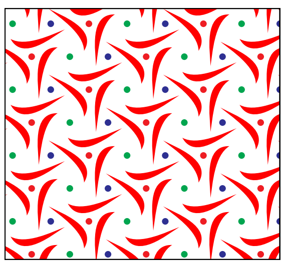


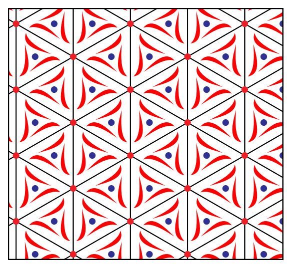
[g(2,2) x]

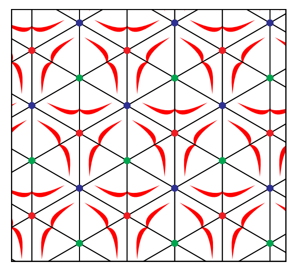

[g(2,2,2,2)]

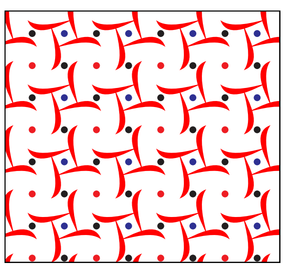

[g(2) s(2,2)]

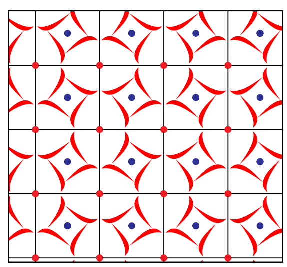

[s(2,2,2,2)]

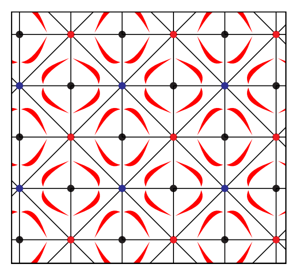

[g(3,3,3)]

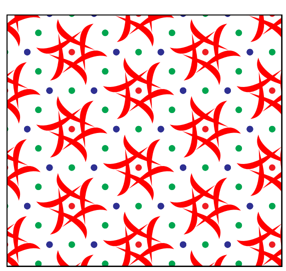

[g(3) s(3)]

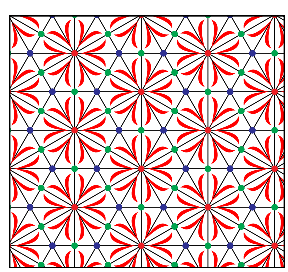

[s(3,3,3)]


[g(4,4,2)]


[g(4) s(2)]


[s(4,4,2)]


[g(6,3,2)]


[s(6,3,2)]

3.3 Enumeration of wallpaper patterns

Isometries in the plane

It is well-known that every isometry in the Euclidean plane is either a translation, or a rotation, or a reflexion in a line, or a glide reflexion. Translations and rotations are direct isometries, line reflexions and glide reflexions are opposite isometries. Any group $\mathcal{G}$ of isometries either consists of direct isometries only, or its direct isometries form a normal subgroup $\mathcal{H}$ of index 2 in $\mathcal{G}$.

A pattern in the plane is called *chiral* if its group of symmetries consists of direct isometries only, otherwise it is called *achiral*. Chirality of patterns can easily be determined by using a mirror.

A rotation has a *center* and a *rotation angle*, which we shall always measure anti-clockwise in radians modulo 2π. Sometimes it is convenient to consider translations as rotations with zero angle and center at infinity in a direction perpendicular to the translation direction.

In a discrete group $\mathcal{G}$ the rotations with a common center form a cyclic subgroup of finite order p, generated by a rotation with angle $2\pi/p$. Its center is called a p-center. Likewise, the translations in $\mathcal{G}$ in a given direction form an infinite cyclic group generated by a translation T with distance $d > 0$.

If R_1 and R_2 are reflections in mirror lines m_1 and m_2, respectively, then R_2R_1 (R_1 followed by R_2) is a translation with distance $2d$ if m_1 and m_2 are parallel with distance d and a rotation with center A and rotation angle 2α if m_1 and m_2 intersect at A with intersection angle α.

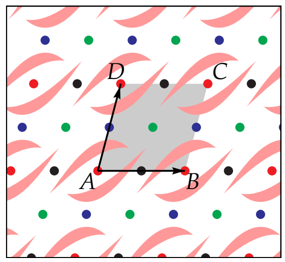
If R_1 and R_2 are rotations with centers A_1 and A_2 and rotation angles $2\alpha_1$ and $2\alpha_2$, respectively, then R_2R_1 is a rotation with angle $2\alpha_1 + 2\alpha_2$, or a translation if $2\alpha_1 + 2\alpha_2 = 0 \bmod 2\pi$. If it is a rotation, its center A_3 can be found by constructing the clockwise oriented triangle $A_1A_2A_3$ with angles α_1 at A_1 and α_2 at A_2.

If both R_1 and R_2 are half-turns, so that both rotation angles are equal to π, then R_2R_1 is a translation with distance $2d(A_1, A_2)$ in the direction from A_1 to A_2.

Chiral patterns

Let be given a wallpaper pattern with a discrete symmetry group $\mathcal{G}$. First we consider the case that the pattern is chiral, so all isometries in $\mathcal{G}$ are translations or rotations. We have already seen that the normal subgroup $\mathcal{T}$ of all translations in $\mathcal{G}$ is generated by two independent translations T_1 and T_2 with minimum translation distances $d_1 \leq d_2$ and angle $\pi/3 \leq \alpha \leq \pi/2$. For any point P the orbit of P under the translation group is a lattice. If there are no rotations, $\mathcal{G}$ coincides with its translation group and its signature is [O]. [O]

Now suppose that all rotations in $\mathcal{G}$ are half-turns, in other words, that there are only 2-centers. Let A be a 2-center and let $ABCD$ be the parallelogram with $B = T_1A$, $C = T_2T_1A$ and $D = T_2A$. Then the other vertices of the parallelogram, the midpoints of its sides and the center of the parallelogram must also be 2-centers, nine centers in total. But centers that differ by a translation in $\mathcal{T}$ are equivalent, so there are only four non-equivalent 2-centers in the parallelogram, which can be taken at A and at the midpoints of AB, AC and AD.

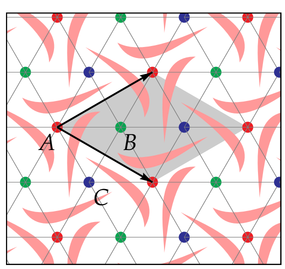


We claim that all half-turns in $\mathcal{G}$ are equivalent to one of these four. In fact, for any other half-turn there is an equivalent one with center E inside or on the boundary of $ABCD$. But if E is not one of the nine 2-centers mentioned above, a combination with the half-turn with center A would result in a translation not generated by T_1 and T_2, which is impossible. Therefore there are exactly four non-equivalent 2-centers, and the signature of this group thus is [g(2,2,2,2)] [g(2,2,2,2)]

If not all rotation centers are 2-centers, we may take two centers A and B of order p and q both greater than 2 and with minimum distance d. Let C be such that triangle ABC has angles π/p and π/q at A and B, respectively. Then C is the center of a rotation in $\mathcal{G}$ with angle $2\pi/p + 2\pi/q = 2\pi - 2\pi/r$ for some positive r, so

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 \tag{5}$$

If $p = q = 3$ then also $r = 3$ must hold, so A, B and C are 3-centers. Then all 3-centers form a lattice of equilateral triangles. By the choice of A and B at minimum distance, no other centers of order greater than 2 occur inside a circle with radius d and center A covering six triangles of this lattice or, for that matter, anywhere in the plane.



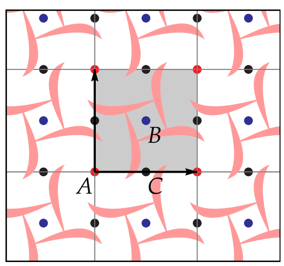
But then also 2-centers are impossible, since they would imply the existence of rotations with angle $2\pi/2 - 2\pi/3 = 2\pi/6$, contradicting the former observation. Combining rotations with angles $\pm 2\pi/3$ around A and $\mp 2\pi/3$ around B yields two translation T_1 and T_2. These two translations generate $\mathcal{T}$. It follows that A, B and C are non-equivalent 3-centers, so the group we just have described must have signature [g(3,3,3)]. [g(3,3,3)]

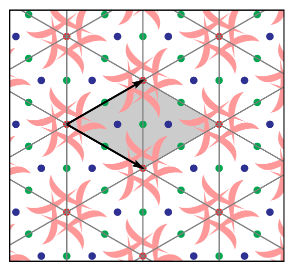
If, in the same notations as above, p and q are not both equal to 3 but still greater than 2, then C is closer to A than B, so C must be a 2-center. It follows that $r = 2$ and

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} \tag{6}$$

which can be written as $(p-2)(q-2) = 4$. Its only solutions with $p \geq q \geq 3$ are $(p,q) = (4,4)$ and $(p,q) = (6,3)$. This implies in particular that in a wallpaper pattern only 2-centers, 3-centers, 4-centers and 6-centers can occur. This is the famous *crystallographic restriction*. *crystallographic restriction*

If $p = q = 4$, then A and B must be adjacent 4-centers, and, again, C is a 2-center. Triangle ABC then is an isosceles right-angled triangle, which can be seen as part of a square with A as one vertex, B as its center and C as the midpoint of a side adjacent to A. Combining rotations around A and B with angles $\pm 2\pi/4$ and $\mp 2\pi/4$ yields translations along the sides of the square. These translations generate $\mathcal{T}$. It follows that A and B are non-equivalent 4-centers, so the signature of this group is [g(4,4,2)]. [g(4,4,2)]





In a similar way, $p = 6$, $q = 3$ leads to a pattern of 6-centers, 3-centers and 2-centers forming a lattice of equilateral triangles, with 6-centers at their vertices, 3-centers at their centers and 2-centers at the midpoints of their sides. The signature of this pattern then is [g(6,3,2)]. [g(6,3,2)]

Note that equation (5) can be written as

$$2 = \left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{q}\right) + \left(1 - \frac{1}{r}\right) \tag{7}$$

which has the same form as equation (3) on page 14. Also pattern [g(2,2,2,2)] satisfies this equation (with four 2-centers of 'weight' $\frac{1}{2} = 1 - \frac{1}{2}$). Even patterns with signature [O] satisfy this equation if we consider the two independent translations T_1 and T_2 as 'rotations' with rotation angle 0 and 'order' $p = q = \infty$. Each then contributes a 'weight' of $1 - 1/\infty = 1$ to the right-hand side.

Achiral patterns without rotations

Now we turn to achiral patterns. Then the group $\mathcal{G}$ must contain at least one line reflection or glide reflection. The direct isometries in $\mathcal{G}$ form a normal subgroup $\mathcal{H}$ of index 2. The translations form a subgroup $\mathcal{T}$ of $\mathcal{H}$ which is also normal in $\mathcal{G}$.

First, suppose that there are no rotations in $\mathcal{G}$, so $\mathcal{H} = \mathcal{T}$. Then mirror lines and glide reflexion axes can occur in only one direction, since intersecting mirrors or glide reflection axes yield nontrivial rotations.

If there is a mirror line m, then there are infinitely many equally spaced mirror lines parallel to m. Let m' be a mirror line at minimum distance from m. If m and m' are not equivalent, there are just two types of mirror lines, either equivalent to m or to m'. The signature of the group then is [s s]. [s s]

If m and m' are equivalent, they must be related by a glide reflection with the midparallel n of m and m' as its axis. Then mirror lines and glide axes alternate at equal distances, and the group has signature [s x]. [s x]

Finally, if there are no mirror lines, all glide axes must be parallel and equidistant. If n and n' are adjacent glide axes, they cannot be equivalent, so the signature is [x x]. [x x]

Achiral patterns in which all rotation centers are on mirrors

If in an achiral pattern rotations occur, there may be rotation centers on mirror lines and rotation centers not on mirror lines (the so-called gyration points). We first consider the case that all rotation points are on mirror lines. The normal subgroup $\mathcal{H}$ of index 2 consisting of all direct isometries in $\mathcal{G}$ then is one of the four types with signatures [g(2,2,2,2)], [g(3,3,3)], [g(4,4,2)] and [g(6,3,2)], and the examples in subsection 3.2 show that in the last three cases it is possible [s(3,3,3)]
to connect all rotation points by mirror lines such that through every q-center [s(4,4,2)]
exactly q mirror lines pass with angles π/q. In this way we obtain the reflexion [s(6,3,2)]
groups with signatures [s(3,3,3)], [s(4,4,2)], [s(6,3,2)].

For patterns with signature [g(2,2,2,2)] this is only possible if the generators T_1 and T_2 of the translation group are orthogonal. Then the mirror lines form a rectangular lattice with a 2-center at each lattice point. The signature of this reflexion group is [s(2,2,2,2)]. [s(2,2,2,2)]

Achiral patterns with gyration points and no mirror lines

Next we consider the case of achiral patterns with gyration points, i.e., rotations centers not on a mirror line. Again, the normal subgroup $\mathcal{H}$ of all direct isometries in $\mathcal{G}$ has signature [g(2,2,2,2)], [g(3,3,3)], [g(4,4,2)] or [g(6,3,2)].

First we suppose that the achiral pattern has no mirror lines. Then all rotation centers are gyration points, and there must be a glide reflexion G in $\mathcal{G}$. Like all isometries in $\mathcal{G}$, the glide reflexion G must map p-centers on p-centers. But if it maps a p-center A on a p-center A' that is equivalent to A in $\mathcal{H}$, then there is a direct isometry D mapping A' back to A. Therefore the opposite isometry DG has a fixed point A, so it must be a reflexion in a line through A, contradicting the assumption that the pattern has no mirror lines.

But if G cannot map gyration points on equivalent gyration points (equivalent in $\mathcal{H}$), the cases that the signature of $\mathcal{H}$ is [g(4,4,2)] or [g(6,3,2)] are excluded since, e.g., 2-centers must be mapped on (necessarily equivalent) 2-centers. Also [g(3,3,3)] is impossible, for suppose that a 3-center A is mapped to a non-equivalent 3-center B, then, since G^2 is a translation, $B = GA$ must be mapped to a 3-center $A' = G^2A$ equivalent to A. But then any 3-center C not equivalent to A or B is mapped to a 3-center C' equivalent to C, contradiction.

However, if the signature of $\mathcal{H}$ is [g(2,2,2,2)] it may happen that there are glide axes but no mirror lines, provided the translation group has orthogonal generators. If A, B, C and D are 2-centers forming a rectangle $ABCD$ of minimum side lengths, then there is a glide reflexion G with $GA = C$ and $GB = D$. In fact, there are two such glide reflexions with perpendicular axes parallel to the sides of the rectangle, both passing through its center. There are two non-equivalent types of 2-centers and the pattern has signature [g(2,2) x]. [g(2,2) x]

Achiral patterns with gyration points and mirror lines

Finally we consider achiral patterns with gyration points and mirror lines. It may happen that all mirror lines are parallel. We take them vertical. Gyration points then can only be of order 2 and they must be situated halfway between two adjacent mirror lines. Let A be such a gyration point. Since there must be vertical translations, there is a vertical translation T with minimum translation distance $2d$. Then a point B above A at distance d is also a gyration point. All other gyration points are equivalent to either A or B and the signature of this pattern is [g(2,2) s]. [g(2,2) s]

If there are mirror lines in more than one direction, the set $\mathcal{M}$ of all mirror lines is a symmetric subset of the set of all mirror lines in a pattern with signature [s(2,2,2,2)], [s(3,3,3)], [s(4,4,2)] or [s(6,3,2)]. Thus, $\mathcal{M}$ divides the plane into congruent rectangles, squares, isosceles rectangular triangles, equilateral triangles or rectangular triangles with angles $\pi/3$ and $\pi/6$. Only divisions into rectangles, equilateral triangles and squares admit gyration points, of order 2, 3 and 4, respectively. This results in three more patterns with gyration points and mirror lines. Their signatures are [g(2) s(2,2)], [g(3) s(3)] and [g(4) s(2)]. See section 3.2 for examples. [g(2) s(2,2)] [g(3) s(3)] [g(4) s(2)]

This completes our enumeration of all types of wallpaper patterns.

The following table summarizes our results on wallpaper patterns. The symmetry groups of chiral patterns are called *rotation groups* (bearing in mind that translations can be viewed of as rotations with angle 0 and center at infinity). Groups generated by (glide) reflections are called *reflexion groups* and groups of achiral patterns with gyration points *mixed groups*.

Rotation groups (chiral patterns)	*Mixed groups (achiral patterns)*	*Reflexion groups (achiral patterns)*
[O]	[g(2) s] [g(2) x]	[s s] [s x] [x x]
[g(2,2,2,2)]	[g(2) s(2,2)]	[s(2,2,2,2)]
[g(3,3,3)]	[g(3) s(3)]	[s(3,3,3)]
[g(4,4,2)]	[g(4) s(2)]	[s(4,4,2)]
[g(6,3,2)]		[s(6,3,2)]

3.4 Conway's 'magic theorems'

One of the most intriguing points in the treatment of symmetry patterns in Conway (2008) is his use of 'magic theorems' to facilitate the enumeration of symmetric patterns. These theorems are used early in the book, but the proofs are postponed many times, which is caused by the fact that they make use of rather advanced topological tools like the concept of 'orbifold' and the classification of surfaces in three-space. In our treatment of symmetric spherical and planar patterns, however, these 'magic theorems' appear as by-products that are obtained in an elementary way, as we shall show presently. It must, however, be stressed that Conway's proof of his 'magic theorems' provides deeper insight and applications in many other symmetry problems, as is amply illustrated in parts II and III of his book (Conway (2008)).

First, for any symmetric feature appearing in the signature of a pattern, Conway defines its 'cost' in the following way. Instead of 'cost', we rather prefer to speak of 'weight'.

- For an equivalence class of p-fold gyration points, its weight is $1 - \frac{1}{p}$.
- For an equivalence class of q-fold rotation centers on mirrors, its weight is $\frac{1}{2} - \frac{1}{2q}$.
- For a symbol "s" (with or without brackets), its weight is 1.
- For a symbol "x", its weight is 1.
- For the symbol "g", its weight is 0.
- For the symbol "O", its weight is 2.

Note that for ∞-fold gyration points or ∞-fold rotation centers on mirrors (as occur in frieze patterns) the weight is 1 or $\frac{1}{2}$, respectively.

The magic theorem for spherical patterns: The total weight of the signature of a pattern with a symmetry group of order n equals $2 - \frac{2}{n}$.

Proof: For chiral patterns, this is equation (1) on page 5. For achiral patterns this is equation (2) on page 7, adding 1 to both sides to account for the symbol "s" or the symbol "x".

The magic theorem for frieze patterns: The total weight of the signature of a frieze pattern is 2.

Proof: For chiral patterns, this is equation (3) on page 14. For achiral patterns this is equation (4) on page 14, adding 1 to both sides to account for the symbol "s" or the symbol "x".

The magic theorem for wallpaper patterns: The total weight of the signature of a wallpaper pattern is 2.

Proof: For chiral patterns, see equation (7) on page 21 and the remarks following it. For achiral patterns, the reader is invited to find the small modifications that are needed to adapt the proof for chiral patterns. Anyhow, the validity of the magic theorem in this case can also be verified directly.

A note on notations

As a final remark on our adapted signature notation, we note that Conway's signature notation is much more compact than ours, omitting all comma's, brackets and the letter "g" for gyration points. For our letters "s" and "x", Conway uses the symbols $*$ and $\times$, respectively. His signatures are put in bold face font. So, for instance, for our notations [g(2) s(2,2)] and [s x], Conway uses **2**$*$**22** and $*\times$. See also Conway (2008), p. 416, for a table of various other notations for wallpaper groups and finite spherical groups, among others the much used notation of the *International Tables for X-ray Crystallography*.

Literature

1. John H. Conway, Heidi Burgiel, Chaim Goodman-Strauss, *The Symmetries of Things*, A.K. Peters, Ltd., Wellesley, Massachusetts, 2008

2. H.S.M. Coxeter, *Introduction to Geometry*, second edition, John Wiley & Sons, Inc., New York, 1969

3. L. Fejes Tóth, *Reguläre Figuren*, Akadémiai Kiadó, Budapest, 1965