Modular Forms: Exercises

Spring 2009

1) a) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ with $A \neq \pm I$ and let H be the upper half plane of $\mathbb{C}$. Show that A is one of the following types:
i) A has two fixed points $\zeta, \overline{\zeta}$ in $\mathbb{C}$ with $\zeta \in H$ (*elliptic)*;
ii) A has two fixed points on the real line $\cup \infty$ (*hyperbolic)*;
iii) A has exactly one fixed point, either ∞ or on the real line (*parabolic)*.
b) Prove
i) If $c = 0$ then A is parabolic.
ii) If $c \neq 0$ then one has

$$\begin{aligned} |a+d| > 2 &\iff A \text{ hyperbolisch} \\ |a+d| = 2 &\iff A \text{ parabolisch} \\ |a+d| < 2 &\iff A \text{ elliptisch.} \end{aligned}$$

2) For $A \in SL_2(\mathbb{R})$ with $A^n \neq \pm I$ show: A is elliptic (resp. hyperbolic, parabolic) if and only if A^n is elliptic (resp. hyperbolic, parabolic).

3) Calculate the isotropy groups in $SL_2(\mathbb{Z})$ of the two points $\tau = i$ and $\tau = \rho = (-1+\sqrt{-3})/2$.

4) Let $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A \in SL_2(\mathbb{Z})$. Show that if A has finite order then A is conjugated with one of the following matrices

$$\pm I, \pm S, \pm(ST), \pm(ST)^2$$

and one has $\mathrm{Tr}(A) < 2$.

5) Show that a 2×2 matrix X with coefficients a, b, c, d from a commutative ring R with 1 satisfies the following equation

$$X^2 - \mathrm{Tr}(X)\, X + \det(X)\, I = 0$$

with $\mathrm{Tr}(X) = a + d$ and $\det(X) = ad - bc$.

6) Show that the map $SL(_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})$ given by reducing the entries of a matrix modulo N is a surjective group homomorphism.

7) Let $\mathbb{F}_q$ be a finite field with q elements. Determine the order of $GL_2(\mathbb{F}_q)$ and of $SL_2(\mathbb{F}_q)$.

8) Let $N = \prod_p p^e$ be the decomposition of a natural number into powers of primes. Prove that

$$SL_2(Z/N\mathbb{Z}) \cong \prod_p SL_2(\mathbb{Z}/p^e\mathbb{Z}).$$

9) Let $j(\gamma, \tau) = (c\tau + d)^{-2}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ en $\tau \in \mathbb{H}$. Show that for $\gamma, \gamma' \in SL_2(\mathbb{Z})$ one has

$$j(\gamma\gamma', \tau) = j(\gamma, \gamma'\tau)\, j(\gamma', \tau).$$

(Such a $j(\gamma, \tau)$ is called a factor of automorphy.)

10) Let f be a modular form of weight k on $SL_2(\mathbb{Z})$. Check that $\mathrm{Im}(\tau)^{k/2}|f(\tau)|$ is invariant under $\tau \mapsto (a\tau + b)/(c\tau + d)$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

11) Prove that the cusp form $\Delta = (E_4^3 - E_6^2)/1728 = \sum \tau(n) q^n$ has integer Fourier coefficients $\tau(n)$. Compute the first 10 Fourier coefficients.

12) Calculate the first 10 Fourier coefficients of $E_{12} - E_6^2$.

13 Let L be an additive subgroup of a finite-dimensional $\mathbb{R}$-vector space V. Prove the equivalence of the following statements:
i) L is discrete and generates the $\mathbb{R}$-vector space V.
ii) L is of the form $L = \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_n$ with $\omega_1, \ldots, \omega_n)$ a $\mathbb{R}$-basis of V.
iii) L is discrete and co-compact (that is, V/L is compact).

14) Check the identity

$$\sum_{n=1}^{\infty} \frac{n^k x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_k(n)\, x^n.$$

15) Prove in the following steps that the matrices S and $R = -ST$ satisfy the relations

$$S^4 = I = R^3, \qquad RS^2 = S^2 R$$

and that these relations define the group $\mathrm{SL}(2, \mathbb{Z})$.
i) Check these relations.
ii) Show that any relation can be written in the form

$$S^{a_1} R^{b_1} S R^{b_2} \cdots S R^{b_n} S^{a_{n+1}} = I$$

with $n \geq 0$, $a_1, a_{n+1} \in \{0,1,2,3\}$ and $b_i \in \{1,2\}$ and hence also in the form

$$R^{b_1} S R^{b_2} \cdots S R^{b_n} S = S^b. \qquad (*)$$

iii) Show that a matrix of the form

$$\begin{pmatrix} a & -b \\ -c & d \end{pmatrix} = R^{b_1} S R^{b_2} \cdots S R^{b_n} S$$

satisfies

$$a, b, c, d \geq 0, \quad b + c > 0 \qquad \text{or} \qquad a, b, c, d \leq 0, \quad b + c < 0.$$

Use induction, starting with RS and $R^2 S$.

iv) Show that this contradicts (*) for $n \geq 1$.

v) Conclude that $\mathrm{PSL}(2, \mathbb{Z})$ is the free product of a cyclic group of order 2 and a cyclic group of order 3.

16) Let ψ be an automorphism of $SL_2(\mathbb{Z})$. Prove i) $\psi(S)$ is conjugated with $S^{\pm 1}$; ii) $\psi(ST)$ is conjugated with $(ST)^{\pm 1}$. Conclude that the quotient group $\mathrm{Aut}(SL_2(\mathbb{Z}))/\mathrm{Inn}(SL_2(\mathbb{Z}))$ is isomorphic with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ (Klein Four group).

17) Let $f(\tau)$ be a modular function (i.e., weakly modular of weight 0). Show that the derivative $f'(\tau)$ has modular weight 2. Furthermore, let $j = E_4^3/\Delta$. Calculate the orders of the functions j, $j - 1728$ and j' in the points $\tau = i, \rho$ and $i\infty$. Deduce that the quotient

$$\frac{j'^a}{j^b\,(j - 1728)^c} \qquad (**)$$

is a modular form of weight $2a$ if and only if

$$a \geq 2, \quad 2c \leq a, \quad 3b \leq 2a, \quad b + c \geq a.$$

18) Prove that E_4 is a multiple of $j'^2/j(j-1728)$. Write Δ in the form (**).

19) Let $M_n = \{A \in \mathrm{Mat}(2 \times 2, \mathbb{Z})\}$. Show that M_n can be written as the following disjoint sum of double cosets

$$M_n = \sqcup_{\{1 \leq a, d,\, ad = n,\, a|d\}} \Gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma$$

with $\Gamma = SL_2(\mathbb{Z})$.

20) Check that the Fourier coefficients $a(n) = \sigma_{k-1}(n)$ of the Eisenstein series G_k satisfy the relations

i) $a(n)a(m) = \sum_{d|(m,n)} d^{k-1} a(nm/d^2)$.

ii) $a(p^{r+1}) = a(p)a(p^r) - p^{k-1} a(p^{r-1})$ for $r \geq 1$.

21) Show that the trace of the Hecke operator $T(n)$ on S_k is an integer.

22) Determine a basis of S_{24}. Determine a basis of common eigenforms for the Hecke operators. Which number field do the Fourier coefficients generate?

23) Same questions for S_k with $k = 26$, 28 and 30.