COMPUTER ALGEBRA:

MATH READY FOR USE

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Talen en Tools

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CONTENTS

1. Characteristics

Goal of computer algebra:

- automate mathematics
- create tools working in style:

  Input: solve(problem);
  Output: answer

A modern computer algebra system is fully integrated environment for technical computing in research and education.

Keywords:

- numerics
- symbolics
- graphics
- documentation
- programming language

– Numerics

Large integer arithmetic

```
In[1]:= 1997^11

Out[1]= 2014460302934265741997296191038900853

In[2]:= FactorInteger[50!]

Out[2]= {{2, 47}, {3, 22}, {5, 12},

>     {7, 8}, {11, 4}, {13, 3}, {17, 2},

>     {19, 2}, {23, 2}, {29, 1}, {31, 1},

>     {37, 1}, {41, 1}, {43, 1}, {47, 1}}
```

Exact results

```
In[3]:= Tan[Pi/12]

Out[3]= 2 - Sqrt[3]
```

```
In[4]:= Sum[1/((3k+1)*(3k+2)*(3k+3)*(3k+4)),

        {k,0,Infinity}]

        6 + Sqrt[3] Pi - 9 Log[3]
Out[4]= -------------------------
                   36
```

User-defined precision

```
In[5]:= N[%, 30]

Out[5]= 0.0431635415191573980340285455573
```

– **Symbolics**

Highlights:

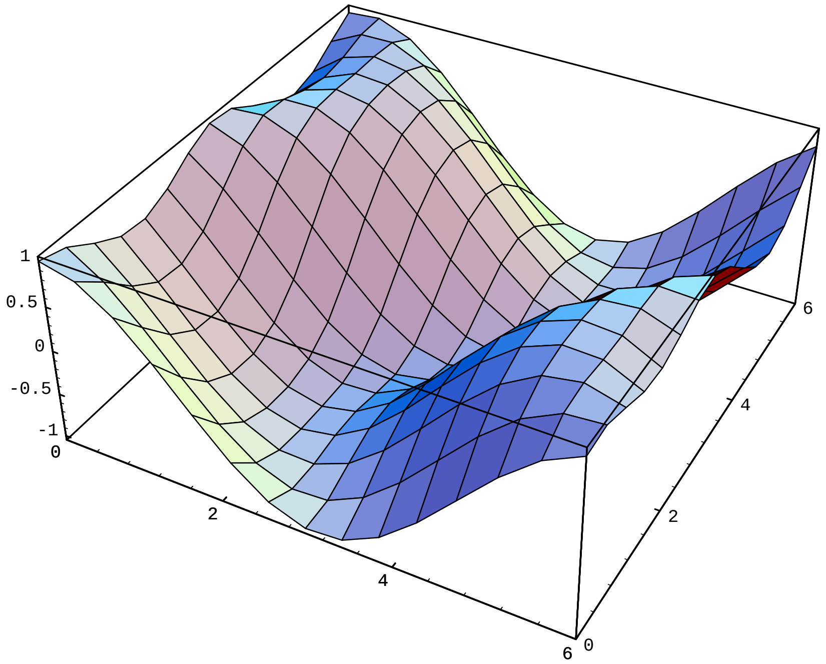
- polynomial calculus
- integration

```
In[6]:= Factor[6*x + 35*x^3 + 56*x^5 +

        36*x^7 + 10*x^9 + x^11]

                   2          2          2
Out[6]= x (1 + x ) (2 + x ) (3 + x )

                        4
>     (1 + 4 x  + x )

In[7]:= GroebnerBasis[{x^2 y - x + y,

        x y^2 - z, x^2 z + y}, {x,y,z}]

                3    5        3
Out[7]= {z + z  + z , y + z , x - z}

In[8]:= Integrate[ Sqrt[x] Sqrt[1+x], x]

                                    3/2
                       Sqrt[x]    x
Out[8]= Sqrt[1 + x] (------- + ----) -
                          4       2

>     ArcSinh[Sqrt[x]] / 4
```

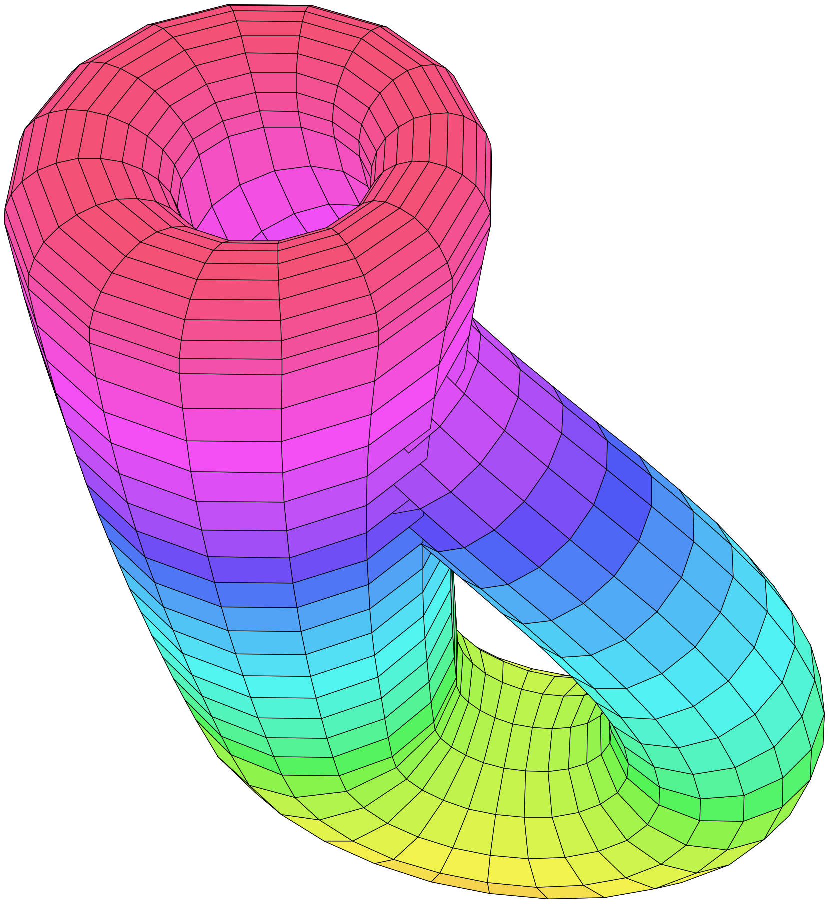
```
In[9]:= D[%, x]
                    -1
Out[9]= --------------------- +
        8 Sqrt[x] Sqrt[1 + x]

          1        3 Sqrt[x]
>    (--------- + ---------) Sqrt[1 + x] +
      8 Sqrt[x]        4

                  3/2
     Sqrt[x]     x
     ------- + ----
        4        2
>    --------------
     2 Sqrt[1 + x]

In[10]:= Simplify[%]

Out[10]= Sqrt[x] Sqrt[1 + x]
```

– Graphics

```
In[11]:= Plot3D[Cos[x+Sin[y]], {x,0,6},{y,0,6}];
```



Klein bottle:

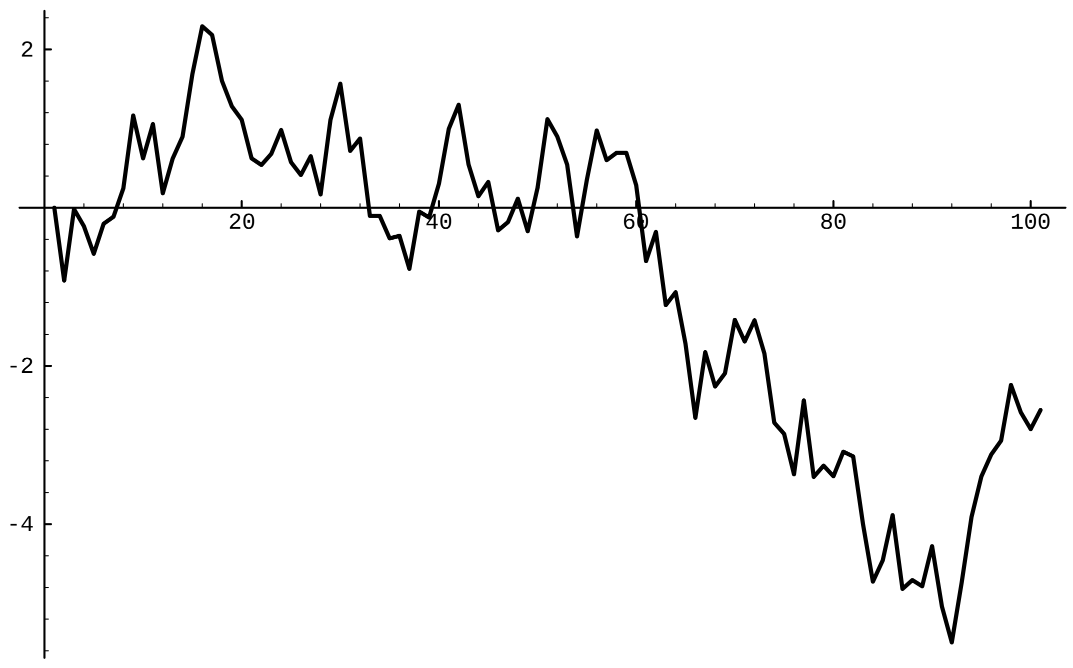


– Documentation

“The interactive book”

– Programming Language

```
In[12]:= FoldList[Plus, 0,
         Table[Random[Real,{-1,1}],{100}]];

In[13]:= ListPlot[%, PlotJoined->True];
```



Many styles:

procedural, functional, rule-based, object-oriented

2. Systems (see `http://www.can.nl`)

General purpose system: "Swiss army knife"

- lots of knowledge built-in
- designed for infinite computer
- big, slow on large problems
- lots of bells and whistles
- does everything except makeing coffee

Axiom, Derive, Macsyma, Maple, *Mathematica*, Reduce

special purpose system: "chef knife"

- only specialized mathematics built-in
- designed for real computer
- optimized for problem solving
- bare bones interfaces
- Only does what is explicitly asked for

FORM, GAP, Macaulay, ...

3. Advantages

– Ready-for-use computational tool

$$\frac{\sin\left(\frac{nz\sqrt{x^2+y^2+z^2}}{\sqrt{y^2+z^2}}\right)}{\sqrt{x^2+y^2+z^2}}$$

is a solution of the PDE

$$\left(\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)+n^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right)f=0.$$

```
> f := sin( n*z*sqrt(x^2+y^2+z^2) /
>   sqrt(y^2+z^2) ) / sqrt(x^2+y^2+z^2):
> radnormal( diff( diff(f,x$2) + diff(f,y$2)
>   + diff(f,z$2),x$2 ) +
>   + n^2 * ( diff(f,x$2) + diff(f,y$2) ) );
```

$$0$$

$$cpu\_time = 2.078\,seconds$$

– Tool for mathemtical experiments

Given a sequence of numbers

$$g_1, g_2, g_3, \cdots$$

define a sequence of f's by

$$f_0 = 1, f_n = g_n s^2 - \sum_{k=1}^{n} \binom{n}{k} f_{n-k}$$

Is there a closed formula for f_n?

```
> f[0] := 1:
> for n from 1 to 5 do
>   f[n] := expand(g[n]*s^2 -
>   sum('binomial(n,k)*f[n-k]', k=1..n));
> od;
```

$$f_1 := g_1\, s^2 - 1$$

$$f_2 := g_2\, s^2 - 2\, g_1\, s^2 + 1$$

$$f_3 := g_3\, s^2 - 3\, g_2\, s^2 + 3\, g_1\, s^2 - 1$$

$$f_4 := g_4\, s^2 - 4\, g_3\, s^2 + 6\, g_2\, s^2 - 4\, g_1\, s^2 + 1$$

$$f_5 := g_5\, s^2 - 5\, g_4\, s^2 + 10\, g_3\, s^2 - 10\, g_2\, s^2 + 5\, g_1\, s^2 - 1$$

Conjecture: $f_n = (-1)^n + s^2 \left(\sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} g_k \right)$

– Access to advanced mathematics

```
> integrate( ...., x );
```

$$\int \frac{7\,x^{13} + 10\,x^8 + 4\,x^7 - 7\,x^6 - 4\,x^3 - 4\,x^2 + 3\,x + 3}{x^{14} - 2\,x^8 - 2\,x^7 - 2\,x^4 - 4\,x^3 - x^2 + 2\,x + 1}\,dx$$

$$= \frac{1}{2}\ln(\,x^7 - \sqrt{2}\,x^2 + (\,-\sqrt{2} - 1\,)\,x - 1\,)\,\sqrt{2}$$

$$+ \frac{1}{2}\ln(\,x^7 - \sqrt{2}\,x^2 + (\,-\sqrt{2} - 1\,)\,x - 1\,)$$

$$+ \frac{1}{2}\ln(\,x^7 + \sqrt{2}\,x^2 + (\,\sqrt{2} - 1\,)\,x - 1\,)$$

$$- \frac{1}{2}\ln(\,x^7 + \sqrt{2}\,x^2 + (\,\sqrt{2} - 1\,)\,x - 1\,)\,\sqrt{2}$$

– Code generation

```
> f := Pi*ln(x^2)-sqrt(2)*ln(x^2)^2;
```

$$f := \pi \ln(x^2) - \sqrt{2} \ln(x^2)^2$$

```
> C(f);
      t0 = 0.3141592653589793E1*log(x*x)
      -sqrt(2.0)*pow(log(x*x),2.0);

> C(f, optimized);
      t1 = x*x;
      t2 = log(t1);
      t4 = sqrt(2.0);
      t5 = t2*t2;
      t7 = 0.3141592653589793E1*t2-t4*t5;

> fortran(f, optimized);
      t1 = x**2
      t2 = alog(t1)
      t5 = t2**2
      t7 = 0.3141593E1*t2-sqrt(2.E0)*t5

> latex(f);
      \pi\,\ln ({x}^{2})-\sqrt{2}
      \left(\ln({x}^{2})\right)^{2}
```

4. Limitations

– Errors of all kinds

mathematical rigor vs. usefulness

```
In[1]:= Integrate[x^a, x]

           1 + a
          x
Out[1]= ------
          1 + a
```

What if $a = -1$?

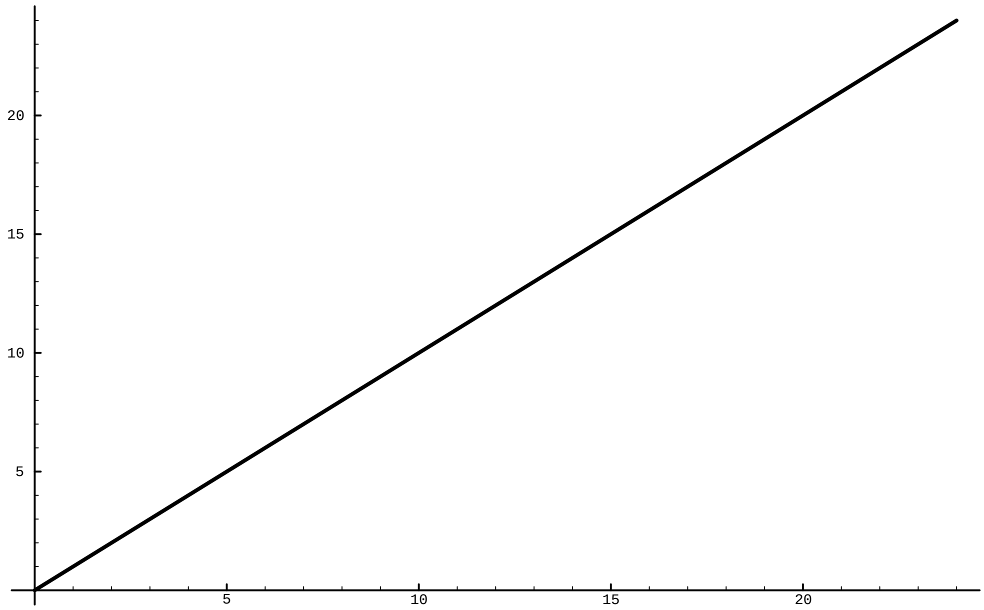
user's help needed

```
In[2]:= delta/(Sin[x]^2+(1+delta)*Cos[x]^2-1)

                          delta
Out[2]= ----------------------------------
                                  2         2
        -1 + (1 + delta) Cos[x]  + Sin[x]
```

```
In[3]:= Limit[%, delta->0]

Out[3]= 0

In[4]:=  Limit[Simplify[%%], delta->0]

                2
Out[4]= Sec[x]
```

plot aliasing

```
In[3]:= Plot[x + Sin[Pi x], {x, 0, 24}];
```



reasons/causes

- implementation mistakes
- zero recognition
- multivalued functions
- automatic simplification
- assumptions

Some limitations are fundamental:

Consider the class of expressions built from

- one variable, say x
- rational constants
- π
- function symbols $+$, $*$, sin, abs

Two expressions are equivalent iff they describe the same real function.

Theorem: For the above class of expressions the equivalence problem is undecidable.

– Intermediate Expression Swell

Example: GCD computation via Euclid's Algorithm.

$$f_1 := 7\,x^7 + 2\,x^6 - 3\,x^5 - 3\,x^3 + x + 5$$

$$f_2 := 9\,x^5 - 3\,x^4 - 4\,x^2 + 7\,x + 7$$

```
> Euclid_gcd :=
> proc( f::polynom, g::polynom, x::name )
>   local r:
>   if g = 0 then  sort( f )
>   else
>     r := sort( rem( f, g, x ) );
>     if r <> 0 then print( r ) fi;
>        Euclid_gcd( g, r, x )
>   fi
> end:

> Euclid_gcd( f[1], f[2], x ):
```

$$\frac{70}{27}\,x^4 - \frac{176}{27}\,x^3 - \frac{770}{81}\,x^2 - \frac{94}{81}\,x + \frac{503}{81}$$

$$\frac{100881}{1225}x^3 + 72x^2 - \frac{14139}{2450}x - \frac{98037}{2450}$$

$$-\frac{16726864175}{10176976161}x^2 - \frac{5255280625}{10176976161}x + \frac{19754564375}{10176976161}$$

$$\frac{35171085032244648729}{45679671041452805 0}x + \frac{66056048950873353 57}{45679671041452805 0}$$

$$\frac{2406814310427212456610119019 25}{121549387831506345564025862481}$$

So, f_1 and f_2 are relatively prime. But look at the coefficients in the computation .....

– **Unmanageable output**

– **Size limitations:**
need for better algorithms

5. Current Trends

The future of mathematics is to computer algebra

What is happening now?

- integration in education
- second wave of special purpose systems and libraries
- interactive environment for computing used in conjuction with other tools:
  - graphics in all kinds of formats: PostScript, GIF, JPG, HPGL)
  - document generation TEX, RTF, HTML
  - hyperlinks to other documents and WWW
  - links to other softwares and libraries

- integrated typesetting of documents
- enhanced manipulation of formulae

- “symbolic engine” in application software