ALGEBRA 3; REPRESENTATIE THEORIE. AANVULLING 1

In this supplement to Serre's [2, §1.5] we discuss tensor products of modules over associative algebras. More information on tensor products can for instance be found in the book "Algebra" of Lang [1, Chpt. XVI].

Definition 0.1. *An associative algebra over $\mathbb{C}$ with unit element 1 is a vector space A over $\mathbb{C}$ endowed with a $\mathbb{C}$-bilinear multiplication map*

$$A \times A \to A, \qquad (a, a') \mapsto a \cdot a'$$

(i.e. for all $a' \in A$, the maps $A \to A$ given by $a \mapsto a \cdot a'$ and $a \mapsto a' \cdot a$ are $\mathbb{C}$-linear), satisfying associativity

$$a \cdot (a' \cdot a'') = (a \cdot a') \cdot a'' \qquad \forall\, a, a', a'' \in A$$

and satisfying $1 \cdot a = a \cdot 1$ for all $a \in A$.

For convenience we will call an associative algebra A with 1 in the remainder of the text a $\mathbb{C}$-algebra. Note that a $\mathbb{C}$-algebra is in particular a unital ring.

We write $\mathrm{End}_{\mathbb{C}}(V) = \{\phi : V \to V \mid \phi \text{ linear }\}$ for the vector space of linear endomorphisms of a complex vector space V.

Example 0.2. (i) *$A = \mathrm{End}_{\mathbb{C}}(V)$ is a $\mathbb{C}$-algebra with multiplication the composition of linear endomorphisms.*
(ii) *The group algebra $A = \mathbb{C}[G]$ of a finite group G is a $\mathbb{C}$-algebra with multiplication*

$$\Big(\sum_{g\in G} \lambda_g e_g, \sum_{g'\in G} \mu_{g'} e_{g'}\Big) \mapsto \sum_{g,g'\in G} \lambda_g \mu_{g'} e_{gg'},$$

where $\lambda_g, \mu_{g'} \in \mathbb{C}$.

Definition 0.3. *Let A be a $\mathbb{C}$-algebra. A left A-module is a vector space V over $\mathbb{C}$ together with a map*

$$A \times V \to V, \qquad (a, v) \mapsto a \cdot v \tag{0.1}$$

satisfying

1. *the action map (0.1) is bilinear,*
2. *for all $a, a' \in A$ and $v \in V$,*

$$a \cdot (a' \cdot v) = (aa') \cdot v, \qquad 1 \cdot v = v.$$

A similar definition can be given for right actions. In this case we have a bilinear map

$$V \times A \to V, \qquad (v, a) \mapsto v \cdot a$$

satisfying $(v \cdot a) \cdot a' = v \cdot (aa')$ and $v \cdot 1 = v$.

Example 0.4. *Let V be a complex vector space. The map*

$$\mathrm{End}_{\mathbb{C}}(V) \times V \to V, \qquad (\phi, v) \mapsto \phi(v)$$

turns V into a left $\mathrm{End}_{\mathbb{C}}(V)$-module.

Exercise 0.5. *Let G be a finite group and V a complex vector space.*

(1) *If $\pi : G \to \mathrm{GL}(V)$ is a representation, then V is a left $\mathbb{C}[G]$-module with action given by*

$$\mathbb{C}[G] \times V \to V, \qquad (\sum_{g \in G} \lambda_g e_g, v) \mapsto \sum_{g \in G} \lambda_g \pi(g) v.$$

Check this.

(2) *Conversely, prove that a left $\mathbb{C}[G]$-action $(a, v) \mapsto a \cdot v$ ($a \in \mathbb{C}[G]$) on V gives rise to a representation $\pi : G \to \mathrm{GL}(V)$ defined by the formula $\pi(g)(v) := e_g \cdot v$ ($g \in G$ and $v \in V$).*

Suppose that V is a right A-module, W is a left A-module and U is a complex vector space. We call a map $\phi : V \times W \to U$ A-bilinear if ϕ is complex bilinear and if

$$\phi(v \cdot a, w) = \phi(v, a \cdot w), \qquad \forall\, v \in V,\ w \in W,\ a \in A. \tag{0.2}$$

We write $\mathrm{Hom}_A^{(2)}(V \times W, U)$ for the space of A-bilinear maps $V \times W \to U$.

Example 0.6. *A vector space V can be viewed as left and right $A = \mathbb{C}$-module by $(v, \lambda) \mapsto \lambda v$ and $(\lambda, v) \mapsto \lambda v$ for $\lambda \in \mathbb{C}$ and $v \in V$. In this case $\mathrm{Hom}_{\mathbb{C}}^{(2)}(V \times W, U)$ is simply the space of $\mathbb{C}$-bilinear maps $V \times W \to U$.*

Proposition 0.7. *Let A be a $\mathbb{C}$-algebra. Let V be a right A-module and W a left A-module.*

(1) *There exists a complex vector space Z and a A-bilinear map $\iota : V \times W \to Z$ satisfying the following universality property: for all A-bilinear maps $\phi : V \times W \to U$ (with U an arbitrary complex vector space) there exists a unique linear map $\overline{\phi} : Z \to U$ such that $\phi = \overline{\phi} \circ \iota$.*

(2) *If (Z', ι') is a second pair satisfying this universality property, then there exists a unique linear isomorphism $\psi : Z \xrightarrow{\sim} Z'$ satisfying $\psi \circ \iota = \iota'$.*

Proof. (1) Consider the (huge!) vector space

$$\widetilde{Z} := \bigoplus_{v \in V, w \in W} \mathbb{C} e_{(v,w)}$$

with linear basis $\{e_{(v,w)}\}_{v \in V, w \in W}$, together with the map

$$\widetilde{\iota} : V \times W \to \widetilde{Z}, \qquad \widetilde{\iota}(v, w) := e_{(v,w)}$$

for $v \in V$ and $w \in W$. Note that the image of $\widetilde{\iota}$ spans $\widetilde{Z}$. But $\widetilde{\iota}$ is not A-bilinear. We can make it A-bilinear by composing $\widetilde{\iota}$ with the canonical map $p : \widetilde{Z} \to Z := \widetilde{Z}/S$ for the vector subspace $S \subset \widetilde{Z}$ spanned by the vectors

$$\begin{aligned} &e_{(v+v',w)} - e_{(v,w)} - e_{(v',w)}, \\ &e_{(\lambda v,w)} - \lambda e_{(v,w)}, \\ &e_{(v,\lambda w)} - \lambda e_{(v,w)}, \\ &e_{(v,w+w')} - e_{(v,w)} - e_{(v,w')}, \\ &e_{(v\cdot a,w)} - e_{(v,a\cdot w)} \end{aligned}$$

for all $a \in A$, $v, v' \in V$, $w, w' \in W$ and $\lambda \in \mathbb{C}$. We write $\iota := p \circ \widetilde{\iota} : V \times W \to Z$ for the resulting A-bilinear map. Note that the image of ι spans Z. We show that the pair (Z, ι) satisfies the desired universality property.

Let $\phi \in \mathrm{Hom}_A^{(2)}(V \times W, U)$. Define a linear map $\widetilde{\phi} : \widetilde{Z} \to U$ by

$$\widetilde{\phi}(e_{(v,w)}) := \phi(v, w), \qquad \forall\, v \in V, w \in W.$$

This is well defined since the $e_{(v,w)}$ ($v \in V$, $w \in W$) form a linear basis of $\widetilde{Z}$. Since ϕ is A-bilinear, $S \subset \mathrm{Ker}(\widetilde{\phi})$, hence there exists a unique linear map $\overline{\phi} : Z \to U$ such that $\widetilde{\phi} = \overline{\phi} \circ p$. Then $\phi = \overline{\phi} \circ \iota$ since for all $v \in V$ and $w \in W$,

$$(\overline{\phi} \circ \iota)(v, w) = \overline{\phi}(e_{(v,w)} + S) = \widetilde{\phi}(e_{(v,w)}) = \phi(v, w).$$

Note that $\overline{\phi}$ is unique since the image of ι spans Z.

(b) Let (Z', ι') be a second pair satisfying the universality property. Then the image of ι' spans Z', because, if this would not be the case, then the choice of a linear map $\overline{\phi} \in \mathrm{Hom}_{\mathbb{C}}(Z', U)$ satisfying $\phi = \overline{\phi} \circ \iota'$ for a given $\phi \in \mathrm{Hom}_A^{(2)}(V \times W, U)$ would not be unique.

Applying the universality property of (Z, ι) to the A-bilinear map $\iota' : V \times W \to Z'$ gives a unique linear map $\sigma : Z \to Z'$ such that

$$\sigma \circ \iota = \iota'.$$

On the other hand, applying the universality property of (Z', ι') to the A-bilinear map $\iota : V \times W \to Z$ gives a unique linear map gives a unique linear map $\sigma' : Z' \to Z$ such that

$$\sigma' \circ \iota' = \iota.$$

The linear map $\sigma' \circ \sigma : Z \to Z$ then satisfies for all $v \in V$ and $w \in W$,

$$(\sigma' \circ \sigma)(\iota(v, w)) = \sigma'(\iota'(v, w)) = \iota(v, w).$$

Since the image of ι spans Z we conclude that $\sigma' \circ \sigma = \mathrm{Id}_Z$. Similarly, $\sigma \circ \sigma' = \mathrm{Id}_{Z'}$. In particular, σ is a linear isomorphism. Take $\psi = \sigma : Z \xrightarrow{\sim} Z'$. Then

$$\psi \circ \iota = \sigma \circ \iota = \iota'.$$

Note that ψ is the unique linear map with this property since the image of ι spans Z. $\square$

By the second part of the proposition, the pair (Z, ι) is unique up to linear isomorphism. It therefore makes sense to denote it by $V \otimes_A W$. It is called the tensor product over A of the right A-module V with the left A-module W. It comes equipped with a A-bilinear map $\iota : V \times W \to V \otimes_A W$. We write

$$v \otimes_A w := \iota(v, w), \qquad v \in V, \, w \in W.$$

The elements $v \otimes_A w$ ($v \in V$ and $w \in W$) are called the pure tensors in $V \otimes_A W$. They form a spanning set of $V \otimes_A W$, cf. the proof of the proposition. Since $\iota \in \mathrm{Hom}_A^{(2)}(V \times W, V \otimes_A W)$ we have in $V \otimes_A W$ the following convenient identities in $V \otimes_A W$,

$$\begin{aligned}
(v + v') \otimes_A w &= v \otimes_A w + v' \otimes_A w, \\
(\lambda v) \otimes_A w &= \lambda(v \otimes_A w) = v \otimes_A (\lambda w), \\
v \otimes_A (w + w') &= v \otimes_A w + v \otimes_A w', \\
(v \cdot a) \otimes_A w &= v \otimes_A (a \cdot w)
\end{aligned}$$

for $a \in A$, $v, v' \in V$, $w, w' \in W$ and $\lambda \in \mathbb{C}$.

Example 0.8. *The proposition gives for two vector spaces V and W their tensor product $V \otimes_{\mathbb{C}} W$, where V and W are viewed as $\mathbb{C}$-modules as in Example 0.6.*

Exercise 0.9. *Let V and W be finite dimensional complex vector spaces. The dual $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ of V is the complex vector space of linear functionals $V \to \mathbb{C}$. Define a complex bilinear map $\iota : V^* \times W \to \mathrm{Hom}_{\mathbb{C}}(V, W)$ by $\iota(f, w) := f(\cdot)w$ (here $f(\cdot)w$ means the linear map $V \to W$ defined by $v \mapsto f(v)w$).*

1. *Suppose $\phi : V^* \times W \to U$ is a complex bilinear map into the vector space U. Prove that there exists a unique linear map $\overline{\phi} : \mathrm{Hom}_{\mathbb{C}}(V, W) \to U$ such that $\overline{\phi} \circ \iota = \phi$.*
2. *Conclude that $V^* \otimes_{\mathbb{C}} W \simeq \mathrm{Hom}_{\mathbb{C}}(V, W)$ as vector spaces.*
3. *Let $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ be linear bases of V and W respectively (with I and J suitable finite index sets). Prove that $\{e_i \otimes_{\mathbb{C}} f_j\}_{i \in I, j \in J}$ is a linear basis of $V \otimes_{\mathbb{C}} W$ (in particular, $\dim_{\mathbb{C}}(V \otimes_{\mathbb{C}} W) = \dim_{\mathbb{C}}(V) \dim_{\mathbb{C}}(W)$).*

Let $\phi : V \to V'$ and $\psi : W \to W'$ be linear maps. Then

$$c_{\phi,\psi} : V \times W \to V' \otimes_{\mathbb{C}} W', \qquad c_{\phi,\psi}(v, w) := \phi(v) \otimes_{\mathbb{C}} \psi(w)$$

defines a $\mathbb{C}$-bilinear map $c_{\phi,\psi} \in \mathrm{Hom}_{\mathbb{C}}^{(2)}(V \times W, V' \otimes_{\mathbb{C}} W')$. Hence, by the universality property of the tensor product, there exists a unique complex linear map

$$\overline{c_{\phi,\psi}} : V \otimes_{\mathbb{C}} W \to V' \otimes_{\mathbb{C}} W'$$

such that $\overline{c_{\phi,\psi}}(v \otimes_{\mathbb{C}} w) = c_{\phi,\psi}(v, w)$ for all $v \in V$ and $w \in W$. We will denote the linear map $\overline{c_{\phi,\psi}}$ by $\phi \otimes_{\mathbb{C}} \psi$. It is called the tensor product of the linear maps ϕ and ψ. Note that

$$(\phi \otimes_{\mathbb{C}} \psi)(v \otimes_{\mathbb{C}} w) = \phi(v) \otimes_{\mathbb{C}} \psi(w)$$

for all $v \in V$ and $w \in W$.

Exercise 0.10. *Let* $\pi : G \to \mathrm{GL}(V)$ *and* $\pi' : G \to \mathrm{GL}(V')$ *be two representations of* G*. Show that*

$$(\pi \otimes \pi')(g)(v \otimes_{\mathbb{C}} v') := (\pi(g)v) \otimes_{\mathbb{C}} (\pi'(g)v')$$

for $g \in G$*,* $v \in V$ *and* $v' \in V'$ *defines a representation*

$$\pi \otimes \pi' : G \to \mathrm{GL}(V \otimes_{\mathbb{C}} V').$$

It is called the tensor product representation of π *and* π'*.*

Exercise 0.11. *Let* V *and* W *be two complex vector spaces. Prove that there exists a unique linear isomorphism* $\theta_{V,W} : V \otimes_{\mathbb{C}} W \xrightarrow{\sim} W \otimes_{\mathbb{C}} V$ *satisfying*

$$\theta_{V,W}(v \otimes_{\mathbb{C}} w) = w \otimes_{\mathbb{C}} v \qquad \forall\, v \in V,\, w \in W.$$

Suppose that V is finite dimensional complex vector space. The linear automorphism $\theta := \theta_{V,V}$ of $V \otimes_{\mathbb{C}} V$ from the previous exercise is an involution, $\theta^2 = \mathrm{Id}_{V \otimes_{\mathbb{C}} V}$. Write

$$\mathbf{Sym}^{(2)}(V) := \{X \in V \otimes_{\mathbb{C}} V \mid \theta(X) = X\},$$
$$\mathbf{Alt}^{(2)}(V) := \{X \in V \otimes_{\mathbb{C}} V \mid \theta(X) = -X\}$$

for the eigenspaces of θ corresponding to its two eigenvalues $+1$ and -1. These subspaces are called the symmetric square and alternating square of V, respectively.

Note that

$$V \otimes_{\mathbb{C}} V = \mathbf{Sym}^{(2)}(V) \oplus \mathbf{Alt}^{(2)}(V).$$

In fact, if $\{e_i\}_{1 \leq i \leq m}$ is a linear basis of V then

$$\{e_i \otimes_{\mathbb{C}} e_j + e_j \otimes_{\mathbb{C}} e_i\}_{1 \leq i \leq j \leq m}$$

is a linear basis of $\mathbf{Sym}^{(2)}(V)$ and

$$\{e_i \otimes_{\mathbb{C}} e_j - e_j \otimes_{\mathbb{C}} e_i\}_{1 \leq i < j \leq m}$$

is a linear basis of $\mathbf{Alt}^{(2)}(V)$. In particular, if $\dim_{\mathbb{C}}(V) = m$ then $\dim_{\mathbb{C}}(\mathbf{Sym}^{(2)}(V)) = m(m+1)/2$ and $\dim_{\mathbb{C}}(\mathbf{Alt}^{(2)}(V)) = m(m-1)/2$.

Exercise 0.12. *Let* $\pi : G \to \mathrm{GL}_{\mathbb{C}}(V)$ *be a representation of the finite group* G*. Show that* $\mathbf{Sym}^{(2)}(V)$ *and* $\mathbf{Alt}^{(2)}(V)$ *are* G*-invariant subspaces of* $V \otimes_{\mathbb{C}} V$ *(the latter viewed as representation space for the tensor product representation* $\pi \otimes \pi$ *of* G*).*

Exercise 0.13. *Let* V *be a finite dimensional vector space and* $\phi \in \mathrm{End}_{\mathbb{C}}(V)$ *(i.e.* ϕ *is a linear map* $V \to V$*). Let* $\{e_i\}_{i=1}^n$ *be a linear basis of* V*, and* $\{e_i^*\}_{i=1}^n$ *the corresponding dual basis of* $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ *(so* $e_i^*(e_j) = \delta_{i,j}$ *with the Kronecker delta function* $\delta_{i,j}$ *being one if* $i = j$ *and zero if* $i \neq j$*). The trace of* ϕ *is defined by*

$$\mathrm{Tr}(\phi) := \sum_{i=1}^n e_i^*(\phi(e_i)).$$

(1) *Show that* $\mathrm{Tr}(\phi)$ *is well defined (i.e. independent of the choice* $\{e_i\}_{i=1}^n$ *of linear basis of* V*).*

(2) *Prove that* $\mathrm{Tr}(\phi \circ \psi) = \mathrm{Tr}(\psi \circ \phi)$ *for all* $\phi, \psi \in \mathrm{End}_{\mathbb{C}}(V)$.

References

[1] S. Lang, *Algebra*, Graduate Texts in Mathematics, **211**. Springer-Verlag, New York, 2002.
[2] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematcs, **42**, Springer-Verlag, New York, 1977.