ALGEBRA 3; REPRESENTATIE THEORIE. AANVULLING 2

1. Homework exercise

The following exercise is the homework exercise of October 10 (hand in at latest thursday october 20 before the start of the exercise class).

Exercise 1.1. *Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group.*

(1) *Determine the conjugacy classes of Q.*
(2) *Give the one-dimensional representations of Q.*
(3) *Prove that $\widehat{Q}$ consists of four one-dimensional representations and one two-dimensional irreducible representation.*
(4) *Give the character table of Q.*

2. Introduction

In this supplement to Serre [2, §2.5] we discuss the center of the group algebra of a finite group G. We also give an alternative apprach to the canonical decomposition of a representation ([2, §2.6]).

We discuss the structure of the group algebra in detail. We relate it explicitly to the structure theory of semisimple algebras. In addition we study the Fourier transform of a finite group and its inverse. This part of the supplement is an addition to Serre [2, §6.1-§6.3].

3. Primitive idempotents in the center of the group algebra

A will always be a finite dimensional associative commutative $\mathbb{C}$-algebra with unit 1. An element $a \in A$ is called an idempotent if $a^2 = a$. Two elements $a, b \in A$ are called mutually orthogonal if $ab = 0$.

Suppose that $a, b \in A$ are mutually orthogonal idempotents. Then

$$(a+b)^2 = a^2 + b^2 + 2ab = a + b + 0 = a + b,$$

so $a + b \in A$ is also an idempotent.

Example 3.1. **(i)** *If $a \in A$ is an idempotent, then so is $1 - a$, since*

$$(1-a)^2 = 1 - 2a + a^2 = 1 - 2a + a = 1 - a.$$

The idempotents a and $1 - a$ are mutually orthogonal since $a(1-a) = a - a = 0$.
(ii) *Consider $p_\pm \in \mathbb{C}[S_n]$ given by*

$$p_+ = \frac{1}{n!} \sum_{\sigma \in S_n} e_\sigma, \qquad p_- = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) e_\sigma,$$

where $\epsilon(\sigma) \in \{\pm 1\}$ is the sign of the permutation σ. Then p_+ and p_- are mutually orthogonal idempotents in $\mathbb{C}[S_n]$. Indeed,

$$p_-^2 = \frac{1}{(n!)^2} \sum_{\sigma,\tau \in S_n} \epsilon(\sigma\tau) e_{\sigma\tau} = \frac{1}{n!} \sum_{\sigma \in S_n} \epsilon(\sigma) e_\sigma = p_-,$$

$$p_+ p_- = \frac{1}{(n!)^2} \sum_{\sigma,\tau \in S_n} \epsilon(\tau) e_{\sigma\tau} = \frac{1}{n!} \#\{\tau \in S_n \,|\, \epsilon(\tau) = 1\} p_+ - \frac{1}{n!} \#\{\tau \in S_n \,|\, \epsilon(\tau) = -1\} p_+ = 0$$

and $p_+^2 = p_+$ is immediate.

The second example suggests that the construction of mutually orthogonal idempotents in group algebras has something to do with representation theory (we have seen that $\mathbb{C}p_+$ realizes the trivial representation in the regular representation of S_n, and $\mathbb{C}p_-$ realizes the sign representation). This is indeed the case, as we shall see in a moment.

An idempotent $a \in A$ is called *primitive* if the following properties hold: $a \neq 0$ and if $a = b + c$ with $b, c \in A$ mutually orthogonal idempotents, then either $b = 0$ or $c = 0$.

Lemma 3.2. *Suppose $a, b \in A$ are primitive idempotents. Prove that $ab = 0$ iff $a \neq b$.*

Proof. We prove $ab \neq 0$ iff $a = b$.
Let $a, b \in A$ be primitive idempotents such that $ab \neq 0$. Then

$$a = ab + a(1 - b)$$

is a decomposition of a in mutually orthogonal idempotents (A is commutative!). Since a is primitive and $ab \neq 0$ we conclude that $a(1 - b) = 0$, hence $a = ab$. Reversing the role of a and b we also conclude that $b = ab$. Hence $a = b$. Thus $ab \neq 0$ implies $a = b$. Conversely, if $a = b$ then $ab = a^2 = a \neq 0$. □

Corollary 3.3. *The set $\{a_i\}_i$ of primitive idempotents of A is a finite, linear independent set.*

Proof. It is a linear independent set for if $\sum_i \lambda_i a_i = 0$ ($\lambda_i \in \mathbb{C}$, all but finitely many zero), then $0 = a_j \sum_i \lambda_i a_i = \lambda_j a_j$ for all j by the previous lemma, hence $\lambda_j = 0$ for all j. Since A is finite dimensional we conclude that $\{a_i\}_i$ is a finite set. □

If B and C are commutative associative algebras with units 1_B and 1_C over $\mathbb{C}$, then the direct sum

$$B \oplus C$$

is a commutative associative algebra with respect to the multiplication $(b, c)(b', c') = (bb', cc')$, and with unit element $1 = (1_B, 1_C)$. The two algebras B and C naturally embed as ideals in $B \oplus C$ via $b \mapsto (b, 0)$ and $c \mapsto (0, c)$ respectively. We write simply b for $(b, 0)$ ($b \in B$) and c for $(0, c)$ ($c \in C$). Note that $BC = 0$ in $B \oplus C$ (meaning $bc = 0$ for all $b \in B$ and $c \in C$).

Exercise 3.4. *Let B and C be two commutative, associative, finite dimensional unital algebras over $\mathbb{C}$. Let $\{b_j\}_j$ and $\{c_k\}_k$ be the set of primitive idempotents of B and C respectively. Show that $\{b_j\}_j \cup \{c_k\}_k$ is the set of primitive idempotents of $B \oplus C$.*

Proposition 3.5. *Let $\{a_i\}_i$ be the set of primitive idempotents in A. Then*

$$1 = \sum_i a_i.$$

Proof. With induction to $\dim(A)$. There is nothing to prove if $\dim(A) = 1$ (then 1 is the only primitive idempotent in A).

Suppose $\dim(A) > 1$. If $1 \in A$ is primitive, then it is the only primitive idempotent. For if $a \in A$ is another primitive idempotent then $0 \neq a = a \cdot 1$, hence by the above lemma $a = 1$.

Thus it suffices to prove the induction step in case $1 \in A$ is not primitive. In that case there exists $0 \neq b, c \in A$ pairwise orthogonal idempotents such that $1 = b + c$. Set $A(b) := Ab := \{ab \,|\, a \in A\}$, and similarly $A(c)$. Then $A(b), A(c) \subset A$ are unital subalgebras with unit elements b and c respectively. In addition, $A = A(b) + A(c)$ since $1 = b + c$ and $A(b) \cap A(c) = \{0\}$ since $bc = 0$. Hence

$$A = A(b) \oplus A(c) \tag{3.1}$$

as vector spaces. Since in addition $A(b)A(c) = 0$, we conclude that A is isomorphic to the direct sum of the two subalgebras $A(b)$ and $A(c)$.

Now $A(b) \neq 0 \neq A(c)$ since $b \in A(b)$ and $c \in A(c)$, hence $\dim(A(b)) < \dim(A)$ and $\dim(A(c)) < \dim(A)$. By the induction hypothesis, $b = \sum_j b_j$ and $c = \sum_k c_k$ with $\{b_j\}_j$ (resp. $\{c_k\}_k$) the set of primitive idempotents of $A(b)$ (resp. $A(c)$). Then Exercise 3.4 completes the proof. □

Lemma 3.6. *Suppose $\{a_i\}_i$ is a set of mutually orthogonal, nonzero idempotents of A. If $\{a_i\}_i$ is a linear basis of A then it is the set of primitive idempotents of A (hence, in particular, $1 = \sum_i a_i$).*

Proof. Just as in the proof of the proposition,

$$A = \bigoplus_i A(a_i)$$

as algebras, and $A(a_i) \neq 0$ since $a_i \in A(a_i)$. The extra assumption that $\{a_i\}_i$ is a linear basis of A gives $A(a_i) = \mathbb{C}a_i$, hence a_i is the only primitive idempotent of $A(a_i)$. Exercise 3.4 completes the proof. □

Recall from Exercise 5 of the extra set of exercises of October 3 that the subspace $F(G)$ of $\mathrm{Fun}(G)$ consisting of class functions is a commutative subalgebra with respect to the convolution product

$$(f * h)(z) := \sum_{\substack{x,y \in G:\\ xy = z}} f(x)h(y), \qquad z \in G$$

on $\mathrm{Fun}(G)$ ($f,h \in \mathrm{Fun}(G)$). The unit element of $\mathrm{Fun}(G)$ with respect to convolution product is $\delta_e(g) := \delta_{e,g}$ ($g \in G$).

Frobenius defined irreducible characters of a finite group G as the set of primitive idempotents of $F(G)$ with respect to convolution product. We relate now Frobenius' notion of an irreducible character to the modern definition using representation theory.

Proposition 3.7. *Define for $\pi \in \widehat{G}$ (i.e. $\pi : G \to \mathrm{GL}_{\mathbb{C}}(V_\pi)$ is an irreducible linear representation of G),*

$$\widetilde{\chi}_\pi := \frac{\dim(V_\pi)}{\#G}\overline{\chi}_\pi \in F(G),$$

where χ_π is the irreducible character associated to π and $\overline{\chi}_\pi(g) = \overline{\chi_\pi(g)}$ ($g \in G$). Then:

- **(i)** *$\{\widetilde{\chi}_\pi\}_{\pi\in\widehat{G}}$ is a linear basis of $F(G)$,*
- **(ii)** *$\delta_e = \sum_{\pi\in\widehat{G}}\widetilde{\chi}_\pi$,*
- **(iii)** *$\{\widetilde{\chi}_\pi\}_{\pi\in\widehat{G}}$ is the set of primitive idempotents of $F(G)$ with respect to the convolution product.*

Proof. **(i)** We proved this last week, see [2, §2.5, Thm. 6].
(ii) For $g \in G$,

$$\begin{aligned}\sum_{\pi\in\widehat{G}}\widetilde{\chi}_\pi(g) &= \frac{1}{\#G}\sum_{\pi\in\widehat{G}}\dim_{\mathbb{C}}(V_\pi)\overline{\chi}_\pi(g)\\ &= \frac{1}{\#G}\overline{r_G(g)} = \delta_e(g),\end{aligned}$$

where r_G is the character of the regular representation of G and the last identity follows from [2, §2.4, Prop. 5].
(iii) In view of **(i)**, **(ii)** and the previous lemma it suffices to prove that $\{\widetilde{\chi}_\pi\}_{\pi\in\widehat{G}}$ is a set of mutually orthogonal idempotents, i.e. that

$$\widetilde{\chi}_\pi * \widetilde{\chi}_{\pi'} = \delta_{\pi,\pi'}\widetilde{\chi}_\pi \tag{3.2}$$

for $\pi,\pi' \in \widehat{G}$. For this we use the following slight generalization

$$\frac{1}{\#G}\sum_{x\in G}\chi_\pi(x)\overline{\chi_{\pi'}(xz^{-1})} = \frac{\delta_{\pi,\pi'}}{\dim_{\mathbb{C}}(V_\pi)}\chi_\pi(z), \qquad z\in G,\ \pi,\pi'\in\widehat{G} \tag{3.3}$$

of Exercise 3 of the extra set of exercises of October 3, which can be proved using the orthogonality relations for the matrix coefficients of the irreducible linear G-representations (cf. also Exercise 3.8). But then (3.3) implies for all $z \in G$,

$$\begin{aligned}\big(\widetilde{\chi}_\pi * \widetilde{\chi}_{\pi'}\big)(z) &= \sum_{x\in G}\widetilde{\chi}_\pi(zx^{-1})\widetilde{\chi}_{\pi'}(x)\\ &= \frac{\dim_{\mathbb{C}}(V_\pi)\dim_{\mathbb{C}}(V_{\pi'})}{(\#G)^2}\overline{\sum_{x\in G}\chi_{\pi'}(x)\overline{\chi_\pi(xz^{-1})}}\\ &= \delta_{\pi,\pi'}\widetilde{\chi}_\pi(z),\end{aligned}$$

where we used that $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$ in the second equality. $\square$

Exercise 3.8. **(i)** *For* $\pi \in \widehat{G}$ *let* $\{\pi_{ij}\}_{i,j}$ *be the matrix coefficients with respect to an orthonormal basis of* V_π *(where we have endowed* V_π *with a scalar product turning* π *into a unitary representation). Set*

$$\widetilde{\pi}_{ij} := \frac{\dim(V_\pi)}{\#G}\overline{\pi}_{ij}. \tag{3.4}$$

Prove that

$$\widetilde{\pi}_{ij} * \widetilde{\pi}'_{kl} = \delta_{\pi,\pi'}\delta_{j,k}\widetilde{\pi}_{il}$$

for $\pi, \pi' \in \widehat{G}$.
(ii) *Derive* (3.2) *as consequence of* **(i)**.

4. The canonical decomposition

Exercise 5 of the extra set of exercises of October 3 shows that

$$f \mapsto \psi_f := \sum_{g\in G} f(g)e_g$$

defines an isomorphism $\psi : \mathrm{Fun}(G) \xrightarrow{\sim} \mathbb{C}[G]$ of algebras (with the convolution product on $\mathrm{Fun}(G)$). It restricts to an isomorphism of commutative algebras

$$\psi : F(G) \xrightarrow{\sim} Z(\mathbb{C}[G]),$$

where $Z(\mathbb{C}[G])$ is the center of the group algebra $\mathbb{C}[G]$. We write for $\pi \in \widehat{G}$,

$$p_\pi := \psi_{\widetilde{\chi}_\pi} = \frac{\dim_{\mathbb{C}}(V_\pi)}{\#G}\sum_{g\in G}\overline{\chi_\pi(g)}e_g \in Z(\mathbb{C}[G]).$$

Proposition 3.7 now immediately gives

Corollary 4.1. **(i)** $\{p_\pi\}_{\pi\in\widehat{G}}$ *is a linear basis of* $Z(\mathbb{C}[G])$,
(ii) $e_e = \sum_{\pi\in\widehat{G}} p_\pi$ *with* $e_e \in \mathbb{C}[G]$ *the unit element,*
(iii) $\{p_\pi\}_{\pi\in\widehat{G}}$ *is the set of primitive orthogonal idempotents of* $Z(\mathbb{C}[G])$.

Lemma 4.2. **(i)** *Let* $\pi \in \widehat{G}$ *and write* V_π *for its representation space. Then*

$$p_\pi|_{V_{\pi'}} = \delta_{\pi,\pi'}\mathrm{id}_{V_{\pi'}}$$

for all $\pi' \in \widehat{G}$.
(ii) *Let* $\sigma : G \to \mathrm{GL}_{\mathbb{C}}(V)$ *be a finite dimensional linear* G*-representation. Then there exists a unique decomposition*

$$V = \bigoplus_{\pi\in\widehat{G}} V(\pi) \tag{4.1}$$

in G*-invariant subspaces such that, for all* $\pi \in \widehat{G}$, $(\chi_{V(\pi)} \,|\, \chi_{\pi'}) = 0$ *for* $\pi' \in \widehat{G}$ *unless* $\pi' = \pi$ *(*$V(\pi)$ *is called the* π*-isotypical component of* σ*. The decomposition* (4.1) *is called*

the canonical decomposition of V).
(iii) *For a representation $\sigma : G \to \mathrm{GL}(V)$ we have $V(\pi) = p_\pi V$ for all $\pi \in \widehat{G}$ (here we use the fact that the representation space V inherits the structure of a $\mathbb{C}[G]$-module).*

Proof. Part **(i)** is an immediate consequence of [2, §2.5, Prop. 6]. By Corollary 4.1, we have a direct sum decomposition

$$V = \bigoplus_{\pi \in \widehat{G}} p_\pi V,$$

and $p_\pi V \subseteq V$ are G-invariant since p_π lies in the center of the group algebra. If $V \simeq \bigoplus_{\pi' \in \widehat{G}} d_{\pi'} V_{\pi'}$ then $p_\pi V \simeq d_\pi V_\pi$ by **(i)**, hence

$$\big(\chi_{p_\pi V} \,|\, \chi_{\pi'}\big) = d_\pi \big(\chi_\pi \,|\, \chi_{\pi'}\big) = \delta_{\pi,\pi'} d_\pi$$

for all $\pi' \in \widehat{G}$. Thus $V = \bigoplus_{\pi \in \widehat{G}} p_\pi V$ is a decomposition satisfying the properties as stated in **(ii)**. It thus remains to prove the uniqueness. Suppose

$$V = \bigoplus_{\pi \in \widehat{G}} V(\pi)$$

is a second decomposition with $V(\pi) \subseteq V$ G-invariant subspaces such that $\big(\chi_{V(\pi)} \,|\, \chi_{\pi'}\big) = 0$ if $\widehat{G} \ni \pi' \neq \pi$. The last condition implies $V(\pi) \simeq d_\pi V_\pi$ for all $\pi \in \widehat{G}$, hence $p_\pi V(\pi') = \delta_{\pi,\pi'} V(\pi')$ for all $\pi, \pi' \in \widehat{G}$ by **(i)**. But then

$$V(\pi) = \sum_{\pi' \in \widehat{G}} p_\pi V(\pi') = p_\pi V$$

for all $\pi \in \widehat{G}$. □

Lemma 4.3. *Let $\pi \in \widehat{G}$ and $\sigma : G \to \mathrm{GL}(V)$ a finite dimensional linear G-representation. Then*

$$V(\pi) = \sum_W W$$

with the sum over G-invariant subspaces $W \subseteq V$ such that $W \simeq V_\pi$ as G-representations.

Proof. Since $V(\pi) \simeq d_\pi V_\pi$, the inclusion $\subseteq$ is clear. Conversely, if $W \subseteq V$ is a G-invariant subspace such that $W \simeq V_\pi$, then $p_\pi|_W = \mathrm{id}_W$, hence $W = p_\pi W \subseteq p_\pi V = V(\pi)$. □

5. The Fourier transform of a finite group

As a special case of the canonical decomposition we have for the regular representation $\rho : G \to \mathrm{End}(\mathbb{C}[G])$,

$$\mathbb{C}[G] = \bigoplus_{\pi \in \widehat{G}} A(\pi) \tag{5.1}$$

with $A(\pi) = p_\pi\mathbb{C}[G] \simeq \dim(V_\pi)V_\pi$ given by

$$A(\pi) = \sum_{I_\pi} I_\pi$$

with the sum over left ideals $I_\pi \subseteq \mathbb{C}[G]$ such that $I_\pi \simeq V_\pi$ as left $\mathbb{C}[G]$-module.

Lemma 5.1. *Let $\pi, \pi' \in \widehat{G}$, then*

- **(i)** *$A(\pi) \subseteq \mathbb{C}[G]$ is a subalgebra with unit p_π.*
- **(ii)** *$A(\pi)A(\pi') = \{0\}$ if $\pi \neq \pi'$.*

In other words, (5.1) *is a direct sum of algebras.*

Proof. Since $A(\pi) = p_\pi\mathbb{C}[G]$ for $\pi \in \widehat{G}$ this follows from the fact that the p_π ($\pi \in \widehat{G}$) are mutually orthogonal idempotents in the center $Z(\mathbb{C}[G])$ of the group algebra $\mathbb{C}[G]$. □

Theorem 5.2. *For all $\pi \in \widehat{G}$ we have*

$$A(\pi) \simeq \mathrm{End}(V_\pi)$$

as algebras.

Proof. Note that $\dim(A(\pi)) = \dim(V_\pi)^2 = \dim(\mathrm{End}(V_\pi))$ since $A(\pi) \simeq \dim(V_\pi)V_\pi$. Hence it suffices to show that the algebra map

$$A(\pi) \to \mathrm{End}(V_\pi), \qquad a \mapsto a|_{V_\pi},$$

with $a|_{V_\pi}$ the action of $a \in A(\pi)$ on the representation space V_π, is surjective. Since $A(\pi')$ acts as zero on V_π if $\pi' \neq \pi$ it suffices to show that for each $f \in \mathrm{End}(V_\pi)$ there exists an $a \in \mathbb{C}[G]$ such that $f(v) = a \cdot v$ for all $v \in V_\pi$. This is a special case of the theorem below. □

Theorem 5.3 (Special case of the density theorem)**.** *Let $\pi \in \widehat{G}$. For each $f \in \mathrm{End}_{\mathbb{C}}(V_\pi)$ there exists an $a \in \mathbb{C}[G]$ such that $f(v) = a \cdot v$ for all $v \in V_\pi$.*

Proof. We give two proofs. The first proof is quick and uses the orthogonality relations of the matrix coefficients of π. It has the disadvantage though that it does not generalize to the more general setup of semisimple algebras. The second proof, which is based on abstract representation theoretic arguments, does generalize to this setting.

First proof: Let $F_\pi : \mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V_\pi)$ be given by $F_\pi(a) := a|_{V_\pi}$ ($a \in \mathbb{C}[G]$). Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of V_π (with respect to the scalar product on V_π turning π into an unitary representation). Then $e_i^*(\cdot)e_j$ ($1 \leq i, j \leq n$) is a linear basis of $\mathrm{End}_{\mathbb{C}}(V_\pi)$ and for $g \in G$,

$$F_\pi(g) = \sum_{i,j=1}^n e_j^*(\pi(g)e_i)e_i^*(\cdot)e_j = \sum_{i,j=1}^n \pi_{ji}(g)e_i^*(\cdot)e_j.$$

Suppose F_π is not surjective. Then there exists a nonzero linear functional η on $\mathrm{End}_{\mathbb{C}}(V_\pi)$ which vanishes on the image of F_π. The linear functional η is characterized by a choice of complex numbers $\lambda_{ij} \in \mathbb{C}$, not all zero, such that $\eta(e_i^*(\cdot)e_j) = \lambda_{ij}$ for all $1 \leq i, j \leq n$. But then for all $g \in G$,

$$0 = \eta(F_\pi(g)) = \sum_{i,j=1}^{n} \lambda_{ij}\pi_{ji}(g),$$

which contradicts the fact that the π_{ij} are linearly independent in $\mathrm{Fun}(G)$ (the linear independence of the π_{ij} is an immediate consequence of the orthogonality relations of the π_{ij}).
Second proof: Fix a linear basis $\{v_1, \ldots, v_n\}$ of V_π and consider the finite dimensional linear G-representation

$$E := V_\pi \oplus V_\pi \oplus \cdots \oplus V_\pi \tag{5.2}$$

(n summands). Define for $1 \leq i, j \leq n$ intertwiners $e_{ij} \in \mathrm{End}^{(G)}(E)$ by

$$e_{ij}(u_1, \ldots, u_n) := u_i^{(j)}, \qquad u_j \in V_\pi$$

where $u_i^{(j)}$ is the r-vector with jth entry u_i and zeros everywhere else. We now use the following

Lemma 5.4. $\mathrm{End}^{(G)}(E) = \bigoplus_{i,j=1}^{n} \mathbb{C}e_{ij}$.

Proof. For $1 \leq i \leq n$ define $\iota_i \in \mathrm{Hom}^{(G)}(V_\pi, E)$ and $p_i \in \mathrm{Hom}^{(G)}(E, V_\pi)$ by

$$\iota_i(u) = u^{(i)}, \qquad u \in V_\pi,$$
$$p_i(u_1, \ldots, u_n) = u_i, \qquad u_j \in V_\pi.$$

Let $h \in \mathrm{End}^{(G)}(E)$. Then for $1 \leq i, j \leq n$ we have $h_{ij} := p_j \circ h \circ \iota_i \in \mathrm{End}^{(G)}(V_\pi)$, hence it equals $\lambda_{ij}\mathrm{id}_{V_\pi}$ for some $\lambda_{ij} \in \mathbb{C}$ by Schur's lemma. But then

$$\begin{aligned} h(u_1, \ldots, u_n) &= \sum_{i=1}^{n} h(u_i^{(i)}) \\ &= \sum_{i=1}^{n} (h \circ \iota_i)(u_i) \\ &= \sum_{i,j=1}^{n} (h_{ij}(u_i))^{(j)} \\ &= \sum_{i,j=1}^{n} \lambda_{ij} u_i^{(j)} \\ &= \sum_{i,j=1}^{n} \lambda_{ij} e_{ij}(u_1, \ldots, u_n). \end{aligned}$$

Hence $h \in \sum_{i,j=1}^{n} \mathbb{C}e_{ij}$. Finally, $\{e_{ij}\}_{i,j=1}^{n}$ is a linear independent set in $\mathrm{End}^{(G)}(E)$. This follows from the fact that

$$p_s \circ e_{ij} \circ \iota_r = \delta_{i,r}\delta_{j,s}\mathrm{id}_{V_\pi}, \qquad 1 \leq i, j, r, s \leq n,$$

which in turn follows from a direct computation,

$$\begin{aligned}(p_s \circ e_{ij} \circ \iota_r)(u) &= (p_s \circ e_{ij})(u^{(r)})\\ &= \delta_{i,r} p_s(u^{(j)})\\ &= \delta_{i,r}\delta_{j,s} u.\end{aligned}$$

□

We continue with the proof of the special case of the density theorem. Fix $f \in \mathrm{End}(V_\pi)$. Define $h \in \mathrm{End}_{\mathbb{C}}(E)$ by

$$h(u_1, \ldots, u_n) = ((f(u_1), \ldots, f(u_n)).$$

Then for all $1 \leq i, j \leq n$,

$$\begin{aligned}h(e_{ij}(u_1, \ldots, u_n)) &= h(u_i^{(j)})\\ &= f(u_i)^{(j)}\\ &= e_{ij}(h(u_1, \ldots, u_n)),\end{aligned}$$

hence $h \circ \xi = \xi \circ h$ for all $\xi \in \mathrm{End}^{(G)}(E)$. Set $x := (v_1, \ldots, v_n) \in E$. Then

$$\mathbb{C}[G]x \subseteq E$$

is a G-invariant subspace, hence there exists a G-invariant subspace $F \subseteq E$ such that $E = \mathbb{C}[G]x \oplus F$. Let $\xi \in \mathrm{End}^{(G)}(E)$ be the projection onto $\mathbb{C}[G]x$ along this decomposition. Then

$$\begin{aligned}(f(v_1), \ldots, f(v_n)) &= h(x)\\ &= h(\xi(x))\\ &= \xi(h(x)) \in \mathbb{C}[G]x,\end{aligned}$$

showing that there exists an $a \in \mathbb{C}[G]$ such that

$$(a \cdot v_1, \ldots, a \cdot v_n) = a \cdot x = h(x) = (f(v_1), \ldots, f(v_n)),$$

hence $f = a|_{V_\pi} = F_\pi(a)$ by linearity. □

Corollary 5.5. *The algebra $\mathbb{C}[G]$ is isomorphic to the direct sum algebra $\bigoplus_{\pi \in \widehat{G}} \mathrm{End}_{\mathbb{C}}(V_\pi)$. The isomorphism $F : \mathbb{C}[G] \xrightarrow{\sim} \bigoplus_{\pi \in \widehat{G}} \mathrm{End}_{\mathbb{C}}(V_\pi)$ is explicitly given by*

$$F(a) := \big(F_\pi(a)\big)_{\pi \in \widehat{G}}$$

with $F_\pi(a) := a|_{V_\pi} \in \mathrm{End}_{\mathbb{C}}(V_\pi)$ for $a \in \mathbb{C}[G]$ (in particular, $F_\pi(e_g) = \pi(g)$ for $g \in G$ and $\pi \in \widehat{G}$). The algebra map F is called the Fourier transform of the finite group G.

Exercise 5.6. **(i)** *Let V be a finite dimensional complex vector space. Show that*

$$Z(\mathrm{End}_{\mathbb{C}}(V)) = \mathbb{C}\mathrm{id}_V.$$

(ii) *Combine* **(i)** *and Corollary 5.5 to give another proof that $\{\widetilde{\chi}_\pi\}_{\pi\in\widehat{G}}$ is the set of primitive idempotents of $Z(\mathbb{C}[G])$.*

Theorem 5.7 (Inversion formula)**.** *The inverse of the algebra isomorphism $F : \mathbb{C}[G] \xrightarrow{\sim} \bigoplus_{\pi\in\widehat{G}} \mathrm{End}_{\mathbb{C}}(V_\pi)$ is explicitly given by*

$$F^{-1}(f) = \sum_{g\in G}\Big(\sum_{\pi\in\widehat{G}} \frac{\dim(V_\pi)}{\#G}\mathrm{Tr}_{V_\pi}(\pi(g^{-1})f_\pi)\Big)e_g \tag{5.3}$$

for $f = (f_\pi)_{\pi\in\widehat{G}}$ with $f_\pi \in \mathrm{End}_{\mathbb{C}}(V_\pi)$.

Proof. Since the right hand side of (5.3) is linear in f it suffices to prove (5.3) for $f = F(e_x)$ ($x \in G$ arbitrary). Then

$$f_\pi = F_\pi(e_x) = \pi(x), \qquad \pi \in \widehat{G}.$$

In that case the right hand side of (5.3) thus becomes

$$\begin{aligned}
\sum_{g\in G}\Big(\sum_{\pi\in\widehat{G}} \frac{\dim(V_\pi)}{\#G}\mathrm{Tr}_{V_\pi}(\pi(g^{-1}x))\Big)e_g &= \sum_{g\in G}\Big(\sum_{\pi\in\widehat{G}} \frac{\dim(V_\pi)}{\#G}\chi_\pi(g^{-1}x)\Big)e_g\\
&= \frac{1}{\#G}\sum_{g\in G} r_G(g^{-1}x)e_g\\
&= \sum_{g\in G}\delta_{g,x}e_g = e_x,
\end{aligned}$$

which yields the desired result. □

For a finite group G, write $(\mathrm{Fun}(G), *)$ for the associative algebra of complex valued functions on G with respect to convolution product and $(\mathrm{Fun}(G), \cdot)$ for the commutative associative algebra of complex valued functions on G with pointwise product.

Exercise 5.8 (Plancherel formula)**.** *Let $u, v \in \mathbb{C}[G]$ and write u and v as $u = \sum_{x\in G} a(x)e_x$ and $v = \sum_{y\in G} b(y)e_y$ with $a(\cdot), b(\cdot) \in \mathrm{Fun}(G)$. Show that*

$$\#G\sum_{x\in G} a(x^{-1})b(x) = \sum_{\pi\in\widehat{G}} \dim(V_\pi)\mathrm{Tr}_{V_\pi}\big(uv|_{V_\pi}\big).$$

Corollary 5.9 (Discrete Fourier transform)**.** *For $a \in \mathrm{Fun}(\mathbb{Z}/n\mathbb{Z})$ define $\widehat{a} \in \mathrm{Fun}(\mathbb{Z}/n\mathbb{Z})$ by*

$$\widehat{a}(\overline{r}) = \sum_{\overline{s}\in\mathbb{Z}/n\mathbb{Z}} a(\overline{s})e^{2\pi i rs/n}.$$

The discrete Fourier transform $\widehat{\ }: (\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}), *) \xrightarrow{\sim} (\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}), \cdot)$, $a \mapsto \widehat{a}$, *is an isomorphism of algebras, and*

$$a(\overline{s}) = \frac{1}{n} \sum_{\overline{r} \in \mathbb{Z}/n\mathbb{Z}} \widehat{a}(\overline{r}) e^{-2\pi i r s/n} \qquad \forall\, a \in \mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}).$$

(Fourier inversion).

Proof. The irreducible representations of $\mathbb{Z}/n\mathbb{Z}$ are one-dimensional since $\mathbb{Z}/n\mathbb{Z}$ is abelian. We can thus identify $\widehat{\mathbb{Z}/n\mathbb{Z}}$ with the set $\{\chi_{\overline{r}}\}_{\overline{r} \in \mathbb{Z}/n\mathbb{Z}}$ of irreducible characters of $\mathbb{Z}/n\mathbb{Z}$, which are given by $\chi_{\overline{r}}(\overline{s}) = e^{2\pi i r s/n}$.

Recall the algebra isomorphism $\psi : (\mathrm{Fun}(\mathbb{Z}/n\mathbb{Z}), *) \xrightarrow{\sim} \mathbb{C}[\mathbb{Z}/n\mathbb{Z}]$ given by $\psi(a) = \sum_{\overline{r} \in \mathbb{Z}/n\mathbb{Z}} a(\overline{r}) e_{\overline{r}}$. We also have an isomorphism

$$\phi : \bigoplus_{\pi \in \widehat{\mathbb{Z}/n\mathbb{Z}}} \mathrm{End}_{\mathbb{C}}(V_\pi) \xrightarrow{\sim} (\mathrm{Fun}(G), \cdot)$$

of algebras as follows. An element $b \in \bigoplus_{\pi \in \widehat{\mathbb{Z}/n\mathbb{Z}}} \mathrm{End}_{\mathbb{C}}(V_\pi)$ is in fact a choice $b = (b_{\overline{r}})_{\overline{r} \in \mathbb{Z}/n\mathbb{Z}}$ of complex numbers $b_{\overline{r}}$, with $b_{\overline{r}} \in \mathbb{C}$ the component in the one-dimensional subalgebra $\mathrm{End}_{\mathbb{C}}(V_{\chi_{\overline{r}}}) \simeq \mathbb{C}$. Then $\phi(b)$ is defined to be the function on $\mathbb{Z}/n\mathbb{Z}$ mapping $\overline{r}$ to $b_{\overline{r}}$ for all $\overline{r} \in \mathbb{Z}/n\mathbb{Z}$.

Let $F : \mathbb{C}[\mathbb{Z}/n\mathbb{Z}] \xrightarrow{\sim} \bigoplus_{\pi \in \widehat{\mathbb{Z}/n\mathbb{Z}}} \mathrm{End}_{\mathbb{C}}(V_\pi)$ be the Fourier transform of $\mathbb{Z}/n\mathbb{Z}$. Then we get an algebra isomorphism

$$\widehat{\ } := \phi \circ F \circ \psi : (\mathrm{Fun}(G), *) \xrightarrow{\sim} (\mathrm{Fun}(G), \cdot), \qquad a \mapsto \widehat{a}.$$

Writing out the explicit formulas we get for $a \in \mathrm{Fun}(\mathbb{Z}/n\mathbb{Z})$,

$$\begin{aligned}
\widehat{a}(\overline{r}) &= F_{\chi_{\overline{r}}}(\psi(a)) \\
&= F_{\chi_{\overline{r}}}\Big(\sum_{\overline{s} \in \mathbb{Z}/n\mathbb{Z}} a(\overline{s}) e_{\overline{s}}\Big) \\
&= \sum_{\overline{s} \in \mathbb{Z}/n\mathbb{Z}} a(\overline{s}) e^{2\pi i r s/n},
\end{aligned}$$

which coincides with the definition of $\widehat{a}$ as given in the statement of the corollary. For the inversion formula, write $f = \psi(a) = \sum_{\overline{s}\in\mathbb{Z}/n\mathbb{Z}} a(\overline{s})e_{\overline{s}}$, then, by the explicit formula for F^{-1},

$$\begin{aligned}\sum_{\overline{s}\in\mathbb{Z}/n\mathbb{Z}} a(\overline{s})e_{\overline{s}} &= f \\ &= F^{-1}(F(f)) \\ &= \sum_{\overline{s}\in\mathbb{Z}/n\mathbb{Z}} \Big(\sum_{\overline{r}\in\mathbb{Z}/n\mathbb{Z}} \frac{1}{n}\chi_{\overline{r}}(-\overline{s})F_{\chi_{\overline{r}}}(f)\Big)e_{\overline{s}} \\ &= \sum_{\overline{s}\in\mathbb{Z}/n\mathbb{Z}} \frac{1}{n}\Big(\sum_{\overline{r}\in\mathbb{Z}/n\mathbb{Z}} \widehat{a}(\overline{r})e^{-2\pi irs/n}\Big)e_{\overline{s}},\end{aligned}$$

hence, for all $\overline{s}\in\mathbb{Z}/n\mathbb{Z}$,

$$a(\overline{s}) = \frac{1}{n}\sum_{\overline{r}\in\mathbb{Z}/n\mathbb{Z}} \widehat{a}(\overline{r})e^{-2\pi irs/n}.$$

□

We have the following generalization of Corollary 5.9:

Theorem 5.10. *Let A be a finite abelian group. Since A is abelian, the irreducible representations of A are one-dimensional, hence $\widehat{A}$ can (and will) be identified with the set of irreducible characters of A. Define for $f\in\mathrm{Fun}(A)$ its Fourier transform $\widehat{f}\in\mathrm{Fun}(\widehat{A})$ by*

$$\widehat{f}(\chi) := \sum_{a\in A} f(a)\chi(a).$$

Then $\widehat{\ }: (\mathrm{Fun}(A), *) \xrightarrow{\sim} (\mathrm{Fun}(A),\cdot)$ is an isomorphism of algebras and we have the inversion formula

$$f(a) = \frac{1}{\#A}\sum_{\chi\in\widehat{A}} \widehat{f}(\chi)\chi(a^{-1})$$

for all $f\in\mathrm{Fun}(A)$ and $a\in A$.

Exercise 5.11. *Prove Theorem 5.10.*

6. Semisimple algebras

If B is a finite dimensional, associative, unital algebra over $\mathbb{C}$ then there are the obvious notions of a B-submodule, intertwiners between B-modules, and irreducible B-modules. The regular representation of B is B itself, viewed as left B-module using the multiplication in B. The submodules of B are precisely the left ideals of B. The irreducible submodules of B are called the simple left ideals of B. Note that a left ideal $I\subseteq B$ is automatically a vector subspace (since B is unital, hence I is closed under multiplication by $\mathbb{C}1$, i.e. it is closed under scalar multiplication).

In case $B = \mathbb{C}[G]$ is the group algebra of a finite group G, a left $\mathbb{C}[G]$-module is the same as a linear representation of G, and $\mathbb{C}[G]$-submodules, intertwiners and irreducible modules are subrepresentations, intertwiners and irreducible linear G-representations respectively.

Definition 6.1. *A finite dimensional, associative, unital algebra B over $\mathbb{C}$ is called semisimple if B is the sum of its simple left ideals.*

In particular, the group algebra $\mathbb{C}[G]$ of a finite group G is a semisimple algebra. There is an important structure theorem for semisimple algebras due to Wedderburn generalizing Lemma 5.1, Theorem 5.2 and Corollary 5.5.

Theorem 6.2. *Let B be a finite dimensional, associative, semisimple, unital algebra over $\mathbb{C}$. Then*

$$B \simeq \mathrm{End}_{\mathbb{C}}(V_1) \oplus \cdots \oplus \mathrm{End}_{\mathbb{C}}(V_r)$$

as algebras for some finite dimensional complex vector spaces V_i. Such an isomorphism can be realized as follows. Let $\{I_i\}_i$ be representatives of the isomorphism classes of simple left ideals of B. This set is finite. Set

$$B(i) := \sum_I I$$

with the sum over left ideals $I \subseteq B$ isomorphic to I_i. Then $B(i) \subseteq B$ is a two-sided ideal, $B(i)B(i') = 0$ if $i \neq i'$ and

$$B = \bigoplus_i B(i)$$

as algebras (this is the canonical decomposition of the regular representation in case $B = \mathbb{C}[G]$). Furthermore, $B(i) \simeq \mathrm{End}_{\mathbb{C}}(I_i)$ as algebras, with map given by

$$b \mapsto b|_{I_i}, \qquad b \in B(i).$$

Proof. We do not give the proof of the theorem. It follows closely the line of arguments which we used for the group algebra in the previous section (for the density theorem, using a slightly adjusted version of the second proof of Theorem 5.3). For details see, e.g., [1, Chpt. XVII]. □

Example 6.3. *The subalgebra*

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$$

of the algebra of 2×2 complex-valued matrices is not semisimple. One can for instance easily show that B is not the sum of simple ideals. Or, if B would be semisimple, then it would be a direct sum of endomorphism spaces. Since B is three dimensional the only possibility would be $B \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. But B is not commutative, which gives the contradiction.

References

[1] S. Lang, *Algebra*, Graduate Texts in Mathematics **211**, Springer Verlag.
[2] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics, **42**, Springer-Verlag, New York, 1977.