# ALGEBRA 3; REPRESENTATIE THEORIE. AANVULLING 3

## 1. Introduction

In this supplement to Serre [1, §3.3 & §7.1] we give an alternative construction of induced representations using tensor products over associative algebras (see the first supplement to the course for a detailed discussion of tensor products).

## 2. Induced representations

Let $G$ be a group and $H$ a subgroup. Let $\pi : G \to \mathrm{GL}_{\mathbb{C}}(V)$ be a linear representation of $G$. The restriction of $\pi$ to $H$, denoted by $\mathrm{Res}_H^G(\pi)$ is the representation

$$\mathrm{Res}_H^G(\pi) : H \to \mathrm{GL}_{\mathbb{C}}(V)$$

of $H$ defined by $\mathrm{Res}_H^G(\pi) := \pi|_H$.

Starting with an irreducible representation $\pi$, it may well happen that $\mathrm{Res}_H^G(\pi)$ is reducible. Take for instance $H = \{e\}$! More subtle examples arise as follows. If $\pi_i : G \to \mathrm{GL}_{\mathbb{C}}(V_i)$ $(i = 1, 2)$ are two irreducible representations of $G$, then we have seen that the product representation $\pi := \pi_1 \otimes \pi_2 : G \times G \to \mathrm{GL}_{\mathbb{C}}(V_1 \otimes_{\mathbb{C}} V_2)$ is irreducible, but $\mathrm{Res}_G^{G\times G}(\pi)$ may be reducible. Here $G$ is viewed as the diagonal subgroup of $G \times G$ via the group embedding $G \hookrightarrow G \times G$ given by $g \mapsto (g, g)$ for all $g \in G$.

We also write $\mathrm{Res}_H^G$ for the linear map $F(G) \to F(H)$ defined by

$$\mathrm{Res}_H^G(\chi) := \chi|_H, \qquad \chi \in F(G)$$

(in other words, it is restricting a class function on $G$ to $H$). If $\chi_\pi \in F(G)$ is the character of the linear $G$-representation $\pi$, then $\mathrm{Res}_H^G(\chi_\pi)$ is the character of $\mathrm{Res}_H^G(\pi)$,

$$\mathrm{Res}_H^G(\chi_\pi) = \chi_{\mathrm{Res}_H^G(\pi)}.$$

In this supplement we are going to make a converse construction: given a representation $\theta : H \to \mathrm{GL}_{\mathbb{C}}(W)$ of the subgroup $H \subseteq G$, we are going to induce ("lift") it to a representation of $G$.

For this we need to recall some facts on left coset spaces from Algebra 1. Let $G/H$ be the left cosets of $H$ in $G$. In other words, $G/H$ is the set of equivalence classes of $G$ with respect to the equivalence relation $g \sim g'$ iff $g^{-1}g' \in H$. The elements of $G/H$ thus are the left cosets $gH = \{gh \,|\, h \in H\}$ $(g \in G)$. Recall that

$$(G : H) := \#(G/H) = \#G/\#H$$

is called the index of $H$ in $G$. We write $\mathcal{R}$ for a complete set of representatives of the left $H$-cosets in $G$. We assume throughout this section that the representative in $\mathcal{R}$ for the left coset $H$ is the unit element $e$ of $G$.

View $\mathbb{C}[G]$ as a right $\mathbb{C}[H]$-module by

$$\mathbb{C}[G] \times \mathbb{C}[H] \to \mathbb{C}[G], \qquad (a,b) \mapsto ab.$$

Then $\{e_r \,|\, r \in \mathcal{R}\}$ is a $\mathbb{C}[H]$-basis of $\mathbb{C}[G]$, i.e. each $a \in \mathbb{C}[G]$ can be uniquely written as $a = \sum_{r\in\mathcal{R}} e_r b_r$ with $b_r \in \mathbb{C}[H]$. Indeed, $a = \sum_{g\in G} \lambda_g e_g$ for unique $\lambda_g \in \mathbb{C}$, hence

$$a = \sum_{r\in\mathcal{R}} e_r b_r, \qquad b_r := \sum_{h\in H} \lambda_{rh} e_h \in \mathbb{C}[H]$$

and clearly this is the only choice for the $b_r \in H$ such that $a = \sum_{r\in\mathcal{R}} e_r b_r$.

**Proposition 2.1.** *Suppose that $\theta : H \to \mathrm{GL}_{\mathbb{C}}(W)$ is a finite dimensional linear representation of $H$. Suppose $\{w_i\}_{i=1}^m$ is a $\mathbb{C}$-basis of $W$. Consider $W$ as left $\mathbb{C}[H]$-module in the usual way,*

$$\mathbb{C}[H] \times W \to W, \qquad \big(\sum_{h\in H} \mu_h e_h, w\big) \mapsto \sum_{h\in H} \mu_h \theta(h) w.$$

- **a.** *The complex vector space $\mathrm{Ind}_H^G(W) := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$ has $\{e_r \otimes_{\mathbb{C}[H]} w_i\}_{r\in\mathcal{R}, 1\leq i\leq m}$ as a $\mathbb{C}$-basis.*
- **b.** *$\mathrm{Ind}_H^G(W)$ is a left $\mathbb{C}[G]$-module with the action defined by*

$$\mathbb{C}[G] \times \mathrm{Ind}_H^G(W) \to \mathrm{Ind}_H^G(W), \qquad \big(a, a' \otimes_{\mathbb{C}[H]} w\big) \mapsto (aa') \otimes_{\mathbb{C}[H]} w.$$

*Proof.* **a.** $\{e_r \otimes_{\mathbb{C}[H]} w_i\}_{r\in\mathcal{R}, 1\leq i\leq m}$ spans $\mathrm{Ind}_H^G(W)$. Indeed, for $a = \sum_{r\in\mathcal{R}} e_r b_r$ with $b_r \in \mathbb{C}[H]$ and for $w \in W$,

$$a \otimes_{\mathbb{C}[H]} w = \sum_{r\in\mathcal{R}} e_r \otimes_{\mathbb{C}[H]} b_r \cdot w = \sum_{r\in\mathcal{R}} \sum_{i=1}^m \lambda_i^{(r)} e_r \otimes_{\mathbb{C}[H]} w_i$$

with $\lambda_i^{(r)} \in \mathbb{C}$ such that $b_r \cdot w = \sum_{i=1}^m \lambda_i^{(r)} w_i$ (here we write $b \cdot w$ for the action of $b \in \mathbb{C}[H]$ on $w \in W$). For the linear independence, define for $r \in \mathcal{R}$ the map $\phi_r : \mathbb{C}[G] \times W \to W$ by

$$\phi_r\big(\sum_{r'\in\mathcal{R}} e_{r'} b_{r'}, w\big) := b_r \cdot w$$

with $b_{r'} \in \mathbb{C}[H]$ and $w \in W$. This is well defined by the remark preceding the proposition. Since $\phi_r$ is a $\mathbb{C}[H]$-bilinear map, there exists a unique linear map

$$\overline{\phi}_r : \mathrm{Ind}_H^G(W) = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to W$$

satisfying $\overline{\phi}_r(a \otimes_{\mathbb{C}[H]} w) = \phi_r(a, w)$ for all $a \in \mathbb{C}[G]$ and $w \in W$ (due to the universal property of $\otimes_{\mathbb{C}[H]}$). Suppose now that

$$\sum_{r'\in\mathcal{R}} \sum_{i=1}^m \mu_i^{(r')} e_{r'} \otimes_{\mathbb{C}[H]} w_i = 0$$

in $\mathrm{Ind}_H^G(W)$ with $\mu_i^{(r')} \in \mathbb{C}$. Write $w^{(r')} = \sum_{i=1}^m \mu_i^{(r')} w_i$. Then for all $r \in \mathcal{R}$,

$$\begin{aligned} 0 &= \overline{\phi}_r\big(\sum_{r' \in \mathcal{R}} \sum_{i=1}^{m} \mu_i^{(r')} e_{r'} \otimes_{\mathbb{C}[H]} w_i\big) \\ &= \overline{\phi}_r\big(\sum_{r' \in \mathcal{R}} e_{r'} \otimes_{\mathbb{C}[H]} w^{(r')}\big) \\ &= w^{(r)}. \end{aligned}$$

Hence $\mu_i^{(r)} = 0$ for all $1 \leq i \leq m$ and $r \in \mathcal{R}$, proving the linear independence.
**b.** Define for $a \in \mathbb{C}[G]$,

$$\widetilde{\pi}(a) : \mathbb{C}[G] \times W \to \mathrm{Ind}_H^G(W)$$

by $\widetilde{\pi}(a)(a', w) := (aa') \otimes_{\mathbb{C}[H]} w$ for $a, a' \in \mathbb{C}[G]$ and $w \in W$. Then $\widetilde{\pi}(a)$ is $\mathbb{C}[H]$-bilinear, hence it gives rise to a complex linear endomorphism $\pi(a)$ of $\mathrm{Ind}_H^G(W)$ defined by

$$\pi(a)(a' \otimes_{\mathbb{C}[H]} w) = (aa') \otimes_{\mathbb{C}[H]} w$$

for $a, a' \in \mathbb{C}[G]$ and $w \in W$. It is straightforward to check that the map $\pi : \mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}\big(\mathrm{Ind}_H^G(W)\big)$ is an algebra homomorphism. □

The $\mathbb{C}[G]$-module structure on $\mathrm{Ind}_H^G(W)$ thus gives rise to a linear representation $\pi := \mathrm{Ind}_H^G(\theta) : G \to \mathrm{GL}_{\mathbb{C}}(\mathrm{Ind}_H^G(W))$, called the representation of $G$ induced from $\theta$. It is explicitly given by

$$\pi(g)\big(a \otimes_{\mathbb{C}[H]} w\big) := (e_g a) \otimes_{\mathbb{C}[H]} w$$

for $g \in G$, $a \in \mathbb{C}[G]$ and $w \in W$. Note that $\mathrm{Dim}_{\mathbb{C}}(\mathrm{Ind}_H^G(W)) = (G : H)\mathrm{Dim}_{\mathbb{C}}(W)$.

The structure of the induced representation $\pi := \mathrm{Ind}_H^G(\theta)$ of $G$ on $V := \mathrm{Ind}_H^G(W)$ is as follows. Define for $g \in G$ the subspace

$$V_g := e_g \otimes_{\mathbb{C}[H]} W \subseteq V.$$

It is linearly isomorphic to $W$ by the linear isomorphism $\psi_g : W \xrightarrow{\sim} V_g$ defined by $\psi_g(w) := e_g \otimes_{\mathbb{C}[H]} w$. Note that $V_g$ only depends on the left coset $gH$. Indeed, for $h \in H$,

$$V_{gh} = e_g e_h \otimes_{\mathbb{C}[H]} W = e_g \otimes_{\mathbb{C}[H]} \theta(h)W = e_g \otimes_{\mathbb{C}[H]} W = V_g.$$

Hence we write $V_{gH} = V_g$ for $g \in G$. Then

$$V = \bigoplus_{gH \in G/H} V_{gH}.$$

Note that $V_e \subseteq V$ is a $H$-invariant subspace with respect to the representation map $\mathrm{Res}_H^G(\pi) = \pi|_H$, isomorphic to $W$ via the bijective $H$-intertwiner $\psi_e : W \xrightarrow{\sim} V_e$. In particular, $\mathrm{Ind}_H^H(W) \simeq W$.

Recall that $G$ acts on $G/H$ by

$$G \times G/H \to G/H, \qquad (g, g'H) \mapsto gg'H.$$

**Corollary 2.2.** *We use the above notations. In particular we write* $\pi = \mathrm{Ind}_H^G(\theta)$ *and* $V = \mathrm{Ind}_H^G(W)$.
**(i)** *If* $g \in G$ *then* $\pi(g) \in \mathrm{GL}_{\mathbb{C}}(V)$ *permutes the subspaces* $V_{g'H}$ $(g'H \in G/H)$. *More precisely,* $\pi(g)$ *restricts to a complex linear isomorphism* $\pi(g)|_{V_{g'H}} : V_{g'H} \xrightarrow{\sim} V_{gg'H}$.
**(ii)** *If* $H \trianglelefteq G$ *is a normal subgroup then* $V_{gH} \subseteq V = \mathrm{Ind}_H^G(W)$ *are* $H$*-invariant subspaces with respect to* $\mathrm{Res}_H^G(\pi) = \pi|_H$ *for all* $gH \in G/H$.

*Proof.* **(i)** For $g, g' \in G$ we have

$$\pi(g)(V_{g'H}) = \pi(g)(e_{g'} \otimes_{\mathbb{C}[H]} W) = e_{gg'} \otimes_{\mathbb{C}[H]} W = V_{gg'H}.$$

Hence $\pi(g)|_{V_{g'H}} : V_{g'H} \to V_{gg'H}$. It is a linear isomorphism since its inverse is given by $\pi(g^{-1})|_{V_{gg'H}}$.
**(ii)** If $H \trianglelefteq G$ then $Hg = gH$ for all $g \in G$ hence, for $h \in H$,

$$\pi(h)|_{V_{gH}} : V_{gH} \xrightarrow{\sim} V_{hgH} = V_{gH}.$$

□

*Remark* 2.3. Let $\theta : H \to \mathrm{GL}_{\mathbb{C}}(W)$ be a linear $H$-representation. If $H \trianglelefteq G$ and $g \in G$, then $V_{gH}$ as linear $H$-representation with respect to $\mathrm{Res}_H^G(\pi) = \pi|H$ is isomorphic to the linear $H$-representation $\theta^g : H \to \mathrm{GL}_{\mathbb{C}}(W)$, defined by $\theta^g(h) := \theta(g^{-1}hg)$ for all $h \in H$.

**Example 2.4.** *Let* $H \subseteq G$ *be an inclusion of finite groups. Let* $\rho_H : H \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}[H])$ *be the regular representation, then* $\{e_r \otimes_{\mathbb{C}[H]} e_h\}_{r \in \mathcal{R}, h \in H}$ *is a* $\mathbb{C}$*-linear basis of the representation space* $\mathrm{Ind}_H^G(\mathbb{C}[H])$ *of the induced representation* $\mathrm{Ind}_H^G(\rho_H)$. *Let* $\rho_G : G \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}[G])$ *be the regular representation of* $G$, *then* $\mathrm{Ind}_H^G(\rho_H) \simeq \rho_G$ *as* $G$*-representations with the bijective intertwiner* $\mathrm{Ind}_H^G(\mathbb{C}[H]) \xrightarrow{\sim} \mathbb{C}[G]$ *defined by* $e_r \otimes_{\mathbb{C}[H]} e_h \mapsto e_{rh}$ *for all* $r \in \mathcal{R}$ *and* $h \in H$.

**Example 2.5.** *Consider the dihedral group* $D_n$ *(*$n$ *odd), generated by* $r, s$ *and satisfying* $r^n = e$, $s^2 = e$ *and* $srs = r^{-1}$. *It has one-dimensional representations* $\rho_\pm$ *and two dimensional representations* $\pi_t$ $(0 \le t < n)$ *defined by*

$$\rho_\pm(r) = 1, \qquad \rho_\pm(s) = \pm 1,$$

*and*

$$\pi_t(r) = \begin{pmatrix} \cos(2\pi t/n) & -\sin(2\pi t/n) \\ \sin(2\pi t/n) & \cos(2\pi t/n) \end{pmatrix}, \qquad \pi_t(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*(we have seen that the* $\pi_t$ $(1 \le t < \frac{n-1}{2})$ *form a complete set of representatives of the equivalence classes of irreducible representations of* $D_n$ *of degree two). Rewriting* $\pi_t(\cdot)$ *in terms of the basis* $v_1 = \begin{pmatrix} 1/2 \\ -i/2 \end{pmatrix}$ *and* $v_2 = \begin{pmatrix} 1/2 \\ i/2 \end{pmatrix}$ *we get* $\pi_t \simeq \pi_t'$ *with* $\pi_t'$ *defined by*

$$\pi_t'(r) = \begin{pmatrix} e^{2\pi i t/n} & 0 \\ 0 & e^{-2\pi i t/n} \end{pmatrix}, \qquad \pi_t'(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

*Note that the subgroup $H := \langle r \rangle$ of $D_n$ generated by $r$ is a normal, index two subgroup of $D_n$, isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Take $\mathcal{R} = \{e, s\}$ as the set of representatives of the left cosets of $\langle r \rangle$ in $D_n$.*

*Consider the one dimensional representation $\chi_t$ of the subgroup $H = \langle r \rangle \subset D_n$, defined by $\chi_t(r) = e^{2\pi i t/n}$. Write $\mathbb{C}_t$ for the representation space of $\chi_t$ and $1_t \in \mathbb{C}_t$ for a basis element of $\mathbb{C}_t$, so that $r \cdot 1_t = \chi_t(r)1_t$. Set $\sigma_t := \mathrm{Ind}_{\langle r \rangle}^{D_n}(\chi_t)$. The representation space $V$ of $\sigma_t$ is two-dimensional. A linear basis of $V$ is given by $u_1 := e_e \otimes_{\langle r \rangle} 1_t$ and $u_2 := e_s \otimes_{\langle r \rangle} 1_t$. We then have*

$$\begin{aligned}\sigma_t(r)u_1 &= e_e \otimes_{\langle r \rangle} \chi_t(r)1_t = e^{2\pi i t/n}u_1,\\ \sigma_t(r)u_2 &= e_{rs} \otimes_{\langle r \rangle} 1_t = e_{sr^{-1}} \otimes_{\langle r \rangle} 1_t = e_s \otimes_{\langle r \rangle} \chi_t(r^{-1})1_t = e^{-2\pi i t/n}u_2,\end{aligned}$$

*and*

$$\sigma_t(s)u_1 = u_2, \qquad \sigma_t(s)u_2 = e_{s^2} \otimes_{\langle r \rangle} 1_t = e_e \otimes_{\langle r \rangle} 1_t = u_1.$$

*Hence $\sigma_t \simeq \pi_t' \simeq \pi_t$.*

**Exercise 2.6.** *Let $n \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$.*
**(i)** *Let $V$ be a complex vector space of dimension $n$ with linear basis $\{v_{\overline{m}}\}_{\overline{m} \in \mathbb{Z}/n\mathbb{Z}}$. Show that there exists a unique group homomorphism $\pi_\epsilon : D_n \to \mathrm{GL}_{\mathbb{C}}(V_\epsilon)$ satisfying*

$$\pi_\epsilon(r)v_{\overline{m}} = v_{\overline{m+1}}, \qquad \pi_\epsilon(s)v_{\overline{m}} = \epsilon v_{\overline{n-m}}.$$

**(ii)** *Let $\langle s \rangle \subset D_n$ be the subgroup of order 2 generated by $s$. Prove that $\pi_\epsilon \simeq \mathrm{Ind}_{\langle s \rangle}^{D_n}(\rho_\epsilon)$, where $\rho_\epsilon$ is the one-dimensional representation of $\langle s \rangle$ defined by $\rho_\epsilon(s) = \epsilon$.*

**Exercise 2.7.** *Let $H \subseteq K \subseteq G$ be an inclusion of finite groups.*
**(i)** *Let $\pi : G \to \mathrm{GL}_{\mathbb{C}}(V)$ be a group homomorphism. Show that*

$$\mathrm{Res}_H^G(\pi) \simeq \mathrm{Res}_H^K(\mathrm{Res}_K^G(\pi))$$

*as linear representations of $H$.*
**(ii)** *Let $\theta : H \to \mathrm{GL}_{\mathbb{C}}(W)$ be a group homomorphism. Show that*

$$\mathrm{Ind}_H^G(\theta) \simeq \mathrm{Ind}_K^G(\mathrm{Ind}_H^K(\theta))$$

*as linear representations of $G$.*

**Exercise 2.8.** *Let $H \subseteq G$ be an inclusion of finite groups and let $\theta : H \to \mathrm{GL}_{\mathbb{C}}(W)$ and $\theta' : H \to \mathrm{GL}_{\mathbb{C}}(W')$ be finite dimensional linear representations of $H$.*
**(i)** *Prove that $\mathrm{Ind}_H^G(\theta \oplus \theta') \simeq \mathrm{Ind}_H^G(\theta) \oplus \mathrm{Ind}_H^G(\theta')$ as $G$-representations.*
**(ii)** *Show that $\mathrm{Ind}_H^G(\theta) \simeq \mathrm{Ind}_H^G(\theta')$ as $G$-representations if $\theta \simeq \theta'$ as $H$-representations.*

## 3. Characters of induced representations

Let $H \subseteq G$ be an inclusion of finite groups. Let $\mathcal{R}$ be a complete set of representatives of the set of left $H$-cosets in $G$. The left $G$-action on $G/H$ gives rise to a left $G$-action on $\mathcal{R}$. Concretely, $g \cdot r$ for $g \in G$ and $r \in \mathcal{R}$ is the representative $r' \in \mathcal{R}$ such that $g \cdot r = r'$. Note that $g \cdot r = r$ iff $gr \in rH$ iff $r^{-1}gr \in H$.

**Theorem 3.1.** *Define for* $\chi \in F(H)$,

$$\mathrm{Ind}_H^G(\chi)(g) := \frac{1}{\#H} \sum_{\substack{s \in G: \\ s^{-1}gs \in H}} \chi(s^{-1}gs). \tag{3.1}$$

**(i)** *This defines a well defined linear map* $\mathrm{Ind}_H^G : F(H) \to F(G)$.
**(ii)** *We have*

$$\mathrm{Ind}_H^G(\chi)(g) = \sum_{r \in \mathcal{R}: g \cdot r = r} \chi(r^{-1}gr).$$

**(iii)** *Let* $(W, \theta)$ *be a finite dimensional linear representation of* $H$. *Let* $\chi_\theta \in F(H)$ *be its character and* $\chi_{\mathrm{Ind}_H^G(\theta)} \in F(G)$ *the character of the corresponding induced representation of* $G$. *Then*

$$\mathrm{Ind}_H^G(\chi_\theta) = \chi_{\mathrm{Ind}_H^G(\theta)}.$$

*Proof.* **(i)** We only need to verify that (3.1) defines a class function on $G$. Let $t, g \in G$. Then

$$\begin{aligned}
\mathrm{Ind}_H^G(\chi)(tgt^{-1}) &= \frac{1}{\#H} \sum_{\substack{s \in G: \\ s^{-1}tgt^{-1}s \in H}} \chi(s^{-1}tgst^{-1}) \\
&= \frac{1}{\#H} \sum_{\substack{u \in G: \\ u^{-1}gu \in H}} \chi(u^{-1}gu) = \mathrm{Ind}_H^G(\chi)(g),
\end{aligned}$$

where the group elements were reparametrized in the second equality by setting $u = t^{-1}s$ ($s \in G$).
**(ii)** Since $\chi$ is a class function on $H$, we have

$$\begin{aligned}
\frac{1}{\#H} \sum_{\substack{s \in G: \\ s^{-1}gs \in H}} \chi(s^{-1}gs) &= \frac{1}{\#H} \sum_{\substack{r \in \mathcal{R}: \\ r^{-1}gr \in H}} \sum_{h \in H} \chi(h^{-1}r^{-1}grh) \\
&= \frac{1}{\#H} \sum_{\substack{r \in \mathcal{R}: \\ r^{-1}gr \in H}} \sum_{h \in H} \chi(r^{-1}gr) \\
&= \sum_{r \in \mathcal{R}: g \cdot r = r} \chi(r^{-1}gr).
\end{aligned}$$

**(iii)** Fix a linear basis $\{w_i\}_{i=1}^m$ of $W$ and consider the corresponding linear basis $v_{r,i} := e_r \otimes_{\mathbb{C}[H]} w_i$ ($1 \leq i \leq m$) of $V_r \subseteq V = \mathrm{Ind}_H^G(W)$ ($r \in \mathcal{R}$). We have seen that

$$\{v_{r,i} \,|\, r \in \mathcal{R}, 1 \leq i \leq m\}$$

is a linear basis of $V = \mathrm{Ind}_H^G(W) = \bigoplus_{r\in\mathcal{R}} V_r$. Write $\pi = \mathrm{Ind}_H^G(\theta)$. Then we have

$$\begin{aligned}\chi_\pi(g) &= \sum_{r\in\mathcal{R}} \sum_{i=1}^m \pi(g) v_{r,i}|_{v_{r,i}} \\ &= \sum_{r\in\mathcal{R}} \sum_{i=1}^m \big(e_{gr} \otimes_{\mathbb{C}[H]} w_i\big)|_{e_r\otimes_{\mathbb{C}[H]} w_i}.\end{aligned}$$

Now observe that the terms for fixed $r \in \mathcal{R}$ will be zero unless $gr \in rH$, i.e. unless $g\cdot r = r$, since $e_{gr} \otimes_{\mathbb{C}[H]} w_i \in V_{g\cdot r}$. Hence

$$\begin{aligned}\chi_\pi(g) &= \sum_{r\in\mathcal{R}: g\cdot r = r} \sum_{i=1}^m (e_{rr^{-1}gr} \otimes_{\mathbb{C}[H]} w_i\big)|_{e_r\otimes_{\mathbb{C}[H]} w_i} \\ &= \sum_{r\in\mathcal{R}: g\cdot r = r} \sum_{i=1}^m \big(e_r \otimes_{\mathbb{C}[H]} \theta(r^{-1}gr) w_i\big)|_{e_r\otimes_{\mathbb{C}[H]} w_i} \\ &= \sum_{r\in\mathcal{R}: g\cdot r = r} \sum_{i=1}^m \theta(r^{-1}gr) w_i|_{w_i} \\ &= \sum_{r\in\mathcal{R}: g\cdot r = r} \chi_\theta(r^{-1}gr) \\ &= \mathrm{Ind}_H^G(\chi_\theta)(g).\end{aligned}$$

□

**Exercise 3.2.** *Let $\chi \in F(H)$ and $\eta \in F(G)$. Denote by $\cdot$ the pointwise multiplication on $F(H)$ and $F(G)$ respectively.*
**(i)** *Show that*

$$\mathrm{Ind}_H^G\big(\chi \cdot \mathrm{Res}_H^G(\eta)\big) = \mathrm{Ind}_H^G(\chi)\cdot\eta$$

*as identity in $F(G)$.*
**(ii)** *Conclude that the image of the induction map $\mathrm{Ind}_H^G : F(H) \to F(G)$ is an ideal in $(F(G), \cdot)$.*

Recall the scalar product

$$\big(f\,|\,f'\big)_G := \frac{1}{\#G} \sum_{g\in G} f(g)\overline{f'(g)}$$

on $F(G) \subseteq \mathrm{Fun}_{\mathbb{C}}(G)$. The following theorem shows that the maps $\mathrm{Ind}_H^G$ and $\mathrm{Res}_H^G$ on class functions are adjoint with respect to these scalar products:

**Theorem 3.3** (Frobenius reciprocity)**.** *Let $\chi \in F(H)$ and $\eta \in F(G)$. Then*

$$\big(\mathrm{Ind}_H^G(\chi)\,|\,\eta\big)_G = \big(\chi\,|\,\mathrm{Res}_H^G(\eta)\big)_H.$$

*Proof.* This is a direct computation,

$$\begin{aligned}
\big(\mathrm{Ind}_H^G(\chi)\,|\,\eta\big)_G &= \frac{1}{\#G}\sum_{g\in G}\mathrm{Ind}_H^G(\chi)(g)\overline{\eta(g)}\\
&= \frac{1}{\#G}\sum_{g\in G}\sum_{r\in\mathcal{R}: g\cdot r=r}\chi(r^{-1}gr)\overline{\eta(g)}\\
&= \frac{1}{\#G}\sum_{r\in\mathcal{R}}\sum_{g\in G: r^{-1}gr\in H}\chi(r^{-1}gr)\overline{\eta(g)}\\
&= \frac{1}{\#G}\sum_{r\in\mathcal{R}}\sum_{g\in G: r^{-1}gr\in H}\chi(r^{-1}gr)\overline{\eta(r^{-1}gr)}\\
&= \frac{1}{\#G}\sum_{r\in\mathcal{R}}\sum_{h\in H}\chi(h)\overline{\eta(h)}\\
&= \frac{(G:H)\#H}{\#G}\frac{1}{\#H}\sum_{h\in H}\chi(h)\overline{\eta(h)}\\
&= \big(\chi\,|\,\mathrm{Res}_H^G(\eta)\big)_H,
\end{aligned}$$

where we used that $\eta$ is a class function on $G$ in the fourth equality. □

**Corollary 3.4.** *Let $\theta\in\widehat{H}$ and $\pi\in\widehat{G}$ with corresponding irreducible characters $\chi_\theta\in F(H)$ and $\eta_\pi\in F(G)$ respectively. The number of times that the irreducible representation $\theta$ appears as constituent in an irreducible decomposition of $\mathrm{Res}_H^G(\pi)$ is equal to the number of times that the irreducible representation $\pi$ appears as constituent in an irreducible decomposition of $\mathrm{Ind}_H^G(\theta)$.*

*Proof.* Since the irreducible characters form an orthonormal basis of the class functions, it follows that

(1) the number of times that the irreducible representation $\theta$ appears as constituent in an irreducible decomposition of $\mathrm{Res}_H^G(\pi)$ is $\big(\chi_\theta\,|\,\mathrm{Res}_H^G(\eta_\pi)\big)_H$,
(2) the number of times that the irreducible representation $\pi$ appears as constituent in an irreducible decomposition of $\mathrm{Ind}_H^G(\theta)$ is $\big(\mathrm{Ind}_H^G(\chi_\theta)\,|\,\eta_\pi\big)_G$.

Hence we need to prove that

$$\big(\chi_\theta\,|\,\mathrm{Res}_H^G(\eta_\pi)\big)_H = \big(\mathrm{Ind}_H^G(\chi_\theta)\,|\,\eta_\pi\big)_G,$$

but this is what Frobenius reciprocity is telling us! □

**Example 3.5.** *Consider again Example 2.5. In particular, $n$ is odd and $0\le t<n$. Note that*

$$\chi_{\sigma_t}(r^m) = e^{2\pi imt/n} + e^{-2\pi imt/n}.$$

*We then have, by Frobenius reciprocity,*

$$\begin{aligned}\big(\chi_{\sigma_t}\,|\,\chi_{\sigma_t}\big)_{D_n} &= \big(\chi_t\,|\,\mathrm{Res}^{D_n}_{\langle r\rangle}(\chi_{\sigma_t})\big)_{\langle r\rangle}\\ &= \frac{1}{n}\sum_{m=0}^{n-1}\chi_t(r^m)\overline{\chi_{\sigma_t}(r^m)}\\ &= \frac{1}{n}\sum_{m=0}^{n-1}e^{2\pi imt/n}\big(e^{2\pi imt/n}+e^{-2\pi imt/n}\big)\\ &= \frac{1}{n}\sum_{m=0}^{n-1}\big(1+e^{4\pi imt/n}\big).\end{aligned}$$

*This equals* 1 *if* $1\le t<n$ *and* 2 *if* $t=0$ *since*

$$\sum_{m=0}^{n-1}\big(e^{2\pi it/n}\big)^m = \begin{cases}\frac{1-(e^{4\pi it/n})^n}{1-e^{4\pi it/n}}=0, & \textit{if } 1\le t<n,\\ n, & \textit{if } t=0\end{cases}$$

*(here we use that $n$ is odd, so that $e^{4\pi it/n}\neq 1$ for all $1\le t<n$). This shows that $\sigma_t\simeq\pi_t\simeq\pi_t'$ is irreducible if $1\le t<n$, and that it decomposes in two one-dimensional irreducible representations if $t=0$.*

We now translate Frobenius reciprocity to the setting of intertwiners. Let $\theta\in\widehat{H}$ and $\pi\in\widehat{G}$ with representation spaces $W_\theta$ and $V_\pi$ and characters $\chi_\theta$ and $\eta_\pi$ respectively. Then Frobenius reciprocity says that

$$\big(\chi_\theta\,|\,\mathrm{Res}^G_H(\eta_\pi)\big)_H = \big(\mathrm{Ind}^G_H(\chi_\theta)\,|\,\eta_\pi\big)_G,$$

on the other hand we have seen that

$$\begin{aligned}\big(\chi_\theta\,|\,\mathrm{Res}^G_H(\eta_\pi)\big)_H &= \mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(H)}(W_\theta,\mathrm{Res}^G_H(V_\pi))\big),\\ \big(\mathrm{Ind}^G_H(\chi_\theta)\,|\,\eta_\pi\big)_G &= \mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(G)}(\mathrm{Ind}^G_H(W_\theta),V_\pi)\big).\end{aligned}$$

The resulting equality of dimensions of intertwiner spaces,

$$\mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(H)}(W_\theta,\mathrm{Res}^G_H(V_\pi))\big) = \mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(G)}(\mathrm{Ind}^G_H(W_\theta),V_\pi)\big) \tag{3.2}$$

lifts to the following explicit linear isomorphism between the intertwiner spaces.

**Proposition 3.6.** *Let $f\in\mathrm{Hom}^{(G)}(\mathrm{Ind}^G_H(W_\theta),V_\pi)$ and set*

$$\widetilde{f}(w) := f(e_e\otimes_{\mathbb{C}[H]}w),\qquad w\in W_\theta. \tag{3.3}$$

*Then $f\mapsto\widetilde{f}$ defines a linear isomorphism*

$$\mathrm{Hom}^{(G)}(\mathrm{Ind}^G_H(W_\theta),V_\pi)\xrightarrow{\sim}\mathrm{Hom}^{(H)}(W_\theta,\mathrm{Res}^G_H(V_\pi)).$$

*In other words, for any $H$-intertwiner $\widetilde{f}:W_\theta\to\mathrm{Res}^G_H(V_\pi)$ there exists a unique $G$-intertwiner $f:\mathrm{Ind}^G_H(W_\theta)\to V_\pi$ such that (3.3) holds true.*

*Proof.* A direct computation shows that $\widetilde{f}$ is an $H$-intertwiner,

$$\begin{aligned}\widetilde{f}(\theta(h)w) &= f(e_e \otimes_{\mathbb{C}[H]} \theta(h)w)\\ &= f(e_h \otimes_{\mathbb{C}[H]} w)\\ &= f(h\cdot(e_e \otimes_{\mathbb{C}[H]} w))\\ &= \pi(h)(f(e_e \otimes_{\mathbb{C}[H]} w))\\ &= \pi(h)\widetilde{f}(w).\end{aligned}$$

Suppose that $\widetilde{f} \equiv 0$. Then

$$f(e_e \otimes_{\mathbb{C}[H]} w) = \widetilde{f}(w) = 0 \qquad \forall w \in W.$$

Consequently, for $r \in \mathcal{R}$ and $w \in W$,

$$f(e_r \otimes_{\mathbb{C}[H]} w) = f(r\cdot(e_e \otimes_{\mathbb{C}[H]} w)) = \pi(r)(f(e_e \otimes_{\mathbb{C}[H]} w)) = 0.$$

Hence $f \equiv 0$. The map $f \mapsto \widetilde{f}$ thus is injective. By (3.2) we conclude that $f \mapsto \widetilde{f}$ is a linear isomorphism. □

**Exercise 3.7.** *Let $H \subseteq G$ be an inclusion of finite groups. Use Frobenius reciprocity to prove that each irreducible representation $\pi \in \widehat{G}$ of $G$ is contained in $\mathrm{Ind}_H^G(\theta)$ for at least one $\theta \in \widehat{H}$. Derive from this fact that $\dim_{\mathbb{C}}(V_\pi) \leq (G:H)$ if $H$ is abelian.*

The following exercise is a preparation to [1, Exerc. 7.2].

**Exercise 3.8.** *Suppose $H \subsetneq G$ is an inclusion of finite groups. Let $\mathcal{R}$ be a complete set of representatives of the left $H$-coset space $G/H$ and suppose that $e \in \mathcal{R}$ (with $e$ the unit element of $G$). Recall that $G$ acts transitively on $G/H$ with action map $G \times G/H \to G/H$ given by $(g, g'H) \mapsto gg'H$.*

(1) *Prove that for all $r \in \mathcal{R}\setminus\{e\}$,*

$$\#\{h \in H \,|\, hr \in rH\} \geq \frac{\#H}{(G:H)-1}.$$

**Hint:** *Loot at the $G$-orbit of $(H, rH) \in G/H \times G/H$ with respect to the diagonal $G$-action on $G/H \times G/H$ (cf.* [1, Exerc. 2.6]*).*

(2) *Prove that the following four statements are equivalent.*

(a) *$G$ acts double transitively on $G/H$ (see* [1, Exerc. 2.6] *for the terminology),*

(b) *there exists an $r \in \mathcal{R}\setminus\{e\}$ such that*

$$\#\{h \in H \,|\, hr \in rH\} = \frac{\#H}{(G:H)-1},$$

(c) *for all $r \in \mathcal{R}\setminus\{e\}$ we have*

$$\#\{h \in H \,|\, hr \in rH\} = \frac{\#H}{(G:H)-1}.$$

(d) $\sum_{r\in\mathcal{R}\setminus\{e\}} \#\{h \in H \,|\, hr \in rH\} = \#H$.

## References

[1] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics, **42**, Springer-Verlag, New York, 1977.