ALGEBRA 3; REPRESENTATIE THEORIE. AANVULLING 4

1. Introduction

In these lecture notes we locate the irreducible representations of the symmetric group. A lot of information about the representation theory of the symmetric group can be found in the book [1]. In the lecture notes of next week we will explicitly construct the irreducible representations.

2. The symmetric group and Young subgroups

Let $n \geq 1$ and write S_n for the symmetric group in n letters. In other words, S_n is the group of bijections $\Omega_n \xrightarrow{\sim} \Omega_n$, where $\Omega_n := \{1, \ldots, n\} \to \Omega_n$. Then $\#S_n = n!$. Let $I = \{i_1, \ldots, i_r\} \subseteq \Omega_n$ be an ordered subset of cardinality r. Then we have the cycle

$$(i_1\, i_2\, \cdots\, i_r) \in S_n,$$

of length r, which is the permutation $i_j \mapsto i_{j+1}$ $(1 \leq j < r)$, $i_r \mapsto i_1$ and $k \mapsto k$ for $k \in \Omega_n \setminus I$. We call I the content of the cycle. Note that a cycle (i) of length one is the identity element e of the symmetric group S_n. We call two cycles disjoint if their contents have trivial intersection. Note that disjoint cycles commute.

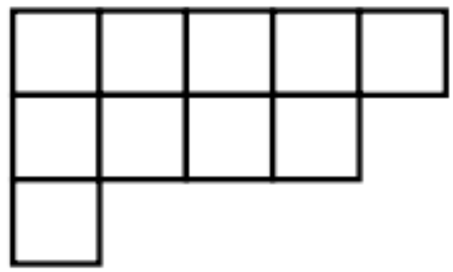
We recall the following basic fact.

Lemma 1. *Any $\sigma \in S_n$ can be written as product of disjoint cycles.*

A composition λ of n is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers λ_i such that $\sum_i \lambda_i = n$. We write in this case $\lambda \models n$, and we write $l(\lambda)$ for the largest index m such that $\lambda_m \neq 0$ (it is called the length of λ). We write $\lambda \vdash n$ if, in addition, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. In this case we call λ a partition of n. Note that $l(\lambda) \leq n$ if $\lambda \vdash n$.

Sometimes it is convenient to write λ as a finite sequence by forgetting the zeros to the right, $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$.

The Young diagram of a composition $\lambda \models$ is the diagram of n boxes placed in $l(\lambda)$ rows, with λ_i boxes in the ith row.



Young diagram van $(5, 4, 1)$

The Young diagram of a partition has the property that the number of boxes per row weakly increase from top to bottom. We will show that the partitions of n naturally parametrize the conjugacy classes of the symmetric group S_n.

Definition 1. *Let $\sigma \in S_n$ and write σ as a product of disjoint cycles, such that each element $i \in \Omega_n$ is in the content of one of the cycles. Collecting the cycle lengths of the disjoint cycles gives a composition of n. Permute the entries such that they are in nondecreasing order. The resulting partition $c(\sigma)$ of n is called the cycle type of σ.*

It is clear that the cycle type $c(\sigma)$ of $\sigma \in S_n$ is well defined.

Consider the group homomorphism $\mathrm{Ad} : S_n \to \mathrm{Aut}(S_n)$ defined by $\mathrm{Ad}(\sigma)(\tau) := \sigma\tau\sigma^{-1}$. This turns S_n into a S_n-set. The corresponding S_n-orbits are denoted by $\mathrm{Ad}(S_n)\tau \in S_n/\sim$ ($\tau \in S_n$). They are the conjugacy classes of S_n. Write $\mathcal{P}_n$ for the set of partitions of n.

Proposition 1. *The map $S_n/\sim \to \mathcal{P}_n$, given by $\mathrm{Ad}(S_n)\tau \mapsto c(\tau)$, is well defined and bijective.*

Proof. Note that

$$\sigma(i_1\, i_2\, \cdots\, i_r)\sigma^{-1} = (\sigma(i_1)\, \sigma(i_2)\, \cdots\, \sigma(i_r)) \tag{1}$$

in S_n for all $\sigma \in S_n$ and for all subsets $I = \{i_1, \ldots, i_r\} \subseteq \Omega_n$ of cardinality r. It follows that

$$\mathrm{Ad}(S_n)\sigma = \{\tau \in S_n \,|\, c(\tau) = c(\sigma)\}.$$

Hence $\mathrm{Ad}(S_n)\sigma \mapsto c(\sigma)$ is a well defined injective map $S_n/\sim \hookrightarrow \mathcal{P}_n$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n ($m = l(\lambda)$). Choose subsets

$$I_j = \{i_1^j, i_2^j \ldots, i_{\lambda_j}^j\} \subset \Omega_n$$

of cardinality λ_j such that $I_j \cap I_{j'} = \emptyset$ if $1 \leq j \neq j' \leq m$. Set

$$\sigma_j := (i_1^j \, i_2^j \, \cdots i_{\lambda_j}^j)$$

for the corresponding cycles of length λ_j $(1 \leq j \leq m)$. Then

$$\sigma := \sigma_1 \sigma_2 \cdots \sigma_m \in S_n$$

is a product of disjoint cycles such that $c(\sigma) = \lambda$. Hence $\mathrm{Ad}(S_n)\sigma \mapsto c(\sigma)$ maps onto $\mathcal{P}_n$. $\square$

Corollary 1. $\#\widehat{S_n} = \#\mathcal{P}_n$.

We will construct representatives of the isomorphy classes of irreducible linear S_n-representations and parametrize them by the partitions of n.

Lemma 2. *Define for $\lambda \vdash n$ and $1 \leq i \leq n$ set*

$$\lambda_i' := \#\{j \in \{1, \ldots, n\} \mid \lambda_j \geq i\}.$$

(1) $\lambda' \vdash n$ *(it is called the conjugate partition of n).*
(2) $l(\lambda') = \lambda_1$.
(3) $(\lambda')' = \lambda$.

Proof. This is immediate by realizing that the transition $\lambda \mapsto \lambda'$ corresponds to reflecting the Young diagram of λ in its main diagonal. For instance, in the example above, $(5, 4, 1)' = (3, 2, 2, 2, 1)$. $\square$

Exercise 1. *Define for $\lambda, \mu \vdash n$,*

$$\lambda \preceq \mu \quad \Leftrightarrow \quad \sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j \quad \forall\, i.$$

(1) *Show that $\preceq$ defines a partial order on $\mathcal{P}_n$.*
(2) *Prove that $\lambda \preceq \mu \Leftrightarrow \mu' \preceq \lambda'$.*

Exercise 2. *For $\lambda \vdash n$ and $i \geq 1$ define*

$$m_i(\lambda) := \#\{j \geq 1 \,|\, \lambda_j = i\}.$$

Let $\sigma \in S_n$ such that $c(\sigma) = \lambda$. Prove that

$$\#\big(\mathrm{Ad}(S_n)\sigma\big) = \frac{n!}{\prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!}$$

Let $\lambda \models n$. A Young tableau T_λ is the Young diagram of λ together with an arbitrary choice of an assignment of the numbers $\{1, \ldots, n\}$ to the boxes of the Young diagram (each number assigned to exactly one box). We write $T_\lambda(i, j)$ for the number assigned to the box in row i and column j $(1 \leq j \leq \lambda_i)$.

Definition 2. *Let $\lambda \models n$ and let T_λ be an λ-tableau. We call T_λ*

1. *row standard if $T_\lambda(i,j) < T_\lambda(i,j+1)$ if $T_\lambda(i,j)$ and $T_\lambda(i,j+1)$ are defined,*
2. *column standard if $T_\lambda(i,j) < T_\lambda(k,j)$ if $i < k$ and if both $T_\lambda(i,j)$ and $T_\lambda(k,j)$ are defined,*
3. *standard if T_λ is both row standard and column standard.*

Example 3. **(i)** *The λ-tableau t^λ defined by*

$$t^\lambda(i,j) := \begin{cases} \sum_{k=1}^{i-1} \lambda_k + j & \text{if } i > 1, \\ j & \text{if } i = 1 \end{cases}$$

for $1 \leq j \leq \lambda_i$, is a standard λ-tableau.
(ii) *The λ-tableau t_λ defined by putting in the jth column the numbers $\{\sum_{k=1}^{j-1} \lambda'_k + 1, \ldots, \sum_{k=1}^{j} \lambda'_k\}$, increasing from top to bottom (and with obvious adjustment for the first column). This is also a standard λ-tableau.*

For $\lambda \vdash n$ let $\mathcal{T}(\lambda)$ be the set of λ-tableaux. If $\sigma \in S_n$ and $T \in \mathcal{T}(\lambda)$ then we define σT to be the λ-tableau with the number $\sigma(T(i,j))$ attached to the (i,j)th box of its underlying Young diagram ($1 \leq j \leq \lambda_i$). This turns $\mathcal{T}(\lambda)$ into a S_n-set (in other words, it defines a left S_n-action $S_n \times \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ on $\mathcal{T}(\lambda)$).

Any λ-tableau T gives rise to a decomposition of Ω_n in disjoint subsets $\Omega_i^h(T)$ of cardinality λ_i ($i \geq 1$), where $\Omega_i^h(T)$ is the collection of numbers in the ith row of the underlying Young tableau ($i \geq 1$). We write $\Omega_i^h(\lambda) := \Omega_i^h(t^\lambda)$, so that

$$\Omega_i^h(\lambda) = \begin{cases} \{1, \ldots, \lambda_1\} & \text{if } i = 1 \text{ and } \lambda_1 \geq 1, \\ \{\sum_{k=1}^{i-1} \lambda_k + 1, \ldots, \sum_{k=1}^{i} \lambda_k\} & \text{if } i > 1 \text{ and } \lambda_i \geq 1, \\ \emptyset & \text{if } \lambda_i = 0 \end{cases}$$

(the superscript h stands for horizontal). In a similar way, a λ-tableau T gives rise to a decomposition of Ω_n in disjoint subsets $\Omega_j^v(T)$ of cardinality λ'_j ($j \geq 1$), where $\Omega_j^v(T)$ is the collection of numbers in the jth column of the underlying Young tableau ($j \geq 1$).

For $T \in \mathcal{T}(\lambda)$ define subgroups $H(T), V(T) \subseteq S_n$ by

$$H(T) = \{\sigma \in S_n \mid \sigma(\Omega_i^h(T)) = \Omega_i^h(T) \quad \forall\, i \geq 1\},$$
$$V(T) = \{\sigma \in S_n \mid \sigma(\Omega_j^v(T)) = \Omega_j^v(T) \quad \forall\, j \geq 1\}.$$

We write $S_\lambda := H(t^\lambda)$.

Subgroups isomorphic to $H(T)$ for some λ-tableau T ($\lambda \vdash n$) are called Young subgroups.

Exercise 4. *Fix* $\lambda \vdash n$.
(i) *Let* $\Omega_n = \Omega'_1 \cup \Omega'_2 \cup \cdots \cup \Omega'_{l(\lambda)}$ *be a disjoint union with* $\#\Omega'_i = \lambda_i$ *and define*

$$S(\{\Omega'_i\}_i) = \{\sigma \in S_n \mid \sigma(\Omega'_i) = \Omega'_i \quad \forall\, i\}.$$

Show that $S(\{\Omega'_i\}_i)$ *is a Young subgroup of* S_n.
(ii) *Let* $\sigma \in S_n$ *and* $T \in \mathcal{T}(\lambda)$. *Show that*

$$H(\sigma T) = \sigma H(T)\sigma^{-1}, \qquad V(\sigma T) = \sigma V(T)\sigma^{-1}.$$

(iii) *Let* $T \in \mathcal{T}(\lambda)$. *Show that*

$$H(T) \simeq S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_{l(\lambda)}} \simeq S(\{\Omega'_i\}_i)$$

as groups, where the Ω'_i *are as in* **(i)**.

To apply Mackey's machinery we need to analyze double (S_λ, S_μ)-cosets of S_n in some detail. Of special importance for us to understand double cosets with the so called trivial intersection property.

Definition 3. *Let* G *be a finite group with unit element* e *and let* $H, K \subseteq G$ *be subgroups. A double* (H,K)*-coset* HgK *is set to have the trivial intersection property if* $H \cap gKg^{-1} = \{e\}$.

Note that the trivial intersection property is well defined (it is independent of the choice of representative of the double coset).

Lemma 3. *Let* $\lambda, \mu \vdash n$ *and* $\sigma \in S_n$. *Then*

$$\tau \in S_\lambda \sigma S_\mu \quad \Leftrightarrow \quad \#\big(\Omega^h_i(\lambda) \cap \sigma(\Omega^h_k(\mu))\big) = \#\big(\Omega^h_i(\lambda) \cap \tau(\Omega^h_k(\mu))\big)$$

for all $i, k \geq 1$.

Proof. We have two disjoint unions

$$\Omega_n = \bigcup_{i,k\geq 1} \Omega_{i,k} = \bigcup_{i,k\geq 1} \Omega'_{i,k}$$

with $\Omega_{i,k} := \Omega^h_i(\lambda) \cap \sigma(\Omega^h_k(\mu))$ and $\Omega'_{i,k} := \Omega^h_i(\lambda) \cap \tau(\Omega^h_k(\mu))$.
$\Leftarrow$: Since $\#\Omega_{i,k} = \#\Omega'_{i,k}$ for all i,k, there exists a $\rho \in S_n$ such that $\rho(\Omega_{i,k}) = \Omega'_{i,k}$ for all $i,k \geq 1$. It follows that $\rho(\Omega^h_i(\lambda)) = \Omega^h_i(\lambda)$ for all $i \geq 1$, hence $\rho \in S_\lambda$. Set $\zeta := \sigma^{-1}\rho^{-1}\tau \in S_n$. Then

$$\begin{aligned}\zeta\big(\tau^{-1}(\Omega^h_i(\lambda)) \cap \Omega^h_k(\mu)\big) &= \sigma^{-1}\rho^{-1}\big(\Omega'_{i,k}\big)\\ &= \sigma^{-1}(\Omega_{i,k}) = \sigma^{-1}(\Omega^h_i(\lambda)) \cap \Omega^h_k(\mu)\end{aligned}$$

for all $i,k \geq 1$. In particular, $\zeta(\Omega^h_k(\mu)) = \Omega^h_k(\mu)$ for all $k \geq 1$, hence $\zeta \in S_\mu$. But then $\tau = \rho\sigma\zeta \in S_\lambda \sigma S_\mu$.

$\Rightarrow$: Write $\tau = \rho\sigma\zeta$ with $\rho \in S_\lambda$ and $\zeta \in S_\mu$. Then for all $i, k \geq 1$,

$$\begin{aligned}\Omega'_{i,k} &= \Omega_i^h(\lambda) \cap \tau(\Omega_k^h(\mu)) \\ &= \rho\big(\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu))\big) = \rho(\Omega_{i,k}),\end{aligned}$$

hence $\#\Omega'_{i,k} = \#\Omega_{i,k}$ for all $i, k \geq 1$. $\square$

Proposition 2. *Let $\lambda, \mu \vdash n$. Let $\mathcal{M}(\lambda,\mu)$ be the set of $n \times n$ matrices $\big(z_{ik}\big)_{i,k=1}^n$ with $z_{ik} \in \mathbb{Z}_{\geq 0}$ satisfying*

$$\sum_{k=1}^n z_{ik} = \lambda_i, \qquad 1 \leq i \leq n,$$

$$\sum_{i=1}^n z_{ik} = \mu_k, \qquad 1 \leq k \leq n.$$

Then the assignment

$$S_\lambda \sigma S_\mu \mapsto \big(\#\big(\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu))\big)\big)_{i,k=1}^n$$

defines a bijection $S_\lambda \setminus S_n / S_\mu \xrightarrow{\sim} \mathcal{M}(\lambda,\mu)$.

Proof. By the previous lemma it is a well defined injective map

$$S_\lambda \setminus S_n / S_\mu \hookrightarrow \mathcal{M}(\lambda,\mu).$$

It thus remains to show that it is a surjective map. Let $(z_{ik})_{i,k=1}^n \in \mathcal{M}(\lambda,\mu)$. We need to construct a $\sigma \in S_n$ such that

$$\#\big(\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu))\big) = z_{ik}$$

for $1 \leq i, k \leq n$. The permutation σ can be constructed as follows. First, for all $i \in \{1, \ldots, n\}$, we decompose $\Omega_i^h(\lambda)$ as a disjoint union

$$\Omega_i^h(\lambda) = \bigcup_{k=1}^n \Omega_{i,k}(\lambda)$$

with $\#\Omega_{i,k}(\lambda) = z_{ik}$ (this is possible since $\sum_{k=1}^n z_{ik} = \lambda_i = \#\Omega_i(\lambda)$). Then

$$\Omega_n = \bigcup_{i,k=1}^n \Omega_{i,k}(\lambda),$$

which is a disjoint union (since $\Omega_i^h(\lambda) \cap \Omega_{i'}^h(\lambda) = \emptyset$ if $i \neq i'$). Now define for $1 \leq k \leq n$,

$$\Omega'_k(\mu) := \bigcup_{i=1}^n \Omega_{i,k}(\lambda).$$

It is a disjoint union, and $\Omega_i^h(\lambda) \cap \Omega'_k(\mu) = \Omega_{i,k}(\lambda)$ for all i, k.

Note that $\#\Omega'_k(\mu) = \sum_{i=1}^n z_{ik} = \mu_k = \#\Omega_k^h(\mu)$ and that

$$\Omega_n = \bigcup_{k=1}^n \Omega'_k(\mu)$$

is a disjoint union, as well as $\Omega_n = \bigcup_{k=1}^n \Omega_k^h(\mu)$. Hence there exists a $\sigma \in S_n$ such that $\sigma(\Omega_k^h(\mu)) = \Omega'_k(\mu)$ for all $k \in \{1, \ldots, n\}$. But then for $1 \le i, k \le n$,

$$\begin{aligned}\#\big(\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu))\big) &= \#\big(\Omega_i^h(\lambda) \cap \Omega'_k(\mu)\big)\\ &= \#\Omega_{i,k}(\lambda) = z_{ik}\end{aligned}$$

as desired. □

Corollary 2. *Let $\lambda, \mu \vdash n$. Then the double (S_λ, S_μ)-coset $S_\lambda \sigma S_\mu$ has the trivial intersection property if and only if the associated matrix $(z_{ik})_{i,j=1}^n \in \mathcal{M}(\lambda, \mu)$ (as defined in the previous proposition) satisfies $z_{ik} \in \{0, 1\}$ for all $1 \le i, k \le n$.*

Proof. Note that

$$\begin{aligned}S_\lambda \cap \sigma S_\mu \sigma^{-1} &= \{\tau \in S_\lambda \mid \tau(\sigma(\Omega_k^h(\mu))) = \sigma(\Omega_k^h(\mu)) \quad \forall\, k\}\\ &= \{\tau \in S_n \mid \tau(\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu))) = \Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu)) \quad \forall\, i, k\}.\end{aligned} \tag{2}$$

We conclude that $S_\lambda \sigma S_\mu$ has the trivial intersection property iff

$$S_\lambda \cap \sigma S_\mu \sigma^{-1} = \{e\}$$

iff

$$z_{ik} = \#(\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu)) \in \{0, 1\}$$

for all $i, k \in \{1, \ldots, n\}$. □

Crucial for our purposes is the following proposition.

Proposition 3. *Let $\lambda \vdash n$ and let $w_\lambda \in S_n$ be the unique element such that $w_\lambda t^\lambda = t_\lambda$. The only double $(S_\lambda, S_{\lambda'})$-coset with the trivial intersection property is $S_\lambda w_\lambda^{-1} S_{\lambda'}$.*

Proof. For $1 \le i, k \le n$ we have

$$\begin{aligned}\#\big(\Omega_i^h(\lambda) \cap w_\lambda^{-1}(\Omega_k^h(\lambda'))\big) &= \#\big(\Omega_i^h(t^\lambda) \cap w_\lambda^{-1}\Omega_k^h(t^{\lambda'})\big)\\ &= \#\big(\Omega_i^h(t^\lambda) \cap w_\lambda^{-1}\Omega_k^v(t_\lambda)\big)\\ &= \#\big(\Omega_i^h(t^\lambda) \cap \Omega_k^v(t^\lambda)\big)\end{aligned}$$

which is one if $1 \le k \le \lambda_i$ and zero otherwise. Hence $S_\lambda w_\lambda^{-1} S_{\lambda'}$ has the trivial intersection property.

To show it is the only one, we prove that there exists a unique matrix $(z_{ik})_{i,k=1}^n$ with $z_{ik} \in \{0, 1\}$ satisfying $\sum_{k=1}^n z_{ik} = \lambda_i$ and $\sum_{i=1}^n z_{ik} = \lambda'_k$.

One such matrix $(z_{ik})_{i,k=1}^n$ is clear (it is the one corresponding to the double coset $S_\lambda w_\lambda^{-1} S_{\lambda'}$): it is given by $z_{ik} = 1$ if $1 \leq k \leq \lambda_i$ and zero otherwise.

Suppose $(z_{ik})_{i,k=1}^n$ is another matrix satisfying these properties. Then the first λ_1 entries of the first row should be ones (and hence the remaining $n - \lambda_1$ entries in the first row zeros). Indeed, if this is not the case then there exists a $\lambda_1 < k \leq n$ such that the $z_{1k} = 1$. But then $\lambda'_k = \sum_{i=1}^n z_{ik} \geq 1$. But $l(\lambda') = \lambda_1 < k$ hence $\lambda'_k = 0$, contradiction. A similar argument shows that in the i'th row of $(z_{ik})_{i,k}$, the first $\lambda_{i'}$ entries should be ones (and the remaining entries zeros). This proves the result. □

Exercise 5. *Complete the argument from the proof of the previous proposition. In other words, show that there exists a unique matrix* $(z_{ik})_{i,k=1}^n$ *with* $z_{ik} \in \{0, 1\}$ *satisfying* $\sum_{k=1}^n z_{ik} = \lambda_i$ *for all* i *and satisfying* $\sum_{i=1}^n z_{ik} = \lambda'_k$ *for all* k.

3. Locating the irreducible S_n-representations

We start with the following general result.

Theorem 6 (The intertwining number theorem)**.** *Let* $H, K \subseteq G$ *be subgroups. Let* $\theta : H \to \mathrm{GL}(V)$ *and* $\rho : K \to \mathrm{GL}(W)$ *be finite dimensional representations of* H *and* K *respectively. Then*

$$\mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(G)}(\mathrm{Ind}_H^G(\theta), \mathrm{Ind}_K^G(\rho)\big) = \sum_{KgH \in K\backslash G/H} \mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(H_g)}(\theta_g, \mathrm{Res}^K_{H_g}(\rho))\big),$$

where $H_g = gHg^{-1} \cap K$ *and* $\theta_g : H_g \to \mathrm{GL}(V)$ *is defined by* $\theta_g(x) := \theta(g^{-1}xg)$ $(x \in H_g)$*.*

Proof. We know that

$$\mathrm{Res}^G_K\big(\mathrm{Ind}_H^G(\theta)\big) \simeq \bigoplus_{g \in \mathcal{S}} \mathrm{Ind}^K_{H_g}(\theta_g),$$

with $\mathcal{S}$ a complete set of representatives of the double coset space $K \setminus G/H$. Hence, by two applications of Frobenius reciprocity,

$$\begin{aligned}
\mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(G)}(\mathrm{Ind}_H^G(\theta), \mathrm{Ind}_K^G(\rho))\big) &= \big(\mathrm{Ind}_H^G(\chi_\theta), \mathrm{Ind}_K^G(\chi_\rho)\big)_G \\
&= \big(\mathrm{Res}_K^G(\mathrm{Ind}_H^G(\chi_\theta)), \chi_\rho\big)_K \\
&= \sum_{g\in\mathcal{S}} \big(\mathrm{Ind}_{H_g}^K(\chi_{\theta_g}), \chi_\rho\big)_K \\
&= \sum_{g\in\mathcal{S}} \big(\chi_{\theta_g}, \mathrm{Res}_{H_g}^K(\chi_\rho)\big)_{H_g} \\
&= \sum_{g\in\mathcal{S}} \mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(H_g)}(\theta_g, \mathrm{Res}_{H_g}^K(\rho))\big).
\end{aligned}$$

□

We apply the intertwining number theorem to the case that $(H, K) = (S_\mu, S_\nu)$ with $\mu, \nu \vdash n$. Let $\epsilon_\mu : S_\mu \to \mathbb{C}^*$ the (restriction of the) alternating representation ($\epsilon_\mu(\sigma) = 1$ if $\sigma \in S_\mu$ is an even permutation and $\epsilon_\mu(\sigma) = -1$ otherwise) and let $\rho_\nu : S_\nu \to \mathbb{C}^*$ be the trivial representation ($\rho_\nu(\sigma) = 1$ for all $\sigma \in S_\nu$). Here we identify these one-dimensional representations with their characters in the usual manner.

Corollary 3. *Let $\mu, \nu \vdash n$. Then*

$$\mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(S_n)}(\mathrm{Ind}_{S_\mu}^{S_n}(\epsilon_\mu), \mathrm{Ind}_{S_\nu}^{S_n}(\rho_\nu))\big)$$

is the number of double (S_ν, S_μ)-cosets with the trivial intersection property.

Proof. Let $\mathcal{S}$ be a complete set of coset representatives of the double (S_ν, S_μ)-cosets in S_n. For $\sigma \in \mathcal{S}$ the twisted one-dimensional representation $\epsilon_{\mu,\sigma}$ of $H_\sigma := \sigma S_\mu \sigma^{-1} \cap S_\nu$ coincides with the restriction of the alternating representation $\epsilon : S_n \to \mathbb{C}^*$ to H_σ, while $\mathrm{Res}_{H_\sigma}^{S_\nu}(\rho_\nu)$ is the restriction of the trivial representation of S_n to H_σ. They are equivalent if and only if they coincide (since they are one-dimensional), and this is if and only if $H_\sigma = \{e\}$ (see the exercise below). Now note that $H_\sigma = \{e\}$ iff the double (S_ν, S_μ)-coset $S_\nu \sigma S_\mu$ has the trivial intersection property.

Finally, if $\epsilon_{\mu,\sigma}$ and $\mathrm{Res}_{H_\sigma}^{S_\nu}(\rho_\nu)$ coincide, then clearly

$$\mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(H_g)}\big(\epsilon_{\mu,g}, \mathrm{Res}_{H_g}^{S_\nu}(\rho_\nu)\big)\big) = 1,$$

and it is zero otherwise. The result now follows from the intertwining number theorem. □

Exercise 7. *Using the notations of the corollary, show that* $\epsilon_{\mu,\sigma} = \mathrm{Res}^{S_\nu}_{H_\sigma}(\rho_\nu)$ *iff* $H_\sigma = \{e\}$.
Hint: *Use formula* (2).

Corollary 4. *Let* $\lambda \vdash n$. *Up to isomorphism there exists a unique irreducible linear* S_n*-representation* $\pi_\lambda : S_n \to \mathrm{GL}_{\mathbb{C}}(V_\lambda)$ *such that*

$$(\chi_\lambda, \mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda))_{S_n} = 1 = (\chi_\lambda, \mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'}))_{S_n}, \tag{3}$$

where $\chi_\lambda := \chi_{\pi_\lambda} \in F(S_n)$ *is the irreducible character of* π_λ.

Proof. By Proposition 3 and Corollary 3 we have

$$\big(\mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'}), \mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda)\big)_{S_n} = \mathrm{Dim}_{\mathbb{C}}\big(\mathrm{Hom}^{(S_n)}\big(\mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'}), \mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda)\big)\big) = 1$$

(we interpret here the induction in the left term as induction of characters, and for the middle term as induction of representations). On the other hand,

$$\big(\mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'}), \mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda)\big)_{S_n} = \sum_{\pi \in \widehat{S_n}} d_\pi e_\pi \tag{4}$$

with $d_\pi, e_\pi \in \mathbb{Z}_{\geq 0}$ $(\pi \in \widehat{S_n})$ such that

$$\mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'}) = \sum_{\pi \in \widehat{S_n}} d_\pi \chi_\pi,$$
$$\mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda) = \sum_{\pi \in \widehat{S_n}} e_\pi \chi_\pi$$

in $F(S_n)$. Then (4) being one thus implies that the representations $\mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'})$ and $\mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda)$ have a unique common irreducible constituent $\pi_\lambda \in \widehat{S_n}$, whose irreducible character χ_λ satisfies (3) (in particular, π_λ occurs with multiplicity one in both $\mathrm{Ind}^{S_n}_{S_{\lambda'}}(\epsilon_{\lambda'})$ and $\mathrm{Ind}^{S_n}_{S_\lambda}(\rho_\lambda)$). □

Next week we prove that $\{\pi_\lambda\}_{\lambda \in \mathcal{P}_n}$ is a complete set of representatives of $\widehat{S}_n$.

References

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[2] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics, **42**, Springer-Verlag, New York, 1977.