ALGEBRA 3; REPRESENTATIE THEORIE. AANVULLING 5

1. Introduction

We keep the notations of Aanvulling 4. Recall that we have constructed for $\lambda \in \mathcal{P}_n$ an irreducible linear representation $\pi_\lambda : S_n \to \mathrm{GL}_{\mathbb{C}}(V_\lambda)$ of S_n as the unique irreducible constituent occurring in both $\mathrm{Ind}_{S_\lambda}^{S_n}(\rho_\lambda)$ and $\mathrm{Ind}_{S_{\lambda'}}^{S_n}(\epsilon_{\lambda'})$. In these lecture notes we show that $\{\pi_\lambda\}_{\lambda\in\mathcal{P}_n}$ is a complete set of representatives of $\widehat{S}_n$.

2. Locating V_λ inside of the group algebra

We know that the regular representation $\rho : S_n \to \mathrm{GL}_{\mathbb{C}}(\mathbb{C}[S_n])$ contains all the irreducible S_n-representations as irreducible components, in particular it contains π_λ.

We first realize $\mathrm{Ind}_{S_\lambda}^{S_n}(\rho_\lambda)$ and $\mathrm{Ind}_{S_{\lambda'}}^{S_n}(\epsilon_{\lambda'})$ inside $\mathbb{C}[S_n]$. Define for $\lambda, \mu \vdash n$,

$$\mathcal{H}_\lambda := \sum_{\sigma\in S_\lambda} e_\sigma \in \mathbb{C}[S_n],$$
$$\mathcal{V}_\mu := \sum_{\sigma\in S_\mu} \epsilon(\sigma) e_\sigma \in \mathbb{C}[S_n]$$

(recall that $\epsilon(\sigma)$ is the sign of $\sigma \in S_n$). The elements $\mathcal{H}_\lambda$ and $\mathcal{V}_\mu$ are called Young projectors. Their basic property is

$$\begin{aligned} \tau\mathcal{H}_\lambda &= \mathcal{H}_\lambda = \mathcal{H}_\lambda\tau, \qquad \forall\, \tau \in S_\lambda,\\ \tau\mathcal{V}_\mu &= \epsilon(\tau)\mathcal{V}_\mu = \mathcal{V}_\mu\tau, \qquad \forall\, \tau \in S_\mu. \end{aligned} \tag{1}$$

Hence $\mathbb{C}\mathcal{H}_\lambda \subseteq \mathrm{Res}_{S_\lambda}^{S_n}(\mathbb{C}[S_n])$ is isomorphic to ρ_λ and $\mathbb{C}\mathcal{V}_\mu \subseteq \mathrm{Res}_{S_\mu}^{S_n}(\mathbb{C}[S_n])$ is isomorphic to ϵ_μ. The left ideals $\mathbb{C}[S_n]\mathcal{H}_\lambda \subseteq \mathbb{C}[S_n]$ and $\mathbb{C}[S_n]\mathcal{V}_\mu \subseteq \mathbb{C}[S_n]$ are subrepresentations of the regular representation. Write $\mathbb{C}_{\rho_\lambda}$ (respectively $\mathbb{C}_{\epsilon_\mu}$) for the (one-dimensional) representation space of ρ_λ (respectively ϵ_μ).

Lemma 1. (i) *$\mathbb{C}[S_n]\mathcal{H}_\lambda$ is isomorphic to $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda})$.*
(ii) *$\mathbb{C}[S_n]\mathcal{V}_\mu$ is isomorphic to $\mathrm{Ind}_{S_\mu}^{S_n}(\mathbb{C}_{\epsilon_\mu})$.*

Proof. **(i)** The unique bilinear map

$$\widetilde{f} : \mathbb{C}[S_n] \times \mathbb{C}_{\rho_\lambda} \to \mathbb{C}[S_n]\mathcal{H}_\lambda$$

satisfying

$$\widetilde{f}(e_\tau, 1) := e_\tau \mathcal{H}_\lambda$$

for all $\tau \in S_n$ is $\mathbb{C}[S_\lambda]$-bilinear, hence it gives rise to a linear map

$$f : \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda}) \to \mathbb{C}[S_n]\mathcal{H}_\lambda$$

satisfying $f(e_\tau \otimes_{\mathbb{C}[S_\lambda]} 1) = e_\tau \mathcal{H}_\lambda$ for all $\tau \in S_n$. It is clear that $f \in \mathrm{Hom}^{(S_n)}(\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda}), \mathbb{C}[S_n]\mathcal{H}_\lambda)$, hence it suffices to prove that f is an isomorphism. Let $\mathcal{S}$ be a complete set of representatives of the left coset space S_n/S_λ. Then $\{e_\tau \otimes_{\mathbb{C}[S_\lambda]} 1\}_{\tau \in \mathcal{S}}$ is a linear basis of $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda})$ which is mapped to $\{e_\tau \mathcal{H}_\lambda\}_{\tau \in \mathcal{S}}$ by f. It thus remains to show that $\{e_\tau \mathcal{H}_\lambda\}_{\tau \in \mathcal{S}}$ is a linear basis of $\mathbb{C}[S_n]\mathcal{H}_\lambda$. But this follows directly from the definition of $\mathcal{H}_\lambda$ and (1).
(ii) The proof is similar to the proof of **(i)**. □

Recall that $w_\lambda \in S_n$ is the unique element such that $w_\lambda t^\lambda = t_\lambda$ (recall from last time that $S_\lambda w_\lambda^{-1} S_{\lambda'}$ is the unique double $(S_\lambda, S_{\lambda'})$-coset with the trivial intersection property).

Exercise 1. *Let* $\lambda \vdash n$.
(i) *Show that* $w_{\lambda'}^{-1} = w_\lambda$.
(ii) *Show that* $w_\lambda^{-1} S_{\lambda'} w_\lambda = V(t^\lambda)$.

We define

$$\widetilde{p}_\lambda := e_{w_\lambda^{-1}} \mathcal{V}_{\lambda'} e_{w_\lambda} \mathcal{H}_\lambda \in \mathbb{C}[S_n]$$

and

$$M_\lambda := \mathbb{C}[S_n]\widetilde{p}_\lambda$$

for the left ideal in $\mathbb{C}[S_n]$ it generates. Note that

$$(2) \qquad \sigma \widetilde{p}_\lambda \tau = \epsilon(\sigma)\widetilde{p}_\lambda, \qquad \forall\, \sigma \in V(t^\lambda), \forall\, \tau \in S_\lambda = H(t^\lambda)$$

(cf. the previous exercise). Clearly $M_\lambda \subseteq \mathbb{C}[S_n]$ is a subrepresentation of the regular S_n-representation. It is called a Specht module.

Lemma 2. **(i)** $M_\lambda \neq \{0\}$.
(ii) $\mathrm{Hom}^{(S_n)}(\mathbb{C}[S_n]\mathcal{H}_\lambda, M_\lambda) \neq 0$.
(iii) $\mathrm{Hom}^{(S_n)}(\mathbb{C}[S_n]\mathcal{V}_{\lambda'}, M_\lambda) \neq 0$.

Proof. **(i)** It suffices to show that $\widetilde{p}_\lambda \neq 0$. By the previous exercise we have

$$\begin{aligned}\widetilde{p}_\lambda &= e_{w_\lambda^{-1}}\mathcal{V}_{\lambda'}e_{w_\lambda}\mathcal{H}_\lambda \\ &= \sum_{\sigma\in H(t^{\lambda'})}\sum_{\tau\in H(t^\lambda)}\epsilon(\sigma)e_{w_\lambda^{-1}\sigma w_\lambda\tau} \\ &= \sum_{\sigma\in V(t^\lambda)}\sum_{\tau\in H(t^\lambda)}\epsilon(\sigma)e_{\sigma\tau}.\end{aligned}$$

The map $V(t^\lambda)\times H(t^\lambda)\to S_n$ defined by $(\sigma,\tau)\mapsto\sigma\tau$ is injective since $V(t^\lambda)\cap H(t^\lambda)=\{e\}$. Hence $\widetilde{p}_\lambda\neq 0$.
(ii) This is clear from **(i)** since $M_\lambda\subset\mathbb{C}[S_n]\mathcal{H}_\lambda$.
(iii) By Frobenius reciprocity,

$$\mathrm{Hom}^{(S_n)}(\mathbb{C}[S_n]\mathcal{V}_{\lambda'},M_\lambda)\simeq\mathrm{Hom}^{(S_{\lambda'})}(\mathbb{C}_{\epsilon_{\lambda'}},\mathrm{Res}^{S_n}_{S_{\lambda'}}(M_\lambda)).$$

By **(i)** and (1),

$$\mathbb{C}_{\epsilon_{\lambda'}}\simeq\mathbb{C}e_{w_\lambda}\widetilde{p}_\lambda\subseteq\mathrm{Res}^{S_n}_{S_{\lambda'}}(M_\lambda)$$

as $S_{\lambda'}$-representations, hence

$$\mathrm{Hom}^{(S_n)}(\mathbb{C}[S_n]\mathcal{V}_{\lambda'},M_\lambda)\neq\{0\}.$$

□

We will show in the next section that $M_\lambda\simeq V_\lambda$ as S_n-representations which, in view of the previous lemma, will immediately follow if we show that M_λ is irreducible.

3. Irreducibility and mutual inequivalence

For $\lambda\vdash n$ we write

$$p_\lambda:=\frac{\dim(M_\lambda)}{n!}\widetilde{p}_\lambda=\frac{\dim(M_\lambda)}{n!}e_{w_\lambda^{-1}}\mathcal{V}_{\lambda'}e_{w_\lambda}\mathcal{H}_\lambda.$$

We will show that the $p_\lambda\in\mathbb{C}[S_n]$ $(\lambda\vdash n)$ are mutually orthogonal primitive idempotents of $\mathbb{C}[S_n]$. We defined before orthogonal idempotents in a commutative ring. It extends to arbitrary rings as follows.

Definition 1. *Let R be a ring. An element $p\in R$ is called an idempotent if $p^2=p$. Two idempotents $p_1,p_2\in R$ are called mutually orthogonal if $p_1p_2=0=p_2p_1$. An idempotent $p\in R$ is called primitive if $p=p_1+p_2$ with p_1,p_2 mutually orthogonal idempotents imply that $p_1=0$ or $p_2=0$.*

Exercise 2. *Let $p\in\mathbb{C}[S_n]$ be an idempotent. Show that $\mathbb{C}[S_n]p$ is an irreducible S_n-representation of the regular representation iff p is a primitive idempotent.*

Lemma 3. *Let $p \in \mathbb{C}[S_n]$ be an idempotent. Then p is primitive iff $p\mathbb{C}[S_n]p = \mathbb{C}p$.*

Proof. $\Rightarrow$ If p is primitive then $M := \mathbb{C}[S_n]p$ is irreducible hence $\mathrm{End}^{(S_n)}(M) = \mathbb{C}\mathrm{Id}_M$. For $h \in \mathbb{C}[S_n]$ the linear map $\phi_h : M \to M$ defined by

$$\phi_h(m) := mphp, \qquad m \in M = \mathbb{C}[S_n]p$$

is an intertwiner, hence $\phi_h = c_h\mathrm{Id}_M$ for some $c_h \in \mathbb{C}$. Then $php = \phi_h(p) = c_h p$.
$\Leftarrow$ Suppose $p = p_1 + p_2$ with p_i pairwise orthogonal idempotents. By the assumption there exist $c_1, c_2 \in \mathbb{C}$ such $pp_ip = c_ip$ $(i = 1, 2)$. On the other hand $(p_1 + p_2)p_i(p_1 + p_2) = p_i$, hence $p_i = c_ip$ $(i = 1, 2)$. Then $0 = p_1p_2 = c_1c_2p$, i.e. $c_1c_2 = 0$. Then $p_1 = c_1p = 0$ or $p_2 = c_2p = 0$, contradiction. □

In addition we will use

Lemma 4. *Let $p_1, p_2 \in \mathbb{C}[S_n]$ be primitive idempotents. Then $\mathbb{C}[S_n]p_1 \simeq \mathbb{C}[S_n]p_2 \Leftrightarrow p_1\mathbb{C}[S_n]p_2 \neq \{0\}$.*

Proof. $\Rightarrow$ Let $T : \mathbb{C}[S_n]p_1 \to \mathbb{C}[S_n]p_2$ be a bijective intertwiner. Then $0 \neq T(p_1) = hp_2$ for some $h \in \mathbb{C}[S_n]$. This implies that

$$0 \neq T(p_1) = T(p_1^2) = p_1T(p_1) = p_1hp_2.$$

$\Leftarrow$ Let $h \in \mathbb{C}[S_n]$ such that $p_1hp_2 \neq 0$. Define $\phi_h : \mathbb{C}[S_n]p_1 \to \mathbb{C}[S_n]p_2$ by

$$\phi_h(m) := mp_1hp_2.$$

Then ϕ_h is an intertwiner, and it is nonzero since $\phi_h(p_1) = p_1hp_2 \neq 0$. Since $\mathbb{C}[S_n]p_i$ $(i = 1, 2)$ is irreducible we conclude that ϕ_h is an isomorphism. □

Lemma 5. *For $\lambda, \mu \vdash n$ and $\sigma \in S_n$ we have*
(i) *$\mathcal{V}_{\lambda'}e_\sigma\mathcal{H}_\mu = 0$ if $S_{\lambda'}\sigma S_\mu$ does not have the trivial intersection property.*
(ii) *$\mathcal{V}_{\lambda'}e_\sigma\mathcal{H}_\lambda = 0$ unless $\sigma \in S_{\lambda'}w_\lambda S_\lambda$.*

Proof. **(i)** If $S_{\lambda'}\sigma S_\mu$ does not have the trivial intersection property then

$$H := S_{\lambda'} \cap \sigma S_\mu \sigma^{-1} \neq \{e\}.$$

But H is a Young subgroup of S_n (cf. formula (2) in Aanvulling 4), hence there exists $1 \leq a < b \leq n$ such that the corresponding transposition $\tau := (a\,b)$ is in $H \subseteq S_{\lambda'}$. Since $\epsilon(\tau) = -1$ we can write

$$\mathcal{V}_{\lambda'} = Y(e_e - e_\tau)$$

with $Y=\sum_{\xi\in\mathcal{S}}\epsilon(\xi)e_\xi$ and $\mathcal{S}$ a complete set of representatives of the left coset space $S_{\lambda'}/\langle\tau\rangle$. Then

$$\begin{aligned}\mathcal{V}_{\lambda'}e_\sigma\mathcal{H}_\mu &= Y(e_e-e_\tau)e_\sigma\mathcal{H}_\mu\\ &= Ye_\sigma(e_e-e_{\sigma^{-1}\tau\sigma})\mathcal{H}_\mu=0\end{aligned}$$

where we use that $\sigma^{-1}\tau\sigma\in S_\mu$ and $h\mathcal{H}_\mu=\mathcal{H}_\mu$ for all $h\in S_\mu$.
(ii) In aanvulling 4, Proposition 3 we have seen that the only double $(S_{\lambda'},S_\lambda)$-coset with the trivial intersection property is $S_{\lambda'}w_{\lambda'}^{-1}S_\lambda=S_{\lambda'}w_\lambda S_\lambda$. For the second equality we use that $w_{\lambda'}^{-1}=w_\lambda$, see Exercise 1. The result now follows from **(i)**. □

Let $\sigma\in S_n$. By the previous lemma, if $\sigma\notin S_{\lambda'}w_\lambda S_\lambda$ then

$$\mathcal{V}_{\lambda'}e_\sigma\mathcal{H}_\lambda=0$$

and if $\sigma\in S_{\lambda'}w_\lambda S_\lambda$ then

$$\mathcal{V}_{\lambda'}e_\sigma\mathcal{H}_\lambda=\pm e_{w_\lambda}p_\lambda\neq 0$$

since $\mathcal{V}_{\lambda'}e_\xi=\epsilon(\xi)\mathcal{V}_{\lambda'}$ for $\xi\in S_{\lambda'}$ and $e_\eta\mathcal{H}_\lambda=\mathcal{H}_\lambda$ for $\eta\in S_\lambda$. In particular,

$$p_\lambda\mathbb{C}[S_n]p_\lambda\subseteq e_{w_\lambda^{-1}}\mathcal{V}_{\lambda'}\mathbb{C}[S_n]\mathcal{H}_\lambda=\mathrm{span}\{p_\lambda\}. \qquad (3)$$

This leads to the following result.

Theorem 3. *Let $\lambda\vdash n$.*
(i) *$p_\lambda\in\mathbb{C}[S_n]$ is a primitive idempotent.*
(ii) *The irreducible S_n-subrepresentation $M_\lambda=\mathbb{C}[S_n]p_\lambda$ of the regular representation is isomorphic to V_λ.*

Proof. **(i) Step 1:** p_λ is an idempotent.
In view of (3) we have

$$p_\lambda^2=cp_\lambda$$

for some $c\in\mathbb{C}$. We need to show that $c=1$. Let $\phi\in\mathrm{End}^{(S_n)}(\mathbb{C}[S_n])$ be the map

$$\phi(h):=h\widetilde{p}_\lambda,\qquad h\in\mathbb{C}[S_n].$$

We compute the trace of ϕ in two different ways. Recall that

$$\widetilde{p}_\lambda=\sum_{\sigma\in V(t^\lambda)}\sum_{\tau\in H(t^\lambda)}\epsilon(\sigma)e_{\sigma\tau}$$

and the map $V(t^\lambda)\times H(t^\lambda)\to S_n$, given by $(\sigma,\tau)\mapsto\sigma\tau$, is injective. Consequently, if $\xi\in S_n$ then $\xi\sigma\tau=\xi$ for $\sigma\in H(t^\lambda)$ and $\tau\in V(t^\lambda)$ iff $\sigma=e=\tau$. Consequently

$$\mathrm{Tr}_{\mathbb{C}[S_n]}(\phi)=\sum_{\xi\in S_n}e_\xi\widetilde{p}_\lambda|_{e_\xi}=n!.$$

On the other hand, there exists a S_n-subrepresentation $V \subset \mathbb{C}[S_n]$ such that

$$\mathbb{C}[S_n] = M_\lambda \oplus V.$$

The intertwiner $\phi : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ clearly maps onto M_λ. Hence, choosing a basis of M_λ and a basis of V and computing the trace $\mathrm{Tr}(\phi)$ with respect to this choice of basis, we get

$$\mathrm{Tr}_{\mathbb{C}[S_n]}(\phi) = \mathrm{Tr}_{M_\lambda}(\phi). \tag{4}$$

But for $h = ap_\lambda \in M_\lambda$ ($a \in \mathbb{C}[S_n]$) we have

$$\phi(h) = ap_\lambda \widetilde{p}_\lambda = \frac{cn!}{\dim(M_\lambda)} ap_\lambda = \frac{cn!}{\dim(M_\lambda)} h, \tag{5}$$

hence $\phi|_{M_\lambda} = \frac{cn!}{\dim(M_\lambda)} \mathrm{Id}_{M_\lambda}$. Combining the two observations (4) and (5) we get

$$\mathrm{Tr}_{\mathbb{C}[S_n]}(\phi) = cn!.$$

Hence $c = 1$.

Step 2: p_λ is primitive.

This follows from (3) and Lemma 3.

(ii) $M_\lambda = \mathbb{C}[S_n]p_\lambda$ is an irreducible S_n-subrepresentation of the regular representation in view of Exercise 2. By Lemma 1 the representation M_λ is an irreducible constituent of both $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda})$ and $\mathrm{Ind}_{S_{\lambda'}}^{S_n}(\mathbb{C}_{\epsilon_{\lambda'}})$. But this was the characterization of V_λ, hence $M_\lambda \simeq V_\lambda$. □

Remark 4. We normalized $\widetilde{p}_\lambda$ using the degree $\mathrm{Dim}(M_\lambda)(= \mathrm{Dim}(V_\lambda))$ of the irreducible representation in order to turn it into the idempotent p_λ. Remarkably we do not need to know the degree explicitly in order to prove that the resulting normalized element p_λ is indeed an idempotent. It is possible to prove that

$$\mathrm{Dim}(V_\lambda) = \#\{\text{standard Young tableaux of shape } \lambda\}.$$

A proof of this formula requires quite some work, see e.g. [1].

Theorem 5. *Let $\lambda, \mu \vdash n$. Then $M_\lambda \simeq M_\mu$ as S_n-representations iff $\lambda = \mu$.*

Proof. Suppose that $\lambda \neq \mu$. Without loss of generality we may assume that $\mu \not\trianglelefteq \lambda$. By Lemma 4 it suffices to show that $p_\lambda \mathbb{C}[S_n] p_\mu = \{0\}$. By the explicit expression of the primitive idempotents p_λ and p_μ this is true iff

$$\mathcal{V}_{\lambda'} \mathbb{C}[S_n] \mathcal{H}_\mu = \{0\}.$$

By Lemma 5 this is equivalent to the condition that there do not exist double $(S_{\lambda'}, S_\mu)$-cosets with the trivial intersection property. This follows from Lemma 6 below (see also Exercise 6). □

Corollary 1. *$\{V_\lambda\}_{\lambda \vdash n}$ is a complete set of representatives of the isomorphy classes of irreducible linear S_n-representations.*

Lemma 6. *Let $\lambda, \mu \vdash n$. If there exists a double (S_λ, S_μ)-coset with the trivial intersection property then $\lambda \preceq \mu'$.*

Proof. Let $\sigma \in S_n$ such that $S_\lambda \sigma S_\mu$ has the trivial intersection property. Then $S_\lambda \cap \sigma S_\mu \sigma^{-1} = \{e\}$ i.e., in view of formula (2) of Aanvulling 4,

$$\Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu))$$

consists of zero or one elements for all i and k. But

$$\begin{aligned} \Omega_i^h(\lambda) \cap \sigma(\Omega_k^h(\mu)) &= \Omega_i^h(t^\lambda) \cap \sigma\Omega_k^h(t^\mu) \\ &= \Omega_i^h(t^\lambda) \cap \sigma\Omega_k^v(t_{\mu'}) \\ &= \Omega_i^h(t^\lambda) \cap \Omega_k^v(\sigma t_{\mu'}). \end{aligned}$$

Hence we conclude that if $1 \leq r < s \leq n$ are two numbers in the *same* row of t^λ, then they are in *different* columns of $\sigma t_{\mu'}$.

Fix now $r \geq 1$. Then we conclude that for all k, the set $\Omega_k^v(\sigma t_{\mu'})$ of numbers in the kth column of $\sigma t_{\mu'}$ contains at most r numbers from the first r rows of t^λ for all $r \geq 1$. Thus we can find a $\tau \in V(\sigma t_{\mu'})$ such that all the numbers in the first r rows of t^λ are contained in the first r rows of $\tau\sigma t_{\mu'}$. In particular

$$\#\{\text{boxes in the first r rows of } t^\lambda\} \leq \#\{\text{boxes in the first r rows of } \tau\sigma t_{\mu'}\},$$

i.e.

$$\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{r} \mu_i'.$$

This is valid for all $r \geq 1$, hence $\lambda \preceq \mu'$. □

Exercise 6. *Complete the last step of the proof of Theorem 5 using Lemma 6.*

Let $\lambda \vdash n$. Remark 4 suggests that it might be possible to make the irreducible S_n-representation V_λ explicit by realizing it in terms of a S_n-action on the formal complex vector space with canonical basis the standard Young tableaux of shape λ. This is indeed possible. We sketch it in the following section.

4. Polytabloids

Let $\lambda \vdash n$. We say that two Young tableaux T and T' of shape $\lambda \vdash n$ are row equivalent, $T \sim T'$, if there exists a $\sigma \in H(T)$ such that $\sigma T = T'$. The corresponding equivalence class $\{T\}$ of a Young tableaux T of shape λ is called a λ-taboid. Write Tabl(λ) for the set of λ-tabloids. Note that $\{T\} \in \mathrm{Tabl}(\lambda)$ has a unique row standard representative T.

Let N_λ be the formal vector space with basis the λ-tabloids $\{T\} \in \mathrm{Tabl}(\lambda)$. An element $m \in N_\lambda$ thus is $m = \sum_{\{T\}\in\mathrm{Tabl}(\lambda)} c_{\{T\}}\{T\}$ for unique $c_{\{T\}} \in \mathbb{C}$. By the previous paragraph,

$$\mathrm{Dim}(N_\lambda) = \#\{\text{row standard Young tableau of shape } \lambda\}.$$

Lemma 7. **(i)** *The assignment* $(\sigma, \{T\}) \mapsto \{\sigma T\}$ *for* $\sigma \in S_n$ *and* $\{T\} \in \mathrm{Tabl}(\lambda)$ *gives rise to a linear* S_n*-representation* $S_n \to \mathrm{GL}_{\mathbb{C}}(N_\lambda)$.
(ii) $N_\lambda \simeq \mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda})$.

Proof. **(i)** It suffices to show that $\{\sigma T\} = \{\sigma T'\}$ if $\{T\} = \{T'\}$. Indeed, $\{T\} = \{T'\}$ implies $\Omega_i^h(T) = \Omega_i^h(T')$ (as unordered sets). Since $\sigma\Omega_i^h(T) = \Omega_i^h(\sigma T)$ this implies $\{\sigma T\} = \{\sigma T'\}$.
(ii) It is clear that the degrees of $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda})$ and N_λ coincide (it is $n!/(\lambda_1!\lambda_2!\cdots\lambda_n!)$). Consider the bilinear map $\mathbb{C}[S_n] \times \mathbb{C}_{\rho_\lambda} \to N_\lambda$ defined by $(e_\sigma, 1) \mapsto \{\sigma t^\lambda\}$. It is surjective and $\mathbb{C}[S_\lambda]$-bilinear, hence it defines a surjective intertwiner $\mathrm{Ind}_{S_\lambda}^{S_n}(\mathbb{C}_{\rho_\lambda}) \to N_\lambda$. This implies the result. □

Definition 2. *Let* T *be a Young tableaux of shape* λ*. The polytabloid* $f_\lambda \in N_\lambda$ *is defined by*

$$f_T := \sum_{\tau \in V(T)} \epsilon(\tau)\{\tau T\}$$

(warning: it is not true that f_T *only depends on* $\{T\}$*).*

Lemma 8. *The span* P_λ *of the polytabloids of shape* λ *is a* S_n*-subrepresentation of* N_λ*. In fact,*

$$\sigma f_T = f_{\sigma T}$$

for $\sigma \in S_n$ *and for a Young tableau* T *of shape* λ*.*

Proof. Let $\sigma \in S_n$ and T a Young tableau of shape λ. Then, since $\sigma V(T)\sigma^{-1} = V(\sigma T)$,

$$\begin{aligned}\sigma f_T &= \sum_{\tau\in V(T)} \epsilon(\sigma\tau\sigma^{-1})\{(\sigma\tau\sigma^{-1})\sigma T\}\\ &= \sum_{\tau\in V(\sigma T)} \epsilon(\tau)\{\tau\sigma T\} = f_{\sigma T}.\end{aligned}$$

□

Theorem 7. P_λ *is isomorphic to the Specht module* $M_\lambda \simeq V_\lambda$*.*

Proof. As in the proof that $p_\lambda \neq 0$, one shows that $f_{t^\lambda} \neq 0$. Hence $P_\lambda \neq \{0\}$. In addition, since $\sigma f_T = f_{\sigma T}$ for $\sigma \in S_n$ and T a Young tableau of shape λ, we have $P_\lambda = \mathbb{C}[S_n] f_{t^\lambda}$. Since

$$p_\lambda = \frac{\dim(M_\lambda)}{n!} \sum_{\sigma \in V(t^\lambda)} \sum_{\tau \in H(t^\lambda)} \epsilon(\sigma) e_{\sigma\tau}$$

and $\tau\{f^\lambda\} = \{f^\lambda\}$ for all $\tau \in H(t^\lambda)$ we have

$$f_{t^\lambda} = \frac{1}{\lambda_1! \cdots \lambda_n!} p_\lambda \{t^\lambda\}.$$

Hence $P_\lambda = \mathbb{C}[S_n] p_\lambda f_{t^\lambda} = M_\lambda f_{t^\lambda}$. Thus the S_n-intertwiner

$$M_\lambda \to P_\lambda, \qquad h \mapsto h f_{t^\lambda}$$

is surjective. It is also injective since M_λ is irreducible and $P_\lambda \neq \{0\}$. Hence $M_\lambda \simeq P_\lambda$. $\square$

The following result we state without proof (for details, see e.g. [1]).

Proposition 1. $\{f_T \,|\, T$ standard Young tableau of shape $\lambda\}$ *is a linear basis of* P_λ*. In particular, the degree of the Specht module* V_λ *is the number of standard tableaux of shape* λ*.*

Exercise 8. (i) *Let* $\mathbb{C}^n$ *be the permutation representation of* S_n*. Show that*

$$N_{(n-1,1)} \simeq \mathbb{C}^n.$$

(ii) *Consider the* S_n*-invariant subspace*

$$U := \{v \in \mathbb{C}^n \mid \sum_{i=1}^n v_i = 0\} \subset V.$$

Show that

$$P_{(n-1,1)} \simeq U.$$

References

[1] G. James, A, Kerber, *The representation theory of the symmetric group*, Encyclopedia of Math. and its Applications **16** (1981).

[2] J.-P. Serre, *Linear Representations of Finite Groups*, Graduate Texts in Mathematics, **42**, Springer-Verlag, New York, 1977.