Extra exercises

Algebra 3; Representation theory

september 5, 2011

Exercise 1. (a) Prove that $\mathrm{Aut}\big((\mathbb{Z}/n\mathbb{Z}, +, \overline{0})\big) \simeq \big(\mathbb{Z}/n\mathbb{Z}\big)^*$ as groups.

(b) Prove that the subgroup of $\mathrm{Aut}\big((\mathbb{Z}/n\mathbb{Z}, +, \overline{0})\big)$ consisting of ring automorphisms of $\mathbb{Z}/n\mathbb{Z}$ is the trivial group.

Exercise 2. Let G be a group and X a set. Write $S(X)$ for the group of bijective maps $X \to X$. Consider the map

$$\{ \text{ group homomorphisms } G \to S(X)\} \to \{ \text{ left actions } G \curvearrowright X\}$$

sending the group homomorphism $\phi : G \to S(X)$ to the left action $(g, x) \mapsto \phi(g)(x)$. Prove that this is a bijection.

Exercise 3. Let G and N be groups with unit elements e_G and e_N respectively. Suppose G acts on N by group automorphisms. In other words, the action is of the form $g \cdot n := \phi(g)(n)$ with group homomorphism $\phi : G \to \mathrm{Aut}(N)$.

(a) Let $G \ltimes N = G \ltimes_\phi N$ be the triple $(G \times N, \cdot, (e_G, e_N))$ with operation

$$(g_1, n_1) \cdot (g_2, n_2) := (g_1 g_2, (g_2^{-1} \cdot n_1)n_2).$$

Prove that $G \ltimes N$ is a group (it is called the semidirect product of G and N).

(b) Prove that $G \times \{e_N\}$ and $\{e_G\} \times N$ are subgroups of $G \ltimes N$, isomorphic to G and N respectively.

(c) Show that $N \trianglelefteq G \ltimes N$.

(d) For which ϕ is it true that $G \ltimes N \simeq G \times N$ as groups?

Exercise 4. Let D_n be the dihedral group of order $2n$.

(a) Give an interpretation of D_n as the group of symmetries of a regular polygon with n sides.

(b) Prove that D_n has two one-dimensional linear representations if n is odd and four one-dimensional linear representations if n is even.

Exercise 5. Let X be a finite set and G a finite group acting on X. Let A be another group. Consider the direct product group

$$K := \prod_{x \in X} A_x$$

with each direct summand A_x isomorphic to A.

(a) Show that

$$g \cdot \{a_x\}_{x \in X} := \{a_{g^{-1}x}\}_{x \in X}$$

defines an action of G on K by group automorphisms.

The associated semidirect product group $G \ltimes K$ is denoted by $A \wr_X G$. It is called the wreath product of A by G.

(b) Consider the semidirect product group $S_n \ltimes A^{\times n}$ with respect to the action

$$\sigma \cdot (a_1, a_2, \ldots, a_n) := (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)})$$

of S_n on $A^{\times n}$ by group automorphisms. Describe the group $S_n \ltimes A^{\times n}$ as a wreath product.

Exercise 6. (a) Show that the symmetric group S_n in n letters ($n \geq 2$) is generated by the neighboring transpositions $\sigma_i := (i, i+1)$ $(1 \leq i < n)$.

(b) Show that

$$\begin{aligned} \sigma_i^2 &= e, \\ \sigma_k \sigma_{k+1} \sigma_k &= \sigma_{k+1} \sigma_k \sigma_{k+1}, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i \end{aligned}$$

for $1 \leq i, j < n$, $1 \leq k < n-1$ and $|i - j| > 1$.

(c) Prove that S_n has precisely two one-dimensional linear representations.

(d) Show that the two one-dimensional linear representations of S_n are subrepresentations of the regular representation of S_n.