Extra exercises

Algebra 3; Representation theory

september 12, 2011

Exercise 1. Let U be a vector space and suppose that $V, W \subseteq U$ are vector subspaces. Prove that the following two statements are equivalent:

(a) $U = V \oplus W$ (in other words, each $u \in U$ can be written in a unique way as $u = v + w$ with $v \in V$ and $w \in W$),

(b) $V \cap W = \{0\}$ and $\dim(U) = \dim(V) + \dim(W)$.

Exercise 2. Let $n \in \mathbb{Z}_{\geq 2}$ and let t be an integer satisfying $0 \leq t < n$. Recall that the assignments

$$\pi_t(r) := \begin{pmatrix} \cos\left(\frac{2\pi t}{n}\right) & -\sin\left(\frac{2\pi t}{n}\right) \\ \sin\left(\frac{2\pi t}{n}\right) & \cos\left(\frac{2\pi t}{n}\right) \end{pmatrix}, \qquad \pi_t(s) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

give rise to a two-dimensional representation $\pi_t : D_n \to \mathrm{GL}(\mathbb{C}^2)$.

(a) For which t is π_t an irreducible representation of D_n?

(b) If π_t is not irreducible, determine then two one-dimensional D_n-invariant subspaces $U, V \subset \mathbb{C}^2$ such that $\mathbb{C}^2 = U \oplus V$.

Exercise 3. Let $\chi, \chi' : G \to \mathbb{C}^*$ be two one-dimensional representations of a fixed finite group G. Show that $\chi \simeq \chi'$ iff $\chi = \chi'$.

Exercise 4. (a) Determine all the one-dimensional representations of the four-group V_4 of Klein.

(b) Give the decomposition of the regular representation of V_4 as a direct sum of irreducible representations.

See the other side for the last exercise

Exercise 5. (a) Let G be a group and $H \subseteq G$ a subgroup. Let $\pi : G \to \mathrm{GL}(V)$ be a representation of G. Show that the restriction $\mathrm{Res}_H^G(\pi) : H \to \mathrm{GL}(V)$ of π to H is a representation of H.

(b) Let $\pi : G \to \mathrm{GL}(V)$ and $\pi' : G' \to \mathrm{GL}(V')$ be two representations. Show that the formula

$$(\pi \boxplus \pi')(g, g')(v, v') := (\pi(g)v, \pi'(g')v'), \qquad g \in G, g' \in G', v \in V, v' \in V'$$

defines a representation $\pi \boxplus \pi' : G \times G' \to \mathrm{GL}(V \oplus V')$.

(c) Let $\pi : G \to \mathrm{GL}(U)$ and $\rho : G \to \mathrm{GL}(W)$ be two representations of the same finite group G. Identify G as group with the diagonal subgroup $\Delta(G) := \{(g, g)\}_{g \in G}$ of $G \times G$ by the map $g \mapsto (g, g)$ $(g \in G)$. Show that

$$\pi \oplus \rho \simeq \mathrm{Res}_{\Delta(G)}^{G \times G}(\pi \boxplus \rho).$$