Extra exercises

Algebra 3; Representation theory

October 3, 2011

Exercise 1. Let $x = \{x_g\}_{g\in G}$ be a collection of complex numbers. Frobenius' group determinant is defined by

$$D_G(x) := \det\bigl(\bigl(x_{gg'}\bigr)_{g,g'\in G}\bigr),$$

where we order the group elements of G by fixing a total order on G.

(a) Show that $D_G(x)$ does not depend on the choice of total order on G.

Consider the linear endomorphism $L_G(x)$ of the group algebra $\mathbb{C}[G]$, uniquely characterized by

$$L_G(x)e_y := \sum_{g\in G} x_g e_{gy}, \qquad y \in G.$$

(b) Show that the matrix of $L_G(x)$ with respect to the linear basis $\{e_g\}_{g\in G}$ of $\mathbb{C}[G]$ is given by $\bigl(x_{g'g^{-1}}\bigr)_{g,g'\in G}$.

(c) Let $\widehat{G}$ be a complete set of representatives of the isomorphism classes of irreducible linear G-representations. Let $\pi \in \widehat{G}$. Define the linear endomorphism $L_{G,\pi}(x)$ of the representation space V_π of π by

$$L_{G,\pi}(x) = \sum_{g\in G} x_g \pi(g)$$

and write $P_\pi(x) = \det(L_{G,\pi}(x))$. Show that

(1) $$\det(L_G(x)) = \prod_{\pi\in\widehat{G}} P_\pi(x)^{\dim(V_\pi)}.$$

Hint: Express L_G in terms of the regular action of G on $\mathbb{C}[G]$ and use the decomposition of the regular representation in irreducible subrepresentations.

(d) Prove that the group determinant for $G := \mathbb{Z}/n\mathbb{Z} = \{\overline{m}\}_{m=0}^{n-1}$ factorizes as

$$D_{\mathbb{Z}/n\mathbb{Z}}(x) = (-1)^{\binom{n-1}{2}} \prod_{r=0}^{n-1} \Bigl(\sum_{s=0}^{n-1} e^{2\pi\sqrt{-1}rs/n} x_{\overline{s}}\Bigr).$$

Historical note: Dedekind defined the group determinant and conjectured that it always factorizes as

$$D_G(x) = \prod_{j=1}^{h} p_j(x)^{\deg(p_j)}$$

with the p_j relative prime irreducible polynomials and h the number of conjugacy classes of G. He wrote a letter to Frobenius in 1896 in which he explained the conjecture. Frobenius' subsequent research on the group determinant led him to the concept of group representations, and ultimately to a proof of the conjecture (from exercise 1(c) it is no surprise that, up to sign, the P_π ($\pi \in \widehat{G}$) will be the irreducible factors p_j ($1 \leq j \leq h$), keeping in mind that $\widehat{G}$ is a set of cardinality h). For further details and historical notes, see [1, §4.10 and §4.11].

Exercise 2. Give a complete set of representatives of the equivalence classes of irreducible representations of the dihedral group D_n if $n \in \mathbb{N}$ is an even integer.

Exercise 3. Let $\pi : G \to \mathrm{GL}(V_\pi)$ be an irreducible linear representation of the finite group G. Let χ_π be the character of π. Show that

$$\frac{\dim(V_\pi)}{\#G} \sum_{x \in G} \chi_\pi(x)\overline{\chi_\pi(xz^{-1})} = \chi_\pi(z) \tag{2}$$

for all $z \in G$.
Hint: Choose a scalar product on V_π turning π into a unitary representation. Let $\{e_i\}_i$ be an orthonormal basis of V_π. Write χ_π in terms of the associated matrix coefficients $\pi_{ij} = \pi_{e_i^*, e_j}$ and use the orthogonality relations of the matrix coefficients.

Exercise 4. Let G be a finite group and let $\widehat{G}$ be the isomorphism classes of irreducible linear G-representations. Let $\mathrm{Fun}(\widehat{G})$ be the space of complex valued functions on $\widehat{G}$, and $F(G)$ the space of class functions on G (i.e. functions $f : G \to \mathbb{C}$ satisfying $f(gg'g^{-1}) = f(g')$ for all $g, g' \in G$). Define a linear map

$$\mathcal{F} : F(G) \to \mathrm{Fun}(\widehat{G})$$

by

$$(\mathcal{F}f)(\pi) := \big(f \,|\, \chi_\pi\big), \qquad f \in F(G).$$

(a) Show that $\mathcal{F}$ is a linear bijection.
(b) Show that the inverse of $\mathcal{F}$ is given by

$$(\mathcal{F}^{-1}h)(g) := \sum_{\pi \in \widehat{G}} h(\pi)\chi_\pi(g), \qquad h \in \mathrm{Fun}(\widehat{G}).$$

(c) Let $\langle \cdot, \cdot \rangle$ be the scalar product on $\mathrm{Fun}(\widehat{G})$ defined by

$$\langle h, h' \rangle := \sum_{\pi \in \widehat{G}} h(\pi)\overline{h'(\pi)}, \qquad h, h' \in \mathrm{Fun}(\widehat{G}).$$

Show that

$$\langle \mathcal{F}f, \mathcal{F}f' \rangle = \big(f \,|\, f'\big), \qquad f, f' \in F(G).$$

(d) For $x \in G$ write $c(x)$ for the number of elements in the conjugatie class $C(x) := \{gxg^{-1} \mid g \in G\}$ containing x. Prove that

$$\sum_{\pi \in \widehat{G}} \chi_\pi(x)\overline{\chi_\pi(y)} = \begin{cases} \frac{\#G}{c(x)} & \text{if } y \in C(x), \\ 0 & \text{if } y \notin C(x). \end{cases}$$

Hint: Write $\{s_1, \ldots, s_h\}$ for a complete set of representatives of the conjugacy classes in G. Let $\{\delta_i\}_{i=1}^h$ be the linear basis of $F(G)$ with $\delta_i \in F(G)$ uniquely characterized by

$$\delta_i(s_j) = \frac{\#G}{c(s_i)}\delta_{i,j}, \qquad 1 \leq j \leq h.$$

Then compute $\big(\delta_i \mid \delta_j\big)$ in two different ways.

Exercise 5. The space $\mathrm{Fun}(G)$ of complex valued functions on a finite group G is a commutative algebra with respect to pointwise multiplication. It has also a (in general noncommutative) associative algebra structure, which will be explored in this exercise.

(a) For $f, h \in \mathrm{Fun}(G)$ set

$$(f * h)(z) := \sum_{x,y \in G: xy = z} f(x)h(y), \qquad z \in G.$$

It is called the convolution product of f and h. Show that $\mathrm{Fun}(G)$ is an associative algebra with respect to the convolution product, with unit element $\delta_e(x) := \delta_{e,x}$ for $x \in G$.

(b) Prove that the assignment

$$f \mapsto \sum_{g \in G} f(g)e_g \tag{3}$$

defines an isomorphism $\mathrm{Fun}(G) \xrightarrow{\sim} \mathbb{C}[G]$ of algebras (here we view $\mathrm{Fun}(G)$ as algebra with respect to the convolution product).

(c) Let $F(G) \subseteq \mathrm{Fun}(G)$ be the subspace of class functions on G. Show that $F(G)$ is a commutative subalgebra of $\mathrm{Fun}(G)$ with respect to the convolution product.

(d) Let $Z(\mathbb{C}[G]) \subset \mathbb{C}[G]$ be the center of $\mathbb{C}[G]$ (i.e. it is the commutative subalgebra of $\mathbb{C}[G]$ consisting of elements $a \in \mathbb{C}[G]$ commuting with all other elements of $\mathbb{C}[G]$). Show that (3) restricts to an isomorphism $F(G) \xrightarrow{\sim} Z(\mathbb{C}[G])$ of commutative algebras.

References

[1] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, E. Yudovina, *Introduction to Representation Theory*, with historical interludes by S. Gerovitch. Student Mathematical Library **59**, AMS, 2011.