QUANTUM GROUPS AND KNOT THEORY: WEEK 39

The basic text of this week's lecture is [2, Chapter 2]. We will concentrate on monoidal categories and on some additional foundational material for this subject. Unexplained notations in this syllabus refer by default to [2]. In [1] one finds more details for this material.

1. **Universal constructions and their applications.**

1.1. *Universal properties.* At various places in this course we (implicitly) use universal properties of objects in categories. This is a common way to characterize an object Y in a category $\mathcal{D}$ by describing the sets of morphisms in $\mathcal{D}$ to (or from) Y. Granted the existence of such an object Y, this information determines Y up to unique isomorphism. It is often elegant and efficient to understand objects through their universal characterization. Examples include tensor products, free algebras, universal enveloping algebras etc. It will be useful to understand and recognize these constructions. Therefore we will pay some attention to *universal properties* here and discuss some examples. There are many variations on these constructions, but we will only discuss here the limited set of examples which are of immediate use to us.

1.2. *Natural transformations.* Before we enter the discussion, let us recall the notion of a natural transformation $\phi : F \to G$ between two functors $F, G : \mathcal{C} \to \mathcal{D}$. By this we mean a family of morphisms $\phi_V \in \mathrm{Hom}_{\mathcal{D}}(F(V), G(V))$ indexed by the objects $V \in \mathcal{C}$ such that for all morphisms $f \in \mathrm{Hom}_C(V, W)$ we have $\phi_W \circ F(f) = G(f) \circ \phi_V$. We say that $\phi_V : F(V) \to G(V)$ is a natural family of morphisms. If these are all isomorphisms we speak of a natural isomorphism from F to G.

1.3. *Tensor products.* To illustrate the characterization of objects by universal properties let us consider the example of the *tensor product* of vector spaces. Let $V_1, V_2, \dots, V_n \in \mathbf{Vect}_k$ be k-vector spaces. A tensor product of $V_1, V_2, \dots, V_n$ is a pair (S, ϕ_0) consisting of a k-vector space S and a multilinear map $\phi_0 : V_1 \times V_2 \times \cdots \times V_n \to S$ such that for any k-vector space X and any multilinear map $\phi : V_1 \times V_2 \times \cdots \times V_n \to X$ there exists a unique linear map $\lambda \in \mathrm{Hom}_k(S, X)$ such that ϕ factorizes as $\phi = \lambda \circ \phi_0$. This is called the universal property of the pair (S, ϕ_0).

1.4. *Representation of functors.* We give an equivalent formulation for the universal property of a pair (S, ϕ_0) as above: The pair (S, ϕ_0) has the above universal property iff the natural family of linear maps Φ_X given by

$$\Phi_X : \mathrm{Hom}_k(S, X) \xrightarrow{\sim} \mathrm{Hom}_k^{(n)}(V_1 \times \cdots \times V_n, X) \qquad (1.1)$$
$$\lambda \to \lambda \circ \phi_0$$

(where $\mathrm{Hom}_k^{(n)}(V_1 \times \cdots \times V_n, X)$ denotes the vector space of k-multilinear maps from $V_1 \times \cdots \times V_n$ to X) consists of linear isomorphisms. Conversely, if $T \in \mathbf{Vect}_k$ and we are given a natural isomorphism

$$\Psi : \mathrm{Hom}_k(T, ?) \xrightarrow{\sim} \mathrm{Hom}_k^{(n)}(V_1 \times \cdots \times V_n, ?) \tag{1.2}$$

of functors from $\mathbf{Vect}_k$ to $\mathbf{Vect}_k$ we set $\psi_0 = \Psi_T(\mathrm{Id}_T)$. It is then easy to see that for all $X \in \mathbf{Vect}_k$ we have $\Psi_X(\lambda) = \lambda \circ \psi_0$, i.e. the pair (T, ψ_0) has the universal property (1.1). Hence the universal property (1.1) for the pair (T, ψ_0) is equivalent to giving the natural isomorphism Ψ of (1.2). One expresses (1.2) by saying that the tensor product T of $V_1, V_2, \ldots, V_n$ should (co-)represent the functor $\mathrm{Hom}_k^{(n)}(V_1 \times \cdots \times V_n, ?)$ from $\mathbf{Vect}_k$ to $\mathbf{Vect}_k$ via a natural isomorphism Ψ.

A tensor product of $V_1, V_2, \ldots, V_n$ always exists (but we will not discuss such constructions here). The tensor product is determined up to unique isomorphism by the above universal property:

Exercise (a). *Prove that if (S, ϕ_0) and (T, ψ_0) are both tensor products of $V_1, V_2, \ldots, V_n$ then there exists a unique isomorphism $\alpha : S \to T$ such that $\psi_0 = \alpha \circ \phi_0$.*

We denote the tensor product as $S = V_1 \otimes V_2 \otimes \cdots \otimes V_n$, and the multilinear map ϕ_0 is denoted by $\phi_0(v_1, v_2 \ldots, v_n) = v_1 \otimes v_2 \otimes \cdots \otimes v_n$. In particular the following rules are satisfied for all i:

$$v_1 \otimes \cdots \otimes (\lambda v_i' + \mu v_i'') \otimes \cdots \otimes v_n = \lambda(v_1 \otimes \cdots \otimes v_i' \otimes \cdots \otimes v_n) + \mu(v_1 \otimes \cdots \otimes v_i'' \otimes \cdots \otimes v_n) \tag{1.3}$$

Exercise (b). *Let (S, ϕ_0) be a tensor product of $V_1, V_2 \ldots, V_n$. Prove that S is equal to the linear span of the image of ϕ_0.*

Exercise (c). *Let U, V be finite dimensional vector space over k, and consider the bilinear map $U^* \times V \to \mathrm{Hom}_k(U, V)$. Prove that the corresponding linear map $U^* \otimes V \to \mathrm{Hom}_k(U, V)$ is a linear isomorphism (hint: prove surjectivity, and use Exercise (b) to conclude the desired result by a dimension count).*

By the previous exercise we see that if U, V are finite dimensional then

$$\dim(U \otimes V) = \dim(U)\dim(V) \tag{1.4}$$

Moreover, if $(e_i)_{i=1}^n$ is a basis of U and $(f_j)_{j=1}^m$ is a basis of V then $e_i \otimes f_j$ is a basis of $U \otimes V$. The main conclusions of the above discussion are:

1. Tensor products of k-vector spaces exist (but we did not discuss the construction).
2. For each multilinear map $\phi : V_1 \times V_2 \times \cdots \times V_n \to X$ there exists a unique linear map $\lambda : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to X$ such that for all $v_i \in V_i$: $\lambda(v_1 \otimes \cdots \otimes v_n) = \phi(v_1, \ldots, v_n)$. This is the "universal characterization" of the tensor product.
3. Equivalently, $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ represents the functor $\mathrm{Hom}_k^{(n)}(V_1 \times \cdots \times V_n, ?)$.

1.5. *Associativity, commutativity and the unit of the tensor product.* A typical application of the universal property of the tensor product is the construction of natural isomorphisms expressing associativity, commutativity of the tensor functor $\otimes : \mathbf{Vect}_k \times \mathbf{Vect}_k \to \mathbf{Vect}_k$. In the paragraph we will discuss these matters. We will use the notion of the Cartesian product $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$. This is the category defined by $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$, and $\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((A,B),(A',B')) = \mathrm{Hom}_{\mathcal{C}}(A,A') \times \mathrm{Hom}_{\mathcal{D}}(B,B')$ with composition $(f,g)\circ(f',g') = (f\circ f', g\circ g')$. We will also use the Cartesian product $\mathcal{C}_1 \times \cdots \times \mathcal{C}_k$ of a finite collection $\mathcal{C}_1,\dots,\mathcal{C}_k$ of categories below.

For instance, let us consider the commutativity of the tensor functor. Let $\sigma : \mathbf{Vect}_k \times \mathbf{Vect}_k \to \mathbf{Vect}_k \times \mathbf{Vect}_k$ be the functor given by $\sigma(V,W) = (W,V)$ and $\sigma(f,g) = (g,f)$. Let $\tau : \otimes \to \otimes \circ \sigma$ ("the flip") characterized by the property that $\tau(v\otimes w) = w \otimes v$. Indeed, the map $\phi : V \times W \to W \otimes V$ defined by $\phi(v,w) = w\otimes v$ is bilinear. Hence there exists a unique linear map $\tau_{V,W} : V\otimes W \to W \otimes V$ such that $\phi = \tau_{V,W} \circ \phi_0$, i.e. $\tau_{V,W}(v\otimes w) = w\otimes v$. The uniqueness assertion in such factorizations makes it clear that $\tau_{W,V}\tau_{V,W} = \mathrm{id}_{V\otimes W}$. Hence $\tau_{V,W}$ is an isomorphism. It is easy to see from this construction that for all linear maps $f : V \to V'$ and $g : W \to W'$ we have $\tau_{V',W'}\circ(f\otimes g) = (g\otimes f)\circ\tau_{V,W}$, i.e. $\tau_{V,W} : V\otimes W \to W\otimes V$ is a natural family of isomorphisms. We call this natural isomorphism the *commutativity constraint* of $\mathbf{Vect}_k$.

Let us now consider associativity of the tensor product of vector spaces. Consider the functors $T^3, T^{2,1}, T^{1,2}$ from $\mathbf{Vect}_k^{\times 3} := \mathbf{Vect}_k \times \mathbf{Vect}_k \times \mathbf{Vect}_k$ to $\mathbf{Vect}_k$ defined by $T^3(U,V,W) = U\otimes V\otimes W$, by $T^{2,1}(U,V,W) = (U\otimes V)\otimes W$, and by $T^{1,2}(U,V,W) = U\otimes(V\otimes W)$. We prove that $\xi^{2,1} : T^3 \xrightarrow{\sim} T^{2,1}$ (characterized by $\xi^{2,1}_{U,V,W}(u\otimes v\otimes w) = (u\otimes v)\otimes w$) and also $\xi^{1,2} : T^3 \xrightarrow{\sim} T^{1,2}$ (characterized by $\xi^{1,2}_{U,V,W}(u\otimes v\otimes w) = u\otimes(v\otimes w)$) are natural families of isomorphisms. The existence of these natural transformations is easily established by the universal properties of the tensor products involved. To show that $\xi^{2,1}$ is an isomorphism we remark that the pair $((U\otimes V)\otimes W, \phi^{2,1})$, where $\phi^{2,1} \in \mathrm{Hom}_k^{(2)}(U\times V, \mathrm{Hom}_k(W,(U\otimes V)\otimes W))$ is defined by $\phi^{2,1} : (u,v) \to \{w \to (u\otimes v)\otimes w\}$, gives a representation $\Phi^{2,1} : \mathrm{Hom}_k((U\otimes V)\otimes W, ?) \xrightarrow{\sim} \mathrm{Hom}_k^{(2)}(U\times V, \mathrm{Hom}_k(W,?))$ of the functor $X \to \mathrm{Hom}_k^{(2)}(U\times V, \mathrm{Hom}_k(W,X))$ from $\mathbf{Vect}_k \to \mathbf{Vect}_k$ via $\Phi^{2,1}(\lambda) = \lambda\circ\phi^{2,1}$. But there exists a natural isomorphism $\psi : \mathrm{Hom}_k^{(2)}(U\times V, \mathrm{Hom}_k(W,?)) \xrightarrow{\sim} \mathrm{Hom}_k^{(3)}(U\times V\times W, ?)$ (the inverse transformation is given by sending a trilinear map α on $U\times V\times W$ to the bilinear map mapping (u,v) to the linear map $w \to \alpha(u,v,w)$). Hence the pair $((U\otimes V)\otimes W, \psi(\phi^{2,1}))$ represents the functor $\mathrm{Hom}_k^{(3)}(U\times V\times W, ?)$. In particular, the trilinear map $\phi(u,v,w) = u\otimes v\otimes w$ factors through the trilinear map $\psi(\phi^{2,1})(u,v,w) = (u\otimes v)\otimes w$, giving rise to the inverse of $\xi^{2,1}_{U,V,W}$. The proof that $\xi^{1,2}$ is a natural isomorphism is similar.

Exercise (d). *Prove the above assertion for* $\xi^{1,2}$.

Combined these natural isomorphisms yield a natural isomorphism $a_{U,V,W} = \xi^{1,2}_{U,V,W} \circ (\xi^{2,1}_{(U,V,W)})^{-1} : (U\otimes V)\otimes W \to U\otimes(V\otimes W)$ called the *associativity constraint* of $\mathbf{Vect}_k$.

In addition we have natural families of isomorphisms $l_V : k \otimes V \xrightarrow{\sim} V$ (given by $l_V(\lambda \otimes v) = \lambda v$) and $r_V : V \otimes k \xrightarrow{\sim} V$ (given by $r_V(v \otimes \lambda) = \lambda v$). These are called the *left and right unit constraints* of $\mathbf{Vect_k}$.

The associativity constraints and the unit constraints satisfy the following compatibility rules: The pentagon axiom says that for all $U, V, W, X \in \mathbf{Vect}_k$

$$a_{U,V,W\otimes X} \circ a_{U\otimes V,W,X} = (\mathrm{Id}_U \otimes a_{V,W,X}) \circ a_{U,(V\otimes W),X} \circ (a_{U,V,W} \otimes \mathrm{Id}_X) \tag{1.5}$$

and the triangle axiom says that for all $U, V \in \mathbf{Vect}_k$ one has

$$r_U \otimes \mathrm{Id}_V = (\mathrm{Id}_U \otimes l_V) \circ a_{U,k,V} \tag{1.6}$$

For the proof of these identities it suffices to verify that both hand sides coincide on elementary tensors, which is trivial. This gives $\mathbf{Vect}_k$ the structure of a monoidal category (see the next section). The commutativity constraint τ on $\mathbf{Vect}_k$ gives $\mathbf{Vect}_k$ the structure of a "braided monoidal category", as it satisfies the so-called braiding axioms or hexagonal axioms (this will be discussed in next week's lecture).

1.6. $\mathbf{Vect}_k$ *as a monoidal category.* A *monoidal category* (or *tensor category*) $(\mathcal{C}, \otimes, I, a, l, r)$ consists of the following structures: A category $\mathcal{C}$, a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the *tensor product*) with a natural isomorphism $a : \otimes \circ (\otimes \times \mathrm{Id}_C) \xrightarrow{\sim} \otimes \circ (\mathrm{Id}_C \times \otimes)$ of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the *associativity constraint*), and a *unit object* $I \in \mathcal{C}$ with natural isomorphisms $l : \otimes(I \times \mathrm{Id}_C) \xrightarrow{\sim} \mathrm{Id}_C$ and $r : \otimes(\mathrm{Id}_C \times I) \xrightarrow{\sim} \mathrm{Id}_C$ of functors $\mathcal{C} \to \mathcal{C}$ (the *left and right unit constraints*). These structures are subject to the *pentagon axiom* (1.5) and the *triangle axiom* (1.6). By what we have seen above it is obvious that $\mathbf{Vect}_k$ is an example of a monoidal category, with it unit object being given by k.

1.7. *The tensor algebra.* Let $V \in \mathbf{Vect}_k$. The bilinear map $\phi_{k,l} : V^{\otimes k} \times V^{\otimes l} \to V^{\otimes(k+l)}$ given by $\phi_{k,l}(a, b) = a \otimes b$ gives rise to a linear map $\mu_{k,l} : V^{\otimes k} \otimes V^{\otimes l} \to V^{\otimes(k+l)}$ which satisfies the associativity rule $\mu_{k+l,m}(\mu_{k,l}(a \otimes b) \otimes c) = \mu_{k,l+m}(a \otimes \mu_{l,m}(b \otimes c))$. We use this to define an associative k-algebra structure on

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n} \tag{1.7}$$

where we put $V^{\otimes 0} := k$. Let us write $i : V \to T(V)$ for the linear embedding of V in $T(V)$ given by $i(v) = v$. Then $T(V)$ is a unital associative algebra (with unit $1 \in k$) which is graded and which is generated by $i(V) \subset T(V)$.

If A is any unital associative algebra and $\alpha : V \to A$ is a linear map, then we define multilinear maps $\alpha_n : V \times \cdots \times V \to A$ by $\alpha_n(v_1, \ldots, v_n) = \alpha(v_1) \ldots \alpha(v_n) \in A$. By the universal property of $V^{\otimes n}$ and the definition of $T(V)$ we obtain a unique linear map $\lambda : T(V) \to A$ such that $\alpha = \lambda \circ i$. We make the important observation that since A is associative, the map $\lambda : T(V) \to A$ is in fact a homomorphism of algebras (check it!). Hence we see that the pair $(T(V), i)$ has the following universal property:

Theorem 1.1. *For all associative, unital algebras A and linear maps $\alpha : V \to A$ there exists a unique homomorphism of algebras $\lambda : T(V) \to A$ such that $\alpha = \lambda \circ i$.*

1.8. *Adjoint functors.* Functorial constructions of universal objects $Y \in \mathcal{D}$ often arise in the context of adjoint functors. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. We call $F : \mathcal{C} \to \mathcal{D}$ a left adjoint of G (and G a right adjoint of F) if there exists a natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathcal{D}}(F(X), ?) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(?)) \tag{1.8}$$

of functors from $\mathcal{D}$ to (a subcategory of) **Set**. In this situation for all $X \in \mathcal{C}$ the pair $(Y = F(X), \phi_0)$, where the morphism ϕ_0 is defined by $\phi_0 = \Phi_Y(\mathrm{Id}_Y) \in \mathrm{Hom}_C(X, G(Y))$, has the following universal property: For each object $W \in \mathcal{D}$ and for each morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, G(W))$ there exists a unique morphism $\lambda \in \mathrm{Hom}_{\mathcal{D}}(F(X), W)$ such that there is a factorization $\phi = G(\lambda) \circ \phi_0 (= \Phi_W(\lambda))$.

We see that for any $X \in \mathcal{C}$, the object $F(X) \in \mathcal{D}$ represents the functor from $\mathcal{D}$ to (a subcategory of) **Set** given by $Y \to \mathrm{Hom}_C(X, G(Y))$. Such a functorial map producing objects in $\mathcal{D}$ satisfying a universal description is very useful. Often the functor $G : \mathcal{D} \to \mathcal{C}$ is a functor forgetting some structure. For instance, if $G : \mathbf{Vect}_k \to \mathbf{Set}$ is the functor associating to a vector space its underlying set, then a left adjoint $\mathcal{F}$ exists, and is the functor associating to a set X the free vector space $k[X]$ over k with basis X.

1.9. *Examples.* There are many important examples of such universal constructions that will be relevant to us:

(1) The tensor algebra $T(V)$. Let $\mathcal{C} = \mathbf{Vect}_k$, the category of k-vector spaces, and let $\mathcal{D} = \mathbf{Alg}_k$, the category of (associative, unital) k-algebras. Let $G : \mathcal{D} \to \mathcal{C}$ be the forgetful functor (forget the algebra structure). By Theorem 1.1 this functor has a left adjoint $F : \mathcal{C} \to \mathcal{D}$ which assigns to a vector space V its tensor algebra $T(V)$.

Indeed, the algebra $T(V)$ comes equipped with a linear map $i : V \to T(V)$, and for each linear map $\alpha : V \to G(A)$ where A is any associative k-algebra, there exists a unique algebra homomorphism $\lambda : T(V) \to A$ such that $\phi = G(\lambda) \circ i$. This determines the pair $(T(V), i)$ uniquely up to unique isomorphism.

(2) The symmetric algebra $S(V)$. As above, but now we let $\mathcal{D} = \mathbf{Alg}_k^{com}$ be the category of associative, unital, *commutative* k-algebras. It is easy to see that we can construct $S(V)$ from $T(V)$ taking the quotient by the two-sided ideal I in $T(V)$ generated by $xy - yx$ (with $x, y \in i(V)$). Indeed, if A is a unital, associative, commutative algebra and $\alpha : V \to A$ is any linear map, then the corresponding algebra homomorphism $\lambda : T(V) \to A$ factors through I.

(3) More generally, let us construct the *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$. Here we let $\mathcal{C}$ be the category of Lie algebras (over k), and $\mathcal{D}$ the category of unital, associative k-algebras. The functor $G : \mathcal{D} \to \mathcal{C}$ associates to an associative algebra A the Lie algebra whose underlying vector space is A, and whose Lie bracket is given by $[X, Y] = XY - YX$ (so G keeps the Lie algebra structure, but forgets the associative algebra structure which gave rise to the Lie bracket). It is known that G has a left adjoint, which assigns to a Lie algebra $\mathfrak{g}$ its "universal enveloping algebra" $U(\mathfrak{g})$. It comes equipped with a Lie algebra map $j : \mathfrak{g} \to U(\mathfrak{g})$ such that for all Lie algebra morphisms $\alpha : \mathfrak{g} \to G(A)$ (where A is an associative algebra) there exists a unique algebra morphism $\lambda : U(\mathfrak{g}) \to A$ such that $\alpha = G(\lambda) \circ j$.

Using the universal property of the tensor algebra $(T(\mathfrak{g}), i)$ it is in fact easy to construct $U(\mathfrak{g})$. Indeed, let $U(\mathfrak{g})$ be the quotient of $T(\mathfrak{g})$ by the two-sided ideal J generated in $T(\mathfrak{g})$ by the elements of the form $i(X)i(Y)-i(Y)i(X)-i([XY])$ (with $X, Y \in \mathfrak{g}$), and let j be the composition of i with the quotient map. If $\alpha : \mathfrak{g} \to A$ is a Lie algebra map then it is easy to see that the corresponding algebra morphism $\tilde{\lambda} : T(\mathfrak{g}) \to A$ factors through $U(\mathfrak{g})$. Hence this defines $\lambda : U(\mathfrak{g}) \to A$, and it is easy to see that the pair $(U(\mathfrak{g}), j)$ satisfies the required universal property.

1.10. *Yoneda's lemma.* In this paragraph we discuss Yoneda's lemma, which is of fundamental importance for understanding natural transformations of functors, and therefore touches on several subjects discussed in this subsection. The class of all functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ between two categories $\mathcal{C}$ and $\mathcal{D}$ can itself be equipped with the structure of a category in which morphisms are natural transformations. We define "Yoneda's functor" $\mathrm{Yon} : X \to h_X$ from $\mathcal{C}^{op}$ to $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ by $h_X(Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y)$ (remark: we need to assume that $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is always a set. Such category $\mathcal{C}$ is called *locally small*).

Theorem 1.2. *(Yoneda's lemma) Given $G \in \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ and $X \in \mathcal{C}$, the map*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(h_X, G) &\to G(X) \\ \Phi &\to \Phi_X(\mathrm{Id}_X) \end{aligned}$$

is a bijection. The inverse map sends $u \in G(X)$ to $\Phi^u \in \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(h_X, G)$ defined by $\Phi^u_Y(\alpha) = G(\alpha)(u)$ for $\alpha \in h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Proof. Let us check that the map $\Phi \to u = \Phi_X(\mathrm{Id}_X)$ is injective. Let $Y \in \mathcal{C}$ and let $\alpha \in h_X(Y)$. Observe that $\alpha = h_X(\alpha)(\mathrm{Id}_X)$. Hence we have the formula

$$\Phi_Y(\alpha) = \Phi_Y(h_X(\alpha)(\mathrm{Id}_X)) = G(\alpha)(u) \tag{1.9}$$

showing that $\Phi = \Phi^u$. The surjectivity of Yoneda's map amounts to showing that the collection of maps $\Phi^u_Y : h_X(Y) \to G(Y)$ (with $X \in \mathcal{C}$) defined as above is natural for any $u \in G(X)$. This is easy and left to the reader. □

Corollary 1.3. *We have a natural bijection*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathbf{Set})}(h_X, h_Y) &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(Y, X) \\ \Phi &\to \Phi_X(\mathrm{Id}_X) \end{aligned}$$

In this situation, the inverse map sends $u \in h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ to $\Phi^u = \mathrm{Yon}(u)$, the natural transformation such that for all $Z \in \mathcal{C}$, $\Phi^u_Z : h_X(Z) \to h_Y(Z)$ is given by $\alpha \to \alpha \circ u$.

An embedding of categories $F : \mathcal{C} \to \mathcal{D}$ is a *fully faithful* functor, which means that the maps $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ to $\mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ are bijective for all $X, Y \in \mathcal{C}$. An *equivalence of categories* is an embedding F which is moreover essentially surjective, i.e. every object of $\mathcal{D}$ is isomorphic to an object in the image of F.

Corollary 1.4. *(Yoneda's embedding) Yoneda's functor* $\mathrm{Yon} : \mathcal{C}^{op} \to \mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ *is an embedding. This defines an equivalence of $\mathcal{C}$ with the category of* representable *functors from $\mathcal{C}$ to* $\mathbf{Set}$.

2. Hopf algebras and moniodal categories

1. **Hopf algebras.** Let k be a field. A Hopf algebra $H = (H, \mu, \eta, \Delta, \epsilon, S)$ over k is a bi-algebra over k in which $\mathrm{Id}_H \in \mathrm{End}_k(H)$ has a (left and right) inverse S with respect to the convolution product on $\mathrm{End}_k(H)$. We call S the *antipode* of H.

The following is a useful lemma (cf. [1, Lemma III.3.6]):

Lemma 1.12. *Suppose that A is a bi-algebra and that A is generated as a k-algebra by a set $X \subset A$. Suppose that $S : A \to A^{op}$ is an algebra homomorphism such that*

$$(S * \mathrm{Id}_A)(x) = (\mathrm{Id}_A * S)(x) = \epsilon(h)1 \tag{1.1}$$

for all $x \in X$. Then A is a Hopf algebra with antipode S.

Proof. If $x, y \in A$ are such that (1.1) holds for $h = x$ and $h = y$. We show that in that case (1.1) holds as well for $h = xy$. Indeed,

$$\begin{aligned}(S * \mathrm{Id}_A)(xy) &= \sum_{(xy)} S((xy)')(xy)'' \\ &= \sum_{(x)(y)} S(x'y')x''y'' \\ &= \sum_{(x)(y)} S(y')S(x')x''y'' \\ &= \epsilon(x)\sum_{(y)} S(y')y'' \\ &= \epsilon(xy)1\end{aligned}$$

and similarly for $\mathrm{Id}_A * S(xy)$. Clearly this implies the result. □

Let us now look at Example **1.11**:

(b) Let G be a group and let $A = k[G]$ with the group algebra of G equipped with the usual bi-algebra structure. It is a Hopf algebra and the antipode is given by $S(x) = x^{-1}$ for all $x \in G$. Indeed, $(S * \mathrm{Id})(x) = xS(x) = \epsilon(x)1 = 1$ for all $x \in G$, and similarly $(\mathrm{Id} * S)(x) = xS(x) = 1$.

(c) Let V be a vector space, and consider the tensor algebra $T(V)$ equipped with its usual bi-algebra structure. It is a Hopf algebra with antipode S determined by $S(v) = -v$ for all $v \in V$. Indeed, the universal property of the tensor algebra $T(V)$ states that there is a natural isomorphism

$$\mathrm{Hom}_{alg}(T(V), A) \approx \mathrm{Hom}_k(V, G(A)) \tag{1.2}$$

for all algebras A (here $G(A)$ denotes the underlying vector space of A). In particular, the exists a unique algebra homomorphism $S : T(V) \to T(V)^{op}$ such that $S(v) = -v$ for all $v \in V$. By the above Lemma all we need to check is that $(S * \mathrm{Id})(v) = (\mathrm{Id} * S)(v) = 0$, which is trivial.

Recall (Week 39; Exercise (e)). If $H = (H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra, and $I \subset H$ is an ideal such that $S(I) \subset I$, $\epsilon(I) = 0$, and $\Delta(I) \subset H \otimes I + I \otimes H$ then $\overline{H}$ is itself a Hopf algebra with respect to the induced structures $\overline{S}$, $\overline{\epsilon}$, and $\overline{\Delta}$. Indeed, it is an easy check that the composition of the structural maps of H with the algebra homomorphisms of the form $\pi^k : H^{\otimes k} \to \overline{H}^{\otimes k}$ factor to define structural maps $\overline{S}$, $\overline{\epsilon}$ and $\overline{\Delta}$ for $\overline{H}$, satisfying the axioms of a Hopf algebra (because H is a Hopf algebra).

This yields some important examples of Hopf algebras:

Example 1.13. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over k. We define the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ by*

$$U(\mathfrak{g}) = T(\mathfrak{g})/I \tag{1.3}$$

where I is the two-sided ideal generated by $xy - yx - [x, y]$ (with $x, y \in \mathfrak{g}$). This ideal satisfies $S(I) \subset I$ (since $S(xy - yx - [x, y]) = yx - xy + [x, y] \in I$), $\epsilon(xy - yx - [x, y]) = 0$, and finally

$$\begin{aligned}\Delta(xy - yx - [x, y]) &= (xy - yx - [x, y]) \otimes +1 \otimes (xy - yx - [x, y]) \\ &\subset I \otimes T(\mathfrak{g}) + T(\mathfrak{g}) \otimes I\end{aligned}$$

This shows that the universal enveloping algebra is a Hopf algebra $U(\mathfrak{g})$. Notice the special case where $\mathfrak{g}$ is an abelian Lie algebra, i.e. $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. In this case we have $U(\mathfrak{g}) = S(\mathfrak{g})$, the symmetric algebra on $\mathfrak{g}$.

2. **Monoidal categories.** We will not discuss monoidal categories and their properties in detail at this point. The two main examples for us are $\mathbf{Vect}_k$ and so called *strict monoidal categories* (in both cases the compatibility axioms are trivially satisfied). The monoidal category $\mathbf{Vect}_k$ was discussed in the previous subsection.

Definition 2.14. *A* strict *monoidal category is a monoidal category in witch the associativity constraint and the unit constraints are identities.*

We remark that $\mathbf{Vect}_k$ is not strict. Strict monoidal categories play a predominant role because of the Coherence Theorem of Mac Lane.

Theorem 2.15. *Every monoidal category is tensor equivalent to a strict monoidal category.*

We will not pay anymore attention to this important theorem at this point, but in the background it plays an important role. It explains why we may restrict our attention to strict monoidal categories for most developments.

We have seen that if $A = (A, \mu, \eta, \Delta, \epsilon)$ is a bi-algebra then the (vector space) tensor product $V \otimes W$ of two A-modules V, W can be equipped in a natural way with the structure of an A-module by putting

$$a(v \otimes w) = \sum_{(a)} a'v \otimes a''w \tag{2.1}$$

Using the co-associativity and co-unit axioms we see easily that the usual vector space associativity constraints and unit constraints $l_V : k \otimes V \xrightarrow{\sim} V$ and $r_V : V \otimes k \to V$ are

A-module maps. For example, the tensor unit k of $\mathbf{Vect}_k$ becomes an A-module by the algebra morphism $\epsilon : A \to k$, and we have

$$\begin{aligned} l_V(a(1 \otimes v)) &= l_V(\sum_{(a)} \epsilon(a') \otimes a''v) \\ &= \sum_{(a)} \epsilon(a')a''v \\ &= av \\ &= al_V(1 \otimes v) \end{aligned}$$

In the same way we can easily show that the other vector space constraints r_V and $a_{U,V,W}$ are A-module maps. Hence we obtain:

Theorem 2.16. *Let A be a bi-algebra over k. The category A-mod naturally becomes a monoidal subcategory of the monoidal category* $\mathbf{Vect}_k$.

3. **Braidings.** The commutativity constraint τ of $\mathbf{Vect}_k$ is a braiding, since

$$\tau_{U \otimes V,W} = (\tau_{U,W} \otimes \mathrm{Id}_W) \circ (\mathrm{Id}_U \otimes \tau_{V,W}) \tag{3.1}$$

and

$$\tau_{U,V \otimes W} = (\mathrm{Id}_V \otimes \tau_{U,W}) \circ (\tau_{U,V} \otimes \mathrm{Id}_W) \tag{3.2}$$

It is *involutive*, which makes is not so interesting for our purposes.

Exercise (e). [2, Exercise 3.5(a)]

Exercise (f). [2, Exercise 3.5(b)]

References

[1] C. Kassel, *Quantum groups*, Springer GTM 155 (1995)
[2] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et syntheses **5**, Soc. Math. de France (1997)