QUANTUM GROUPS AND KNOT THEORY LECTURE: WEEK 41

This week we discuss braided Hopf algebras and braided monoidal categories. See [2, Chapter 2, sections 3 and 4].

1. braided monoidal categories, braided bi-algebras and Hopf algebras

1. **Bi-algebras and monoidal categories.** Let k be a field and let A be a k-algebra. Recall that an A-module is an ordered pair (V, m) where V is a k-vector space and $m : A \to \mathrm{End}_k(V)$ is a homomorphism of k-algebras. Often we do not write the homomorphism m explicitly, so we simple speak of the A-module V. Given an A-module V we define a "scalar multiplication" map $\mu : A \otimes V \to V$ by $\mu(a \otimes v) := m(a)(v)$. It is common to use the shorthand notation $av := \mu(a \otimes v)$. Notice that $1v = v$ for all $v \in V$, and $a(bv) = (ab)v$ for all $a, b \in A$ and $v \in V$. Conversely, if we are given a linear map $\mu : A \otimes V \to V$ such that $1v = v$ for all $v \in V$ and also $a(bv) = (ab)v$ for all $a, b \in A$ and $v \in V$ then (V, m), with $m : A \to \mathrm{End}_k(V)$ defined by $m(a)(v) = \mu(a \otimes v) = av$, is an A-module.

If V, W are A-modules and $f : V \to W$ is a k-linear map then we say that f is A-linear (notation $f \in \mathrm{End}_A(V, W)$) if $f(av) = af(v)$ for all $a \in A$. Alternatively we say that f is a homomorphism of A-modules. For example the identity map id_V is always an A-linear map if V is an A-module.

Let $\mathbf{Mod}_A$ denote the category of A-modules, i.e. the category whose objects are A-modules and whose morphisms are A-linear maps between A-modules. Let $\mathbf{Mod}_A^f$ be the full subcategory whose objects are the A-modules which are finite dimensional as a k-vector space.

We denote $\mathbf{Mod}_k$ by $\mathbf{Vect_k}$; recall that we have equipped this category with a (non-strict) monoidal structure with tensor unit k (see week 40). Recall the notion of a bi-algebra from week 40 (in particular week 40, Theorem 2.16):

Theorem 1.1. *Let A be a bi-algebra over a field k. The category $\mathbf{Mod}_A$ is a monoidal subcategory of the monoidal category $\mathbf{Vect}_k$. Here the tensor unit $k = (k, \epsilon)$ is equipped with the A-module structure obtained from the counit $\epsilon : A \to k$, and if $V, W \in \mathbf{Mod}_A$ then the A-action on $V \otimes W$ is defined by*

$$a(v \otimes w) = \Delta(a)(v \otimes w) = \sum_{(a)} a'v \otimes a''w \tag{1.1}$$

The converse is also true: if $\mathbf{Mod}_A$ is a monoidal subcategory of $\mathbf{Vect_k}$ then there exists a unique bi-algebra structure on A such that the monoidal structure of $\mathbf{Mod}_A$ arises from the bi-algebra structure of A (but we will largely ignore this and similar "converse statements" in this syllabus).

2. **Braidings.**

2.1. *Braided monoidal categories.* We refer to the textbook [2, Chapter 2, section 3].

2.2. *Braided Hopf algebras.* Now let us assume in addition that our Hopf algebra A is braided and has an *invertible* antipode S. Braided means that A has a universal R-matrix, i.e. an invertible element $R \in A \otimes A$ such that for all $a \in A$ we have

$$\Delta^{op}(a) = R\Delta(a)R^{-1} \tag{2.1}$$

and such that R satisfies the braiding axioms

$$(\Delta \otimes \mathrm{id}_A)(R) = R_{1,3}R_{2,3} \tag{2.2}$$

$$(\mathrm{id}_A \otimes \Delta)(R) = R_{1,3}R_{1,2} \tag{2.3}$$

Let us write $R = \sum_i s_i \otimes t_i$. Then the braiding relations take the following form in the Sweedler notation:

$$\sum_i \Delta(s_i) \otimes t_i = \sum_{i,(s_i)} s_i' \otimes s_i'' \otimes t_i = \sum_{i,j} s_i \otimes s_j \otimes t_i t_j \tag{2.4}$$

$$\sum_i s_i \otimes \Delta(t_i) = \sum_{i,(t_i)} s_i \otimes t_i' \otimes t_i'' = \sum_{i,j} s_i s_j \otimes t_j \otimes t_i \tag{2.5}$$

Recall [2, Proposition 4.2, Ch. 2]:

Proposition 2.2. *If A is a braided Hopf algebra with invertible antipode S then*

(a) $R_{1,2}R_{1,3}R_{2,3} = R_{2,3}R_{1,3}R_{1,2}$.
(b) $(\epsilon \otimes \mathrm{id}_A)(R) = 1 = (\mathrm{id}_A \otimes \epsilon)(R)$.
(c) $(S \otimes \mathrm{id}_A)(R) = R^{-1} = (\mathrm{id}_A \otimes S^{-1})(R)$.
(d) $R = (S \otimes S)(R)$.

Proof. (a): We compute

$$\begin{aligned} R_{1,2}R_{1,3}R_{2,3} &= R_{1,2}(\Delta \otimes \mathrm{id}_A)(R) \\ &= (\Delta^{op} \otimes \mathrm{id}_A)(R)R_{1,2} \\ &= (\tau \otimes \mathrm{id}_A)(\Delta \otimes \mathrm{id}_A)(R)R_{1,2} \\ &= (\tau \otimes \mathrm{id}_A)(R_{1,3}R_{2,3})R_{1,2} \\ &= R_{2,3}R_{1,3}R_{1,2} \end{aligned}$$

where τ denotes the flip of the two tensor legs involved.

(b): We have by the co-unit axiom:

$$\begin{aligned} R &= (\epsilon \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(\Delta \otimes \mathrm{id}_A)(R) \\ &= (\epsilon \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(R_{1,3}R_{2,3}) \\ &= (\epsilon \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(R_{1,3})(\epsilon \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(R_{2,3}) \\ &= (1 \otimes (\epsilon \otimes \mathrm{id}_A)(R))R \end{aligned}$$

and since R is invertible this implies that $(\epsilon \otimes \mathrm{id}_A)(R) = 1$. The other identity has a similar proof (left to the reader).

(c): By (b) and the definition of the antipode:

$$\begin{aligned} 1 \otimes 1 &= (\mu \otimes \mathrm{id}_A)(S \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(\Delta \otimes \mathrm{id}_A)(R) \\ &= (\mu \otimes \mathrm{id}_A)(S \otimes \mathrm{id}_A \otimes \mathrm{id}_A)(R_{1,3}R_{2,3}) \\ &= (S \otimes \mathrm{id}_A)(R)R \end{aligned}$$

hence the invertibility of R implies the result.

Similarly we have

$$\begin{aligned} 1 \otimes 1 &= (\mathrm{id}_A \otimes \mu)(\mathrm{id}_A \otimes \mathrm{id}_A \otimes S^{-1})(\mathrm{id}_A \otimes \Delta^{op})(R) \\ &= (\mathrm{id}_A \otimes \mu)(\mathrm{id}_A \otimes \mathrm{id}_A \otimes S^{-1})(R_{1,2}R_{1,3}) \\ &= R(\mathrm{id}_A \otimes S^{-1})(R) \end{aligned}$$

proving the second identity.

(d): We combine the two identities of (c):

$$\begin{aligned} (S \otimes S)(R) &= (\mathrm{id}_A \otimes S)(S \otimes \mathrm{id}_A)(R) \\ &= (\mathrm{id}_A \otimes S)(R^{-1}) \\ &= (\mathrm{id}_A \otimes S)(\mathrm{id}_A \otimes S^{-1})(R) \\ &= R \end{aligned}$$

□

2.3. *The braiding of* $\mathbf{Mod}_A$. Let A be a braided Hopf algebra with universal R-matrix R. For $V, W \in \mathbf{Mod}_A$ we define k-linear maps

$$\begin{aligned} c_{V,W} : V \otimes W &\to W \otimes V \\ v \otimes w &\to \tau(R(v \otimes w)) \end{aligned}$$

Here τ denotes the flip of the two tensor legs as usual, and the action of $A \otimes A$ on $V \otimes W$ is defined by $(a \otimes b)(v \otimes w) = av \otimes bw$.

Proposition 2.3. *The k-linear maps $c_{V,W}$ are A-linear isomorphims. The family $c_{V,W}$ defines a commutativity constraint on $\mathbf{Mod}_A$. In other words, c is a natural family of A-module isomorphisms in the sense that for any two A-module morphisms $f : V \to X$ and $g : W \to Y$ we have*

$$(g \otimes f)c_{V,W} = c_{X,Y}(f \otimes g) \tag{2.6}$$

Proof. Let us check the A-linearity:

$$\begin{aligned} c_{V,W}(a(v\otimes w)) &= c_{V,W}(\Delta(a)(v\otimes w)) \\ &= \tau(R\Delta(a)(v\otimes w)) \\ &= \tau(\Delta^{op}(a)R(v\otimes w)) \\ &= \Delta(a)\tau(R(v\otimes w)) \\ &= ac_{V,W}(v\otimes w) \end{aligned}$$

Because R is invertible it is clear that $c_{V,W}$ is an isomorphism. Finally we need to show the naturality:

$$\begin{aligned} (g\otimes f)c_{V,W}(v\otimes w) &= (g\otimes f)\tau(R(v\otimes w)) \\ &= \tau((f\otimes g)(R(v\otimes w))) \\ &= \tau(R(f(v)\otimes g(w))) \\ &= c_{X,Y}(f(v)\otimes g(w)) \end{aligned}$$

□

Theorem 2.4. *The commutativity constraint c is a braiding of* $\mathbf{Mod}_A$.

Proof. In a strict monoidal category we need to show the following identities:

$$\begin{aligned} c_{U,V\otimes W} &= (\mathrm{id}_V\otimes c_{U,W})(c_{U,V}\otimes \mathrm{id}_W) \\ c_{U\otimes V,W} &= (c_{U,W}\otimes \mathrm{id}_V)(\mathrm{id}_U\otimes c_{V,W}) \end{aligned}$$

In the situation at hand we need to adapt these identities by inserting associativity constraints (as before, this can be done in a unique way). Let us write $R=\sum_i s_i\otimes t_i$. Observe that the braiding axioms imply that

$$\sum_{i,(t_i)} s_i\otimes t_i'\otimes t_i'' = \sum_{i,j} s_is_j\otimes t_j\otimes t_i \tag{2.7}$$

Using this equality we have

$$\begin{aligned} a^{-1}_{V,W,U}(\mathrm{id}_V\otimes c_{U,W})a_{V,U,W}(c_{U,V}\otimes \mathrm{id}_W)a^{-1}_{U,V,W}(u\otimes(v\otimes w)) \\ &= \sum_{i,j}(t_iv\otimes t_jw)\otimes s_js_iu \\ &= \tau_{U,V\otimes W}(\sum_{i,j} s_js_iu\otimes(t_iv\otimes t_jw)) \\ &= \tau_{U,V\otimes W}(\sum_{i} s_iu\otimes(t_i'v\otimes t_i''w)) \\ &= \tau_{U,V\otimes W}(R(u\otimes(v\otimes w)) \\ &= c_{U,V\otimes W}(u\otimes(v\otimes w)) \end{aligned}$$

The other braiding identity is handled in a similar fashion. □

Exercise (a). *See* [2, Chapter 2, Section 4.4, Exercise (a)].

Exercise (b). *See* [2, Chapter 2, Section 4.4, Exercise (b)].

2.4. *The square of the antipode.* Let A be a braided (or quasi triangular) Hopf algebra with invertible antipode S and universal R-matrix $R = \sum_i s_i \otimes t_i$. We define

$$u := \sum_i S(t_i)s_i \in A \tag{2.8}$$

Theorem 2.5. *(see* [1, Proposition VIII.4.1]*) The element u is invertible with inverse*

$$u^{-1} = \sum_i S^{-1}(t_i)S(s_i) \tag{2.9}$$

The square of the antipode is the inner automorphism of A obtained by conjugating with u: we have $S^2(a) = uau^{-1}$ for all $a \in A$.

Proof. We first show that $S^2(a)u = ua$ for all $a \in A$. Using the commutativity constraint we have:

$$\sum_{i,(a)} s_i a' \otimes t_i a'' \otimes a''' = \sum_{i,(a)} a'' s_i \otimes a' t_i \otimes a''' \tag{2.10}$$

We apply to this identity the linear map $A \otimes A \otimes A \to A$ defined by $(a \otimes b \otimes c) \to S^2(c)S(b)a$ to obtain:

$$\sum_{i,(a)} S^2(a''')S(t_i a'')s_i a' = \sum_{i,(a)} S^2(a''')S(a't_i)a''s_i \tag{2.11}$$

or

$$\sum_{i,(a)} S(a''S(a'''))S(t_i)s_i a' = \sum_{i,(a)} S^2(a''')S(t_i)S(a')a''s_i \tag{2.12}$$

Using the defining property of S this gives

$$\sum_{i,(a)} S(\epsilon(a'')1)S(t_i)s_i a' = \sum_{i,(a)} S^2(a'')S(t_i)\epsilon(a')s_i \tag{2.13}$$

and by co-unitality and the fact that S is an anti-algebra isomorphism we get

$$\sum_i S(t_i)s_i a = \sum_i S^2(a)S(t_i)s_i \tag{2.14}$$

or $S^2(a)u = ua$. It remains to show that u is invertible. Let $R^{-1} = \sum_i x_i \otimes y_i$ and put $v = \sum_i S^{-1}(y_i)x_i$. Then

$$\begin{aligned} uv &= \sum_i uS^{-1}(y_i)x_i = \sum_i S(y_i)ux_i \\ &= \sum_{i,j} S(y_i)S(t_j)s_j x_i = \sum_{i,j} S(t_j y_i)s_j x_i = 1 \end{aligned}$$

where the last equality follows from the observation that $1 \otimes 1 = RR^{-1} = \sum_{i,j} s_j x_i \otimes t_j y_i$. Now apply the linear map $A \otimes A \to A$ defined by $a \otimes b \to S(b)a$ to this identity.

Finally we use $R^{-1} = (S \otimes \mathrm{id}_A)(R)$ to obtain the explicit expression for u^{-1}. □

Corollary 2.6. *The element $D = uS(u) = S(u)u$ is central in A.*

Proof. For all $a \in A$ we have $uS^{-1}(a) = S(a)u$. Now apply S to this identity to get $aS(u) = S(u)S^2(a)$. By the previous theorem we have $aS(u) = S(u)uau^{-1}$, or $a(S(u)u) = (S(u)u)a$. Thus $S(u)u$ is central. In particular, $S(u)u = u(S(u)u)u^{-1} = uS(u)$, as was claimed. □

The element u and the central element D play an important role in the theory of ribbon algebras. We discuss some useful properties of these elements:

Proposition 2.7. *(see* [1, Proposition VIII.4.5]*).*

(a) $\epsilon(u) = 1$.
(b) $\Delta(u) = (R_{2,1}R)^{-1}(u \otimes u) = (u \otimes u)(R_{2,1}R)^{-1}$.
(c) $\Delta(S(u)) = (R_{2,1}R)^{-1}(S(u) \otimes S(u)) = (S(u) \otimes S(u))(R_{2,1}R)^{-1}$.
(d) $\Delta(D) = (R_{2,1}R)^{-2}(D \otimes D) = (D \otimes D)(R_{2,1}R)^{-2}$.

Proof. (a): By Proposition 2.2(b) it follows that

$$\begin{aligned}\epsilon(u) &= \sum_i \epsilon(S(t_i))\epsilon(s_i) = \sum_i \epsilon(t_i)\epsilon(s_i) \\ &= \epsilon(\sum_i \epsilon(s_i)t_i) = \epsilon((\epsilon \otimes \mathrm{id}_A)(R)) = 1\end{aligned}$$

(b): This is a rather involved computation. We want to show that

$$(R_{2,1}R)\Delta(u) = \Delta(u)(R_{2,1}R) = u \otimes u \tag{2.15}$$

First observe that τ applied to (2.1) yields the relation $\Delta(a) = R_{2,1}\Delta^{op}(a)R_{2,1}^{-1}$. Hence the element $R_{2,1}R \in A \otimes A$ commutes with $\Delta(A)$. This proves the first equality of (2.15). Moreover, using this property of $R_{2,1}R$ we see that

$$\begin{aligned}\Delta(u)(R_{2,1}R) &= \sum_i \Delta(S(t_i))\Delta(s_i)R_{2,1}R \\ &= \sum_i (S \otimes S)(\Delta^{op}(t_i))\Delta(s_i)R_{2,1}R \\ &= \sum_i (S \otimes S)(\Delta^{op}(t_i))R_{2,1}R\Delta(s_i)\end{aligned}$$

Let us define a *right* action $\diamond$ of $A \otimes A \otimes A \otimes A$ on $A \otimes A$ as follows: If $X, Y, Z \in A \otimes A$ we define

$$X \diamond (Y \otimes Z) := (S \otimes S)(Z)XY \tag{2.16}$$

(this is indeed a right action because $S \otimes S$ is an anti-algebra homomorphism of $A \otimes A$). Then it is clear that we can rewrite the preceding expression for $\Delta(u)(R_{2,1}R)$ in terms of this action as follows:

$$\Delta(u)(R_{2,1}R) = R_{2,1} \diamond (R_{1,2}(\Delta \otimes \Delta^{op})(R)) \tag{2.17}$$

First we compute $(\Delta \otimes \Delta^{op})(R)$ using (2.4),(2.5):

$$\begin{aligned}
(\Delta \otimes \Delta^{op})(R) &= (\Delta \otimes \mathrm{id}_{A\otimes A})\tau_{2,3}(\sum_i s_i \otimes \Delta(t_i)) \\
&= (\Delta \otimes \mathrm{id}_{A\otimes A})(\sum_{i,j} s_i s_j \otimes t_i \otimes t_j) \\
&= \sum_{i,j} \Delta(s_i)\Delta(s_j) \otimes t_i \otimes t_j \\
&= \sum_{i,j,k,l} s_i s_j \otimes s_k s_l \otimes t_i t_k \otimes t_j t_l \\
&= R_{1,3}R_{1,4}R_{2,3}R_{2,4} \\
&= R_{1,3}R_{2,3}R_{1,4}R_{2,4}
\end{aligned}$$

Next we list some useful identities for the diamond operation. These are simple reformulations of Proposition 2.2 and of the definition of the element u:

$$\begin{aligned}
((a \otimes 1)R) \diamond R_{1,4} &= a \otimes 1 \ \forall a \in A \\
((1 \otimes b)R_{2,1}) \diamond R_{2,3} &= 1 \otimes b \ \forall b \in A \\
(1 \otimes b) \diamond R_{1,3} &= u \otimes b \ \forall b \in A \\
(a \otimes 1) \diamond R_{2,4} &= a \otimes u \ \forall a \in A
\end{aligned}$$

For instance, the first of these is proved as follows using Proposition 2.2(c):

$$\begin{aligned}
((a \otimes 1)R) \diamond R_{1,4} &= \sum_{i,j} a s_i s_j \otimes S(t_j) t_i \\
&= (\mathrm{id}_A \otimes S)(\sum_{i,j} a s_i s_j \otimes S^{-1}(t_i) t_j) \\
&= (\mathrm{id}_A \otimes S)(a \otimes 1) \\
&= a \otimes 1
\end{aligned}$$

and the others have similar proofs. Using all this we compute:

$$\begin{aligned}
\Delta(u)(R_{2,1}R) &= \sum_i (S\otimes S)(\Delta^{op}(t_i))R_{2,1}R\Delta(s_i) \\
&= R_{2,1}\diamond(R_{1,2}(\Delta\otimes\Delta^{op})(R)) \\
&= R_{2,1}\diamond(R_{1,2}R_{1,3}R_{2,3}R_{1,4}R_{2,4}) \\
&= R_{2,1}\diamond(R_{2,3}R_{1,3}R_{1,2}R_{1,4}R_{2,4}) \\
&= (R_{2,1}\diamond R_{2,3})\diamond(R_{1,3}R_{1,2}R_{1,4}R_{2,4}) \\
&= (1\otimes 1)\diamond(R_{1,3}R_{1,2}R_{1,4}R_{2,4}) \\
&= (u\otimes 1)\diamond(R_{1,2}R_{1,4}R_{2,4}) \\
&= ((u\otimes 1)R)\diamond(R_{1,4}R_{2,4}) \\
&= (u\otimes 1)\diamond R_{2,4} \\
&= u\otimes u
\end{aligned}$$

which is what we had set out to prove.

(c): Using that $S : A \to A^{op,coop}$ is a bi-algebra homomorphism and using Proposition 2.2(d) we see that this identity arises from the preceding one by applying $\tau\circ(S\otimes S)$:

$$\begin{aligned}
S(u)\otimes S(u) &= \tau((S\otimes S)(u\otimes u)) \\
&= \tau((S\otimes S)(R_{2,1}R)\Delta(u)) \\
&= \tau(\Delta^{op}(S(u))(S\otimes S)(R)(S\otimes S)(R_{2,1})) \\
&= \tau(\Delta^{op}(S(u))(RR_{2,1})) \\
&= \Delta(S(u))(R_{2,1}R)
\end{aligned}$$

(d): Take the product of identities (b) and (c). □

References

[1] C. Kassel, *Quantum groups*, Springer GTM 155 (1995)
[2] C. Kassel, M. Rosso, and V. Turaev, *Quantum groups and knot invariants*, Panoramas et syntheses **5**, Soc. Math. de France (1997)