Continuity Controlled Hybrid Automata

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Abstract

We investigate the connections between the process algebra for hybrid systems of
Bergstra and Middelburg and the formalism of hybrid automata of Henzinger et al.
We give interpretations of hybrid automata in the process algebra for hybrid systems
and compare them with the standard interpretation of hybrid automata as timed
transition systems. We also relate the synchronized product operator on hybrid
automata to the parallel composition operator of the process algebra. It turns out
that the formalism of hybrid automata matches a fragment of the process algebra
for hybrid systems closely. We present an adaptation of the formalism of hybrid
automata that yields an exact match.

Key words: hybrid systems, hybrid automata, hybrid transition systems, timed
transition systems, process algebra, continuous relative timing, continuity control

1 Introduction

Hybrid systems are systems that exhibit both discrete and continuous be-
haviour. They typically consist of a controlling subsystem made up of digital
components and a controlled subsystem made up of analog components. The
controlling subsystem exhibits discrete behaviour and the controlled subsys-
tem exhibits continuous behaviour. In general, the controlling subsystem is

embedded in the controlled subsystem without being accessible from the outside. Moreover, the behaviour of the controlling subsystem generally depends on the behaviour of the controlled subsystem and cannot be considered in isolation. Hybrid systems constitute a topic that is vital to computer science, for they are found in many areas.

It was proposed almost at the outset of the interest for hybrid systems in computer science to model them as hybrid automata [1–3]. Hybrid automata are automata equipped with variables that evolve continuously with time. They can be viewed as a generalization of timed automata [4,5]. The study of hybrid systems in computer science is up to now largely focussed on hybrid automata, in particular on model checking for hybrid automata (see e.g. Refs. [6–9]). Satisfaction of properties expressed in an expressive temporal logic can be automatically verified for a restricted subclass of hybrid automata, known as linear hybrid automata. Conservative approximations are needed for other hybrid automata to make automatic verification possible.

In Ref. [10], we have introduced a process algebra for hybrid systems. This process algebra comprises:

- mathematical expressions for hybrid systems;
- equational axioms for equational reasoning about hybrid systems;
- rules for lifting results from real analysis to equations about hybrid systems;
- a structural operational semantics of the expressions.

The expressions are constructed by means of operators, each of which corresponds to a distinct and natural way in which hybrid systems can be combined or adapted. The axioms and lifting rules make fully precise how to establish whether two expressions constructed in different ways represent the same hybrid system. The structural operational semantics induces a transition system for each expression. The transition systems concerned are similar to the ones used for model checking in the setting of hybrid automata.

The process algebra for hybrid systems introduced in Ref. [10] can be regarded as originating from the formalism of hybrid automata. Both adopt the view that a hybrid system is a system in which an instantaneous state transition takes place when the system performs an action and a continuous state evolution takes place while the system is idling between performing successive actions. The process algebra for hybrid systems from Ref. [10] is an extension of the process algebra with continuous relative timing from the collection of process algebras with timing, each dealing with timing in a different way, presented in Refs. [11,12]. All the process algebras with timing presented in Refs. [11,12] are extensions of ACP [13,14].

To the best of our knowledge, the other existing process algebras for hybrid systems are a variant of timed CSP [15], called hybrid CSP, introduced in

Ref. [16], a variant of the π-calculus [17], called the ϕ-calculus, introduced in Ref. [18], and another extension of ACP [14], called HyPA, introduced in Ref. [19]. In comparison with the process algebra for hybrid systems proposed in Ref. [10], hybrid CSP and the ϕ-calculus have certain limitations with regard to the description and/or analysis of hybrid systems. Hybrid CSP, the ϕ-calculus and HyPA are further discussed in Ref. [10].

In this paper, we take a closer look at the connections between the process algebra for hybrid systems from Ref. [10] and the formalism of hybrid automata. The purpose of this is twofold. Firstly, we want to substantiate our claim made in Ref. [10] that hybrid automata can be faithfully interpreted in the proposed process algebra for hybrid systems in a uniform and direct way. Secondly, we want to lay the foundation of the adaptation of model checking tools developed for hybrid automata to restricted versions of the proposed process algebra for hybrid systems. We show that the formalism of hybrid automata matches a fragment of the process algebra for hybrid systems from Ref. [10] closely. Hybrid automata can be faithfully interpreted in a uniform and direct way, but a faithful interpretation of synchronized products of hybrid automata cannot be obtained in a compositional way. We introduce an adaptation of the formalism of hybrid automata, which does not lead to any loss of generality, and show that the adaptation yields an exact match.

The structure of this paper is as follows. First of all, we give a brief summary of the version of process algebra for hybrid systems from Ref. [10] (Section 2). Next, we describe the form and meaning of the propositions used in the process algebra for hybrid systems and the formalism of hybrid automata (Section 3). Then, we give a summary of the formalism of hybrid automata (Section 4). Following, we discuss the semantic issues concerning the process algebra for hybrid systems that are relevant to the rest of the paper (Section 5). Thereupon, we investigate the connections between the process algebra for hybrid systems and the formalism of hybrid automata (Section 6). After that, we present an adaptation of the formalism of hybrid automata that yields a better match than the old one (Section 7). Finally, we make some concluding remarks (Section 8). For reference, the operational semantics of the process algebra for hybrid systems is given in Appendix A. For a comprehensive overview of this process algebra, the reader is referred to Ref. [10].

Various constants and operators of the process algebra with continuous relative timing from Refs. [11,12] have counterparts in the other versions from the above-mentioned collection. A notational distinction is made between a constant or operator of one version and its counterparts in another version, by means of different decorations of a common symbol, if they should not be identified in case versions are integrated. So long as one uses a single version, one can safely omit those decorations. However, we refrain from omitting them in this paper because we think that change of notation in a series of technical

publications is undesirable.

2 Process Algebra for Hybrid Systems

In this section, we give an overview of the process algebra for hybrid systems proposed in Ref. [10]. For an extensive treatment, the reader is referred to that paper. We distinguish between $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$, the process algebra that is the mere adaptation of $\mathrm{ACP}^{\mathrm{srt}}$ to the description and analysis of hybrid systems, and two extensions that are useful in many applications: integration, which provides for alternative composition over a continuum of differently timed alternatives, and guarded recursion, which allows for the description of (potentially) non-terminating processes.

2.1 $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$

$\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ is obtained by extending a combination of two existing extensions of ACP [13], namely $\mathrm{ACP}^{\mathrm{srt}}$, the process algebra with continuous relative timing from Ref. [12], and $\mathrm{ACP}_{\mathrm{ps}}$, the process algebra with propositional signals from Ref. [20], with two new operators. A process may idle for some period of time before it performs its next action (instantaneously), in which case the next action is performed after a delay. $\mathrm{ACP}^{\mathrm{srt}}$ covers this aspect of process behaviour. The state of processes is kept invisible. In $\mathrm{ACP}_{\mathrm{ps}}$, a process can have its state to some extent visible. The basic idea is that the visible part of the state of a process, called the signal emitted by the process, is a proposition. Only discrete state changes, caused by performing actions, are covered.

One of the new operators, called *signal evolution*, makes it possible to deal with continuous state changes during delays as well. With this new operator, we can have signals at all points of time during a delay instead of only at its begin and end. Algebraic and differential equations and inequalities concerning named state components are taken as the atomic propositions from which the signals are generated. The other new operator, called *signal transition*, makes it possible to deal better with instantaneous state changes where the state immediately after the change depends upon the state immediately before the change. The resulting process algebra has, in addition to equational axioms, some rules to derive further equations with the help of real analysis. These lifting rules permit to cast the effects of continuous state changes into equations about processes.

An extensive treatment of $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ can be found in Ref. [10]. In this section, we only give a brief overview of the constants and operators of $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$.

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As usual in ACP-style process algebras, we assume that a fixed but arbitrary set A of actions and a fixed but arbitrary partial commutative and associative *communication* function $\gamma : \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ have been given. The function γ is regarded to give the result of synchronously performing any two actions for which this is possible, and to be undefined otherwise.

$\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ has the following constants and operators in common with $\mathrm{ACP}^{\mathrm{srt}}$:

- for each action a in A, the *undelayable action a*, written $\tilde{\underline{a}}$, is the process that immediately performs action a at the current point of time, and then terminates successfully;
- the *undelayable deadlock*, written $\tilde{\underline{\delta}}$, is the process that is neither capable of performing any action nor capable of idling beyond the current point of time;
- the *relative delay* of P for a period of time r, written $\sigma_{\mathrm{rel}}^r(P)$, is the process that idles for a period of time r and then behaves like P;
- the *alternative composition* of P_1 and P_2, written $P_1 + P_2$, is the process that either behaves like P_1 or behaves like P_2 (but not both, unless P_1 equals P_2);
- the *sequential composition* of P_1 and P_2, written $P_1 \cdot P_2$, is the process that first behaves like P_1, but when P_1 terminates successfully it continues by behaving like P_2;
- the *parallel composition* of P_1 and P_2, written $P_1 \parallel P_2$, is the process that proceeds with P_1 and P_2 in parallel;
- the *left merge* of P_1 and P_2, written $P_1 \lfloor\!\lfloor P_2$, is the same as $P_1 \parallel P_2$ except that $P_1 \lfloor\!\lfloor P_2$ starts with performing an action of P_1;
- the *communication merge* of P_1 and P_2, written $P_1 \mid P_2$, is the same as $P_1 \parallel P_2$ except that $P_1 \mid P_2$ starts with performing an action of P_1 and an action of P_2 synchronously;
- the *encapsulation* of P with respect to H, written $\partial_H(P)$, keeps P from performing actions in H;
- the *relative undelayable time-out* of P, written $\nu_{\mathrm{rel}}(P)$, keeps P entirely from idling.

In $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$, propositions are used as signals that are emitted by processes. The intuition is that the signal emitted by a process, as well as each of its logical consequences, holds at the start of the process. The propositions concerned, called *state propositions*, are constructed in the usual way from atomic propositions that are algebraic and differential equations and inequalities concerning named state components. The named state components, called *state variables*, are real-valued functions of time. Their values may change both instantaneously at the points of time at which an action is performed and continuously during the periods in between. In order to deal with instantaneous state transitions, propositions concerning the values of the state variables immediately before and after a transition are used as well. Those propositions

are called *transition propositions*. The form and meaning of state propositions and transition propositions are described in Section 3. It is assumed that a fixed but arbitrary set V of state variables has been given.

$\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$, has, in addition to the constants and operators in common with $\mathrm{ACP}^{\mathrm{srt}}$, the following constants and operators:

- the *non-existent process*, written $\bot$, is a process that emits a signal that cannot hold;
- *P emitting signal* ψ, written $\psi \wedge P$, is the process that behaves like P, but moreover emits the signal ψ;
- *P proceeding conditionally* on ψ, written $\psi :\rightarrow P$, is the process that behaves like P if state proposition ψ holds at its start, and otherwise behaves like undelayable deadlock;
- *P in evolution* according to ϕ with V smooth (V a finite subset of V), written $\phi \,\vec{\cap}_V\, P$, is the process P of which the emitted signal changes continuously till it performs its first action in such a way that state proposition ϕ is satisfied and without discontinuities for the state variables in V; signal evolution does not take its signal changing effect if the first action is performed immediately, but what remains in such cases is that P emits the signal ϕ;
- *P in transition* according to χ, written $\chi \,\vec{\cap}\, P$, is the process P of which the signal changes instantaneously over performing its first action in such a way that transition proposition χ is satisfied if it performs its first action immediately; otherwise signal transition does not take its signal changing effect, and in either case the process $\chi \,\vec{\cap}\, P$ behaves like undelayable deadlock if there is no transition satisfying χ possible at the start of P.

The operational semantics of $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ is described in a mathematically precise way in Appendix A. Here, we only point at the most important issues:

- In $P_1 + P_2$, there is an arbitrary choice between P_1 and P_2. The choice is resolved on one of them performing its first action, and not otherwise. Consequently, the choice between two idling processes will always be postponed until at least one of the processes can perform its first action. Only when both processes cannot idle any longer, further postponement is not an option. If the choice has not yet been resolved when one of the processes cannot idle any longer, the choice will simply not be resolved in its favour. As long as both processes idle their emitted signals change jointly.
- $P_1 \parallel P_2$ can behave in the following ways: (i) first either P_1 or P_2 performs its first action and next it proceeds in parallel with the process following that action and the process that did not perform an action; (ii) if their first actions can be performed synchronously, first P_1 and P_2 perform their first actions synchronously and next it proceeds in parallel with the processes following those actions. However, P_1 and P_2 may have to idle before they

can perform their first action. Therefore, $P_1 \parallel P_2$ can only start with: (i) performing an action of P_1 or P_2 if it can do so before or at the ultimate point of time for the other process to start performing actions or to deadlock; (ii) performing an action of P_1 and an action of P_2 synchronously if both processes can do so at the same point of time. Moreover, the state transition caused by performing the first action of P_1 or P_2 must not be precluded by the other process: (i) the signal emitted by the other process must hold in the state immediately before the transition and the state immediately after the transition; (ii) if the other process is idling when the action is performed, a state evolution with discontinuities for the state variables of which the value changes by the transition must be possible for the other process. There is only one action left when actions are performed synchronously.

The axioms and lifting rules of $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$, as well as the additional axioms for integration and guarded recursion, can be found in Ref. [10]. In Section 3.3, after the description of the form and meaning of state propositions and transition propositions, we give simple equations, derivable from those axioms and lifting rules, that illustrate the kind of calculations that are possible with them. For examples of the use of this process algebra for the description and analysis of hybrid systems, the reader is referred to Ref. [10] as well.

We use the notation $\sum_{i \in \mathcal{I}} t_i$, where $\mathcal{I} = \{i_1, \ldots, i_n\}$ and $t_{i_1}, \ldots, t_{i_n}$ are terms of $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$, for $t_{i_1} + \ldots + t_{i_n}$. The convention is that $\sum_{i \in \mathcal{I}} t_i$ stands for $\tilde{\tilde{\delta}}$ if $\mathcal{I} = \emptyset$. Throughout this paper, the need to use parentheses is reduced by using the associativity of the operators $+$ and $\cdot$, and by ranking the precedence of the binary operators. We adhere to the following precedence rules: (i) the operator $+$ has lower precedence than all others, (ii) the operator $\cdot$ has higher precedence than all others, and (iii) all other operators have the same precedence.

2.2 Integration

In order to allow for alternative composition over an infinite set of differently timed alternatives, we add integration to $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$. Integration was first introduced for a timed variant of ACP in Ref. [21].

Integration is represented by the variable-binding operator $\int$. Let P be an expression, possibly containing variable u, such that $P[p/u]$ (P with p substituted for u) represents a process for all $p \in \mathbb{R}^{\geq}$; and let $U \subseteq \mathbb{R}^{\geq}$. Then the *integration* $\int_{u \in U} P$ behaves like one of the processes $P[p/u]$ for $p \in U$. Hence, integration is a form of alternative composition over a set of alternatives that may be infinite and even uncountable.

The operational semantics for integration is described in Appendix A.

In order to allow for the description of (potentially) non-terminating processes, we add guarded recursion to $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$.

A *recursive specification* over $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ is a set of *recursive equations* $E = \{X = t_X \mid X \in V\}$ where V is a set of variables and each t_X is a term of $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ that only contains variables from V. We write $\mathrm{V}(E)$ for the set of all variables that occur on the left-hand side of an equation in E. A *solution* of a recursive specification E is a set of processes (in some model of $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$) $\{P_X \mid X \in \mathrm{V}(E)\}$ such that the equations of E hold if, for all $X \in \mathrm{V}(E)$, X stands for P_X. Let t be a term of $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ containing a variable X. We call an occurrence of X in t *guarded* if t has a subterm of the form $\tilde{\tilde{a}} \cdot t'$ or $\sigma^r_{\mathrm{rel}}(t')$, where $a \in \mathsf{A}$, $r > 0$ and t' a term of $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$, with t' containing this occurrence of X. A recursive specification over $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ is called a *guarded* recursive specification if all occurrences of variables in the right-hand sides of its equations are guarded or it can be rewritten to such a recursive specification using the axioms of $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ and the equations of the recursive specification. A guarded recursive specification has a unique solution.

For each guarded recursive specification E and each variable $X \in \mathrm{V}(E)$, we introduce a constant $\langle X|E \rangle$ which is interpreted as the unique solution of E for X. We often write X for $\langle X|E \rangle$ if E is clear from the context. In such cases, it should also be clear from the context that we use X as a constant. The constants $\langle X|E \rangle$ were first introduced in Ref. [22] under the name R-expressions.

The operational semantics for guarded recursion is described in Appendix A.

3   State Propositions and Transition Propositions

The propositions used in $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ and the formalism of hybrid automata to describe state evolutions and state transitions are roughly the same. We describe in this section the form and meaning of those propositions. We also give simple examples of their use in $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$. Actually, the sets of propositions available in $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ are slightly richer than described here, but the differences are irrelevant to the purpose of this paper.

3.1   Formation of State and Transition Propositions

We assume that a fixed but arbitrary set V of *state variables* has been given. For each state variable $v \in \mathsf{V}$, we introduce an additional state variable $\dot{v}$, standing for the derivative of v. We write $\dot{\mathsf{V}}$ for $\{\dot{v} \mid v \in \mathsf{V}\}$. For each state variable $v \in \mathsf{V} \cup \dot{\mathsf{V}}$, we further introduce two additional state variables $^\bullet v$ and $v^\bullet$, standing for the state variable v immediately before and immediately after a transition. We write $^\bullet\mathsf{V}$ for $\{^\bullet v \mid v \in \mathsf{V} \cup \dot{\mathsf{V}}\}$ and $\mathsf{V}^\bullet$ for $\{v^\bullet \mid v \in \mathsf{V} \cup \dot{\mathsf{V}}\}$. We further assume that a set of constants, arithmetic operators and relational operators of real arithmetic, including the basic ones $(0, 1, +, -, \cdot, ^{-1}, <)$, has been given.

The set of *state expressions* is inductively defined by the following formation rules:

- each state variable $v \in \mathsf{V} \cup \dot{\mathsf{V}}$ is a state expression;
- each constant c is a state expression;
- if o is an arithmetic operator of arity n and $s_1, \ldots, s_n$ are state expressions, then $o(s_1, \ldots, s_n)$ is a state expression.

The set of *atomic state propositions* is inductively defined by the following formation rules:

- T and F are atomic state propositions;
- if s_1 and s_2 are state expressions, then $s_1 = s_2$ is an atomic state proposition;
- if π is a relational operator of arity n, and $s_1, \ldots, s_n$ are state expressions, then $\pi(s_1, \ldots, s_n)$ is an atomic state proposition.

State propositions are constructed from atomic state propositions in the usual way with the various logical connectives.

The set of *transition expressions* is inductively defined by the following formation rules:

- each state variable $v \in {}^\bullet\mathsf{V} \cup \mathsf{V}^\bullet$ is a transition expression;
- each constant c is a transition expression;
- if o is an arithmetic operator of arity n and $t_1, \ldots, t_n$ are transition expressions, then $o(t_1, \ldots, t_n)$ is a transition expression.

The set of *atomic transition propositions* is inductively defined by the following formation rules:

- T and F are atomic transition propositions;
- if t_1 and t_2 are transition expressions, then $t_1 = t_2$ is an atomic transition proposition;

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- if π is a relational operator of arity n, and $t_1, \ldots, t_n$ are transition expressions, then $\pi(t_1, \ldots, t_n)$ is an atomic transition proposition.

Transition propositions are constructed from atomic transition propositions in the usual way with the various logical connectives.

We write $\mathcal{P}_{\mathsf{st}}$ for the set of all state propositions, and $\mathcal{P}_{\mathsf{tr}}$ for the set of all transition propositions. Let $V \subseteq \mathsf{V}$. Then we write $\mathcal{P}_{\mathsf{st}}(V)$ for the set of all state propositions that only contain variables from V, and $\mathcal{P}_{\mathsf{tr}}(V)$ for the set of all transition propositions that only contain variables from V.

We shall henceforth use $v, v', \ldots$ to stand for arbitrary elements of V, $\psi, \psi', \ldots$ and $\phi, \phi', \ldots$ to stand for arbitrary state propositions, $\chi, \chi', \ldots$ to stand for arbitrary transition propositions, $V, V', \ldots$ to stand for arbitrary subsets of V.

3.2 Satisfaction of State and Transition Propositions

A *valuation* of state variables is a function $\alpha : \mathsf{V} \cup \dot{\mathsf{V}} \to \mathbb{R}$ or $\beta : {}^{\bullet}\mathsf{V} \cup \mathsf{V}^{\bullet} \to \mathbb{R}$. We write $\mathcal{V}_{\mathsf{st}}$ for the set of all valuations $\alpha : \mathsf{V} \cup \dot{\mathsf{V}} \to \mathbb{R}$ and $\mathcal{V}_{\mathsf{tr}}$ for the set of all valuations $\beta : {}^{\bullet}\mathsf{V} \cup \mathsf{V}^{\bullet} \to \mathbb{R}$. In $\mathrm{ACP}_{\mathsf{hs}}^{\mathrm{srt}}$, a valuation $\alpha \in \mathcal{V}_{\mathsf{st}}$ is called a *state*.

A valuation $\alpha \in \mathcal{V}_{\mathsf{st}}$ can be extended to state expressions, atomic state propositions and state propositions in the usual homomorphic way, and a valuation $\beta \in \mathcal{V}_{\mathsf{tr}}$ can be extended to transition expressions, atomic transition propositions and transition propositions in the usual homomorphic way. We will use the same name for a valuation and its extensions.

Let $\rho : [0, r] \to (\mathsf{V} \to \mathbb{R})$, where $r \in \mathbb{R}^{>}$, and $V \subseteq \mathsf{V}$. Then, for every $v \in \mathsf{V}$, we write ρ_v for the function $\rho_v : [0, r] \to \mathbb{R}$ defined by $\rho_v(t) = \rho(t)(v)$. We say that ρ is a *state evolution* if ρ_v is piecewise of class C^{∞} in $[0, r)$ for all $v \in \mathsf{V}$. We say that ρ is *smooth* for V if ρ_v is of class C^{∞} in $[0, r]$ for all $v \in V$. We write $\mathcal{E}_r$ for the set of all state evolutions $\rho : [0, r] \to (\mathsf{V} \to \mathbb{R})$.

If we replace C^{∞} by C^1, the soundness results for $\mathrm{ACP}_{\mathsf{hs}}^{\mathrm{srt}}$ and its extension with integration and guarded recursion, which can be found in Ref. [10], go through. In other words, we could have chosen for state variables that are functions from $\mathbb{R}^{\geq}$ to $\mathbb{R}$ that are piecewise of class C^1 in $\mathbb{R}^{\geq}$. However, that choice would complicate the theory and might inhibit useful extensions.

If $\rho \in \mathcal{E}_r$, we say that a valuation $\alpha \in \mathcal{V}_{\mathsf{st}}$ *agrees* with ρ at time t, $t \in [0, r]$, if for all $v \in \mathsf{V}$:

$$\alpha(v) = \rho_v(t) , \quad \alpha(\dot{v}) = \dot{\rho}_v(t) .$$

For a given state evolution $\rho \in \mathcal{E}_r$ and a given time $t \in [0, r]$, there is a unique valuation from $\mathcal{V}_{\text{st}}$ that agrees with ρ at t. We write α_t^ρ for this unique valuation.

If $(\alpha, \alpha') \in \mathcal{V}_{\text{st}} \times \mathcal{V}_{\text{st}}$, we say that a valuation $\beta \in \mathcal{V}_{\text{tr}}$ agrees with (α, α') if for all $v \in \mathsf{V}$:

$$\beta({}^\bullet v) = \alpha(v) \,, \quad \beta({}^\bullet \dot{v}) = \alpha(\dot{v}) \,, \quad \beta(v^\bullet) = \alpha'(v) \,, \quad \beta(\dot{v}^\bullet) = \alpha'(\dot{v}) \,.$$

For a given pair $(\alpha, \alpha') \in \mathcal{V}_{\text{st}} \times \mathcal{V}_{\text{st}}$, there is a unique valuation from $\mathcal{V}_{\text{tr}}$ that *agrees* with (α, α'). We write $\beta_{\alpha'}^\alpha$ for this unique valuation.

Satisfaction of state propositions and transition propositions is defined as follows:

- state proposition ψ is *satisfied* by $\alpha \in \mathcal{V}_{\text{st}}$, written $\alpha \models \psi$, if

$$\alpha(\psi) = \mathsf{T} \,;$$

- state proposition ϕ is *satisfied* by $\rho \in \mathcal{E}_r$, written $\rho \models \phi$, if

$$\alpha_t^\rho(\phi) = \mathsf{T} \text{ for all } t \in [0, r] \,;$$

- transition proposition χ is *satisfied* by $(\alpha, \alpha') \in \mathcal{V}_{\text{st}} \times \mathcal{V}_{\text{st}}$, written $\alpha \to \alpha' \models \chi$, if

$$\beta_{\alpha'}^\alpha(\chi) = \mathsf{T} \,.$$

We write $\alpha \overset{r,\rho}{\longmapsto} \alpha' \models_V \phi$ for

$$\rho \in \mathcal{E}_r, \ \alpha_0^\rho = \alpha, \ \alpha_r^\rho = \alpha', \ \rho \text{ is smooth for } V \text{ and } \rho \models \phi \,.$$

3.3 Use of State and Transition Propositions in $\text{ACP}_{\text{hs}}^{\text{srt}}$

In this subsection, we give some examples of the use of state propositions and transition propositions in $\text{ACP}_{\text{hs}}^{\text{srt}}$. The examples concern simple equations, derivable from the axioms and lifting rules of $\text{ACP}_{\text{hs}}^{\text{srt}}$, that illustrate the kind of calculations that are possible with those axioms and lifting rules.

The following derivable equation illustrates how signal transition changes the signal emitted by a process over performing an action:

$$(v = 0) \, {}^{\curlywedge} \, (({}^\bullet v + v^\bullet = 1) \, {}^{\ulcorner}\!\!{}_\blacktriangledown \, \tilde{\tilde{a}} \cdot \tilde{\tilde{b}})$$
$$= (v = 0) \, {}^{\curlywedge} \, (({}^\bullet v + v^\bullet = 1) \, {}^{\ulcorner}\!\!{}_\blacktriangledown \, \tilde{\tilde{a}} \cdot ((v = 1) \, {}^{\curlywedge} \, \tilde{\tilde{b}})) \,.$$

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The following derivable equation illustrates how signal emission changes the signal emitted by a process over idling:

$$(v = 0) \curlywedge ((v \leq 5 \wedge \dot{v} = 1) \curvearrowright \sigma_{\mathsf{rel}}^4(\tilde{\tilde{a}}))$$
$$= (v = 0) \curlywedge ((v \leq 5 \wedge \dot{v} = 1) \curvearrowright \sigma_{\mathsf{rel}}^4((v = 4) \curlywedge \tilde{\tilde{a}})) \;.$$

The following derivable equation illustrates that signal emission forces an idling process to deadlock as soon as the state proposition concerned cannot be satisfied any longer:

$$(v = 0) \curlywedge ((v \leq 5 \wedge \dot{v} = 1) \curvearrowright \sigma_{\mathsf{rel}}^6(\tilde{\tilde{a}}))$$
$$= (v = 0) \curlywedge ((v \leq 5 \wedge \dot{v} = 1) \curvearrowright \sigma_{\mathsf{rel}}^5(\tilde{\tilde{\delta}})) \;.$$

The following derivable equation illustrates that signal emission turns an idling process into the non-existent process if the state proposition concerned is not satisfiable:

$$((v = 0) \wedge (\dot{v} = 1)) \curvearrowright \sigma_{\mathsf{rel}}^1(\tilde{\tilde{a}}) = \bot \;.$$

The following derivable equations illustrate how signal transition and signal emission may interact in parallel composition:

$$\sigma_{\mathsf{rel}}^2((v^\bullet = {}^\bullet v + 1) \curvearrowright \tilde{\tilde{a}} \cdot \sigma_{\mathsf{rel}}^1(\tilde{\tilde{b}})) \parallel ((\dot{v} = 0) \curvearrowright_{\{v\}} \sigma_{\mathsf{rel}}^3(\tilde{\tilde{c}}))$$
$$= (\dot{v} = 0) \curvearrowright_{\{v\}} \sigma_{\mathsf{rel}}^2(\tilde{\tilde{\delta}}) \;,$$

$$\sigma_{\mathsf{rel}}^2((v^\bullet = {}^\bullet v + 1) \curvearrowright \tilde{\tilde{a}} \cdot \sigma_{\mathsf{rel}}^1(\tilde{\tilde{b}})) \parallel ((\dot{v} = 0) \curvearrowright_{\emptyset} \sigma_{\mathsf{rel}}^3(\tilde{\tilde{c}}))$$
$$= (\dot{v} = 0) \curvearrowright_{\emptyset}$$
$$\sigma_{\mathsf{rel}}^2((v^\bullet = {}^\bullet v + 1) \curvearrowright \tilde{\tilde{a}} \cdot ((\dot{v} = 0) \curvearrowright_{\emptyset} \sigma_{\mathsf{rel}}^1(\tilde{\tilde{b}} \cdot (\dot{v} = 0) \curlywedge \tilde{\tilde{c}} + \tilde{\tilde{c}} \cdot \tilde{\tilde{b}}))) \;.$$

Note the difference on the left hand side of these equations: $(\dot{v} = 0) \curvearrowright_{\{v\}} \sigma_{\mathsf{rel}}^3(\tilde{\tilde{c}})$ precludes discontinuities for v, but $(\dot{v} = 0) \curvearrowright_{\emptyset} \sigma_{\mathsf{rel}}^3(\tilde{\tilde{c}})$ does not preclude discontinuities for v.

4 The Formalism of Hybrid Automata

In this section, we give a brief summary of the formalism of hybrid automata. For a more extensive treatment, the reader is referred to Refs. [8,23]. First, we define the notion of hybrid automaton and related notions, including the interpretation of a hybrid automaton as a timed transition system. Next, we

define the notion of synchronized product of hybrid automata. We also show how the interpretation of the synchronized product of two hybrid automata can be expressed in terms of the interpretations of the two hybrid automata.

4.1   Hybrid Automata

Informally, a hybrid automaton is a labelled multigraph equipped with a finite set of state variables. The edges of the graph, called control switches, are used to model discrete state changes. Each control switch is labelled with a condition on the values of the state variables immediately before and immediately after the discrete state change concerned. The vertices of the graph, called control modes, are used to model continuous state changes. Each control mode is labelled with a condition on the values and derivatives of the state variables during the continuous state change concerned. The conditions on discrete state changes and the conditions on continuous state changes are called jump conditions and flow conditions, respectively. In addition, each control mode is labelled with a condition on the initial values and derivatives of the state variables in case of a start in that control mode, and each control switch is labelled with the event on which that control switch takes place. Like in Ref. [8], we make invariant conditions implicit within flow conditions.

A *hybrid automaton H* consists of

- a finite set V of *state variables*;
- a finite set M of *control modes*;
- a finite set E of *events*;
- a finite set S of *control switches*;
- a *source* function $\mu : S \to M$,
- a *target* function $\nu : S \to M$,
- an *event* function $\epsilon : S \to E$,
- a *jump* function $\chi : S \to \mathcal{P}_{\mathsf{tr}}(V)$,
- a *flow* function $\phi : M \to \mathcal{P}_{\mathsf{st}}(V)$,
- an *init* function $\psi : M \to \mathcal{P}_{\mathsf{st}}(V)$.

We often write m_s for $\mu(s)$, m'_s for $\nu(s)$, e_s for $\epsilon(s)$, χ_s for $\chi(s)$, ϕ_m for $\phi(m)$, and ψ_m for $\psi(m)$.

Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a hybrid automaton. Then we write $\mathrm{V}(H)$ for V, $\mathrm{M}(H)$ for M, $\mathrm{E}(H)$ for E and $\mathrm{S}(H)$ for S.

The definition of hybrid automata given above is essentially the same as the definition given in Ref. [8], except for leaving out the labeling of control modes

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with *final* conditions. [1] The omitted feature allows for the description of (negations of) safety properties, but is considered to be irrelevant to the use of hybrid automata for modeling hybrid systems.

In most applications of hybrid automata, there is only one control mode of which the initial condition can be satisfied. However, this is not a requirement. Consequently, there may be two or more initial control modes.

A hybrid automaton $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ has *initial non-determinism* if there exist more than one $m \in M$ such that ψ_m is satisfiable. In the case where H has no initial non-determinism, we will refer by m^0 to the unique control mode m for which ψ_m is satisfiable.

The meaning of hybrid automata is given in terms of timed transition systems, i.e. labelled transition systems of which each transition is labelled with an action or a non-negative real number. A transition is labelled with an action to indicate that the transition takes place on performing that action. Transitions of this kind are called *jump transitions*. A transition is labelled with an non-negative real number to indicate that the transition takes place on idling for that number of time units. Transitions of this kind are called *flow transitions*. We use transition systems of which each state is labelled with an observation. The labelling of states is to anticipate that later on we have to prevent states from being identified if they show differences that are relevant to the behaviour of hybrid systems.

A *timed transition system T* consists of

- a set Q of *states*;
- a set $Q^0 \subseteq Q$ of *initial states*;
- a set A of *actions*;
- a set $\xrightarrow{\ell} \subseteq Q \times Q$ of *ℓ-transitions*, for each $\ell \in A \cup \mathbb{R}^>$;
- a set O of *observations*;
- a set $\|o\| \subseteq Q$ of *o-states*, for each $o \in O$.

Instead of $(q, q') \in \xrightarrow{\ell}$, we write $q \xrightarrow{\ell} q'$ in the case where $\ell \in A$ and $q \xmapsto{\ell} q'$ in the case where $\ell \in \mathbb{R}^>$. We write $\rightarrow$ for the family of sets $(\xrightarrow{\ell})_{\ell \in A \cup \mathbb{R}^>}$ and $\|_-\|$ for the family of sets $(\|o\|)_{o \in O}$.

Let $T = (Q, Q^0, A, \rightarrow, O, \|_-\|)$ be a timed transition system. Then the set $\twoheadrightarrow \subseteq Q \times (A \cup \mathbb{R}^>)^* \times Q$ of *generalized transitions* of T is the smallest subset of $Q \times (A \cup \mathbb{R}^>)^* \times Q$ satisfying:

---

[1] In Ref. [8], control modes are identified with pairs of control switches. Consequently, a multiset of control modes is needed. We circumvent the need for a multiset by leaving out the identification.

- $q \xrightarrow{\epsilon} q$ for each $q \in Q$;
- if $q \xrightarrow{\ell} q'$, then $q \xrightarrow{\ell} q'$;
- if $q \xrightarrow{\sigma} q'$ and $q' \xrightarrow{\sigma'} q''$, then $q \xrightarrow{\sigma\sigma'} q''$.

A state $q \in Q$ is called a *reachable* state of T if there is a $q^0 \in Q^0$ and a $\sigma \in (A \cup \mathbb{R}^>)^*$ such that $q_0 \xrightarrow{\sigma} q$.

A version of bisimilarity is used to identify timed transition systems that only differ in details that are considered to be irrelevant to the behaviour of any system.

Let $T_i = (Q_i, Q_i^0, A, \rightarrow_i, O, \|_-\|_i)$, for $i = 1, 2$, be timed transition systems with the same set of actions and the same set of observations. Then a *bisimulation* between T_1 and T_2 is a binary relation $B \subseteq Q_1 \times Q_2$ such that for all $q_1 \in Q_1$ and $q_2 \in Q_2$:

- if $q_1 \in Q_1^0$, then there is a $q_2 \in Q_2^0$ such that $B(q_1, q_2)$;
- if $q_2 \in Q_2^0$, then there is a $q_1 \in Q_1^0$ such that $B(q_1, q_2)$;
- if $B(q_1, q_2)$ and $q_1 \xrightarrow{\ell}_1 q_1'$, then there is a q_2' such that $q_2 \xrightarrow{\ell}_2 q_2'$ and $B(q_1', q_2')$;
- if $B(q_1, q_2)$ and $q_2 \xrightarrow{\ell}_2 q_2'$, then there is a q_1' such that $q_1 \xrightarrow{\ell}_1 q_1'$ and $B(q_1', q_2')$;
- if $B(q_1, q_2)$ and $q_1 \in \|o\|_1$, then $q_2 \in \|o\|_2$;
- if $B(q_1, q_2)$ and $q_2 \in \|o\|_2$, then $q_1 \in \|o\|_1$.

We say that T_1 and T_2 are *bisimilar*, written $T_1 \leftrightarrow T_2$, if there exists a bisimulation B between T_1 and T_2.

Note that, if timed transition systems T_1 and T_2 are bisimilar, then there exists a bisimulation B between T_1 and T_2 such that $B(q_1, q_2)$ only if q_1 and q_2 are reachable states of T_1 and T_2, respectively.

Below, we define the transition system interpretation of hybrid automata. Here, we use the notations introduced in Section 3.2 for the first time. We start to define what the states of the timed transition system associated with a hybrid automaton are. Like in Ref. [8], we include the derivatives of state variables in the states.

Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a hybrid automaton. Then a *state* of H is a pair $(m, \alpha) \in M \times \mathcal{V}_{\mathsf{st}}$. A state (m, α) of H is *admissible* if $\alpha \models \phi_m$. A state (m, α) of H is *initial* if it is admissible and $\alpha \models \psi_m$. We usually write $\langle m, \alpha \rangle$ instead of (m, α).

The *transition system interpretation* of H, written $[\![H]\!]$, is the timed transition system $(Q, Q^0, E, \rightarrow, \mathcal{V}_{\mathsf{st}}, \|_-\|)$ where

- Q is the set of admissible states of H;
- Q^0 is the set of initial states of H;

- the $\xrightarrow{\ell}$, one for each $\ell \in E \cup \mathbb{R}^>$, are the smallest subsets of $Q \times Q$ such that:
  - · if $s \in S$, $\langle m_s, \alpha \rangle \in Q$, $\langle m'_s, \alpha' \rangle \in Q$ and $\alpha \rightarrow \alpha' \models \chi_s$, then $\langle m_s, \alpha \rangle \xrightarrow{e_s} \langle m'_s, \alpha' \rangle$;
  - · if $m \in M$, $r \in \mathbb{R}^>$ and there exists a $\rho \in \mathcal{E}_r$ such that $\alpha \xmapsto{r,\rho} \alpha' \models_V \phi_m$, then $\langle m, \alpha \rangle \xmapsto{r} \langle m, \alpha' \rangle$;
- $\|\alpha\| = \{ \langle m, \alpha \rangle \mid \langle m, \alpha \rangle \in Q \}$, for each $\alpha \in \mathcal{V}_{\mathsf{st}}$.

We say that state evolution $\rho \in \mathcal{E}_r$ is a *witness* of flow transition $\langle m, \alpha \rangle \xmapsto{r} \langle m, \alpha' \rangle$ if $\alpha \xmapsto{r,\rho} \alpha' \models_V \phi_m$. Note that, in the case of transition system interpretations of hybrid automata, a bisimulation does not relate states of which the valuation components differ.

Let H_1 and H_2 be hybrid automata with $\mathrm{E}(H_1) = \mathrm{E}(H_2)$. Then we say that H_1 and H_2 are *bisimilar* if $\llbracket H_1 \rrbracket \leftrightarroweq \llbracket H_2 \rrbracket$.

We may have $\llbracket H_1 \rrbracket \leftrightarroweq \llbracket H_2 \rrbracket$, but not $\mathrm{V}(H_1) = \mathrm{V}(H_2)$. This can, for example, be the case if both H_1 and H_2 do not have reachable states from which a flow transition is possible. In the literature on hybrid automata, bisimilarity is only defined for hybrid automata with the same set of state variables.

We have the following result concerning bisimulations and the witnesses of flow transitions.

Proposition 1 (Bisimulations and Witnesses of Flow Transitions)
Let H_1 and H_2 be hybrid automata with $\llbracket H_1 \rrbracket = (Q_1, Q_1^0, A_1, \rightarrow_1, \mathcal{V}_{\mathsf{st}}, \|_-\|_1)$ and $\llbracket H_2 \rrbracket = (Q_2, Q_2^0, A_2, \rightarrow_2, \mathcal{V}_{\mathsf{st}}, \|_-\|_2)$. Let B be a bisimulation between $\llbracket H_1 \rrbracket$ and $\llbracket H_2 \rrbracket$. Suppose $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, $\langle m_1, \alpha \rangle \xmapsto{r}_1 \langle m_1, \alpha' \rangle$ and $\langle m_2, \alpha \rangle \xmapsto{r}_2 \langle m_2, \alpha' \rangle$. Then, for all $\rho \in \mathcal{E}_r$, ρ is a witness of $\langle m_1, \alpha \rangle \xmapsto{r}_1 \langle m_1, \alpha' \rangle$ iff ρ is a witness of $\langle m_2, \alpha \rangle \xmapsto{r}_2 \langle m_2, \alpha' \rangle$.

PROOF. Because $\langle m_1, \alpha \rangle \xmapsto{r}_1 \langle m_1, \alpha' \rangle$ and $\langle m_2, \alpha \rangle \xmapsto{r}_2 \langle m_2, \alpha' \rangle$, it follows from the definitions of transition system interpretation and bisimulation between timed transition systems that $\mathrm{V}(H_1) = \mathrm{V}(H_2)$. Suppose that $\mathrm{V}(H_1) = \mathrm{V}(H_2) = V$. We proceed by distinguishing the case where $\alpha_0^\rho = \alpha$, $\alpha_r^\rho = \alpha'$ and ρ is smooth for V and the case where $\alpha_0^\rho \neq \alpha$, $\alpha_r^\rho \neq \alpha'$ or ρ is not smooth for V. In the first case, it follows from the definitions of transition system interpretation and satisfaction of state propositions by state evolutions that, for $i = 1, 2$, ρ is a witness of $\langle m_i, \alpha \rangle \xmapsto{r}_i \langle m_i, \alpha' \rangle$ iff for all s such that $0 < s < r$ we have $\langle m_i, \alpha \rangle \xmapsto{s}_i \langle m_i, \alpha_s^\rho \rangle$. Moreover, because $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, it follows from the definitions of transition system interpretation and bisimulation between timed transition systems that for all s such that $0 < s < r$ we have $\langle m_1, \alpha \rangle \xmapsto{s}_1 \langle m_1, \alpha_s^\rho \rangle$ iff $\langle m_2, \alpha \rangle \xmapsto{s}_2 \langle m_2, \alpha_s^\rho \rangle$. Hence, we conclude: ρ is a witness of $\langle m_1, \alpha \rangle \xmapsto{r}_1 \langle m_1, \alpha' \rangle$ iff ρ is a witness of $\langle m_2, \alpha \rangle \xmapsto{r}_2 \langle m_2, \alpha' \rangle$. In the

second case, neither flow transition can have ρ as a witness; and we immediately conclude: ρ is a witness of $\langle m_1, \alpha \rangle \xmapsto{r}_1 \langle m_1, \alpha' \rangle$ iff ρ is a witness of $\langle m_2, \alpha \rangle \xmapsto{r}_2 \langle m_2, \alpha' \rangle$. $\quad\square$

4.2 Synchronized Product of Hybrid Automata

Hybrid systems are generally composed of several components that act concurrently and interact with each other. In order to deal with such composition in the formalism of hybrid automata, the synchronized product of hybrid automata has been introduced. In the synchronized product of two hybrid automata, control modes of the two component automata are conjoined. The conjunction of the flow conditions and the conjunction of the initial conditions apply. Control switches of the two component automata that take place on joint events occur simultaneously, others are interleaved. In the former case, the conjunction of the jump conditions applies. The case where the two component automata have shared state variables is not excluded.

Let $H_i = (V_i, M_i, E_i, S_i, \mu_i, \nu_i, \epsilon_i, \chi_i, \phi_i, \psi_i)$, for $i = 1, 2$, be hybrid automata. Then the *synchronized product* of H_1 and H_2, written $H_1 \times H_2$, is the hybrid automaton

$$H = (V_1 \cup V_2, M_1 \times M_2, E_1 \cup E_2, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$$

where

$$S = \{(s, m) \in S_1 \times M_2 \mid \epsilon_1(s) \notin E_2\} \cup \{(m, s) \in M_1 \times S_2 \mid \epsilon_2(s) \notin E_1\}$$
$$\cup \{(s_1, s_2) \in S_1 \times S_2 \mid \epsilon_1(s_1) = \epsilon_2(s_2)\},$$

$$\mu(s, m) = (\mu_1(s), m), \mu(m, s) = (m, \mu_2(s)), \mu(s_1, s_2) = (\mu_1(s_1), \mu_2(s_2)),$$
$$\nu(s, m) = (\nu_1(s), m), \nu(m, s) = (m, \nu_2(s)), \nu(s_1, s_2) = (\nu_1(s_1), \nu_2(s_2)),$$
$$\epsilon(s, m) = \epsilon_1(s), \qquad \epsilon(m, s) = \epsilon_2(s), \qquad \epsilon(s_1, s_2) = \epsilon_1(s_1),$$
$$\chi(s, m) = \chi_1(s), \qquad \chi(m, s) = \chi_2(s), \qquad \chi(s_1, s_2) = \chi_1(s_1) \wedge \chi_2(s_2),$$

$$\phi(m_1, m_2) = \phi_1(m_1) \wedge \phi_2(m_2),$$
$$\psi(m_1, m_2) = \psi_1(m_1) \wedge \psi_2(m_2).$$

The synchronized product of hybrid automata is defined here like in Ref. [23]. It is the most general definition. It does not require, as already mentioned, that the sets V_1 and V_2 are disjunct. In Ref. [3] is only dealt with the case where this disjunctness requirement is met. Moreover, not the synchronized

product operator on hybrid automata is described, but rather the corresponding operator on the transition system interpretations of hybrid automata. We have the following result concerning the composition of transition system interpretations of hybrid automata.

Proposition 2 (TS Interpretation of Synchronized Products) *For all hybrid automata H_1, H_2 such that $[\![H_1]\!] = (Q_1, Q_1^0, A_1, \rightarrow_1, \mathcal{V}_{\mathsf{st}}, \|_-\|_1)$ and $[\![H_2]\!] = (Q_2, Q_2^0, A_2, \rightarrow_2, \mathcal{V}_{\mathsf{st}}, \|_-\|_2)$:*

$$[\![H_1 \times H_2]\!] \leftrightarrows (Q, Q^0, A_1 \cup A_2, \rightarrow, \mathcal{V}_{\mathsf{st}}, \|_-\|) \,,$$

where

- $Q = \{\langle(m_1, m_2), \alpha\rangle \mid \langle m_1, \alpha\rangle \in Q_1, \langle m_2, \alpha\rangle \in Q_2\}$;
- $Q^0 = \{\langle(m_1, m_2), \alpha\rangle \mid \langle m_1, \alpha\rangle \in Q_1^0, \langle m_2, \alpha\rangle \in Q_2^0\}$;
- *the $\xrightarrow{\ell}$, one for each $\ell \in A \cup \mathbb{R}^>$, are the smallest subsets of $Q \times Q$ such that:*
  - *if $\langle m_1, \alpha\rangle \xrightarrow{a}_1 \langle m_1', \alpha'\rangle$, $\langle m_2, \alpha\rangle \in Q_2$, $\langle m_2, \alpha'\rangle \in Q_2$ and $a \notin A_2$, then $\langle(m_1, m_2), \alpha\rangle \xrightarrow{a} \langle(m_1', m_2), \alpha'\rangle$;*
  - *if $\langle m_1, \alpha\rangle \in Q_1$, $\langle m_1, \alpha'\rangle \in Q_1$, $a \notin A_1$ and $\langle m_2, \alpha\rangle \xrightarrow{a}_2 \langle m_2', \alpha'\rangle$, then $\langle(m_1, m_2), \alpha\rangle \xrightarrow{a} \langle(m_1, m_2'), \alpha'\rangle$;*
  - *if both $\langle m_1, \alpha\rangle \xrightarrow{a}_1 \langle m_1', \alpha'\rangle$ and $\langle m_2, \alpha\rangle \xrightarrow{a}_2 \langle m_2', \alpha'\rangle$, then $\langle(m_1, m_2), \alpha\rangle \xrightarrow{a} \langle(m_1', m_2'), \alpha'\rangle$;*
  - *if both $\langle m_1, \alpha\rangle \xmapsto{r}_1 \langle m_1, \alpha'\rangle$ and $\langle m_2, \alpha\rangle \xmapsto{r}_2 \langle m_2, \alpha'\rangle$, then $\langle(m_1, m_2), \alpha\rangle \xmapsto{r} \langle(m_1, m_2), \alpha'\rangle$;*
- $\|\alpha\| = \{\langle m, \alpha\rangle \mid \langle m, \alpha\rangle \in Q\}$, *for each $\alpha \in \mathcal{V}_{\mathsf{st}}$.*

PROOF. We check the definitions of Q, Q^0, $\rightarrow$ and $\|_-\|$ in turn.

- It follows from the definitions of transition system interpretation and synchronized product that $\langle(m_1, m_2), \alpha\rangle \in Q$ iff $\alpha \models \phi(m_1, m_2)$ iff $\alpha \models \phi(m_1) \wedge \alpha \models \phi(m_2)$ iff $\langle m_1, \alpha\rangle \in Q_1 \wedge \langle m_2, \alpha\rangle \in Q_2$.
- Analogously, it is proved that $\langle(m_1, m_2), \alpha\rangle \in Q^0$ iff $\langle m_1, \alpha\rangle \in Q_1^0 \wedge \langle m_2, \alpha\rangle \in Q_2^0$.
- Suppose that $H_i = (V_i, M_i, E_i, S_i, \mu_i, \nu_i, \epsilon_i, \chi_i, \phi_i, \psi_i)$, for $i = 1, 2$. By means of the definition of synchronized product, we can rewrite the rules of the inductive definition of $\rightarrow$ from the definition of transition system interpretation. For jump transitions, the rule has to be split up in three rules because there are three cases to consider. We obtain the following rules:
  - *if $s_1 \in S_1$, $m_2 \in M_2$, $\epsilon_1(s_1) \notin A_2$, $\langle \mu_1(s_1), \alpha\rangle \in Q_1$, $\langle m_2, \alpha\rangle \in Q_2$, $\langle \nu_1(s_1), \alpha'\rangle \in Q_1$, $\langle m_2, \alpha'\rangle \in Q_2$, $\alpha \rightarrow \alpha' \models \chi_1(s_1)$, then $\langle(\mu_1(s_1), m_2), \alpha\rangle \xrightarrow{\epsilon_1(s_1)} \langle(\nu_1(s_1), m_2), \alpha'\rangle$;*
  - *if $m_1 \in M_1$, $s_2 \in S_2$, $\epsilon_2(s_2) \notin A_1$, $\langle m_1, \alpha\rangle \in Q_1$, $\langle \mu_2(s_2), \alpha\rangle \in Q_2$, $\langle m_1, \alpha'\rangle \in Q_1$, $\langle \nu_2(s_2), \alpha'\rangle \in Q_2$, $\alpha \rightarrow \alpha' \models \chi_2(s_2)$, then*

$\langle(m_1, \mu_2(s_2)), \alpha\rangle \xrightarrow{\epsilon_2(s_2)} \langle(m_1, \nu_2(s_2)), \alpha'\rangle$;

· if $s_1 \in S_1$, $s_2 \in S_2$, $\epsilon_1(s_1) = \epsilon_2(s_2)$, $\langle\mu_1(s_1), \alpha\rangle \in Q_1$, $\langle\mu_2(s_2), \alpha\rangle \in Q_2$, $\langle\nu_1(s_1), \alpha'\rangle \in Q_1$, $\langle\nu_2(s_2), \alpha'\rangle \in Q_2$, $\alpha \to \alpha' \models \chi_1(s_1)$, $\alpha \to \alpha' \models \chi_2(s_2)$, then $\langle(\mu_1(s_1), \mu_2(s_2)), \alpha\rangle \xrightarrow{\epsilon_1(s_1)} \langle(\nu_1(s_1), \nu_2(s_2)), \alpha'\rangle$;

· if $m_1 \in M_1$, $m_2 \in M_2$, $r \in \mathbb{R}^>$, and there exists a $\rho \in \mathcal{E}_r$ such that $\alpha \xmapsto{r,\rho} \alpha' \models_{V_1} \phi_1(m_1)$ and $\alpha \xmapsto{r,\rho} \alpha' \models_{V_2} \phi_2(m_2)$, then $\langle(m_1, m_2), \alpha\rangle \xrightarrow{r} \langle(m_1, m_2), \alpha'\rangle$.

By means of the definition of transition system interpretation, we can rewrite the rules once more. For the first rule, we obtain the following rule:

· if $\langle\mu_1(s_1), \alpha\rangle \xrightarrow{\epsilon_1(s_1)}_1 \langle\nu_1(s_1), \alpha'\rangle$, $\epsilon_1(s_1) \notin A_2$, $\langle m_2, \alpha\rangle \in Q_2$, $\langle m_2, \alpha'\rangle \in Q_2$, then $\langle(\mu_1(s_1), m_2), \alpha\rangle \xrightarrow{\epsilon_1(s_1)} \langle(\nu_1(s_1), m_2), \alpha'\rangle$.

The other rules are rewritten analogously.

The inductive definition of $\to$ given in the theorem follows immediately because $\langle m, \alpha\rangle \xrightarrow{a}_i \langle m', \alpha'\rangle$ only if there exists a $s \in S_i$ such that $\mu_i(s) = m$, $\nu_i(s) = m'$ and $\epsilon_i(s) = a$ ($i = 1, 2$).

• By definition, $\|\alpha\| = \{\langle m, \alpha\rangle \mid \langle m, \alpha\rangle \in Q\}$, for each $\alpha \in \mathcal{V}_{\mathsf{st}}$.

□

It is worth mentioning that bisimilarity of hybrid automata is not preserved by the synchronized product operator. It is unknown to us whether this fact is mentioned earlier in the literature on hybrid automata.

Theorem 3 (Bisimilarity not Preserved by Synchronized Product)
There exist hybrid automata H_1, H_2, H_1' and H_2' such that $[\![H_1]\!] \leftrightarrows [\![H_1']\!]$ and $[\![H_2]\!] \leftrightarrows [\![H_2']\!]$, but $[\![H_1 \times H_2]\!] \not\leftrightarrows [\![H_1' \times H_2']\!]$.

PROOF. Consider a hybrid automaton H_1 with state variable v, control modes m_0 and m_1 and control switches s_0 and s_1. Control switch s_0 is from m_0 to m_1 and control switch s_1 is from m_1 to m_0. The jump condition of both s_0 and s_1 is $v^\bullet = 0$. The flow conditions of m_0 and m_1 are $v \leq 1 \wedge \dot{v} = 1$ and $v \geq 0 \wedge \dot{v} = -1$, respectively. The initial conditions of m_0 and m_1 are $v = 0$ and F, respectively. The events associated with s_0 and s_1 are a and b, respectively.

Consider the hybrid automaton H_1' obtained from H_1 by first changing the flow condition of m_1 into $v \geq 0 \wedge \dot{v} = 2v - 1$ and after that replacing control modes m_0 and m_1 by m_0' and m_1', respectively, and control switches s_0 and s_1 by s_0' and s_1', respectively.

$[\![H_1]\!]$ and $[\![H_1']\!]$ are bisimilar. Only the different flow conditions associated with control modes m_1 and m_1' may preclude bisimilarity. It is easy to see that the states $\langle m_1, \alpha\rangle$ with $\alpha(v) = 0$ and $\alpha(\dot{v}) = -1$ are the only reachable states of

$[\![H_1]\!]$ with control mode component m_1 and the states $\langle m_1', \alpha \rangle$ with $\alpha(v) = 0$ and $\alpha(\dot{v}) = -1$ are the only reachable states of $[\![H_1']\!]$ with control mode component m_1'. Because no flow transitions are possible from these states, the different flow conditions do not matter.

Consider also a hybrid automaton H_2 with state variables v and w, control modes m_0'' and m_1'' and control switches s_0'' and s_1''. Control switch s_0'' is from m_0'' to m_1'' and control switch s_1'' is from m_1'' to m_0''. The jump conditions of s_0'' and s_1'' are $v^\bullet = 1 \wedge w^\bullet = 1$ and $w^\bullet = 0$, respectively. The flow conditions of m_0'' and m_1'' are $w \leq 1 \wedge \dot{w} = 1$ and $w \geq 0 \wedge \dot{w} = -1$, respectively. The initial conditions of m_0'' and m_1'' are $w = 0$ and F, respectively. The events associated with s_0'' and s_1'' are c and d, different from a and b, respectively.

Take furthermore H_2' identical to H_2. Thus, $[\![H_2]\!]$ and $[\![H_2']\!]$ are trivially bisimilar.

Then $[\![H_1 \times H_2]\!]$ and $[\![H_1' \times H_2']\!]$ are not bisimilar. It is not difficult to see that a bisimulation between $[\![H_1 \times H_2]\!]$ and $[\![H_1' \times H_2']\!]$ must relate states $\langle (m_1, m_0''), \alpha \rangle$ and $\langle (m_1', m_0''), \alpha \rangle$ with $\alpha(v) = 0$, $\alpha(\dot{v}) = -1$, $\alpha(w) = 1$ and $\alpha(\dot{w}) = 1$. Only one jump transition, with c as associated event, is possible from these states. The resulting states $\langle (m_1, m_1''), \alpha' \rangle$ and $\langle (m_1', m_1''), \alpha'' \rangle$ cannot be related because $\alpha' \neq \alpha''$ ($\alpha'(\dot{v}) = -1$ and $\alpha''(\dot{v}) = 1$). Hence, a bisimulation between $[\![H_1 \times H_2]\!]$ and $[\![H_1' \times H_2']\!]$ does not exist. $\quad\square$

5 $\text{ACP}_{\text{hs}}^{\text{srt}}$ Terms: Operational Semantics and Bisimilarity

In Section 6, we will investigate the connections between $\text{ACP}_{\text{hs}}^{\text{srt}}$ and the formalism of hybrid automata. Among other things, hybrid automata will be interpreted in $\text{ACP}_{\text{hs}}^{\text{srt}}$. Important is whether the interpretation concerned is faithfull, i.e. whether two hybrid automata are interpreted as terms that are identified in $\text{ACP}_{\text{hs}}^{\text{srt}}$ if and only if they are identified in the formalism of hybrid automata. In $\text{ACP}_{\text{hs}}^{\text{srt}}$, like in the formalism of hybrid automata, a version of bisimilarity is used to identify terms of which the operational semantics only differ in details that are considered to be irrelevant.

The operational semantics of $\text{ACP}_{\text{hs}}^{\text{srt}}$, and its extensions with integration and recursion, is described by the transition rules given in Tables A.1–A.6 (Appendix A). The following transition relations are used:

- a binary relation $\langle \_, \alpha \rangle \xrightarrow{a} \langle \_, \alpha' \rangle$ for each $a \in \mathsf{A}$, $\alpha, \alpha' \in \mathcal{V}_{\text{st}}$;
- a unary relation $\langle \_, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ for each $a \in \mathsf{A}$, $\alpha, \alpha' \in \mathcal{V}_{\text{st}}$;
- a binary relation $\langle \_, \alpha \rangle \xmapsto{r,\rho} \langle \_, \alpha' \rangle$ for each $r \in \mathbb{R}^>$, $\rho \in \mathcal{E}_r$, $\alpha, \alpha' \in \mathcal{V}_{\text{st}}$ such that $\alpha = \alpha_0^\rho$ and $\alpha' = \alpha_r^\rho$;

- a unary relation $\alpha \in [\mathsf{s}(\_)]$ for each $\alpha \in \mathcal{V}_{\mathsf{st}}$;
- a unary relation $\alpha \to \alpha' \in [\mathsf{d}(\_)]$ for each $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$.

The five kinds of transition relations are called the *action step*, *action termination*, *time step*, *signal* and *discontinuity* relations, respectively. They can be explained as follows:

- $\langle t, \alpha \rangle \xrightarrow{a} \langle t', \alpha' \rangle$: in state α, process t is capable of first performing action a at the current point of time and then proceeding as process t' in state α';
- $\langle t, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$: in state α, process t is capable of first performing action a at the current point of time and then terminating successfully in state α';
- $\langle t, \alpha \rangle \xmapsto{r,\rho} \langle t', \alpha' \rangle$: in state α, process t is capable of first idling for a period of time r, meanwhile evolving its state according to ρ, and then proceeding as process t' in state α';
- $\alpha \in [\mathsf{s}(t)]$: in state α, the signal emitted by process t holds;
- $\alpha \to \alpha' \in [\mathsf{d}(t)]$: in state α, discontinuities for the state variables of which the value changes by an instantaneous transition to state α' are possible for process t.

Recall that in $\mathrm{ACP}_{\mathsf{hs}}^{\mathrm{srt}}$, a valuation $\alpha \in \mathcal{V}_{\mathsf{st}}$ is called a state. Henceforth, we write $\mathcal{PT}$ for the set of closed terms of $\mathrm{ACP}_{\mathsf{hs}}^{\mathrm{srt}}$ extended with integration and recursion.

We have the following corollary of the definition of the operational semantics of $\mathrm{ACP}_{\mathsf{hs}}^{\mathrm{srt}}$ extended with integration and recursion.

Corollary 4 (Signal Emission is Essential) *For all $t, t' \in \mathcal{PT}$, $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$, $a \in \mathsf{A}$, $r \in \mathbb{R}^{>}$ and $\rho \in \mathcal{E}_r$:*

- *if $\langle t, \alpha \rangle \xrightarrow{a} \langle t', \alpha' \rangle$ or $\langle t, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ or $\langle t, \alpha \rangle \xmapsto{r,\rho} \langle t', \alpha' \rangle$ or $\alpha \to \alpha' \in [\mathsf{d}(t)]$, then $\alpha \in [\mathsf{s}(t)]$.*

Let $\mathsf{L} = \mathsf{A} \cup \{(r, \rho) \mid r \in \mathbb{R}^{>} \land \rho \in \mathcal{E}_r\}$. The *generalized transition* relations $\langle \_, \alpha \rangle \xrightarrow{\sigma} \langle \_, \alpha' \rangle$ for each $\sigma \in \mathsf{L}^*$ and $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$ are the smallest binary relations on $\mathcal{PT}$ satisfying:

- if $\alpha \in [\mathsf{s}(t)]$, then $\langle t, \alpha \rangle \xrightarrow{\epsilon} \langle t, \alpha \rangle$;
- if $\langle t, \alpha \rangle \xrightarrow{a} \langle t', \alpha' \rangle$, then $\langle t, \alpha \rangle \xrightarrow{a} \langle t', \alpha' \rangle$;
- if $\langle t, \alpha \rangle \xmapsto{r,\rho} \langle t', \alpha' \rangle$, then $\langle t, \alpha \rangle \xrightarrow{r,\rho} \langle t', \alpha' \rangle$;
- if $\langle t, \alpha \rangle \xrightarrow{\sigma} \langle t', \alpha' \rangle$ and $\langle t', \alpha' \rangle \xrightarrow{\sigma'} \langle t'', \alpha'' \rangle$, then $\langle t, \alpha \rangle \xrightarrow{\sigma \sigma'} \langle t'', \alpha'' \rangle$.

Elements of $\mathcal{PT} \times \mathcal{V}_{\mathsf{st}}$ are called *configurations*. A configuration $\langle t', \alpha' \rangle$ is a *reachable* configuration from configuration $\langle t, \alpha \rangle$ if there is a $\sigma \in \mathsf{L}^*$ such that $\langle t, \alpha \rangle \xrightarrow{\sigma} \langle t', \alpha' \rangle$.

Bisimilarity of closed terms is defined as follows.

A *bisimulation* is a symmetric binary relation B on $\mathcal{PT} \times \mathcal{V}_{\mathsf{st}}$ such that for all $t_1, t_2 \in \mathcal{PT}$ and $\alpha \in \mathcal{V}_{\mathsf{st}}$:

- if $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t_1', \alpha' \rangle$, then there is a t_2' such that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t_2', \alpha' \rangle$ and $B(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$;
- if $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$, then $\langle t_2, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$;
- if $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $\langle t_1, \alpha \rangle \xmapsto{r, \rho} \langle t_1', \alpha' \rangle$, then there is a t_2' such that $\langle t_2, \alpha \rangle \xmapsto{r, \rho} \langle t_2', \alpha' \rangle$ and $B(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$;
- if $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$;
- if $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$.

Two configurations $\langle t_1, \alpha_1 \rangle$ and $\langle t_2, \alpha_2 \rangle$ are *bisimilar*, written $\langle t_1, \alpha_1 \rangle \leftrightarrow \langle t_2, \alpha_2 \rangle$, if $\alpha_1 = \alpha_2$ and there exists a bisimulation B such that $B(\langle t_1, \alpha_1 \rangle, \langle t_2, \alpha_2 \rangle)$. Two closed terms t_1 and t_2 are *bisimilar*, written $t_1 \leftrightarrow t_2$, if $\langle t_1, \alpha \rangle \leftrightarrow \langle t_2, \alpha \rangle$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$.

Note that, if $t_1 \leftrightarrow t_2$, then there exists a bisimulation B with $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. If B is a bisimulation and $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$, then we say that B is a bisimulation *witnessing* $t_1 \leftrightarrow t_2$

Note further that, if $\langle t_1, \alpha \rangle \leftrightarrow \langle t_2, \alpha \rangle$, then there exists a bisimulation B with $B(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ such that $B(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$ only if $\langle t_1', \alpha' \rangle$ and $\langle t_2', \alpha' \rangle$ are reachable from $\langle t_1, \alpha \rangle$ and $\langle t_2, \alpha \rangle$, respectively.

Lemma 5 (Bisimilarity and Action Prefixing) *For all closed* $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ *terms* t_1, t_2 *and* $a \in \mathsf{A}$:

$$t_1 \leftrightarrow t_2 \Leftrightarrow \tilde{\tilde{a}} \cdot t_1 \leftrightarrow \tilde{\tilde{a}} \cdot t_2 \ .$$

PROOF. From left to right, suppose that B is a bisimulation witnessing $t_1 \leftrightarrow t_2$. Then consider the relation $B'' = B \cup B'$ where

$$B' = \{ (\langle \tilde{\tilde{a}} \cdot t_1, \alpha \rangle, \langle \tilde{\tilde{a}} \cdot t_2, \alpha \rangle) \mid \alpha \in \mathcal{V}_{\mathsf{st}} \} \ .$$

It is easy to see that B'' is a bisimulation witnessing $\tilde{\tilde{a}} \cdot t_1 \leftrightarrow \tilde{\tilde{a}} \cdot t_2$.

From right to left, suppose that B is a bisimulation witnessing $\tilde{\tilde{a}} \cdot t_1 \leftrightarrow \tilde{\tilde{a}} \cdot t_2$. This means that $B(\langle \tilde{\tilde{a}} \cdot t_1, \alpha \rangle, \langle \tilde{\tilde{a}} \cdot t_2, \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. Because $\langle \tilde{\tilde{a}} \cdot t_1, \alpha \rangle \xrightarrow{a} \langle t_1, \alpha' \rangle$ and $\langle \tilde{\tilde{a}} \cdot t_2, \alpha \rangle \xrightarrow{a} \langle t_2, \alpha' \rangle$ for all $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$, it follows that $B(\langle t_1, \alpha' \rangle, \langle t_2, \alpha' \rangle)$ for all $\alpha' \in \mathcal{V}_{\mathsf{st}}$. So $t_1 \leftrightarrow t_2$. $\square$

6 Relating the Formalism of Hybrid Automata to ACP$_{\mathrm{hs}}^{\mathrm{srt}}$

In this section, we study the connections between ACP$_{\mathrm{hs}}^{\mathrm{srt}}$ and the formalism of hybrid automata. First of all, we investigate the interpretation of hybrid automata in ACP$_{\mathrm{hs}}^{\mathrm{srt}}$. Next, we investigate the relationship between the synchronized product operator on hybrid automata and the parallel composition operator of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$. We illustrate the interpretation of hybrid automata in ACP$_{\mathrm{hs}}^{\mathrm{srt}}$ by means of an example taken from the literature on hybrid automata.

6.1 Interpretation of Hybrid Automata in ACP$_{\mathrm{hs}}^{\mathrm{srt}}$

We give two interpretations of hybrid automata in ACP$_{\mathrm{hs}}^{\mathrm{srt}}$: a strong interpretation, in which discontinuities during state evolutions are impossible for all state variables, and a weak interpretation, in which discontinuities during state evolutions are possible for all state variables. Only the strong interpretation agrees with the transition system interpretation from Section 4.1. The weak interpretation is introduced because of its usefulness in relating the synchronized product operator on hybrid automata to the parallel composition operator of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$.

In both interpretations, we use a special *initialize* action ι to deal with initial non-determinism. We assume that $\iota \notin \mathrm{E}(H)$ for any hybrid automaton H. The idea to deal with initial non-determinism in this way is taken from Ref. [24].

Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a hybrid automaton. Then the *strong process algebra interpretation* of H, written $[\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}$, is the term

$$\sum_{m \in M} \tilde{\tilde{\iota}} \cdot (\psi_m \, {}^{\curlyvee\!\blacktriangle}\, \langle X_m | F \rangle) \,,$$

where the guarded recursive specification F consists of the following equation for each $m \in M$:

$$X_m = \phi_m \, {}^{\curlyvee\!\blacktriangledown}_V \left(\sum_{s \in \{s \in S \mid m_s = m\}} \chi_s \, {}^{\ulcorner\blacktriangledown}\, \tilde{\tilde{e}}_s \cdot X_{m'_s} + \int_{u \in (0, \infty)} \sigma_{\mathsf{rel}}^u(X_m) \right) \,.$$

Variable X_m corresponds to control mode m, so the guarded recursive specification F contains one equation for each control mode of the hybrid automaton. The right-hand side of the equation for control mode m has one alternative for each control switch that may occur from m as well as an alternative for the case where the control does not switch. This process algebra interpretation draws attention to the fact that, different from what is said in Ref. [19],

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time-determinism is in line with the hybrid automata approach. Because each control mode has just one alternative to proceed with idling, time-determinism is just not an issue.

Note that an equivalent guarded recursive specification is obtained with the following equation for each $m \in M$:

$$X_m = \phi_m \,\reflectbox{$\blacktriangledown$}_V \int_{u \in [0, \infty)} \sigma_{\mathsf{rel}}^u \left(\sum_{s \in \{s \in S \mid m_s = m\}} \chi_s \,\ulcorner\blacktriangleright\, \widetilde{\widetilde{e}}_s \cdot X_{m_s'} \right) .$$

However, we believe that the close correspondence with the transition system interpretation is less clear with such equations.

We have the following results concerning this process algebra interpretation and the transition system interpretation of hybrid automata.

Theorem 6 (Relation between TS and Strong PA Interpretations)
Let H be a hybrid automaton, Q be the set of admissible states of H and Q^0 be the set of initial states of H. Then the states and transitions of $\llbracket H \rrbracket$ and $\llbracket H \rrbracket_{\mathsf{s}}^{\mathrm{pa}}$ are related as follows:

$$\langle m, \alpha \rangle \in Q \Leftrightarrow \alpha \in [\mathsf{s}(X_m)] \,,$$

$$\langle m, \alpha \rangle \in Q^0 \Leftrightarrow \alpha \in [\mathsf{s}(\psi_m \,{}^{\curlywedge} X_m)] \,,$$

$$\langle m, \alpha \rangle \xrightarrow{e} \langle m', \alpha' \rangle \Leftrightarrow \langle X_m, \alpha \rangle \xrightarrow{e} \langle X_{m'}, \alpha' \rangle \,,$$

$$\langle m, \alpha \rangle \in Q^0 \wedge \langle m, \alpha \rangle \xrightarrow{e} \langle m', \alpha' \rangle \Leftrightarrow \langle \psi_m \,{}^{\curlywedge} X_m, \alpha \rangle \xrightarrow{e} \langle X_{m'}, \alpha' \rangle \,,$$

$$\langle m, \alpha \rangle \xmapsto{r} \langle m, \alpha' \rangle$$
$$\Leftrightarrow \exists \rho \in \mathcal{E}_r \bullet \exists t \in \mathcal{PT} \bullet \langle X_m, \alpha \rangle \xmapsto{r, \rho} \langle t, \alpha' \rangle \wedge t \leftrightarroweq X_m \,,$$

$$\langle m, \alpha \rangle \in Q^0 \wedge \langle m, \alpha \rangle \xmapsto{r} \langle m, \alpha' \rangle$$
$$\Leftrightarrow \exists \rho \in \mathcal{E}_r \bullet \exists t \in \mathcal{PT} \bullet \langle \psi_m \,{}^{\curlywedge} X_m, \alpha \rangle \xmapsto{r, \rho} \langle t, \alpha' \rangle \wedge t \leftrightarroweq X_m \,,$$

for all $m, m' \in \mathrm{M}(H)$, $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$, $e \in \mathrm{E}(H)$ and $r \in \mathbb{R}^{>}$.

PROOF. These bi-implications follow easily from the definitions of transition system interpretation and strong process algebra interpretation. We present the proof of the last bi-implication. The proofs of the other bi-implications are similar, but simpler.
From the definition of transition system interpretation, it follows that

$$\langle m, \alpha \rangle \in Q^0 \Leftrightarrow \alpha \models \phi(m) \wedge \alpha \models \psi(m) \tag{1}$$

$$\langle m, \alpha \rangle \overset{r}{\longmapsto} \langle m, \alpha' \rangle \Leftrightarrow \exists \rho \in \mathcal{E}_r \bullet \alpha \overset{r,\rho}{\longmapsto} \alpha' \models_{\mathrm{V}(H)} \phi(m) \,. \tag{2}$$

From the definition of strong process algebra interpretation, it follows that

$$\exists \rho \in \mathcal{E}_r \bullet \exists t \in \mathcal{PT} \bullet \langle \psi_m \overset{\curvearrowright}{} X_m, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t, \alpha' \rangle \wedge t \leftrightarrow X_m$$
$$\Leftrightarrow \exists \rho \in \mathcal{E}_r \bullet \alpha \overset{r,\rho}{\longmapsto} \alpha' \models_{\mathrm{V}(H)} \phi(m) \wedge \alpha \models \psi(m) \,. \tag{3}$$

Clearly, the conjunction of the right-hand sides of 1 and 2 is equivalent to the right-hand side of 3. Hence, the conjunction of the left-hand sides of 1 and 2 is equivalent to the left-hand side of 3. $\quad\square$

Theorem 7 (Faithfulness of Strong PA Interpretation) *For all hybrid automata H_1 and H_2 with $\mathrm{V}(H_1) = \mathrm{V}(H_2)$:*

$$\llbracket H_1 \rrbracket_{\mathrm{s}}^{\mathrm{pa}} \leftrightarrow \llbracket H_2 \rrbracket_{\mathrm{s}}^{\mathrm{pa}} \Leftrightarrow \llbracket H_1 \rrbracket \leftrightarrow \llbracket H_2 \rrbracket \,. \; ^2$$

PROOF. See Appendix B.1. $\quad\square$

Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a hybrid automaton. Then the *weak process algebra interpretation* of H, written $\llbracket H \rrbracket_{\mathrm{w}}^{\mathrm{pa}}$, is the term

$$\sum_{m \in M} \tilde{\tilde{\imath}} \cdot (\psi_m \overset{\curvearrowright}{} \langle X_m | F' \rangle) \,,$$

where the guarded recursive specification F' consists of the following equation for each $m \in M$:

$$X_m = \phi_m \overset{\curvearrowright}{}_\emptyset \left(\sum_{s \in \{s \in S | m_s = m\}} \chi_s \overset{\text{\Small \textbf{r}}}{} \tilde{\tilde{e}}_s \cdot X_{m'_s} + \int_{u \in (0,\infty)} \sigma_{\mathrm{rel}}^u (X_m) \right) \,.$$

Note that V has been replaced by $\emptyset$. In this way, discontinuities during state evolutions become possible for all state variables.

The strong and weak process algebra interpretations can be regarded as two instances of a generic process algebra interpretation that has the set of state variables for which discontinuities during state evolutions must be impossible

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[2] Note that on the left-hand side $\leftrightarrow$ is bisimilarity on $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ terms and on the right-hand side $\leftrightarrow$ is bisimilarity on timed transition systems.

as a parameter. However, for a given hybrid automaton H, other choices for this set than $V(H)$ and $\emptyset$ appear to be absolutely arbitrary.

Unlike the strong process algebra interpretation, the weak process algebra interpretation does not agree with the transition system interpretation from Section 4.1: in the case of the weak process algebra interpretation, the last two bi-implications of Theorem 6 do not hold from right to left. That is the reason why we prefer the strong process algebra interpretation.

Proposition 8 (Weakness of Weak PA Interpretations) *There exists a hybrid automaton H such that in the case of weak process algebra interpretation:*

$$\langle m, \alpha \rangle \overset{r}{\longmapsto} \langle m, \alpha' \rangle \not\Leftarrow \exists \rho \in \mathcal{E}_r \bullet \exists t \in \mathcal{PT} \bullet \langle X_m, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t, \alpha' \rangle \wedge t \leftrightharpoons X_m$$

for some $m \in M(H)$, $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$ and $r \in \mathbb{R}^>$.

PROOF. Consider a hybrid automaton H with state variable v, control mode m and control switch s. Control switch s is from m to m. The jump condition of s is $v^\bullet = 0$. The flow condition of m is $v \leq 1 \wedge \dot{v} = 1$. The initial condition of m is $v = 0$. The event associated with s is a.

Let ϕ_m be the flow condition of m. In $[\![H]\!]_{\mathsf{w}}^{\mathrm{pa}}$, because a finite number of discontinuities may occur, there exist transitions $\langle X_m, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle \phi_m \curvearrowright_\emptyset X_m, \alpha' \rangle$ with $r > 1$, and $\phi_m \curvearrowright_\emptyset X_m \leftrightharpoons X_m$; but in $[\![H]\!]$, because no discontinuities may occur, there do not exist transitions $\langle m, \alpha \rangle \overset{r}{\longmapsto} \langle m, \alpha' \rangle$ with $r > 1$. $\square$

We have the following result concerning the connection between the two process algebra interpretations of hybrid automata.

Proposition 9 (Coarseness of Strong PA Interpretation) *For all hybrid automata H_1 and H_2 with $V(H_1) = V(H_2)$:*

$$[\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \leftrightharpoons [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}} \Rightarrow [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}} \leftrightharpoons [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}} .$$

PROOF. See Appendix B.2. $\square$

In Proposition 9, "$\Rightarrow$" cannot be replaced by "$\Leftrightarrow$".

Proposition 10 (Coarseness of Strong PA Interpretation is Proper)
There exist hybrid automata H_1 and H_2 such that:

$$[\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}} \nLeftarrow [\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}} .$$

PROOF. Consider a hybrid automaton H_1 with state variable v, control modes m_0 and m_1 and control switches s_0 and s_1. Control switch s_0 is from m_0 to m_1 and control switch s_1 is from m_1 to m_0. The jump conditions of s_0 and s_1 are $v^{\bullet} = {}^{\bullet}v$ and ${}^{\bullet}v = 1 \wedge v^{\bullet} = 0$, respectively. The flow condition of both m_0 and m_1 is $\dot{v} = 0$. The initial conditions of m_0 and m_1 are $v = 0$ and F, respectively. The events associated with s_0 and s_1 are a and b, respectively.

Consider a hybrid automaton H_2 with state variable v, control modes m_0' and m_1' and control switch s_0'. Control switch s_0' is from m_0' to m_1'. The jump condition of s_0' is $v^{\bullet} = {}^{\bullet}v$. The flow condition of both m_0' and m_1' is $\dot{v} = 0$. The initial conditions of m_0' and m_1' are $v = 0$ and F, respectively. The event associated with s_0 is a.

$[\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}}$ and $[\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}}$ are bisimilar. Only the presence or absence of a control switch from control modes m_1 and m_1' may preclude bisimilarity. It is easy to see that the configurations $\langle X_{m_1}, \alpha \rangle$ with $\alpha(v) = 0$ and $\alpha(\dot{v}) = 0$ are the only reachable configurations of $[\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}}$ with term component X_{m_1} and the configurations $\langle X_{m_1'}, \alpha \rangle$ with $\alpha(v) = 0$ and $\alpha(\dot{v}) = 0$ are the only reachable configurations of $[\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}}$ with term component $X_{m_1'}$. Because no action steps are possible from these configurations, the presence or absence of a control switch does not matter.

$[\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}}$ and $[\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}$ are not bisimilar. It is not difficult to see that a witnessing bisimulation must relate configurations $\langle X_{m_1}, \alpha \rangle$ and $\langle X_{m_1'}, \alpha \rangle$ with $\alpha(v) = 1$ and $\alpha(\dot{v}) = 0$. An action step is possible from the former configuration, but not from the latter configuration. Hence, a bisimulation between $[\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}}$ and $[\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}$ does not exist. $\square$

In Ref. [10], $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ is also extended with localization. The localization operator makes it possible to keep discontinuities of a state variable local, in other words to inhibit discontinuities of the state variable caused by the environment. The *localization* of P with respect to v, written $v \nabla P$, behaves like P, but with its state evolving without discontinuities for v whenever it is idling. The operational semantics for localization is described in Appendix A. We use the notation $\{v_1, \ldots, v_n\} \nabla t$ for $v_1 \nabla (v_2 \nabla \ldots (v_n \nabla t) \ldots)$. For hybrid automata H, we can express, using localization, $[\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}$ in terms of $[\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}$.

Proposition 11 (Strengthening of Weak PA Interpretation) *For all*

hybrid automata H:

$$[\![H]\!]^{\text{pa}}_{\text{s}} \leftrightarroweq \text{V}(H) \nabla [\![H]\!]^{\text{pa}}_{\text{w}} \,.$$

PROOF. See Appendix B.3. □

In the frequently occurring case where the hybrid automata under consideration have no initial non-determinism, we can give simpler strong and weak interpretations.

Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a hybrid automaton that has no initial non-determinism. Then the *restricted strong process algebra interpretation* of H, written $[\![H]\!]^{\text{pa}}_{\text{rs}}$, and the *restricted weak process algebra interpretation* of H, written $[\![H]\!]^{\text{pa}}_{\text{rw}}$, are the terms

$$\psi_{m^0} \,^{\wedge\!\!\blacktriangle} \langle X_{m^0} | F \rangle \quad \text{and} \quad \psi_{m^0} \,^{\wedge\!\!\blacktriangle} \langle X_{m^0} | F' \rangle \,,$$

respectively, where the guarded recursive specification F and F' are the same as above.

For hybrid automata H without initial non-determinism, we can express both $[\![H]\!]^{\text{pa}}_{\text{s}}$ in terms of $[\![H]\!]^{\text{pa}}_{\text{rs}}$ and $[\![H]\!]^{\text{pa}}_{\text{w}}$ in terms of $[\![H]\!]^{\text{pa}}_{\text{rw}}$.

Proposition 12 (Lifting Restricted Interpretations) *For all hybrid automata H that have no initial non-determinism:*

$$[\![H]\!]^{\text{pa}}_{\text{s}} = \tilde{\tilde{\iota}} \cdot [\![H]\!]^{\text{pa}}_{\text{rs}} \quad \text{and} \quad [\![H]\!]^{\text{pa}}_{\text{w}} = \tilde{\tilde{\iota}} \cdot [\![H]\!]^{\text{pa}}_{\text{rw}} \,.$$

PROOF. Follows immediately from the definitions of the process algebra interpretations concerned. □

We have the following corollary of Lemma 5, Theorem 7 and Proposition 12.

Corollary 13 (Faithfulness of Restricted Strong PA Interpretation) *For all hybrid automata H_1 and H_2 with $\text{V}(H_1) = \text{V}(H_2)$ that have no initial non-determinism:*

$$[\![H_1]\!]^{\text{pa}}_{\text{rs}} \leftrightarroweq [\![H_2]\!]^{\text{pa}}_{\text{rs}} \Leftrightarrow [\![H_1]\!] \leftrightarroweq [\![H_2]\!] \,.$$

In this section, we consider a thermostat with delay. The behaviour of the thermostat can be described informally as follows.

Initially, the temperature is $18°\,$C and the heating is on. While the heating is on, the temperature T in the room goes up according to the the differential equation $\dot{T} = -T + 21$. When the temperature becomes $20°\,$C, the heating will be turned off after a delay of 1 second. While the heating is off, the temperature T in the room goes down according to the the differential equation $\dot{T} = -T + 17$. When the temperature becomes $18°\,$C, the heating will be turned on again after a delay of 1 second.

The example is taken from Ref. [9], but it has been adapted to more realistic room temperatures. There, the thermostat is described by a hybrid automaton. Here, we give the strong process algebra interpretation of that hybrid automaton:

$$\tilde{\iota} \cdot \left((T = 18) \mathrel{\text{\^{}\!\blacktriangle}} \langle\, Th_{\text{on}} | F \,\rangle \right),$$

where the recursive specification F consists of the following equations:

$$Th_{\text{on}} = (T \le 20 \wedge \dot{T} = -T + 21) \mathrel{\text{\reflectbox{\textsf{V}}}_{\{T,c\}}}$$
$$\left(({}^{\bullet}T = 20 \wedge T^{\bullet} = {}^{\bullet}T \wedge c^{\bullet} = 0) \mathrel{\text{\reflectbox{\textsf{V}}}} \widetilde{high} \cdot Th_{\text{d1}} + \sigma^{+}_{\text{rel}}(Th_{\text{on}}) \right),$$

$$Th_{\text{d1}} = (c \le 1 \wedge \dot{T} = -T + 21 \wedge \dot{c} = 1) \mathrel{\text{\reflectbox{\textsf{V}}}_{\{T,c\}}}$$
$$\left(({}^{\bullet}c = 1 \wedge T^{\bullet} = {}^{\bullet}T) \mathrel{\text{\reflectbox{\textsf{V}}}} \widetilde{turn\text{-}off} \cdot Th_{\text{off}} + \sigma^{+}_{\text{rel}}(Th_{\text{d1}}) \right),$$

$$Th_{\text{off}} = (T \ge 18 \wedge \dot{T} = -T + 17) \mathrel{\text{\reflectbox{\textsf{V}}}_{\{T,c\}}}$$
$$\left(({}^{\bullet}T = 18 \wedge T^{\bullet} = {}^{\bullet}T \wedge c^{\bullet} = 0) \mathrel{\text{\reflectbox{\textsf{V}}}} \widetilde{low} \cdot Th_{\text{d2}} + \sigma^{+}_{\text{rel}}(Th_{\text{off}}) \right),$$

$$Th_{\text{d2}} = (c \le 1 \wedge \dot{T} = -T + 17 \wedge \dot{c} = 1) \mathrel{\text{\reflectbox{\textsf{V}}}_{\{T,c\}}}$$
$$\left(({}^{\bullet}c = 1 \wedge T^{\bullet} = {}^{\bullet}T) \mathrel{\text{\reflectbox{\textsf{V}}}} \widetilde{turn\text{-}on} \cdot Th_{\text{on}} + \sigma^{+}_{\text{rel}}(Th_{\text{d2}}) \right).$$

Here and in Section 7.3, we use the notation $\sigma^{+}_{\text{rel}}(t)$ for $\int_{u \in (0,\infty)} \sigma^{u}_{\text{rel}}(t)$ with u a variable not occurring free in t.

The hybrid automaton for the thermostat has no initial non-determinism. Hence, the restricted strong process algebra interpretation $(T = 18) \mathrel{\text{\^{}\!\blacktriangle}} \langle\, Th_{\text{on}} | F \,\rangle$ makes sense as well.

As usual when a hybrid system is described by a hybrid automaton, the delays of the thermostat are modelled by means of a state variable c with $\dot{c} = 1$. In

$\text{ACP}_{\text{hs}}^{\text{srt}}$, the relative delay operator is available for that purpose. This means that we can replace the recursive specification given above by the following one:

$$
\begin{aligned}
Th_{\text{on}} &= (T \leq 20 \wedge \dot{T} = -T + 21) \,\overline{\mathbb{v}}_{\{T\}} \\
&\qquad \left(({}^{\bullet}T = 20 \wedge T^{\bullet} = {}^{\bullet}T) \,\overline{\mathbb{v}}\, \widetilde{\widetilde{high}} \cdot Th_{\text{d1}} + \sigma_{\text{rel}}^{+}(Th_{\text{on}}) \right), \\[4pt]
Th_{\text{d1}} &= (\dot{T} = -T + 21) \,\overline{\mathbb{v}}_{\{T\}}\, \sigma_{\text{rel}}^{1}\left((T^{\bullet} = {}^{\bullet}T) \,\overline{\mathbb{v}}\, \widetilde{\widetilde{turn\text{-}off}} \cdot Th_{\text{off}} \right), \\[4pt]
Th_{\text{off}} &= (T \geq 18 \wedge \dot{T} = -T + 17) \,\overline{\mathbb{v}}_{\{T\}} \\
&\qquad \left(({}^{\bullet}T = 18 \wedge T^{\bullet} = {}^{\bullet}T) \,\overline{\mathbb{v}}\, \widetilde{\widetilde{low}} \cdot Th_{\text{d2}} + \sigma_{\text{rel}}^{+}(Th_{\text{off}}) \right), \\[4pt]
Th_{\text{d2}} &= (\dot{T} = -T + 17) \,\overline{\mathbb{v}}_{\{T\}}\, \sigma_{\text{rel}}^{1}\left((T^{\bullet} = {}^{\bullet}T) \,\overline{\mathbb{v}}\, \widetilde{\widetilde{turn\text{-}on}} \cdot Th_{\text{on}} \right).
\end{aligned}
$$

6.3 Relating Synchronized Product to Parallel Composition

In order to relate the synchronized product operator on hybrid automata to the parallel composition operator of $\text{ACP}_{\text{hs}}^{\text{srt}}$, we have to extend $\text{ACP}_{\text{hs}}^{\text{srt}}$ with action renaming. This operator provides for change of actions. For $f : \mathsf{A} \to \mathsf{A}$, the *action renaming* of P according to f, written $\rho_f(P)$, behaves like P, but with undelayable actions $\tilde{a}$ replaced by $\widetilde{f(a)}$. The operational semantics for action renaming is described in Appendix A.

We have the following result concerning the synchronized product operator of the formalism of hybrid automata and the parallel composition operator of $\text{ACP}_{\text{hs}}^{\text{srt}}$.

Theorem 14 (Weak PA Interpretation of Synchronized Products)
For all hybrid automata H_1, H_2:

$$
[\![H_1 \times H_2]\!]_{\text{w}}^{\text{pa}} \leftrightarrow \rho_f(\partial_{A'}([\![H_1]\!]_{\text{w}}^{\text{pa}} \parallel [\![H_2]\!]_{\text{w}}^{\text{pa}})),
$$

where $A' = (\mathrm{E}(H_1) \cap \mathrm{E}(H_2)) \cup \{\iota\}$, the renaming function f is such that $f(\bar{a}) = a$ if $a \in A'$ and $f(a) = a$ if $a \notin \{\bar{a} \mid a \in A'\}$, and the communication function γ is such that $\gamma(a, a) = \bar{a}$ if $a \in A'$ and it is undefined otherwise.

PROOF. See Appendix B.4. □

Recall that we prefer strong process algebra interpretations. Therefore, Theorem 14 does not give us the compositionality result that we really want. However, a similar compositionality result does not hold in the case of strong process algebra interpretations. The main point is that, in the case of the strong

process algebra interpretations of the two hybrid automata, jump transitions of one of them cannot take place during flow transitions of the other. This is closely related to the fact that in the strong process algebra interpretation of a hybrid automaton only state evolutions in which no discontinuities occur are associated with flow transitions. In parallel composition, this precludes parallel discontinuities due to both jump transitions and flow transitions. For a compositionality result, the definitions of hybrid automata and synchronized product have to be adapted. This is worked out in Sections 7.1 and 7.2. We have the following corollary of Proposition 11 and Theorem 14.

Corollary 15 (Strengthening of Weak PA Interpretation) *For all hybrid automata H_1, H_2:*

$$[\![H_1 \times H_2]\!]_{\text{s}}^{\text{pa}} \leftrightarrows \text{V}(H_1 \times H_2) \nabla \rho_f(\partial_{A'}([\![H_1]\!]_{\text{w}}^{\text{pa}} \parallel [\![H_2]\!]_{\text{w}}^{\text{pa}})) \,,$$

where A', f and γ are as in Theorem 14.

7 Adapting the Formalism of Hybrid Automata

In Section 6, we found that the formalism of hybrid automata matches a fragment of the process algebra for hybrid systems closely, but not exactly. The mismatch manifests itself entirely with the strong process algebra interpretation of synchronous products. It cannot be expressed in terms of the strong process algebra interpretations of the hybrid automata being composed: $[\![H_1 \times H_2]\!]_{\text{s}}^{\text{pa}} \leftrightarrows \rho_f(\partial_{A'}([\![H_1]\!]_{\text{s}}^{\text{pa}} \parallel [\![H_2]\!]_{\text{s}}^{\text{pa}}))$, for appropriate f and A', does not hold for all hybrid automata H_1 and H_2 (see Section 6.3). In this section we adapt the definitions of hybrid automaton, transition system interpretation of a hybrid automaton and synchronized product of hybrid automata such that an exact match results, as is witnessed by the main theorems: Theorems 18 and 19. We illustrate the use of the adapted formalism of hybrid automata by means of an example taken from the literature on hybrid automata. We also add localization to the adapted formalism.

7.1 Continuity Controlled Hybrid Automata

We adapt the definition of hybrid automaton. The underlying idea is that the continuous changes of some state variables may be interrupted from the environment, but the continuous changes of other state variables not. Such a partition of state variables with respect to the interruptability of continuous changes is impossible with original hybrid automata.

A *continuity controlled hybrid automaton* is a tuple

$$(V, W, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$$

where V, M, E, S, μ, ν, ϵ, χ, ϕ and ψ are as in the definition of hybrid automaton in Section 4.1, and $W \subseteq V$. The set W is called the set of *robust state variables*. We write $W(H)$ for W.

The difference between continuity controlled hybrid automata and original hybrid automata is that in the case of continuity controlled hybrid automata evolutions with a finite number of discontinuities for certain state variables may take place. The meaning of continuity controlled hybrid automata is given in terms of hybrid transition systems instead of timed transition systems. Hybrid transition systems are less abstract than timed transition systems: they have flow transitions $q \xmapsto{r,\rho} q'$, where $\rho \in \mathcal{E}_r$, instead of $q \xmapsto{r} q'$.

A *hybrid transition system* T is a tuple $(Q, Q^0, A, \rightarrow, O, \|_-\|)$ where Q, Q^0, A, O and $\|_-\|$ are as in the definition of timed transition system in Section 4.1, but $\rightarrow$ consists of

- a set $\xrightarrow{\ell} \subseteq Q \times Q$ of *ℓ-transitions*, for each $\ell \in A \cup \{(r, \rho) \mid r \in \mathbb{R}^>, \rho \in \mathcal{E}_r\}$.

Bisimilarity on hybrid transition systems is defined as on timed transition systems in Section 4.1, on the understanding that the range of ℓ is changed: $\ell \in A \cup \{(r, \rho) \mid r \in \mathbb{R}^>, \rho \in \mathcal{E}_r\}$ instead of $\ell \in A \cup \mathbb{R}^>$.

States, *admissible states* and *initial states* of continuity controlled hybrid automata are defined as for original hybrid automata.

Let $H = (V, W, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a continuity controlled hybrid automaton. The *transition system interpretation* of H, written $[\![H]\!]_{\mathrm{cc}}$, is the hybrid transition system $(Q, Q^0, E, \rightarrow, \mathcal{V}_{\mathsf{st}}, \|_-\|)$ where Q, Q^0 and $\|_-\|$ are as in the definition of transition system interpretation in Section 4.1, but the $\xrightarrow{\ell}$, one for each $\ell \in E \cup \{(r, \rho) \mid r \in \mathbb{R}^>, \rho \in \mathcal{E}_r\}$, are the smallest subsets of $Q \times Q$ such that:

- if $s \in S$, $\langle m_s, \alpha \rangle \in Q$, $\langle m'_s, \alpha' \rangle \in Q$ and $\alpha \rightarrow \alpha' \models \chi_s$, then $\langle m_s, \alpha \rangle \xrightarrow{e_s} \langle m'_s, \alpha' \rangle$;
- if $m \in M$, $r \in \mathbb{R}^>$, $\rho \in \mathcal{E}_r$ and $\alpha \xmapsto{r,\rho} \alpha' \models_W \phi_m$, then $\langle m, \alpha \rangle \xmapsto{r,\rho} \langle m, \alpha' \rangle$.

Note that evolutions with a finite number of discontinuities for state variables in $V \setminus W$ may take place. An interesting special case occurs if $V = W$.

Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be an original hybrid automaton. Then we write $\mathrm{cc}(H)$ for the corresponding continuity controlled hybrid automaton $(V, V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$.

Proposition 16 (Relation Original and CC Hybrid Automata) *For all hybrid automata H_1, H_2:*

$$[\![\mathrm{cc}(H_1)]\!]_{\mathrm{cc}} \leftrightarrow [\![\mathrm{cc}(H_2)]\!]_{\mathrm{cc}} \Leftrightarrow [\![H_1]\!] \leftrightarrow [\![H_2]\!] \; . \;\; ^3$$

PROOF. From the definitions of transition system interpretation for original hybrid automata and continuity controlled hybrid automata, it is easy to see that for any hybrid automaton H, $[\![\mathrm{cc}(H)]\!]_{\mathrm{cc}}$ and $[\![H]\!]$ only differ in their flow transitions. For flow transitions, we have:

$$\exists \rho \in \mathcal{E}_r \bullet \langle m, \alpha \rangle \xmapsto{r,\rho} \langle m', \alpha' \rangle \Leftrightarrow \langle m, \alpha \rangle \xmapsto{r} \langle m', \alpha' \rangle \; .$$

From left to right, suppose that B is a bisimulation between $[\![\mathrm{cc}(H_1)]\!]_{\mathrm{cc}}$ and $[\![\mathrm{cc}(H_2)]\!]_{\mathrm{cc}}$. Using the bi-implication given above, it follows immediately that B is a bisimulation between $[\![H_1]\!]$ and $[\![H_2]\!]$ as well. The proof from right to left is similar, using Proposition 1. $\square$

Let $H = (V, W, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a continuity controlled hybrid automaton. Then the *process algebra interpretation* of H, written $[\![H]\!]_{\mathrm{cc}}^{\mathrm{pa}}$, is the term

$$\sum_{m \in M} \tilde{\tilde{\iota}} \cdot (\psi_m \,^{\wedge}\!\!\blacktriangle\, \langle X_m | F \rangle) \; ,$$

where the guarded recursive specification F consists of the following equation for each $m \in M$:

$$X_m = \phi_m \,^{\wedge}\!\!\blacktriangledown_W \left(\sum_{s \in \{s \in S | m_s = m\}} \chi_s \,^{\ulcorner\!\blacktriangledown}\, \tilde{\tilde{e}}_s \cdot X_{m'_s} + \int_{u \in (0,\infty)} \sigma_{\mathrm{rel}}^u(X_m) \right) \; .$$

Note that only evolutions without discontinuities for state variables in W may take place. Note further that this process algebra interpretation is reminiscent of the generic process algebra interpretation of original hybrid automata mentioned in Section 6.1.

We have the following results concerning the process algebra interpretation and transition system interpretation of continuity controlled hybrid automata.

---

^{3} Note that on the left-hand side $\leftrightarrow$ is bisimilarity on hybrid transition systems and on the right-hand side $\leftrightarrow$ is bisimilarity on timed transition systems.

Theorem 17 (Relation between TS and PA Interpretations) *Let H be a continuity controlled hybrid automaton, Q be the set of admissible states of H and Q^0 be the set of initial states of H. Then the state and transitions of $[\![H]\!]_{\mathrm{cc}}$ and $[\![H]\!]_{\mathrm{cc}}^{\mathrm{pa}}$ are related as follows:*

$$\langle m, \alpha \rangle \in Q \Leftrightarrow \alpha \in [\mathsf{s}(X_m)] \ ,$$

$$\langle m, \alpha \rangle \in Q^0 \Leftrightarrow \alpha \in [\mathsf{s}(\psi_m \curlywedge X_m)] \ ,$$

$$\langle m, \alpha \rangle \xrightarrow{e} \langle m', \alpha' \rangle \Leftrightarrow \langle X_m, \alpha \rangle \xrightarrow{e} \langle X_{m'}, \alpha' \rangle \ ,$$

$$\langle m, \alpha \rangle \in Q^0 \wedge \langle m, \alpha \rangle \xrightarrow{e} \langle m', \alpha' \rangle \Leftrightarrow \langle \psi_m \curlywedge X_m, \alpha \rangle \xrightarrow{e} \langle X_{m'}, \alpha' \rangle \ ,$$

$$\langle m, \alpha \rangle \xmapsto{r, \rho} \langle m, \alpha' \rangle$$
$$\Leftrightarrow \exists t \in \mathcal{PT} \bullet \langle X_m, \alpha \rangle \xmapsto{r, \rho} \langle t, \alpha' \rangle \wedge t \leftrightarroweq X_m \ ,$$

$$\langle m, \alpha \rangle \in Q^0 \wedge \langle m, \alpha \rangle \xmapsto{r, \rho} \langle m, \alpha' \rangle$$
$$\Leftrightarrow \exists t \in \mathcal{PT} \bullet \langle \psi_m \curlywedge X_m, \alpha \rangle \xmapsto{r, \rho} \langle t, \alpha' \rangle \wedge t \leftrightarroweq X_m \ ,$$

for all $m, m' \in \mathrm{M}(H)$, $\alpha, \alpha' \in \mathcal{V}_{\mathsf{st}}$, $e \in \mathrm{E}(H)$ and $r \in \mathbb{R}^>$.

PROOF. The proof is analogous to the proof of Theorem 6. □

Theorem 18 (Faithfulness of PA Interpretation) *For all continuity controlled hybrid automata H_1 and H_2 with $\mathrm{W}(H_1) = \mathrm{W}(H_2)$:*

$$[\![H_1]\!]_{\mathrm{cc}}^{\mathrm{pa}} \leftrightarroweq [\![H_2]\!]_{\mathrm{cc}}^{\mathrm{pa}} \Leftrightarrow [\![H_1]\!]_{\mathrm{cc}} \leftrightarroweq [\![H_2]\!]_{\mathrm{cc}} \ . \ ^4$$

PROOF. The proof is analogous to the proof of Theorem 7. □

7.2 Synchronized Product of Continuity Controlled Hybrid Automata

The definition of synchronized product has to be adapted to take care of the constraints with respect to interruption of continuous changes of state variables.

---

[4] Note that on the left-hand side $\leftrightarroweq$ is bisimilarity on $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ terms and on the right-hand side $\leftrightarroweq$ is bisimilarity on hybrid transition systems.

Let $H_i = (V_i, W_i, M_i, E_i, S_i, \mu_i, \nu_i, \epsilon_i, \chi_i, \phi_i, \psi_i)$, for $i = 1, 2$, be continuity controlled hybrid automata. Then the *synchronized product* of H_1 and H_2, written $H_1 \times H_2$, is the continuity controlled hybrid automaton

$$H = (V_1 \cup V_2, W_1 \cup W_2, M_1 \times M_2, E_1 \cup E_2, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$$

where S, μ, ν, ϵ, ϕ and ψ are as in the definition of synchronized product in Section 4.2, and

$$\chi(s, m) = \chi_1(s) \wedge \bigwedge_{v \in W_2}(v^\bullet = {}^\bullet v \wedge \dot{v}^\bullet = {}^\bullet \dot{v}) \ ,$$
$$\chi(m, s) = \bigwedge_{v \in W_1}(v^\bullet = {}^\bullet v \wedge \dot{v}^\bullet = {}^\bullet \dot{v}) \wedge \chi_2(s) \ ,$$
$$\chi(s_1, s_2) = \chi_1(s_1) \wedge \chi_2(s_2) \ .$$

Note that only for the state variables in $V_1 \setminus W_1$, continuous changes originating from H_1 may be interrupted by instantaneous changes originating from H_2; and vice versa. This distinction is impossible with original hybrid automata.

We have the following result concerning the synchronized product of continuity controlled hybrid automata and the parallel composition of $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ terms.

Theorem 19 (PA Interpretation of Synchronized Products) *For all continuity controlled hybrid automata H_1, H_2:*

$$[\![H_1 \times H_2]\!]_{\mathrm{cc}}^{\mathrm{pa}} \leftrightarrow \rho_f(\partial_{A'}([\![H_1]\!]_{\mathrm{cc}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathrm{cc}}^{\mathrm{pa}})) \ ,$$

where $A' = (\mathrm{E}(H_1) \cap \mathrm{E}(H_2)) \cup \{\iota\}$, the renaming function f is such that $f(\overline{a}) = a$ if $a \in A'$ and $f(a) = a$ if $a \notin \{\overline{a} \mid a \in A'\}$, and the communication function γ is such that $\gamma(a, a) = \overline{a}$ if $a \in A'$ and it is undefined otherwise.

PROOF. The proof is similar to the proof of Theorem 14. $\square$

Recall that Theorem 14 did not give us the compositionality result that we really wanted. In the case of continuity controlled hybrid automata, Theorem 19 gives us the desired result.

In the setting of continuity controlled hybrid automata, bisimilarity is not preserved by synchronized product too.

Theorem 20 (Bisimilarity not Preserved by Synchronized Products) *There exist continuity controlled hybrid automata H_1, H_2, H_1' and H_2' such that $[\![H_1]\!]_{\mathrm{cc}} \leftrightarrow [\![H_1']\!]_{\mathrm{cc}}$ and $[\![H_2]\!]_{\mathrm{cc}} \leftrightarrow [\![H_2']\!]_{\mathrm{cc}}$, but $[\![H_1 \times H_2]\!]_{\mathrm{cc}} \not\leftrightarrow [\![H_1' \times H_2']\!]_{\mathrm{cc}}$.*

PROOF. With continuity controlled hybrid automata without robust state variables, the counterexample of preservation of bisimilarity given in the proof of Theorem 3 goes through. □

The following is a corollary of Theorems 18 and 20.

Corollary 21 (Bisimilarity not Preserved by Synchronized Products)
There exist continuity controlled hybrid automata H_1, H_2, H_1' and H_2' such that $[\![H_1]\!]_{\mathrm{cc}}^{\mathrm{pa}} \leftrightarrows [\![H_1']\!]_{\mathrm{cc}}^{\mathrm{pa}}$ and $[\![H_2]\!]_{\mathrm{cc}}^{\mathrm{pa}} \leftrightarrows [\![H_2']\!]_{\mathrm{cc}}^{\mathrm{pa}}$, but $[\![H_1 \times H_2]\!]_{\mathrm{cc}}^{\mathrm{pa}} \not\leftrightarrows [\![H_1' \times H_2']\!]_{\mathrm{cc}}^{\mathrm{pa}}$.

A positive result can be obtained for a variant of bisimilarity on $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ terms that is finer than bisimilarity on $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ terms. In Ref. [10], we consider such a variant, called interference-compatible bisimilarity. The idea behind interference-compatible bisimulation is the following. A process proceeding in parallel with a process P can change the state of P at any time. Interference-compatible bisimulation offers resistance to such changes. For example, if a configuration $\langle t_1, \alpha \rangle$ is related to a configuration $\langle t_2, \alpha \rangle$ and $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t_1', \alpha' \rangle$, then there is a t_2' such that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t_2', \alpha' \rangle$ and $\langle t_1', \alpha'' \rangle$ and $\langle t_2', \alpha'' \rangle$ are related for all $\alpha'' \in \mathcal{V}_{\mathsf{st}}$.

An *interference-compatible bisimulation* is a symmetric binary relation B on $\mathcal{PT}$ such that for all $t_1, t_2 \in \mathcal{PT}$:

- if $B(t_1, t_2)$ and $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t_1', \alpha' \rangle$, then there is a t_2' such that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t_2', \alpha' \rangle$ and $B(t_1', t_2')$;
- if $B(t_1, t_2)$ and $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$, then $\langle t_2, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$;
- if $B(t_1, t_2)$ and $\langle t_1, \alpha \rangle \xmapsto{r,\rho} \langle t_1', \alpha' \rangle$, then there is a t_2' such that $\langle t_2, \alpha \rangle \xmapsto{r,\rho} \langle t_2', \alpha' \rangle$ and $B(t_1', t_2')$;
- if $B(t_1, t_2)$ and $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$;
- if $B(t_1, t_2)$ and $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$.

Two closed terms t_1 and t_2 are *interference-compatible bisimilar*, written $t_1 \leftrightarrows t_2$, if there exists an interference-compatible bisimulation B such that $B(t_1, t_2)$.

The following is a corollary of the definitions of bisimilarity (Section 5) and interference-compatible bisimilarity.

Corollary 22 (Interference-Compatible Bisimilarity as Bisimilarity)
For all $t_1, t_2 \in \mathcal{PT}$, $t_1 \leftrightarrows t_2$ if there exists a bisimulation B witnessing $t_1 \leftrightarrows t_2$ such that for all $t_1', t_2' \in \mathcal{PT}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$:

- *if $B(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$, then $B(\langle t_1', \alpha \rangle, \langle t_2', \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$.*

We say that bisimulation B is *closed under changes of valuation* if the condition on B given above holds. We can strengthen Theorem 19 as follows.

Proposition 23 (PA Interpretation of Synchronized Products) *For all continuity controlled hybrid automata H_1, H_2:*

$$\llbracket H_1 \times H_2 \rrbracket_{\text{cc}}^{\text{pa}} \Leftrightarrow \rho_f(\partial_{A'}(\llbracket H_1 \rrbracket_{\text{cc}}^{\text{pa}} \parallel \llbracket H_2 \rrbracket_{\text{cc}}^{\text{pa}})) \ ,$$

where A', f and γ are as in Theorem 19.

PROOF. Consider the relation

$$
\begin{aligned}
B' = B \cup \{ (\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle) \mid \\
\exists \alpha' \in \mathcal{V}_{\text{st}} \bullet B(\langle t_1, \alpha' \rangle, \langle t_2, \alpha' \rangle) \wedge \alpha \notin [\mathsf{s}(t_1)] \wedge \alpha \notin [\mathsf{s}(t_2)] \} \ ,
\end{aligned}
$$

where B is the bisimulation used in the proof of Theorem 19. It follows immediately from Corollary 4 that B' is a bisimulation as well. Moreover, B' is closed under changes of valuation. Hence, using Corollary 22, we conclude that $\llbracket H_1 \times H_2 \rrbracket_{\text{cc}}^{\text{pa}} \Leftrightarrow \rho_f(\partial_{A'}(\llbracket H_1 \rrbracket_{\text{cc}}^{\text{pa}} \parallel \llbracket H_2 \rrbracket_{\text{cc}}^{\text{pa}}))$. □

We have the following positive result concerning preservation of interference-compatible bisimilarity.

Proposition 24 (Synchronized Products Preserve IC-Bisimilarity)
For all continuity controlled hybrid automata H_1, H_2, H_1' and H_2' such that $\llbracket H_1 \rrbracket_{\text{cc}}^{\text{pa}} \Leftrightarrow \llbracket H_1' \rrbracket_{\text{cc}}^{\text{pa}}$ and $\llbracket H_2 \rrbracket_{\text{cc}}^{\text{pa}} \Leftrightarrow \llbracket H_2' \rrbracket_{\text{cc}}^{\text{pa}}$, we have $\llbracket H_1 \times H_2 \rrbracket_{\text{cc}}^{\text{pa}} \Leftrightarrow \llbracket H_1' \times H_2' \rrbracket_{\text{cc}}^{\text{pa}}$.

PROOF. In Ref. [10], it is shown that $\Leftrightarrow$ is preserved by parallel composition and encapsulation. It is easy to see that $\Leftrightarrow$ is also preserved by action renaming. Then the preservation of $\Leftrightarrow$ by synchronized product follows immediately from Proposition 23. □

It is worth mentioning that general preservation results for both bisimilarity and interference-compatible bisimilarity are given in Ref. [25]. There, they are called initially stateless bisimilarity and stateless bisimilarity, respectively. Interference-compatible bisimilarity is called robust bisimilarity in Ref. [19].

7.3 Example: Nuclear Reactor

In this section, we consider a simple nuclear reactor in which the temperature of the reactor core is controlled by two control rods. The behaviour of the reactor can be described informally as follows.

Initially, the temperature of the reactor core is $510\,^\circ$ C and the control rods are outside the reactor core. With the control rods outside the reactor core, the temperature T increases according to the differential equation $\dot{T} = 0.1T - 50$. The reactor must be shut down if the temperature becomes higher than $550\,^\circ$ C. To prevent a shutdown, one of the control rods should be put into the reactor core once the temperature becomes $550\,^\circ$ C. With control rod 1 inside the reactor core, the temperature T decreases according to the differential equation $\dot{T} = 0.1T - 56$. With control rod 2 inside the reactor core, the temperature T decreases according to the differential equation $\dot{T} = 0.1T - 60$. The control rod inside the reactor is removed from the reactor core once the temperature becomes $510\,^\circ$ C. When it is removed, it cannot be put back in the reactor core for the next k seconds. To prevent that the reactor ever needs to be shut down, the time k must be short enough to guarantee that, whenever the temperature of the reactor core becomes $550\,^\circ$ C, one of the control rods can be put back in the reactor core.

The example is taken from Ref. [6]. There, the reactor core and the two control rods are described by hybrid automata. Here, we give the strong process algebra interpretation of the continuity controlled hybrid automata that are obtained from those hybrid automata by designating state variables as robust state variables as follows: the temperature T in the case of the automaton for the reactor core, the clock c_1 in the case of the automaton for control rod 1 and the clock c_2 in the case of the automaton for control rod 2.

The process algebra interpretation of the continuity controlled hybrid automaton that describes the reactor core is as follows:

$$\tilde{\tilde{\iota}} \cdot \left((T = 510) \mathbin{\text{\rotatebox[origin=c]{45}{$\blacktriangle$}}} \langle C_{\text{out}} | F \rangle \right),$$

where the recursive specification F consists of the following equations:

$$
\begin{aligned}
C_{\text{out}} = {} & (T \le 550 \wedge \dot{T} = 0.1T - 50) \mathbin{\curvearrowright_{\{T\}}} \\
& \left((\,^\bullet T = 550 \wedge T^\bullet = {}^\bullet T) \mathbin{\text{\rotatebox{-30}{$\blacktriangleright$}}} \widetilde{\widetilde{add_1}} \cdot C_{\text{in1}} \right. \\
& \left. + (\,^\bullet T = 550 \wedge T^\bullet = {}^\bullet T) \mathbin{\text{\rotatebox{-30}{$\blacktriangleright$}}} \widetilde{\widetilde{add_2}} \cdot C_{\text{in2}} + \sigma^+_{\text{rel}}(C_{\text{out}}) \right),
\end{aligned}
$$

$$
\begin{aligned}
C_{\text{in1}} = {} & (T \ge 510 \wedge \dot{T} = 0.1T - 56) \mathbin{\curvearrowright_{\{T\}}} \\
& \left((\,^\bullet T = 510 \wedge T^\bullet = {}^\bullet T) \mathbin{\text{\rotatebox{-30}{$\blacktriangleright$}}} \widetilde{\widetilde{rmv_1}} \cdot C_{\text{out}} + \sigma^+_{\text{rel}}(C_{\text{in1}}) \right),
\end{aligned}
$$

$$
\begin{aligned}
C_{\text{in2}} = {} & (T \ge 510 \wedge \dot{T} = 0.1T - 60) \mathbin{\curvearrowright_{\{T\}}} \\
& \left((\,^\bullet T = 510 \wedge T^\bullet = {}^\bullet T) \mathbin{\text{\rotatebox{-30}{$\blacktriangleright$}}} \widetilde{\widetilde{rmv_2}} \cdot C_{\text{out}} + \sigma^+_{\text{rel}}(C_{\text{in2}}) \right).
\end{aligned}
$$

The process algebra interpretation of the continuity controlled hybrid automa-

ton that describes control rod 1 is as follows:

$$\tilde{\iota} \cdot (\mathsf{T} \curvearrowleft \langle R_{\mathrm{out}}^1 | F_1 \rangle) ,$$

where the recursive specification F_1 consists of the following equations:

$$R_{\mathrm{out}}^1 = (\dot{c}_1 = 1) \curvearrowright_{\{c_1\}} \left((^\bullet c_1 \geq k \wedge c_1{}^\bullet = {}^\bullet c_1) \rightthreetimes \widetilde{\widetilde{add_1}} \cdot R_{\mathrm{in}}^1 + \sigma_{\mathrm{rel}}^+(R_{\mathrm{out}}^1) \right) ,$$

$$R_{\mathrm{in}}^1 = \mathsf{T} \curvearrowright_{\{c_1\}} \left((c_1{}^\bullet = 0) \rightthreetimes \widetilde{\widetilde{rmv_1}} \cdot R_{\mathrm{out}}^1 + \sigma_{\mathrm{rel}}^+(R_{\mathrm{in}}^1) \right) .$$

The process algebra interpretation of the continuity controlled hybrid automaton that describes control rod 2 is as follows:

$$\tilde{\iota} \cdot (\mathsf{T} \curvearrowleft \langle R_{\mathrm{out}}^2 | F_2 \rangle) ,$$

where the recursive specification F_2 consists of the following equations:

$$R_{\mathrm{out}}^2 = (\dot{c}_2 = 1) \curvearrowright_{\{c_2\}} \left((^\bullet c_2 \geq k \wedge c_2{}^\bullet = {}^\bullet c_2) \rightthreetimes \widetilde{\widetilde{add_2}} \cdot R_{\mathrm{in}}^2 + \sigma_{\mathrm{rel}}^+(R_{\mathrm{out}}^2) \right) ,$$

$$R_{\mathrm{in}}^2 = \mathsf{T} \curvearrowright_{\{c_2\}} \left((c_2{}^\bullet = 0) \rightthreetimes \widetilde{\widetilde{rmv_2}} \cdot R_{\mathrm{out}}^2 + \sigma_{\mathrm{rel}}^+(R_{\mathrm{in}}^2) \right) .$$

A continuity controlled hybrid automaton for the whole system is obtained by constructing the synchronized product of the continuity controlled hybrid automata for the reactor core and the two control rods. Because Theorem 19 applies here, the process algebra interpretation of the continuity controlled hybrid automaton for the whole system is bisimilar to the following term:

$$\rho_f(\partial_H(C_{\mathrm{out}} \parallel \rho_{f'}(\partial_{H'}(R_{\mathrm{out}}^1 \parallel R_{\mathrm{out}}^2)))) ,$$

where $H = \{add_1, rmv_1, add_2, rmv_2, \iota\}$, $H' = \{\iota\}$, the renaming function f is such that $f(\bar{a}) = a$ if $a \in H$ and $f(a) = a$ if $a \notin \{\bar{a} \mid a \in H\}$, the renaming function f' is such that $f'(\bar{a}) = a$ if $a \in H'$ and $f'(a) = a$ if $a \notin \{\bar{a} \mid a \in H'\}$, and the communication function γ is such that $\gamma(a, a) = \bar{a}$ if $a \in H$ and it is undefined otherwise.

In the continuity controlled hybrid automata for the nuclear reactor and control rods, just like in the hybrid automaton for the thermostat of Section 6.2, delays are modelled by means of state variables with derivative 1. Such state variables are called clock variables. Because the relative delay operator is available in $\mathrm{ACP}_{\mathrm{hs}}^{\mathrm{srt}}$ for that purpose, we can replace the recursive specifications given above by ones without clock variables. Such recursive specifications are given in Section 4.1 of Ref. [10].

For continuity controlled hybrid automata, it is useful to introduce localization. With localization extra state variables can be made robust.

Let $H = (V, W, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a continuity controlled hybrid automaton and $V' \subseteq V$. Then the *localization* of H with respect to V', written $V' \, \nabla \, H$, is the continuity controlled hybrid automaton

$$H = (V, V' \cup W, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi) \ .$$

We have the following result concerning the localization of continuity controlled hybrid automata and the localization of $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ terms.

Theorem 25 (PA Interpretation of Localization) *For all continuity controlled hybrid automata H and $V' \subseteq \mathrm{V}(H)$:*

$$[\![V' \, \nabla \, H]\!]^{\mathrm{pa}}_{\mathrm{cc}} \leftrightarrow V' \, \nabla \, [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}} \ .$$

PROOF. See Appendix B.5. □

Continuity controlled hybrid automata H for which $\mathrm{V}(H) \nabla H = H$ are closely related to original hybrid automata.

Proposition 26 (Relation Original and CC Hybrid Automata) *For all continuity controlled hybrid automata H, there exists a hybrid automaton H' such that:*

$$\mathrm{V}(H) \, \nabla \, H = \mathrm{cc}(H') \ .$$

PROOF. Follows immediately from the definitions of hybrid automaton and continuity controlled hybrid automaton and the definition of localization of continuity controlled hybrid automata. □

In continuity controlled hybrid automata, evolutions with a finite number of discontinuities for certain state variables may take place. In synchronized products of continuity controlled hybrid automata, only the continuous changes of those state variables are interruptable. Thus, continuity controlled hybrid automata offer controllability of interruption of continuous changes of state variables in synchronized products. Proposition 26 shows that, after composition of continuity controlled hybrid automata by means of synchronized products,

a continuity controlled hybrid automaton that is essentially an original hybrid automaton can always be obtained by localization.

8   Concluding Remarks

The connections between the process algebra for hybrid systems introduced in Ref. [10] and the formalism of hybrid automata have been investigated. It has been shown that there is a fragment of the process algebra for hybrid systems that gets near a symbolic counterpart of the formalism of hybrid automata. However, an exact match is not attainable. This has brought us to introduce an adaptation of the formalism of hybrid automata that yields an exact match. In continuation of the work presented in this paper, an interesting option for future work is to investigate the adaptation of model checking tools developed for hybrid automata to a suitable fragment of our process algebra for hybrid systems.

Hybrid automata and related notions are defined in different ways in the literature. The following are some examples of the differences. Hybrid automata are interpreted as trajectories in e.g. Ref. [6] and as timed transition systems in e.g. Ref. [3]. The state variables are interpreted as functions from $\mathbb{R}^{\geq}$ to $\mathbb{R}$ that are piecewise of class C^{∞} in e.g. Ref. [1], as functions from $\mathbb{R}^{\geq}$ to $\mathbb{R}$ that are piecewise of class C^1 in e.g. Ref. [8], and as functions from $\mathbb{R}^{\geq}$ to $\mathbb{R}$ that are piecewise differentiable in e.g. Ref. [3]. Control switches are labelled with a set of events in e.g. Ref. [6] and they are labelled with a single event in e.g. Ref. [23]. Stutter control switches for each control mode are required in e.g. Ref. [1] and they are not required in e.g. Ref. [7]. Because of these differences, we have taken the liberty to choose the definitions that result in the closest match with our process algebra for hybrid systems.

In some papers on hybrid automata, e.g. in Ref. [8], the events of a hybrid automaton include a silent event τ and bisimilarity of hybrid automata is weak bisimilarity in the sense of Ref. [26]. Our process algebra for hybrid systems does not incorporate silent actions and weak (or branching) bisimilarity. This issue is not even fully understood in process algebras with timing. The version of branching bisimilarity for processes with discrete relative timing proposed in Ref. [27] for this purpose, and adapted to continuous relative timing in Ref. [12], is too fine for many applications. A slightly coarser equivalence is proposed in Ref. [28].

We have given process algebra interpretations of a hybrid automaton that describes a thermostat and continuity controlled hybrid automata that describe the components of a simple nuclear reactor. The hybrid automaton for the thermostat can be found in Ref. [8] and the hybrid automata from which the

continuity controlled hybrid automata for the components of the simple nuclear reactor are obtained can be found in Ref. [6]. More examples of the use of hybrid automata for describing hybrid systems can be found there, and in the remaining literature on hybrid automata. For example, hybrid automata describing the components of a railroad crossing system can be found in Ref. [6]. Process algebra interpretations for those hybrid automata are essentially given in Section 4.7 of Ref. [10]: instead of equations of the form

$$X_m = \phi_m \,{}^{\text{\tiny\ding{228}}}\!_V \left(\sum_{s \in \{s \in S | m_s = m\}} \chi_s \,{}^{\text{\tiny\ding{228}}}\, \widetilde{\widetilde{e}}_s \cdot X_{m'_s} + \int_{u \in (0,\infty)} \sigma^u_{\text{rel}}(X_m) \right)$$

equivalent equations of the form

$$X_m = \phi_m \,{}^{\text{\tiny\ding{228}}}\!_V \int_{u \in [0,\infty)} \sigma^u_{\text{rel}} \left(\sum_{s \in \{s \in S | m_s = m\}} \chi_s \,{}^{\text{\tiny\ding{228}}}\, \widetilde{\widetilde{e}}_s \cdot X_{m'_s} \right)$$

are used.

The term hybrid system is sometimes, e.g. in Refs. [29,30], used for a hybrid automaton with the initial, flow and jump conditions replaced by the sets, functions and relations defined by them. In those cases, hybrid automata are regarded as concrete syntactic descriptions of such hybrid systems. For the study of connections with $\text{ACP}^{\text{srt}}_{\text{hs}}$, such hybrid systems are essentially the same as hybrid automata.

The term hybrid automaton is used in a rather uncommon way in the HIOA framework [31]. The hybrid automata from that framework are similar to the hybrid transition systems introduced in Section 7.1 of the current paper. The main difference is that it is stipulated in the HIOA framework that the states must be valuations of state variables.

As mentioned in Section 1, the process algebra for hybrid systems introduced in Ref. [10] includes among other things equational axioms for reasoning about hybrid systems. It is worth mentioning that the propositions and theorems that assert bisimilarity of process algebra interpretations of hybrid automata can alternatively be proved by means of those axioms.

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A Structural Operational Semantics of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$

We assume that a fixed but arbitrary set A of *actions* and a fixed but arbitrary partial commutative and associative *communication* function $\gamma : \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ have been given. We also assume that a fixed but arbitrary set V of *state variables* has been given. Furthermore, it is assumed that each first-order definable set of non-negative real numbers can be denoted by a closed term.

We shall henceforth use $x, y, x', y', \ldots$ and $X, Y, \ldots$ as variables ranging over processes, $t_X, t_Y, \ldots$ to stand for arbitrary terms of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$, $a, b, c, \ldots$ to stand for arbitrary elements of A, $H, H', \ldots$ to stand for arbitrary subsets of A, $u, u', \ldots$ as variables ranging over $\mathbb{R}^{\geq}$, $p, q, r, \ldots$ to stand for arbitrary closed terms denoting elements of $\mathbb{R}^{\geq}$, $U, U', \ldots$ to stand for arbitrary closed terms denoting first-order definable subsets of $\mathbb{R}^{\geq}$, $E, E', \ldots$ to stand for arbitrary guarded recursive specifications. Moreover, we shall henceforth use F and G as variables ranging over functions that map each non-negative real number to a process and can be represented by terms containing a designated free variable ranging over $\mathbb{R}^{\geq}$. For more information on such second-order variables, see e.g. Refs. [32,33].

We write A_δ for $\mathsf{A} \cup \{\delta\}$. Let t be a term of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$ and E be a guarded recursive specification. Then we write $\langle t|E \rangle$ for t with, for all $X \in \mathrm{V}(E)$, all occurrences of X in t replaced by $\langle X|E \rangle$. Let $V \subseteq \mathsf{V}$. Then we write $\mathcal{C}_V$ for $\bigwedge_{v \in V}(v^\bullet = {}^\bullet v \wedge \dot{v}^\bullet = {}^\bullet \dot{v})$.

The structural operational semantics of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$ is described by the rules given in Tables A.1, A.2, A.3 and A.4. We write $\langle t, \alpha \rangle \overset{\gamma}{\nrightarrow}$ for the set of all transition formulas $\neg(\langle t, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t', \alpha' \rangle)$ where t' is a closed term of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$, $\alpha' \in \mathcal{V}_{\mathsf{st}}$ and $\rho \in \mathcal{E}_r$. We write $\langle t, \alpha \rangle \nrightarrow$ for the set of all transition formulas $\neg(\langle t, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t', \alpha' \rangle)$ where t' is a closed term of ACP$_{\mathrm{hs}}^{\mathrm{srt}}$, $\alpha' \in \mathcal{V}_{\mathsf{st}}$, $r \in \mathbb{R}^{>}$ and $\rho \in \mathcal{E}_r$. We write $\rho \trianglerighteq r$, where $\rho \in \mathcal{E}_{r+s}$ ($r, s > 0$), for the $\rho' \in \mathcal{E}_s$ such that $\rho'(s') = \rho(r+s')$ for all $s' \in [0, s]$. The five kinds of transition relations used are further explained in Section 5.

The structural operational semantics for integration is described by the rules given in Table A.5. The complexity of the rule concerning the time-related capabilities of a process $\int_{u \in U} F(u)$ is caused by the fact that the processes $F(p)$ with $p \in U$ that are capable of idling need not change uniformly while idling. For more information on this phenomenon, see e.g. Refs. [12,34]. The structural operational semantics for recursion is described by the rules given in Table A.6. The structural operational semantics for localization is described by the rules given in Table A.7. The structural operational semantics for action renaming is described by the rules given in Table A.8.

Table A.1

Rules for operational semantics of $\text{BPA}_{\text{hs}}^{\text{srt}}$ $(a \in \mathsf{A},\, r, s > 0)$

$$\frac{}{\langle \tilde{\tilde{a}}, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle \sigma_{\text{rel}}^0(x), \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle} \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle \sigma_{\text{rel}}^0(x), \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle} \qquad \frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}{\langle \sigma_{\text{rel}}^0(x), \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}$$

$$\frac{}{\langle \sigma_{\text{rel}}^{r+s}(x), \alpha \rangle \xmapsto{r,\rho} \langle \sigma_{\text{rel}}^s(x), \alpha' \rangle} \qquad \frac{\alpha' \in [\mathsf{s}(x)]}{\langle \sigma_{\text{rel}}^r(x), \alpha \rangle \xmapsto{r,\rho} \langle x, \alpha' \rangle} \qquad \frac{\langle x, \alpha' \rangle \xmapsto{s,\rho \rhd r} \langle x', \alpha'' \rangle}{\langle \sigma_{\text{rel}}^r(x), \alpha \rangle \xmapsto{r+s,\rho} \langle x', \alpha'' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle,\ \alpha \in [\mathsf{s}(y)]}{\langle x + y, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle} \qquad \frac{\alpha \in [\mathsf{s}(x)],\ \langle y, \alpha \rangle \xrightarrow{a} \langle y', \alpha' \rangle}{\langle x + y, \alpha \rangle \xrightarrow{a} \langle y', \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle,\ \alpha \in [\mathsf{s}(y)]}{\langle x + y, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle} \qquad \frac{\alpha \in [\mathsf{s}(x)],\ \langle y, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle x + y, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle,\ \langle y, \alpha \rangle \not\xmapsto{},\ \alpha \in [\mathsf{s}(y)]}{\langle x + y, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle} \qquad \frac{\langle x, \alpha \rangle \not\xmapsto{},\ \alpha \in [\mathsf{s}(x)],\ \langle y, \alpha \rangle \xmapsto{r,\rho} \langle y', \alpha' \rangle}{\langle x + y, \alpha \rangle \xmapsto{r,\rho} \langle y', \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle,\ \langle y, \alpha \rangle \xmapsto{r,\rho} \langle y', \alpha' \rangle}{\langle x + y, \alpha \rangle \xmapsto{r,\rho} \langle x' + y', \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle x \cdot y, \alpha \rangle \xrightarrow{a} \langle x' \cdot y, \alpha' \rangle} \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle,\ \alpha' \in [\mathsf{s}(y)]}{\langle x \cdot y, \alpha \rangle \xrightarrow{a} \langle y, \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}{\langle x \cdot y, \alpha \rangle \xmapsto{r,\rho} \langle x' \cdot y, \alpha' \rangle}$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle \psi :\rightarrow x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}\ \alpha \models \psi \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle \psi :\rightarrow x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}\ \alpha \models \psi$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}{\langle \psi :\rightarrow x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}\ \alpha \models \psi$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle \psi \curlywedge x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}\ \alpha \models \psi \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle \psi \curlywedge x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}\ \alpha \models \psi$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}{\langle \psi \curlywedge x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}\ \alpha \models \psi$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle \phi \curvearrowright_V x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}\ \alpha \models \phi \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle \phi \curvearrowright_V x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}\ \alpha \models \phi$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}{\langle \phi \curvearrowright_V x, \alpha \rangle \xmapsto{r,\rho} \langle \phi \curvearrowright_V x', \alpha' \rangle}\ \alpha \xmapsto{r,\rho} \alpha' \models_V \phi$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle \chi \curvearrowleft x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}\ \alpha \rightarrow \alpha' \models \chi \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle \chi \curvearrowleft x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}\ \alpha \rightarrow \alpha' \models \chi$$

$$\frac{\langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}{\langle \chi \curvearrowleft x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha' \rangle}\ \exists \alpha'' \bullet \alpha \rightarrow \alpha'' \models \chi$$

$$\frac{\langle x, \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}{\langle \nu_{\text{rel}}(x), \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle} \qquad \frac{\langle x, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}{\langle \nu_{\text{rel}}(x), \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle}$$

Table A.2
Additional rules for $\mathrm{ACP}^{\mathrm{srt}}_{\mathrm{hs}}$ $(a,b,c \in \mathsf{A},\ r > 0)$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle,\ \alpha\rightarrow\alpha'\in[\mathsf{d}(y)],\ \alpha'\in[\mathsf{s}(y)]}{\langle x \parallel y,\alpha\rangle \xrightarrow{a} \langle x' \parallel y,\alpha'\rangle} \qquad \frac{\alpha\rightarrow\alpha'\in[\mathsf{d}(x)],\ \alpha'\in[\mathsf{s}(x)],\ \langle y,\alpha\rangle \xrightarrow{a} \langle y',\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xrightarrow{a} \langle x \parallel y',\alpha'\rangle}$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle,\ \alpha\rightarrow\alpha'\in[\mathsf{d}(y)],\ \alpha'\in[\mathsf{s}(y)]}{\langle x \parallel y,\alpha\rangle \xrightarrow{a} \langle y,\alpha'\rangle} \qquad \frac{\alpha\rightarrow\alpha'\in[\mathsf{d}(x)],\ \alpha'\in[\mathsf{s}(x)],\ \langle y,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xrightarrow{a} \langle x,\alpha'\rangle}$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle y',\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xrightarrow{c} \langle x' \parallel y',\alpha'\rangle}\ \gamma(a,b)=c \qquad \frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle\surd,\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xrightarrow{c} \langle x',\alpha'\rangle}\ \gamma(a,b)=c$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle y',\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xrightarrow{c} \langle y',\alpha'\rangle}\ \gamma(a,b)=c \qquad \frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle\surd,\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xrightarrow{c} \langle\surd,\alpha'\rangle}\ \gamma(a,b)=c$$

$$\frac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xmapsto{r,\rho} \langle y',\alpha'\rangle}{\langle x \parallel y,\alpha\rangle \xmapsto{r,\rho} \langle x' \parallel y',\alpha'\rangle}$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle,\ \alpha\rightarrow\alpha'\in[\mathsf{d}(y)],\ \alpha'\in[\mathsf{s}(y)]}{\langle x \,\rule[-0.5ex]{0.4pt}{1.6ex}\!\parallel y,\alpha\rangle \xrightarrow{a} \langle x' \parallel y,\alpha'\rangle} \qquad \frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle,\ \alpha\rightarrow\alpha'\in[\mathsf{d}(y)],\ \alpha'\in[\mathsf{s}(y)]}{\langle x \,\rule[-0.5ex]{0.4pt}{1.6ex}\!\parallel y,\alpha\rangle \xrightarrow{a} \langle y,\alpha'\rangle}$$

$$\frac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xmapsto{r,\rho} \langle y',\alpha'\rangle}{\langle x \,\rule[-0.5ex]{0.4pt}{1.6ex}\!\parallel y,\alpha\rangle \xmapsto{r,\rho} \langle x' \,\rule[-0.5ex]{0.4pt}{1.6ex}\!\parallel y',\alpha'\rangle}$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle y',\alpha'\rangle}{\langle x \mid y,\alpha\rangle \xrightarrow{c} \langle x' \parallel y',\alpha'\rangle}\ \gamma(a,b)=c \qquad \frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle\surd,\alpha'\rangle}{\langle x \mid y,\alpha\rangle \xrightarrow{c} \langle x',\alpha'\rangle}\ \gamma(a,b)=c$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle y',\alpha'\rangle}{\langle x \mid y,\alpha\rangle \xrightarrow{c} \langle y',\alpha'\rangle}\ \gamma(a,b)=c \qquad \frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle,\ \langle y,\alpha\rangle \xrightarrow{b} \langle\surd,\alpha'\rangle}{\langle x \mid y,\alpha\rangle \xrightarrow{c} \langle\surd,\alpha'\rangle}\ \gamma(a,b)=c$$

$$\frac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle,\ \langle y,\alpha\rangle \xmapsto{r,\rho} \langle y',\alpha'\rangle}{\langle x \mid y,\alpha\rangle \xmapsto{r,\rho} \langle x' \mid y',\alpha'\rangle}$$

$$\frac{\langle x,\alpha\rangle \xrightarrow{a} \langle x',\alpha'\rangle}{\langle\partial_H(x),\alpha\rangle \xrightarrow{a} \langle\partial_H(x'),\alpha'\rangle}\ a\notin H \qquad \frac{\langle x,\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle}{\langle\partial_H(x),\alpha\rangle \xrightarrow{a} \langle\surd,\alpha'\rangle}\ a\notin H$$

$$\frac{\langle x,\alpha\rangle \xmapsto{r,\rho} \langle x',\alpha'\rangle}{\langle\partial_H(x),\alpha\rangle \xmapsto{r,\rho} \langle\partial_H(x'),\alpha'\rangle}$$

Table A.3
Rules for $\alpha \in [\mathsf{s}(\_)]$ $(a \in \mathsf{A}_\delta,\ r > 0)$

$$\frac{}{\alpha\in[\mathsf{s}(\tilde{a})]} \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\sigma^0_{\mathrm{rel}}(x))]} \qquad \frac{}{\alpha\in[\mathsf{s}(\sigma^r_{\mathrm{rel}}(x))]} \qquad \frac{\alpha\in[\mathsf{s}(x)],\ \alpha\in[\mathsf{s}(y)]}{\alpha\in[\mathsf{s}(x+y)]} \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(x\cdot y)]}$$

$$\frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\psi :\!\rightarrow x)]} \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\psi :\!\rightarrow x)]}\ \alpha\not\models\psi \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\psi \,^{\wedge\!\blacktriangle}\, x)]}\ \alpha\models\psi$$

$$\frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\phi \,^{\wedge\!\blacktriangledown}_V\, x)]}\ \alpha\models\phi \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\chi \,^{\ulcorner\!\blacktriangledown}\, x)]} \qquad \frac{}{\alpha\in[\mathsf{s}(\chi \,^{\ulcorner\!\blacktriangledown}\, x)]}\ \neg\exists\alpha'\bullet\alpha\rightarrow\alpha'\models\chi \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\nu_{\mathrm{rel}}(x))]}$$

$$\frac{\alpha\in[\mathsf{s}(x)],\ \alpha\in[\mathsf{s}(y)]}{\alpha\in[\mathsf{s}(x \parallel y)]} \qquad \frac{\alpha\in[\mathsf{s}(x)],\ \alpha\in[\mathsf{s}(y)]}{\alpha\in[\mathsf{s}(x \,\rule[-0.5ex]{0.4pt}{1.6ex}\!\parallel y)]} \qquad \frac{\alpha\in[\mathsf{s}(x)],\ \alpha\in[\mathsf{s}(y)]}{\alpha\in[\mathsf{s}(x \mid y)]} \qquad \frac{\alpha\in[\mathsf{s}(x)]}{\alpha\in[\mathsf{s}(\partial_H(x))]}$$

Table A.4

Rules for $\alpha \to \alpha' \in [\mathsf{d}(\_)]$ $(a \in \mathsf{A}_\delta,\ r > 0)$

$$\frac{}{\alpha \to \alpha' \in [\mathsf{d}(\tilde{a})]} \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\sigma_{\mathsf{rel}}^0(x))]} \qquad \frac{}{\alpha \to \alpha' \in [\mathsf{d}(\sigma_{\mathsf{rel}}^r(x))]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)],\ \alpha \to \alpha' \in [\mathsf{d}(y)]}{\alpha \to \alpha' \in [\mathsf{d}(x + y)]} \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(x \cdot y)]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\psi :\to x)]} \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\psi :\to x)]}\ \alpha \not\models \psi \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\psi \wedge x)]}\ \alpha \models \psi$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)],\ \langle x, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha'' \rangle}{\alpha \to \alpha' \in [\mathsf{d}(\phi \curvearrowright_V x)]}\ \alpha \to \alpha' \models \mathcal{C}_V,\ \alpha \models \phi \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)],\ \langle x, \alpha \rangle \not\mapsto}{\alpha \to \alpha' \in [\mathsf{d}(\phi \curvearrowright_V x)]}\ \alpha \models \phi$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\chi \ulcorner\!\blacktriangledown x)]} \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(\chi \ulcorner\!\blacktriangledown x)]}{} \neg \exists \alpha'' \bullet \alpha \to \alpha'' \models \chi \qquad \frac{\alpha \in [\mathsf{s}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\nu_{\mathsf{rel}}(x))]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)],\ \alpha \to \alpha' \in [\mathsf{d}(y)],\ \langle x \parallel y, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha'' \rangle}{\alpha \to \alpha' \in [\mathsf{d}(x \parallel y)]} \qquad \frac{\alpha \in [\mathsf{s}(x)],\ \alpha \in [\mathsf{s}(y)],\ \langle x \parallel y, \alpha \rangle \not\mapsto}{\alpha \to \alpha' \in [\mathsf{d}(x \parallel y)]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)],\ \alpha \to \alpha' \in [\mathsf{d}(y)],\ \langle x \mathbin{\underline{\parallel}} y, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha'' \rangle}{\alpha \to \alpha' \in [\mathsf{d}(x \mathbin{\underline{\parallel}} y)]} \qquad \frac{\alpha \in [\mathsf{s}(x)],\ \alpha \in [\mathsf{s}(y)],\ \langle x \mathbin{\underline{\parallel}} y, \alpha \rangle \not\mapsto}{\alpha \to \alpha' \in [\mathsf{d}(x \mathbin{\underline{\parallel}} y)]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)],\ \alpha \to \alpha' \in [\mathsf{d}(y)],\ \langle x \mid y, \alpha \rangle \xmapsto{r,\rho} \langle x', \alpha'' \rangle}{\alpha \to \alpha' \in [\mathsf{d}(x \mid y)]} \qquad \frac{\alpha \in [\mathsf{s}(x)],\ \alpha \in [\mathsf{s}(y)],\ \langle x \mid y, \alpha \rangle \not\mapsto}{\alpha \to \alpha' \in [\mathsf{d}(x \mid y)]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\partial_H(x))]}$$

Table A.5

Additional rules for integration $(a \in \mathsf{A},\ p, q \geq 0,\ r > 0)$

$$\frac{\langle F(p), \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle,\ \{\alpha \in [\mathsf{s}(F(q))] \mid q \in U\}}{\langle \int_{u \in U} F(u), \alpha \rangle \xrightarrow{a} \langle x', \alpha' \rangle}\ p \in U$$

$$\frac{\langle F(p), \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle,\ \{\alpha \in [\mathsf{s}(F(q))] \mid q \in U\}}{\langle \int_{u \in U} F(u), \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle}\ p \in U$$

$$\frac{\begin{array}{c} \{\langle F(q), \alpha \rangle \xmapsto{r,\rho} \langle F_1(q), \alpha' \rangle \mid q \in U_1\}, \\ \ldots, \\ \{\langle F(q), \alpha \rangle \xmapsto{r,\rho} \langle F_n(q), \alpha' \rangle \mid q \in U_n\}, \\ \{\langle F(q), \alpha \rangle \not\mapsto,\ \alpha \in [\mathsf{s}(F(q))] \mid q \in U_{n+1}\} \end{array}}{\langle \int_{u \in U} F(u), \alpha \rangle \xmapsto{r,\rho} \langle \int_{u \in U_1} F_1(u) + \ldots + \int_{u \in U_n} F_n(u), \alpha' \rangle}\ \begin{array}{l} \{U_1, \ldots, U_n\}\ \text{partition} \\ \text{of } U \setminus U_{n+1}, U_{n+1} \subset U \end{array}$$

$$\frac{\{\alpha \in [\mathsf{s}(F(q))] \mid q \in U\}}{\alpha \in [\mathsf{s}(\int_{u \in U} F(u))]} \qquad \frac{\{\alpha \to \alpha' \in [\mathsf{d}(F(q))] \mid q \in U\}}{\alpha \to \alpha' \in [\mathsf{d}(\int_{u \in U} F(u))]}$$

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Table A.6
Additional rules for recursion ($a \in \mathsf{A}$, $r > 0$)

$$\frac{\langle\langle t_X|E\rangle, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle}{\langle\langle X|E\rangle, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle} \ X = t_X \in E \qquad \frac{\langle\langle t_X|E\rangle, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle}{\langle\langle X|E\rangle, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle} \ X = t_X \in E$$

$$\frac{\langle\langle t_X|E\rangle, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}{\langle\langle X|E\rangle, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle} \ X = t_X \in E$$

$$\frac{\alpha \in [\mathsf{s}(\langle t_X|E\rangle)]}{\alpha \in [\mathsf{s}(\langle X|E\rangle)]} \ X = t_X \in E \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(\langle t_X|E\rangle)]}{\alpha \to \alpha' \in [\mathsf{d}(\langle X|E\rangle)]} \ X = t_X \in E$$

Table A.7
Additional rules for localization ($a \in \mathsf{A}$, $r > 0$)

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle}{\langle v \nabla x, \alpha\rangle \xrightarrow{a} \langle v \nabla x', \alpha'\rangle} \qquad \frac{\langle a, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle}{\langle v \nabla x, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle}$$

$$\frac{\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}{\langle v \nabla x, \alpha\rangle \xmapsto{r,\rho} \langle v \nabla x', \alpha'\rangle} \ \alpha \xmapsto{r,\rho} \alpha' \models_{\{v\}} \mathsf{T} \qquad \frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(v \nabla x)]}$$

$$\frac{\alpha \to \alpha' \in [\mathsf{d}(x)], \ \langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha''\rangle}{\alpha \to \alpha' \in [\mathsf{d}(v \nabla x)]} \ \alpha \to \alpha' \models \mathcal{C}_{\{v\}} \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)], \ \langle x, \alpha\rangle \nrightarrow}{\alpha \to \alpha' \in [\mathsf{d}(v \nabla x)]}$$

Table A.8
Additional rules for action renaming ($a \in \mathsf{A}$, $r > 0$)

$$\frac{\langle x, \alpha\rangle \xrightarrow{a} \langle x', \alpha'\rangle}{\langle \rho_f(x), \alpha\rangle \xrightarrow{f(a)} \langle \rho_f(x'), \alpha'\rangle} \qquad \frac{\langle x, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle}{\langle \rho_f(x), \alpha\rangle \xrightarrow{f(a)} \langle\sqrt{}, \alpha'\rangle}$$

$$\frac{\langle x, \alpha\rangle \xmapsto{r,\rho} \langle x', \alpha'\rangle}{\langle \rho_f(x), \alpha\rangle \xmapsto{r,\rho} \langle \rho_f(x'), \alpha'\rangle} \qquad \frac{\alpha \in [\mathsf{s}(x)]}{\alpha \in [\mathsf{s}(\rho_f(x))]} \qquad \frac{\alpha \to \alpha' \in [\mathsf{d}(x)]}{\alpha \to \alpha' \in [\mathsf{d}(\rho_f(x))]}$$

B Proofs

In this appendix, we give the proofs of Theorem 7, Proposition 9, Proposition 11, Theorem 14 and Theorem 25.

In the proofs, we write $\langle t, \alpha \rangle \mapsto$ to indicate that there exist an $r \in \mathbb{R}^>$, $\rho \in \mathcal{E}_r$, $t' \in \mathcal{PT}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$ such that $\langle t, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t', \alpha' \rangle$); and $\langle t, \alpha \rangle \not\mapsto$ to indicate that not $\langle t, \alpha \rangle \mapsto$.

B.1 Proof of Theorem 7

In this section, we prove the following theorem.

Theorem 7. *For all hybrid automata H_1 and H_2 with $\mathrm{V}(H_1) = \mathrm{V}(H_2)$:*

$$\llbracket H_1 \rrbracket_{\mathsf{s}}^{\mathrm{pa}} \leftrightarrows \llbracket H_2 \rrbracket_{\mathsf{s}}^{\mathrm{pa}} \Leftrightarrow \llbracket H_1 \rrbracket \leftrightarrows \llbracket H_2 \rrbracket \ .$$

PROOF. Let $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$ be a hybrid automaton. The following facts concerning $\llbracket H \rrbracket_{\mathsf{s}}^{\mathrm{pa}}$, which follow easily from the definition of strong process algebra interpretation, are used in the proof of Theorem 7. For action steps, the following implications hold:

$$\langle X_m, \alpha \rangle \overset{a}{\to} \langle t, \alpha' \rangle \Rightarrow \exists m' \in M \bullet t \equiv X_{m'} \land \alpha' \in [\mathsf{s}(X_{m'})] \tag{B.1}$$

$$\langle \psi_m \curlywedge X_m, \alpha \rangle \overset{a}{\to} \langle t, \alpha' \rangle \Rightarrow \exists m' \in M \bullet t \equiv X_{m'} \land \alpha' \in [\mathsf{s}(X_{m'})] \tag{B.2}$$

For time steps, the following implications hold:

$$\langle X_m, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t, \alpha' \rangle \Rightarrow t \leftrightarrows X_m \land \alpha' \in [\mathsf{s}(X_m)] \tag{B.3}$$

$$\langle \psi_m \curlywedge X_m, \alpha \rangle \overset{r,\rho}{\longmapsto} \langle t, \alpha' \rangle \Rightarrow t \leftrightarrows X_m \land \alpha' \in [\mathsf{s}(X_m)] \tag{B.4}$$

For signals, the following bi-implications hold:

$$\alpha \in [\mathsf{s}(X_m)] \land (\langle X_m, \alpha \rangle \mapsto \Rightarrow \alpha \to \alpha' \models \mathcal{C}_V) \Leftrightarrow \alpha \to \alpha' \in [\mathsf{d}(X_m)] \tag{B.5}$$

$$\alpha \in [\mathsf{s}(\psi_m \curlywedge X_m)] \land (\langle \psi_m \curlywedge X_m, \alpha \rangle \mapsto \Rightarrow \alpha \to \alpha' \models \mathcal{C}_V)$$
$$\Leftrightarrow \alpha \to \alpha' \in [\mathsf{d}(\psi_m \curlywedge X_m)] \tag{B.6}$$

The following three facts, which follow easily from the definitions of transition system interpretation and strong process algebra interpretation are also used:

$$\exists \rho \in \mathcal{E}_r \bullet \alpha \stackrel{r,\rho}{\longmapsto} \alpha' \models_V \phi(m) \Leftrightarrow \langle m, \alpha \rangle \stackrel{r}{\longmapsto} \langle m, \alpha' \rangle \tag{B.7}$$

$$\alpha \stackrel{r,\rho}{\longmapsto} \alpha' \models_V \phi(m) \Leftrightarrow \exists t \in \mathcal{PT} \bullet \langle X_m, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t, \alpha' \rangle \tag{B.8}$$

$$\alpha \models \psi(m) \wedge \alpha \stackrel{r,\rho}{\longmapsto} \alpha' \models_V \phi(m) \Leftrightarrow \exists t \in \mathcal{PT} \bullet \langle \psi_m \stackrel{\wedge}{\wedge} X_m, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t, \alpha' \rangle \tag{B.9}$$

We proceed with the proof of Theorem 7. Suppose that

$$H_i = (V, M_i, E_i, S_i, \mu_i, \nu_i, \epsilon_i, \chi_i, \phi_i, \psi_i) ,$$

$$[\![H_i]\!] = (Q_i, Q_i^0, E_i, \rightarrow_i, \mathcal{V}_{\mathsf{st}}, \|_{\text{-}}\|_i) ,$$

for $i = 1, 2$.

We prove the implication from left to right as follows. Consider the relation $B = B_0 \cup B_1$ where

$$B_0 = \{(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle) \mid \langle \psi_{m_1} \stackrel{\wedge}{\wedge} X_{m_1}, \alpha \rangle \leftrightarrow \langle \psi_{m_2} \stackrel{\wedge}{\wedge} X_{m_2}, \alpha \rangle \} ,$$

$$B_1 = \{(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle) \mid \langle X_{m_1}, \alpha \rangle \leftrightarrow \langle X_{m_2}, \alpha \rangle \} .$$

We show that B is a bisimulation. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:

- Because $[\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}}$, it follows from the definition of strong process algebra interpretation that, for all $\alpha \in \mathcal{V}_{\mathsf{st}}$, for all $m_1 \in M_1$ with $\alpha \in [\mathsf{s}(\psi_{m_1} \stackrel{\wedge}{\wedge} X_{m_1})]$, there exists an $m_2 \in M_2$ such that $\langle \psi_{m_1} \stackrel{\wedge}{\wedge} X_{m_1}, \alpha \rangle \leftrightarrow \langle \psi_{m_2} \stackrel{\wedge}{\wedge} X_{m_2}, \alpha \rangle$. Therefore, if $\alpha \in [\mathsf{s}(\psi_{m_1} \stackrel{\wedge}{\wedge} X_{m_1})]$, then there exists an $m_2 \in M_2$ such that $\langle \psi_{m_1} \stackrel{\wedge}{\wedge} X_{m_1}, \alpha \rangle \leftrightarrow \langle \psi_{m_2} \stackrel{\wedge}{\wedge} X_{m_2}, \alpha \rangle$ and $\alpha \in [\mathsf{s}(\psi_{m_2} \stackrel{\wedge}{\wedge} X_{m_2})]$. Using Theorem 6, we conclude: if $\langle m_1, \alpha \rangle \in Q_1^0$, then there exists a $\langle m_2, \alpha \rangle \in Q_2^0$ such that $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$. The proof for the other direction goes analogous.
- Suppose $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$. We proceed by distinguishing the two subrelations:
  - $B_1(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$: In this case, we may assume $\langle X_{m_1}, \alpha \rangle \leftrightarrow \langle X_{m_2}, \alpha \rangle$. We distinguish between jump and flow transitions:
    - Suppose $\langle X_{m_1}, \alpha \rangle \stackrel{a}{\rightarrow} \langle t_1', \alpha' \rangle$. It follows, using (B.1), that $t_1' \equiv X_{m_1'}$ for some $m_1' \in M_1$. Because $\langle X_{m_1}, \alpha \rangle \leftrightarrow \langle X_{m_2}, \alpha \rangle$, it also follows that there exists a $t_2' \in \mathcal{PT}$ such that $\langle X_{m_2}, \alpha \rangle \stackrel{a}{\rightarrow} \langle t_2', \alpha' \rangle$ and $\langle t_1', \alpha' \rangle \leftrightarrow \langle t_2', \alpha' \rangle$. It follows, using (B.1), that $t_2' \equiv X_{m_2'}$ and $\alpha' \in [\mathsf{s}(X_{m_2'})]$ for some $m_2' \in M_2$. Note that, because $t_1' \equiv X_{m_1'}$ and $t_2' \equiv X_{m_2'}$, $\langle X_{m_1'}, \alpha' \rangle \leftrightarrow \langle X_{m_2'}, \alpha' \rangle$. Using Theorem 6, we conclude: if $\langle m_1, \alpha \rangle \stackrel{a}{\rightarrow}_1 \langle m_1', \alpha' \rangle$, then there exists a $\langle m_2', \alpha' \rangle \in Q_2$ such that $\langle m_2, \alpha \rangle \stackrel{a}{\rightarrow}_2 \langle m_2', \alpha' \rangle$ and $B(\langle m_1', \alpha' \rangle, \langle m_2', \alpha' \rangle)$.

Suppose $\langle X_{m_1}, \alpha \rangle \xmapsto{r,\rho} \langle t_1, \alpha' \rangle$. It follows, using (B.3), that $t_1 \leftrightarrow X_{m_1}$. Because $\langle X_{m_1}, \alpha \rangle \leftrightarrow \langle X_{m_2}, \alpha \rangle$, it also follows that there exists a $t_2 \in \mathcal{PT}$ such that $\langle X_{m_2}, \alpha \rangle \xmapsto{r,\rho} \langle t_2, \alpha' \rangle$ and $\langle t_1, \alpha' \rangle \leftrightarrow \langle t_2, \alpha' \rangle$. It follows, using (B.3), that $t_2 \leftrightarrow X_{m_2}$ and $\alpha' \in [\mathsf{s}(X_{m_2})]$. Note that, because $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, $\langle X_{m_1}, \alpha' \rangle \leftrightarrow \langle X_{m_2}, \alpha' \rangle$. Using Theorem 6, we conclude: if $\langle m_1, \alpha \rangle \xmapsto{r}_1 \langle m_1, \alpha' \rangle$, then there exists a $\langle m_2, \alpha' \rangle \in Q_2$ such that $\langle m_2, \alpha \rangle \xmapsto{r}_2 \langle m_2, \alpha' \rangle$ and $B(\langle m_1, \alpha' \rangle, \langle m_2, \alpha' \rangle)$.

The proof for the other direction goes analogous.

· $B_0(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$: The proof for this case goes similar to the proof for the case $B_1(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, using (B.2) and (B.4) instead of (B.1) and (B.3).

• Because $\langle m, \alpha \rangle \in \|\alpha'\|$ iff $\alpha = \alpha'$, the conditions on observations are trivially satisfied.

We prove the implication from right to left as follows. Suppose that B is a bisimulation between $[\![H_1]\!]$ and $[\![H_2]\!]$. Then consider the relation

$$B' = B'_* \cup B'_0 \cup B'_1 \cup {B'_*}^{-1} \cup {B'_0}^{-1} \cup {B'_1}^{-1} \, ,$$

where

$$B'_* = \{(\langle [\![H_1]\!]^{\mathrm{pa}}_{\mathsf{s}}, \alpha \rangle, \langle [\![H_2]\!]^{\mathrm{pa}}_{\mathsf{s}}, \alpha \rangle) \mid \alpha \in \mathcal{V}_{\mathsf{st}}\} \, ,$$

$$B'_0 = \{(\langle \psi_{m_1} \curlywedge X_{m_1}, \alpha \rangle, \langle \psi_{m_2} \curlywedge X_{m_2}, \alpha \rangle) \mid \\ B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle) \wedge \alpha \models \psi_1(m_1) \wedge \alpha \models \psi_2(m_2)\} \, ,$$

$$B'_1 = \{(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle) \mid \exists m_1 \in M_1, m_2 \in M_2 \bullet \\ B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle) \wedge t_1 \leftrightarrow X_{m_1} \wedge t_2 \leftrightarrow X_{m_2}\} \, .$$

Note that, by definition, B' is a symmetric relation. First, we show that B' is a bisimulation. Suppose $B'(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$. We proceed by distinguishing the six subrelation:

• $B'_1(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$: In this case, we may assume that $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$ for some $m_1 \in M_1$ and $m_2 \in M_2$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:
  · Suppose $\langle m_1, \alpha \rangle \xrightarrow{a}_1 \langle m'_1, \alpha' \rangle$. Because $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, it follows that there exists a $\langle m'_2, \alpha' \rangle \in Q_2$ such that $\langle m_2, \alpha \rangle \xrightarrow{a}_2 \langle m'_2, \alpha' \rangle$ and $B(\langle m'_1, \alpha' \rangle, \langle m'_2, \alpha' \rangle)$. Using Theorem 6 and (B.1), and also $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t'_1, \alpha' \rangle$, then there exists a $t'_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t'_2, \alpha' \rangle$ and $B'(\langle t'_1, \alpha' \rangle, \langle t'_2, \alpha' \rangle)$.
  · It follows immediately from the definition of strong process algebra interpretation that not $\langle X_{m_1}, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ for all $a \in A$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \leftrightarrow X_{m_1}$, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ then $\langle t_2, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$.

· Suppose $\langle m_1, \alpha \rangle \stackrel{r}{\longmapsto}_1 \langle m_1, \alpha' \rangle$. Because $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, it follows that $\langle m_2, \alpha' \rangle \in Q_2$, $\langle m_2, \alpha \rangle \stackrel{r}{\longmapsto}_2 \langle m_2, \alpha' \rangle$ and $B(\langle m_1, \alpha' \rangle, \langle m_2, \alpha' \rangle)$. First, using Proposition 1 and (B.7), we conclude: if $\alpha \stackrel{r,\rho}{\longmapsto} \alpha' \models_V \phi_1(m_1)$, then $\alpha \stackrel{r,\rho}{\longmapsto} \alpha' \models_V \phi_2(m_2)$ and $B(\langle m_1, \alpha' \rangle, \langle m_2, \alpha' \rangle)$. Next, using (B.8), and also $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, we conclude: if $\langle t_1, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t_1', \alpha' \rangle$, then there exists a $t_2' \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t_2', \alpha' \rangle$ and $B'(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$.

· Suppose $\langle m_1, \alpha \rangle \in Q_1$. Because $B(\langle m_1, \alpha \rangle, \langle m_2, \alpha \rangle)$, it follows that $\langle m_2, \alpha \rangle \in Q_2$. Using Theorem 6, and also $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$.

· Satisfaction of the condition concerning $\alpha \to \alpha' \in [\mathsf{d}(\_)]$ is proved separately below.

- $B_0'(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$: The proof for this case goes similar to the proof for the case $B_1'(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$, using (B.2) and (B.9) instead of (B.1) and (B.8).

- $B_*'(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$: In this case, we may assume that $t_1 \equiv [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}$ and $t_2 \equiv [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}}$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:

  · Because $[\![H_1]\!] \leftrightarrow [\![H_2]\!]$, it follows from the definition of transition system interpretation that, if $\langle m_1, \alpha' \rangle \in Q_1^0$, then there exists a $\langle m_2, \alpha' \rangle \in Q_2^0$ such that $B(\langle m_1, \alpha' \rangle, \langle m_2, \alpha' \rangle)$. Using Theorem 6, we conclude: if $\alpha' \in [\mathsf{s}(\psi_{m_1} \wedge X_{m_1})]$, then there exists an $m_2 \in M_2$ such that $\alpha' \in [\mathsf{s}(\psi_{m_2} \wedge X_{m_2})]$ and $B'(\langle \psi_{m_1} \wedge X_{m_1}, \alpha' \rangle, \langle \psi_{m_2} \wedge X_{m_2}, \alpha' \rangle)$. Moreover, it follows from the definition of strong process algebra interpretation that, for $i = 1, 2$, $\langle [\![H_i]\!]_{\mathsf{s}}^{\mathrm{pa}}, \alpha \rangle \stackrel{a}{\to} \langle t_i', \alpha' \rangle$ iff $a = \iota$, $t_i' \equiv \psi_{m_i} \wedge X_{m_i}$ and $\alpha \models \psi_i(m_i)$ for some $m_i \in M_i$. Because $t_1 \equiv [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}$ and $t_2 \equiv [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}}$, we conclude: if $\langle t_1, \alpha \rangle \stackrel{a}{\to} \langle t_1', \alpha' \rangle$, then there exists a $t_2' \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \stackrel{a}{\to} \langle t_2', \alpha' \rangle$ and $B'(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$.

  · It follows immediately from the definition of strong process algebra interpretation that not $\langle [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}, \alpha \rangle \stackrel{a}{\to} \langle \sqrt{}, \alpha' \rangle$ for all $a \in A$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \equiv [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}$, we conclude: if $\langle t_1, \alpha \rangle \stackrel{a}{\to} \langle \sqrt{}, \alpha' \rangle$, then $\langle t_2, \alpha \rangle \stackrel{a}{\to} \langle \sqrt{}, \alpha' \rangle$.

  · It follows immediately from the definition of strong process algebra interpretation that not $\langle [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t_1', \alpha' \rangle$ for all $r \in \mathbb{R}^>$, $\rho \in \mathcal{E}_r$, $t_1' \in \mathcal{PT}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \equiv [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}$, we conclude: if $\langle t_1, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t_1', \alpha' \rangle$, then there exists a $t_2' \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \stackrel{r,\rho}{\longmapsto} \langle t_2', \alpha' \rangle$ and $B(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$.

  · It follows immediately from the definition of strong process algebra interpretation that $\alpha \in [\mathsf{s}([\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}})]$ and $\alpha \in [\mathsf{s}([\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}})]$. Because $t_1 \equiv [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}$ and $t_2 \equiv [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}}$, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_1)]$.

  · Satisfaction of the condition concerning $\alpha \to \alpha' \in [\mathsf{d}(\_)]$ is proved separately below.

- The symmetric cases $B_1'^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$, $B_0'^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $B_*'^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ go analogous.

Having proved that all other conditions are satisfied, we can easily prove that the condition concerning $\alpha \to \alpha' \in [\mathsf{d}(\_)]$ is satisfied by B' as well. We proceed by distinguishing again the six subrelation:

- $B'_1(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: In this case, we may assume that $B(\langle m_1, \alpha\rangle, \langle m_2, \alpha\rangle)$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Because all other conditions are satisfied, we conclude immediately from (B.5): if $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$.
- $B'_0(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: The proof for this case goes similar to the proof for the case $B'_1(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$, using (B.6) instead of (B.5).
- $B'_*(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: In this case, we may assume that $t_1 \equiv [\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}}$ and $t_2 \equiv [\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}}$. We conclude immediately from the definition of strong process algebra interpretation: if $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$.
- The symmetric cases $B'_1{}^{-1}(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$, $B'_0{}^{-1}(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$ and $B'_*{}^{-1}(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$ go analogous.

This finishes the proof that B' is a bisimulation. By definition, we have $B'(\langle [\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha\rangle, \langle [\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha\rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. So, we immediately conclude that $[\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}}$. $\square$

B.2 Proof of Proposition 9

In this section, we prove the following proposition.

Proposition 9. *For all hybrid automata H_1 and H_2 with $\mathrm{V}(H_1) = \mathrm{V}(H_2)$:*

$$[\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}} \Rightarrow [\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}} \ .$$

PROOF. Suppose that $\mathrm{V}(H_1) = \mathrm{V}(H_2) = V$. Moreover, suppose that B is a bisimulation witnessing $[\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}} \leftrightarrow [\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}$. Without loss of generality, we assume that B only relates terms reachable from $[\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}}$ and $[\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}$. Then consider the relation

$$B' = \{(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle) \mid B(\langle t_1^\diamond, \alpha\rangle, \langle t_2^\diamond, \alpha\rangle)\} \ ,$$

where $t^\diamond$ is the term t with, in each subterm of the form $\phi \curvearrowright_V t'$, V replaced by $\emptyset$. Note that, by definition, B' is a symmetric relation and $B'(\langle [\![H_1]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha\rangle, \langle [\![H_2]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha\rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. First, we show that B' is a bisimulation. Suppose $B'(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$. Then we may assume that $B(\langle t_1^\diamond, \alpha\rangle, \langle t_2^\diamond, \alpha\rangle)$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:

- Suppose $\langle t_1^\diamond, \alpha\rangle \xrightarrow{a} \langle t_1'^\diamond, \alpha'\rangle$. Because $B(\langle t_1^\diamond, \alpha\rangle, \langle t_2^\diamond, \alpha\rangle)$, it follows that there exists a $t_2' \in \mathcal{PT}$ such that $\langle t_2^\diamond, \alpha\rangle \xrightarrow{a} \langle t_2'^\diamond, \alpha'\rangle$ and $B(\langle t_1'^\diamond, \alpha'\rangle, \langle t_2'^\diamond, \alpha'\rangle)$. It follows from the definitions of strong and weak process algebra interpretation that, for $i = 1, 2$, $\langle t_i, \alpha\rangle \xrightarrow{a} \langle t_i', \alpha'\rangle$ iff $\langle t_i^\diamond, \alpha\rangle \xrightarrow{a} \langle t_i'^\diamond, \alpha'\rangle$. Hence, we conclude: if $\langle t_1, \alpha\rangle \xrightarrow{a} \langle t_1', \alpha'\rangle$, then $\langle t_2, \alpha\rangle \xrightarrow{a} \langle t_2', \alpha'\rangle$ and $B'(\langle t_1', \alpha'\rangle, \langle t_2', \alpha'\rangle)$.

- It follows immediately from the definition of strong process algebra interpretation that not $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ for all $a \in \mathsf{A}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Hence, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$, then $\langle t_2, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$.

- Suppose $\langle t_1^\diamond, \alpha \rangle \xmapsto{r,\rho} \langle {t_1'}^\diamond, \alpha' \rangle$ for some ρ that is smooth for V. Because $B(\langle t_1^\diamond, \alpha \rangle, \langle t_2^\diamond, \alpha \rangle)$, it follows that there exists a $t_2' \in \mathcal{PI}$ such that $\langle t_2^\diamond, \alpha \rangle \xmapsto{r,\rho} \langle {t_2'}^\diamond, \alpha' \rangle$ and $B(\langle {t_1'}^\diamond, \alpha' \rangle, \langle {t_2'}^\diamond, \alpha' \rangle)$. It follows from the definitions of strong and weak process algebra interpretation that, for $i = 1, 2$, $\langle t_i, \alpha \rangle \xmapsto{r,\rho} \langle t_i', \alpha' \rangle$ iff $\langle t_i^\diamond, \alpha \rangle \xmapsto{r,\rho} \langle {t_i'}^\diamond, \alpha' \rangle$ and ρ is smooth for V. Hence, we conclude: if $\langle t_1, \alpha \rangle \xmapsto{r,\rho} \langle t_1', \alpha' \rangle$, then $\langle t_2, \alpha \rangle \xmapsto{r,\rho} \langle t_2', \alpha' \rangle$ and $B'(\langle t_1', \alpha' \rangle, \langle t_2', \alpha' \rangle)$.

- Suppose $\alpha \in [\mathsf{s}(t_1^\diamond)]$. Because $B(\langle t_1^\diamond, \alpha \rangle, \langle t_2^\diamond, \alpha \rangle)$, it follows that $\alpha \in [\mathsf{s}(t_2^\diamond)]$. It follows from the definitions of strong and weak process algebra interpretation that, for $i = 1, 2$, $\alpha \in [\mathsf{s}(t_i)]$ iff $\alpha \in [\mathsf{s}(t_i^\diamond)]$. Hence, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$.

- Suppose $\alpha \to \alpha' \in [\mathsf{d}(t_1^\diamond)]$. Because $B(\langle t_1^\diamond, \alpha \rangle, \langle t_2^\diamond, \alpha \rangle)$, it follows that $\alpha \to \alpha' \in [\mathsf{d}(t_2^\diamond)]$. It follows from the definitions of strong and weak process algebra interpretation that, for $i = 1, 2$, $\alpha \to \alpha' \in [\mathsf{d}(t_i)]$ iff $\alpha \to \alpha' \in [\mathsf{d}(t_i^\diamond)]$ and either $\alpha \to \alpha' \models \mathcal{C}_V$ or $\langle t_i^\diamond, \alpha \rangle \not\xmapsto{}$. Hence, we conclude: if $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$.

This finishes the proof that B' is a bisimulation. By definition, we have $B'(\langle [\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}}, \alpha \rangle, \langle [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}}, \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. So, we immediately conclude that $[\![H_1]\!]_{\mathsf{s}}^{\mathrm{pa}} \underline{\leftrightarrow} [\![H_2]\!]_{\mathsf{s}}^{\mathrm{pa}}$. $\square$

B.3   Proof of Proposition 11

In this section, we prove the following proposition.

Proposition 11. *For all hybrid automata H:*

$$[\![H]\!]_{\mathsf{s}}^{\mathrm{pa}} \underline{\leftrightarrow} \mathrm{V}(H) \, \nabla \, [\![H]\!]_{\mathsf{w}}^{\mathrm{pa}} \, .$$

PROOF. In order to preclude confusion between the variables from the different guarded recursive specifications in contexts where they are used as constants, we decorate the variables from the guarded recursive specification that forms part of $[\![H]\!]_{\mathsf{s}}^{\mathrm{pa}}$ with the superscript "$'$" and the variables from the guarded recursive specification that forms part of $[\![H]\!]_{\mathsf{w}}^{\mathrm{pa}}$ with the superscript "$''$" wherever they are used as constants.

Suppose that $H = (V, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$. Consider the relation

$$B = B_* \cup B_0 \cup B_1 \cup B_*{}^{-1} \cup B_0{}^{-1} \cup B_1{}^{-1} \, ,$$

where

$$B_* = \{(\langle[\![H]\!]^{\mathrm{pa}}_{\mathsf{s}}, \alpha\rangle, \langle V \nabla [\![H]\!]^{\mathrm{pa}}_{\mathsf{w}}, \alpha\rangle) \mid \alpha \in \mathcal{V}_{\mathsf{st}}\} \ ,$$

$$B_0 = \{((\langle\psi_m \curlywedge X'_m, \alpha\rangle, \langle V \nabla (\psi_m \curlywedge X''_m), \alpha\rangle) \mid \alpha \models \phi(m) \wedge \alpha \models \psi(m)\} \ ,$$

$$B_1 = \{(\langle t', \alpha\rangle, \langle V \nabla t'', \alpha\rangle) \mid \exists m \in M \bullet t' \leftrightarrow X'_m \wedge t'' \leftrightarrow X''_m \wedge \alpha \models \phi(m)\} \ .$$

Note that, by definition, B is a symmetric relation. First, we show that B is a bisimulation. Suppose $B(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$. We proceed by distinguishing the six subrelations:

- $B_1(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: In this case, we may assume that $t_1 \leftrightarrow X'_m$, $t_2 \equiv V \nabla t''$, $t'' \leftrightarrow X''_m$ and $\alpha \models \phi(m)$ for some $t'' \in \mathcal{PT}$ and $m \in M$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:
  - Suppose $\langle t_1, \alpha\rangle \xrightarrow{a} \langle t'_1, \alpha'\rangle$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows from the definitions of strong and weak process algebra interpretation that $\langle t'', \alpha\rangle \xrightarrow{a} \langle t'_2, \alpha'\rangle$, $t'_1 \leftrightarrow X'_{m'}$, $t'_2 \leftrightarrow X''_{m'}$ and $\alpha' \models \phi(m')$ for some $t'_2 \in \mathcal{PT}$ and $m' \in M$. Then also $\langle V \nabla t'', \alpha\rangle \xrightarrow{a} \langle V \nabla t'_2, \alpha'\rangle$. Because $t_2 \equiv V \nabla t''$, we conclude: if $\langle t_1, \alpha\rangle \xrightarrow{a} \langle t'_1, \alpha'\rangle$, then there exists a $t''_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha\rangle \xrightarrow{a} \langle t''_2, \alpha'\rangle$ and $B(\langle t'_1, \alpha'\rangle, \langle t''_2, \alpha'\rangle)$.
  - It follows immediately from the definition of strong process algebra interpretation that not $\langle t_1, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle$ for all $a \in \mathsf{A}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Hence, we conclude: if $\langle t_1, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle$, then $\langle t_2, \alpha\rangle \xrightarrow{a} \langle\sqrt{}, \alpha'\rangle$.
  - Suppose $\langle t_1, \alpha\rangle \overset{r,\rho}{\longmapsto} \langle t'_1, \alpha'\rangle$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows from the definitions of strong and weak process algebra interpretation that $\langle t'', \alpha\rangle \overset{r,\rho}{\longmapsto} \langle t'_2, \alpha'\rangle$, $t'_1 \leftrightarrow X'_m$, $t'_2 \leftrightarrow X''_m$, $\alpha' \models \phi(m)$ and $\alpha \overset{r,\rho}{\longmapsto} \alpha' \models_V \mathsf{T}$ for some $t'_2 \in \mathcal{PT}$. Then also $\langle V \nabla t'', \alpha\rangle \overset{r,\rho}{\longmapsto} \langle V \nabla t'_2, \alpha'\rangle$. Because $t_2 \equiv V \nabla t''$, we conclude: if $\langle t_1, \alpha\rangle \overset{r,\rho}{\longmapsto} \langle t'_1, \alpha'\rangle$, then there exists a $t''_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha\rangle \overset{r,\rho}{\longmapsto} \langle t''_2, \alpha'\rangle$ and $B(\langle t'_1, \alpha'\rangle, \langle t''_2, \alpha'\rangle)$.
  - Suppose $\alpha \in [\mathsf{s}(t_1)]$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows from the definitions of strong and weak process algebra interpretation that $\alpha \in [\mathsf{s}(t'')]$. Then also $\alpha \in [\mathsf{s}(V \nabla t'')]$. Because $t_2 \equiv V \nabla t''$, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$.
  - Suppose $\alpha \rightarrow \alpha' \in [\mathsf{d}(t_1)]$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows from the definitions of strong and weak process algebra interpretation that $\alpha \rightarrow \alpha' \in [\mathsf{d}(t'')]$ and either $\alpha \rightarrow \alpha' \models \mathcal{C}_V$ or $\langle t'', \alpha\rangle \not\rightarrow$. Then also $\alpha \rightarrow \alpha' \in [\mathsf{d}(V \nabla t'')]$. Because $t_2 \equiv V \nabla t''$, we conclude: if $\alpha \rightarrow \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \rightarrow \alpha' \in [\mathsf{d}(t_2)]$.
- $B_0(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: The proof for this case goes similar to the proof for the case $B_1(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$.
- $B_*(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: In this case, we may assume that $t_1 \equiv [\![H]\!]^{\mathrm{pa}}_{\mathsf{s}}$ and $t_2 \equiv V \nabla [\![H]\!]^{\mathrm{pa}}_{\mathsf{w}}$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:
  - Suppose $\langle t_1, \alpha\rangle \xrightarrow{a} \langle t'_1, \alpha'\rangle$. It follows from the definition of strong process

algebra interpretation that $\langle [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle t'_1, \alpha' \rangle$ iff $a = \iota$, $t'_1 \equiv \psi_m \mathbin{\!\!^{\blacktriangle}\!\!} X'_m$, $\alpha' \models \phi(m)$ and $\alpha' \models \psi(m)$ for some $m \in M$. Moreover, it follows from the definitions of weak process algebra interpretation that $\langle [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle t'_2, \alpha' \rangle$ iff $a = \iota$, $t'_2 \equiv \psi_m \mathbin{\!\!^{\blacktriangle}\!\!} X''_m$, $\alpha' \models \phi(m)$ and $\alpha' \models \psi(m)$ for some $m \in M$. Then also $\langle V \mathbin{\nabla} [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle V \mathbin{\nabla} t'_2, \alpha' \rangle$ iff $a = \iota$, $t'_2 \equiv \psi_m \mathbin{\!\!^{\blacktriangle}\!\!} X''_m$, $\alpha' \models \phi(m)$ and $\alpha' \models \psi(m)$ for some $m \in M$. Because $t_1 \equiv [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}$ and $t_2 \equiv V \mathbin{\nabla} [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}$, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t'_1, \alpha' \rangle$, then there exists a $t''_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t''_2, \alpha' \rangle$ and $B(\langle t'_1, \alpha' \rangle, \langle t''_2, \alpha' \rangle)$.

· It follows immediately from the definition of strong process algebra interpretation that not $\langle [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$ for all $a \in \mathsf{A}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \equiv [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}$, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$, then $\langle t_2, \alpha \rangle \xrightarrow{a} \langle \surd, \alpha' \rangle$.

· It follows immediately from the definition of strong process algebra interpretation that not $\langle [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha \rangle \xmapsto{r,\rho} \langle t'_1, \alpha' \rangle$ for all $r \in \mathbb{R}^{>}$, $\rho \in \mathcal{E}_r$, $t'_1 \in \mathcal{PT}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \equiv [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}$, we conclude: if $\langle t_1, \alpha \rangle \xmapsto{r,\rho} \langle t'_1, \alpha' \rangle$, then there exists a $t'_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \xmapsto{r,\rho} \langle t'_2, \alpha' \rangle$ and $B(\langle t'_1, \alpha' \rangle, \langle t'_2, \alpha' \rangle)$.

· It follows immediately from the definitions of strong and weak process algebra interpretation that $\alpha \in [\mathsf{s}([\![H]\!]_{\mathrm{s}}^{\mathrm{pa}})]$ and $\alpha \in [\mathsf{s}([\![H]\!]_{\mathrm{w}}^{\mathrm{pa}})]$. Then also $\alpha \in [\mathsf{s}(V \mathbin{\nabla} [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}})]$. Because $t_1 \equiv [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}$ and $t_2 \equiv V \mathbin{\nabla} [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}$, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$.

 · The case $\alpha \rightarrow \alpha' \in [\mathsf{d}(t_1)]$ goes analogous to the previous case.

• The symmetric cases $B_1^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$, $B_0^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $B_*^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ are easy because the line of reasoning used for each condition in the previous cases can be reversed.

This finishes the proof that B is a bisimulation. By definition, we have $B(\langle [\![H]\!]_{\mathrm{s}}^{\mathrm{pa}}, \alpha \rangle, \langle \mathrm{V}(H) \mathbin{\nabla} [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}, \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. So, we immediately conclude that $[\![H]\!]_{\mathrm{s}}^{\mathrm{pa}} \underset{\hspace{-0.6em}\leftrightarrow}{} \mathrm{V}(H) \mathbin{\nabla} [\![H]\!]_{\mathrm{w}}^{\mathrm{pa}}$. □

B.4 Proof of Theorem 14

In this section, we prove the following theorem.

Theorem 14. *For all hybrid automata H_1, H_2:*

$$[\![H_1 \times H_2]\!]_{\mathrm{w}}^{\mathrm{pa}} \underset{\hspace{-0.6em}\leftrightarrow}{} \rho_f(\partial_{A'}([\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}})) \;,$$

where $A' = (\mathrm{E}(H_1) \cap \mathrm{E}(H_2)) \cup \{\iota\}$, the renaming function f is such that $f(\overline{a}) = a$ if $a \in A'$ and $f(a) = a$ if $a \notin \{\overline{a} \mid a \in A'\}$, and the communication function γ is such that $\gamma(a, a) = \overline{a}$ if $a \in A'$ and it is undefined otherwise.

PROOF. Suppose that

$$H_i = (V_i, M_i, E_i, S_i, \mu_i, \nu_i, \epsilon_i, \chi_i, \phi_i, \psi_i), \text{ for } i = 1, 2 .$$

From the definitions of synchronized product and weak process algebra interpretation, we obtain for $[\![H_1 \times H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}$:

$$[\![H_1 \times H_2]\!]_{\mathrm{w}}^{\mathrm{pa}} = \sum_{(m_1, m_2) \in M_1 \times M_2} \tilde{\tilde{\imath}} \cdot ((\psi_{m_1} \wedge \psi_{m_2}) \,{}^{\blacktriangle} \langle X_{(m_1, m_2)} | F' \rangle) , \qquad \text{(B.10)}$$

where the guarded recursive specification F' consists of the following equation for each $(m_1, m_2) \in M_1 \times M_2$:

$$
\begin{aligned}
X_{(m_1, m_2)} \\
= (\phi_{m_1} \wedge \phi_{m_2}) \,{}^{\curvearrowright}\!\!\!{}_{\emptyset} \\
\left(\sum_{s \in S_1'} \chi_s \,{}^{\curvearrowright\!\blacktriangledown} \tilde{\tilde{e}}_s \cdot X_{(m_s', m_2)} + \sum_{s \in S_2'} \chi_s \,{}^{\curvearrowright\!\blacktriangledown} \tilde{\tilde{e}}_s \cdot X_{(m_1, m_s')} \right. \\
\left. + \sum_{(s_1, s_2) \in S_3'} (\chi_{s_1} \wedge \chi_{s_2}) \,{}^{\curvearrowright\!\blacktriangledown} \tilde{\tilde{\tilde{e}}}_{s_1} \cdot X_{(m_{s_1}', m_{s_2}')} + \int_{u \in (0, \infty)} \sigma_{\mathrm{rel}}^u (X_{(m_1, m_2)}) \right)
\end{aligned}
$$

with

$$
\begin{aligned}
S_1' &= \{s \in S_1 \mid m_s = m_1, e_s \notin E_2\} , \\
S_2' &= \{s \in S_2 \mid m_s = m_2, e_s \notin E_1\} , \\
S_3' &= \{(s_1, s_2) \in S_1 \times S_2 \mid m_{s_1} = m_1, m_{s_2} = m_2, e_{s_1} = e_{s_2}\} .
\end{aligned}
$$

From the definition of weak process algebra interpretation, we obtain for $[\![H_i]\!]_{\mathrm{w}}^{\mathrm{pa}}$ $(i = 1, 2)$:

$$[\![H_i]\!]_{\mathrm{w}}^{\mathrm{pa}} = \sum_{m_i \in M_i} \tilde{\tilde{\imath}} \cdot (\psi_{m_i} \,{}^{\blacktriangle} \langle X_{m_i} | F_i' \rangle) , \qquad \text{(B.11)}$$

where the guarded recursive specification F_i' consists of the following equation for each $m_i \in M_i$:

$$X_{m_i} = \phi_{m_i} \,{}^{\curvearrowright\!\blacktriangledown}\!\!\!{}_{\emptyset} \left(\sum_{s_i \in S_i^*} \chi_{s_i} \,{}^{\curvearrowright\!\blacktriangledown} \tilde{\tilde{\tilde{e}}}_{s_i} \cdot X_{m_{s_i}'} + \int_{u \in (0, \infty)} \sigma_{\mathrm{rel}}^u (X_{m_i}) \right)$$

with

$$S_i^* = \{s \in S_i \mid m_s = m_i\} .$$

We prove the bisimilarity of $[\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}$ and $\rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}))$ as follows. Consider the relation

$$B = B_* \cup B_0 \cup B_1 \cup B_*{}^{-1} \cup B_0{}^{-1} \cup B_1{}^{-1} \ ,$$

where

$$B_* = \{(\langle [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}, \alpha \rangle, \langle \rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}})), \alpha \rangle) \mid \alpha \in \mathcal{V}_{\mathsf{st}}\} \ ,$$

$$\begin{aligned}
B_0 = \{(\langle ((\psi_{m_1} \wedge \psi_{m_2}) \,^{\blacktriangle\!}\, X_{(m_1,m_2)}, \alpha \rangle, \\
\langle \rho_f(\partial_{A'}((\psi_{m_1} \,^{\blacktriangle\!}\, X_{m_1}) \parallel (\psi_{m_2} \,^{\blacktriangle\!}\, X_{m_2}))), \alpha \rangle) \mid \\
\alpha \models \phi_1(m_1) \wedge \phi_2(m_2) \wedge \alpha \models \psi_1(m_1) \wedge \psi_2(m_2)\} \ ,
\end{aligned}$$

$$\begin{aligned}
B_1 = \{(\langle t, \alpha \rangle, \langle \rho_f(\partial_{A'}(t_1 \parallel t_2)), \alpha \rangle) \mid \exists (m_1, m_2) \in M_1 \times M_2 \bullet \\
t \leftrightarrow X_{(m_1,m_2)} \wedge t_1 \leftrightarrow X_{m_1} \wedge t_2 \leftrightarrow X_{m_2} \wedge \alpha \models \phi_1(m_1) \wedge \phi_2(m_2)\} \ .
\end{aligned}$$

Note that, by definition, B is a symmetric relation. First, we show that B is a bisimulation. Suppose $B(\langle t, \alpha \rangle, \langle t', \alpha \rangle)$. We proceed by distinguishing the six subrelations:

- $B_1(\langle t, \alpha \rangle, \langle t', \alpha \rangle)$: In this case, we may assume that $t' \equiv \rho_f(\partial_{A'}(t_1 \parallel t_2))$, $t \leftrightarrow X_{(m_1,m_2)}$, $t_1 \leftrightarrow X_{m_1}$, $t_2 \leftrightarrow X_{m_2}$, $\alpha \models \phi_1(m_1)$ and $\alpha \models \phi_2(m_2)$ for some $m_1 \in M_1$ and $m_2 \in M_2$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:
  - Suppose $\langle t, \alpha \rangle \xrightarrow{a} \langle t'', \alpha' \rangle$. We proceed by distinguishing the three possibilities for a:

    $a \in E_1$ and $a \notin E_2$: It follows, using (B.10), that $t'' \leftrightarrow X_{(m_1',m_2)}$ and $\alpha' \models \phi_1(m_1') \wedge \phi_2(m_2)$ for some $m_1' \in M_1$. Because $t \leftrightarrow X_{(m_1,m_2)}$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, it also follows, using (B.10) and (B.11), that $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t_1', \alpha' \rangle$ for some t_1' with $t_1' \leftrightarrow X_{m_1'}$ and that $\alpha' \in [\mathsf{s}(t_2)]$. Moreover, because $\alpha \models \phi_2(m_2)$, we have $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$. Hence, it follows that $\langle \rho_f(\partial_{A'}(t_1 \parallel t_2)), \alpha \rangle \xrightarrow{a} \langle \rho_f(\partial_{A'}(t_1' \parallel t_2)), \alpha' \rangle$. Because $t' \equiv \rho_f(\partial_{A'}(t_1 \parallel t_2))$, we conclude: if $\langle t, \alpha \rangle \xrightarrow{a} \langle t'', \alpha' \rangle$, then there exists a $t''' \in \mathcal{PT}$ such that $\langle t', \alpha \rangle \xrightarrow{a} \langle t''', \alpha' \rangle$ and $B(\langle t'', \alpha' \rangle, \langle t''', \alpha' \rangle)$.

    $a \notin E_1$ and $a \in E_2$: This case is analogous to the previous case.

    $a \in E_1$ and $a \in E_2$: It follows, using (B.10), that $t'' \leftrightarrow X_{(m_1',m_2')}$ and $\alpha' \models \phi_1(m_1') \wedge \phi_2(m_2')$ for some $m_1' \in M_1$ and $m_2' \in M_2$. Because $t \leftrightarrow X_{(m_1,m_2)}$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, it also follows, using (B.10) and (B.11), that $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t_1', \alpha' \rangle$ for some t_1' with $t_1' \leftrightarrow X_{m_1'}$ and that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t_2', \alpha' \rangle$ for some t_2' with $t_2' \leftrightarrow X_{m_2'}$. Hence, it follows that $\langle \rho_f(\partial_{A'}(t_1 \parallel t_2)), \alpha \rangle \xrightarrow{a} \langle \rho_f(\partial_{A'}(t_1' \parallel t_2')), \alpha' \rangle$. Because $t' \equiv \rho_f(\partial_{A'}(t_1 \parallel t_2))$, we conclude: if $\langle t, \alpha \rangle \xrightarrow{a} \langle t'', \alpha' \rangle$, then there exists a $t''' \in \mathcal{PT}$ such that $\langle t', \alpha \rangle \xrightarrow{a} \langle t''', \alpha' \rangle$ and $B(\langle t'', \alpha' \rangle, \langle t''', \alpha' \rangle)$.

- It follows immediately from (B.10) that not $\langle t, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ for all $a \in A$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Hence, we conclude: if $\langle t, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$, then $\langle t', \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$.

- Suppose $\langle t, \alpha \rangle \xmapsto{r,\rho} \langle t'', \alpha' \rangle$. It follows, using (B.10), that $t'' \leftrightarrow X_{(m_1,m_2)}$ and $\alpha' \models \phi_1(m_1) \wedge \phi_2(m_2)$. Because $t \leftrightarrow X_{(m_1,m_2)}$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, it also follows, using (B.10) and (B.11), that $\langle t_1, \alpha \rangle \xmapsto{r,\rho} \langle t_1', \alpha' \rangle$ for some t_1' with $t_1' \leftrightarrow X_{m_1}$ and that $\langle t_2, \alpha \rangle \xmapsto{r,\rho} \langle t_2', \alpha' \rangle$ for some t_2' with $t_2' \leftrightarrow X_{m_2}$. Hence, it follows that $\langle \rho_f(\partial_{A'}(t_1 \parallel t_2)), \alpha \rangle \xmapsto{r,\rho} \langle \rho_f(\partial_{A'}(t_1' \parallel t_2')), \alpha' \rangle$. Because $t' \equiv \rho_f(\partial_{A'}(t_1 \parallel t_2))$, we conclude: if $\langle t, \alpha \rangle \xmapsto{r,\rho} \langle t'', \alpha' \rangle$, then there exists a $t''' \in \mathcal{PT}$ such that $\langle t', \alpha \rangle \xmapsto{r,\rho} \langle t''', \alpha' \rangle$ and $B(\langle t'', \alpha' \rangle, \langle t''', \alpha' \rangle)$.

- Suppose $\alpha \in [\mathsf{s}(t)]$. Because $t \leftrightarrow X_{(m_1,m_2)}$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, it follows, using (B.10) and (B.11), that $\alpha \in [\mathsf{s}(t_1)]$ and $\alpha \in [\mathsf{s}(t_2)]$. Then also $\alpha \in [\mathsf{s}(\rho_f(\partial_{A'}(t_1 \parallel t_2)))]$. Because $t' \equiv \rho_f(\partial_{A'}(t_1 \parallel t_2))$, we conclude: if $\alpha \in [\mathsf{s}(t)]$, then $\alpha \in [\mathsf{s}(t')]$.

- Suppose $\alpha \to \alpha' \in [\mathsf{d}(t)]$. Because $t \leftrightarrow X_{(m_1,m_2)}$, $t_1 \leftrightarrow X_{m_1}$ and $t_2 \leftrightarrow X_{m_2}$, it follows, using (B.10) and (B.11), that $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$ and $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$ and also that $\alpha \in [\mathsf{s}(t_1)]$ and $\alpha \in [\mathsf{s}(t_2)]$. Then also $\alpha \to \alpha' \in [\mathsf{d}(\rho_f(\partial_{A'}(t_1 \parallel t_2)))]$. Because $t' \equiv \rho_f(\partial_{A'}(t_1 \parallel t_2))$, we conclude: if $\alpha \to \alpha' \in [\mathsf{d}(t)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t')]$.

- $B_0(\langle t, \alpha \rangle, \langle t', \alpha \rangle)$: The proof for this case goes similar to the proof for the case $B_1(\langle t, \alpha \rangle, \langle t', \alpha \rangle)$.

- $B_*(\langle t, \alpha \rangle, \langle t', \alpha \rangle)$: In this case, we may assume that $t \equiv [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}$ and $t' \equiv \rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}))$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:

  - Suppose $\langle t, \alpha \rangle \xrightarrow{a} \langle t'', \alpha' \rangle$. It follows, using (B.10), that $\langle [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle t'', \alpha' \rangle$ iff $a = \iota$, $t'' \equiv (\psi_{m_1} \wedge \psi_{m_2}) \mathbin{\wedge\!\!\!\blacktriangle} X_{(m_1,m_2)}$, $\alpha' \models \phi_1(m_1) \wedge \phi_2(m_2)$ and $\alpha' \models \psi_1(m_1) \wedge \psi_2(m_2)$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Moreover, it follows, using (B.11), that, for $i = 1, 2$, $\langle [\![H_i]\!]_{\mathsf{w}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle t_i', \alpha' \rangle$ iff $a = \iota$, $t_i' \equiv \psi_{m_i} \mathbin{\wedge\!\!\!\blacktriangle} X_{m_i}$, $\alpha' \models \phi_i(m_i)$ and $\alpha' \models \psi_i(m_i)$ for some $m_i \in M_i$. Then also $\langle \rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}})), \alpha \rangle \xrightarrow{a} \langle t''', \alpha' \rangle$ iff $a = \iota$, $t''' \equiv \rho_f(\partial_{A'}((\psi_{m_1} \mathbin{\wedge\!\!\!\blacktriangle} X_{m_1}) \parallel (\psi_{m_2} \mathbin{\wedge\!\!\!\blacktriangle} X_{m_2})))$, $\alpha' \models \phi_1(m_1)$, $\alpha' \models \psi_1(m_1)$, $\alpha' \models \phi_2(m_2)$ and $\alpha' \models \psi_2(m_2)$ for some $m_1 \in M_1$ and $m_2 \in M_2$. Because $t \equiv [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}$ and $t' \equiv \rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}))$, we conclude: if $\langle t, \alpha \rangle \xrightarrow{a} \langle t'', \alpha' \rangle$, then there exists a $t''' \in \mathcal{PT}$ such that $\langle t', \alpha \rangle \xrightarrow{a} \langle t''', \alpha' \rangle$ and $B(\langle t'', \alpha' \rangle, \langle t''', \alpha' \rangle)$.

  - It follows immediately from (B.10) that not $\langle [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$ for all $a \in A$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t \equiv [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}$, we conclude: if $\langle t, \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$, then $\langle t', \alpha \rangle \xrightarrow{a} \langle \sqrt{}, \alpha' \rangle$.

  - It follows immediately from (B.10) that not $\langle [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}, \alpha \rangle \xmapsto{r,\rho} \langle t'', \alpha' \rangle$ for all $r \in \mathbb{R}^{>}$, $\rho \in \mathcal{E}_r$, $t'' \in \mathcal{PT}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t \equiv [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}$, we conclude: if $\langle t, \alpha \rangle \xmapsto{r,\rho} \langle t'', \alpha' \rangle$, then there exists a $t''' \in \mathcal{PT}$ such that $\langle t', \alpha \rangle \xmapsto{r,\rho} \langle t''', \alpha' \rangle$ and $B(\langle t'', \alpha' \rangle, \langle t''', \alpha' \rangle)$.

  - It follows immediately from (B.10) and (B.11) that $\alpha \in [\mathsf{s}([\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}})]$, $\alpha \in [\mathsf{s}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}})]$ and $\alpha \in [\mathsf{s}([\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}})]$. Then also $\alpha \in [\mathsf{s}(\rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}})))]$. Because $t \equiv [\![H_1 \times H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}$ and $t' \equiv \rho_f(\partial_{A'}([\![H_1]\!]_{\mathsf{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathsf{w}}^{\mathrm{pa}}))$, we conclude: if $\alpha \in [\mathsf{s}(t)]$, then $\alpha \in [\mathsf{s}(t')]$.

· The case $\alpha \to \alpha' \in [\mathsf{d}(t)]$ goes analogous to the previous case.

- The symmetric cases $B_1^{-1}(\langle t, \alpha \rangle, \langle t'', \alpha \rangle)$, $B_0^{-1}(\langle t, \alpha \rangle, \langle t'', \alpha \rangle)$ and $B_*^{-1}(\langle t, \alpha \rangle, \langle t'', \alpha \rangle)$ are easy because the line of reasoning used for each condition in the previous cases can be reversed.

This finishes the proof that B is a bisimulation. By definition, we have $B(\langle [\![H_1 \times H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}, \alpha \rangle, \langle \rho_f(\partial_{A'}([\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}})), \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. So, we immediately conclude that $[\![H_1 \times H_2]\!]_{\mathrm{w}}^{\mathrm{pa}} \leftrightarrow \rho_f(\partial_{A'}([\![H_1]\!]_{\mathrm{w}}^{\mathrm{pa}} \parallel [\![H_2]\!]_{\mathrm{w}}^{\mathrm{pa}}))$. □

B.5 Proof of Theorem 25

In this section, we prove the following theorem.

Theorem 25. *For all continuity controlled hybrid automata H and $V' \subseteq \mathrm{V}(H)$:*

$$[\![V' \nabla H]\!]_{\mathrm{cc}}^{\mathrm{pa}} \leftrightarrow V' \nabla [\![H]\!]_{\mathrm{cc}}^{\mathrm{pa}} .$$

PROOF. Suppose that $H = (V, W, M, E, S, \mu, \nu, \epsilon, \chi, \phi, \psi)$. From the definitions of localization and process algebra interpretation, we obtain for $[\![V' \nabla H]\!]_{\mathrm{cc}}^{\mathrm{pa}}$:

$$[\![V' \nabla H]\!]_{\mathrm{cc}}^{\mathrm{pa}} = \sum_{m \in M} \tilde{\iota} \cdot (\psi_m \mathbin{^{\wedge}\!\!\blacktriangle} \langle X_m | F' \rangle) , \tag{B.12}$$

where the guarded recursive specification F' consists of the following equation for each $m \in M$:

$$X_m = \phi_m \mathbin{^{\curvearrowright}\!\!\blacktriangledown}_{V' \cup W} \left(\sum_{s \in S} \chi_s \mathbin{^{\ulcorner}\!\blacktriangledown} \tilde{\bar{e}}_s \cdot X_{m'_s} + \int_{u \in (0, \infty)} \sigma_{\mathsf{rel}}^u(X_m) \right) .$$

From the definition of process algebra interpretation, we obtain for $[\![H]\!]_{\mathrm{cc}}^{\mathrm{pa}}$:

$$[\![H]\!]_{\mathrm{cc}}^{\mathrm{pa}} = \sum_{m \in M} \tilde{\iota} \cdot (\psi_m \mathbin{^{\wedge}\!\!\blacktriangle} \langle X_m | F \rangle) , \tag{B.13}$$

where the guarded recursive specification F consists of the following equation for each $m \in M$:

$$X_m = \phi_m \mathbin{^{\curvearrowright}\!\!\blacktriangledown}_W \left(\sum_{s \in S} \chi_s \mathbin{^{\ulcorner}\!\blacktriangledown} \tilde{\bar{e}}_s \cdot X_{m'_s} + \int_{u \in (0, \infty)} \sigma_{\mathsf{rel}}^u(X_m) \right) .$$

In order to preclude confusion between the variables from the different guarded recursive specifications in contexts where they are used as constants, we decorate the variables from the guarded recursive specification that forms part of $[\![V' \nabla H]\!]^{\text{pa}}_{\text{cc}}$ with the superscript "′" and the variables from the guarded recursive specification that forms part of $[\![H]\!]^{\text{pa}}_{\text{cc}}$ with the superscript "″" wherever they are used as constants.

We prove the bisimilarity of $[\![V' \nabla H]\!]^{\text{pa}}_{\text{cc}}$ and $V' \nabla [\![H]\!]^{\text{pa}}_{\text{cc}}$ as follows. Consider the relation

$$B = B_* \cup B_0 \cup B_1 \cup B_*^{-1} \cup B_0^{-1} \cup B_1^{-1} \ ,$$

where

$$B_* = \{(\langle [\![V' \nabla H]\!]^{\text{pa}}_{\text{cc}}, \alpha\rangle, \langle V' \nabla [\![H]\!]^{\text{pa}}_{\text{cc}}, \alpha\rangle) \mid \alpha \in \mathcal{V}_{\text{st}}\} \ ,$$

$$B_0 = \{(\langle \psi_m \curlywedge X'_m, \alpha\rangle, \langle V' \nabla (\psi_m \curlywedge X''_m), \alpha\rangle) \mid \alpha \models \phi(m) \wedge \alpha \models \psi(m)\} \ ,$$

$$B_1 = \{(\langle t', \alpha\rangle, \langle V' \nabla t'', \alpha\rangle) \mid \exists m \in M \bullet t' \leftrightarrow X'_m \wedge t'' \leftrightarrow X''_m \wedge \alpha \models \phi(m)\} \ .$$

Note that, by definition, B is a symmetric relation. First, we show that B is a bisimulation. Suppose $B(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$. We proceed by distinguishing the six subrelations:

- $B_1(\langle t_1, \alpha\rangle, \langle t_2, \alpha\rangle)$: In this case, we may assume that $t_1 \leftrightarrow X'_m$, $t_2 \equiv V' \nabla t''$, $t'' \leftrightarrow X''_m$ and $\alpha \models \phi(m)$ for some $t'' \in \mathcal{PT}$ and $m \in M$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:
  - Suppose $\langle t_1, \alpha\rangle \xrightarrow{a} \langle t'_1, \alpha'\rangle$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows, using (B.12) and (B.13), that $\langle t'', \alpha\rangle \xrightarrow{a} \langle t'_2, \alpha'\rangle$, $t'_1 \leftrightarrow X'_{m'}$, $t'_2 \leftrightarrow X''_{m'}$ and $\alpha' \models \phi(m')$ for some $t'_2 \in \mathcal{PT}$ and $m' \in M$. Then also $\langle V' \nabla t'', \alpha\rangle \xrightarrow{a} \langle V' \nabla t'_2, \alpha'\rangle$. Because $t_2 \equiv V' \nabla t''$, we conclude: if $\langle t_1, \alpha\rangle \xrightarrow{a} \langle t'_1, \alpha'\rangle$, then there exists a $t''_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha\rangle \xrightarrow{a} \langle t''_2, \alpha'\rangle$ and $B(\langle t'_1, \alpha'\rangle, \langle t''_2, \alpha'\rangle)$.
  - It follows immediately from (B.12) that not $\langle t_1, \alpha\rangle \xrightarrow{a} \langle \sqrt{}, \alpha'\rangle$ for all $a \in \mathsf{A}$ and $\alpha' \in \mathcal{V}_{\text{st}}$. Hence, we conclude: if $\langle t_1, \alpha\rangle \xrightarrow{a} \langle \sqrt{}, \alpha'\rangle$, then $\langle t_2, \alpha\rangle \xrightarrow{a} \langle \sqrt{}, \alpha'\rangle$.
  - Suppose $\langle t_1, \alpha\rangle \xmapsto{r,\rho} \langle t'_1, \alpha'\rangle$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows, using (B.12) and (B.13), that $\langle t'', \alpha\rangle \xmapsto{r,\rho} \langle t'_2, \alpha'\rangle$, $t'_1 \leftrightarrow X'_m$, $t'_2 \leftrightarrow X''_m$, $\alpha' \models \phi(m)$ and $\alpha \xmapsto{r,\rho} \alpha' \models_{V'} \mathsf{T}$ for some $t'_2 \in \mathcal{PT}$. Then also $\langle V' \nabla t'', \alpha\rangle \xmapsto{r,\rho} \langle V' \nabla t'_2, \alpha'\rangle$. Because $t_2 \equiv V' \nabla t''$, we conclude: if $\langle t_1, \alpha\rangle \xmapsto{r,\rho} \langle t'_1, \alpha'\rangle$, then there exists a $t''_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha\rangle \xmapsto{r,\rho} \langle t''_2, \alpha'\rangle$ and $B(\langle t'_1, \alpha'\rangle, \langle t''_2, \alpha'\rangle)$.
  - Suppose $\alpha \in [\mathsf{s}(t_1)]$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows, using (B.12) and (B.13), that $\alpha \in [\mathsf{s}(t'')]$. Then also $\alpha \in [\mathsf{s}(V' \nabla t'')]$. Because $t_2 \equiv V' \nabla t''$, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$.
  - Suppose $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$. Because $t_1 \leftrightarrow X'_m$ and $t'' \leftrightarrow X''_m$, it follows, using (B.12) and (B.13), that $\alpha \to \alpha' \in [\mathsf{d}(t'')]$ and either $\alpha \to \alpha' \models \mathcal{C}_{V'}$ or

$\langle t'', \alpha \rangle \nrightarrow$. Then also $\alpha \to \alpha' \in [\mathsf{d}(V' \nabla t'')]$. Because $t_2 \equiv V' \nabla t''$, we conclude: if $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$, then $\alpha \to \alpha' \in [\mathsf{d}(t_2)]$.

- $B_0(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$: The proof for this case goes similar to the proof for the case $B_1(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$.
- $B_*(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$: In this case, we may assume that $t_1 \equiv [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$ and $t_2 \equiv V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$. We proceed by distinguishing the different conditions to be satisfied by a bisimulation:
  - Suppose $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t'_1, \alpha' \rangle$. It follows, using (B.12), that $\langle [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle \xrightarrow{a} \langle t'_1, \alpha' \rangle$ iff $a = \iota$, $t'_1 \equiv \psi_m \wedge X'_m$, $\alpha' \models \phi(m)$ and $\alpha' \models \psi(m)$ for some $m \in M$. Moreover, it follows, using (B.13), that $\langle [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle \xrightarrow{a} \langle t'_2, \alpha' \rangle$ iff $a = \iota$, $t'_2 \equiv \psi_m \wedge X''_m$, $\alpha' \models \phi(m)$ and $\alpha' \models \psi(m)$ for some $m \in M$. Then also $\langle V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle \xrightarrow{a} \langle V' \nabla t'_2, \alpha' \rangle$ iff $a = \iota$, $t'_2 \equiv \psi_m \wedge X''_m$, $\alpha' \models \phi(m)$ and $\alpha' \models \psi(m)$ for some $m \in M$. Because $t_1 \equiv [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$ and $t_2 \equiv V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle t'_1, \alpha' \rangle$, then there exists a $t'_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \xrightarrow{a} \langle t'_2, \alpha' \rangle$ and $B(\langle t'_1, \alpha' \rangle, \langle t'_2, \alpha' \rangle)$.
  - It follows immediately from (B.12) that not $\langle [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle \xrightarrow{a} \langle \checkmark, \alpha' \rangle$ for all $a \in \mathsf{A}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \equiv [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$, we conclude: if $\langle t_1, \alpha \rangle \xrightarrow{a} \langle \checkmark, \alpha' \rangle$, then $\langle t_2, \alpha \rangle \xrightarrow{a} \langle \checkmark, \alpha' \rangle$.
  - It follows immediately from (B.12) that not $\langle [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle \xmapsto{r, \rho} \langle t'_1, \alpha' \rangle$ for all $r \in \mathbb{R}^>$, $\rho \in \mathcal{E}_r$, $t'_1 \in \mathcal{PT}$ and $\alpha' \in \mathcal{V}_{\mathsf{st}}$. Because $t_1 \equiv [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$, we conclude: if $\langle t_1, \alpha \rangle \xmapsto{r, \rho} \langle t'_1, \alpha' \rangle$, then there exists a $t'_2 \in \mathcal{PT}$ such that $\langle t_2, \alpha \rangle \xmapsto{r, \rho} \langle t'_2, \alpha' \rangle$ and $B(\langle t'_1, \alpha' \rangle, \langle t'_2, \alpha' \rangle)$.
  - It follows immediately from (B.12) and (B.13) that $\alpha \in [\mathsf{s}([\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}})]$ and $\alpha \in [\mathsf{s}([\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}})]$. Then also $\alpha \in [\mathsf{s}(V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}})]$. Because $t_1 \equiv [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$ and $t_2 \equiv V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$, we conclude: if $\alpha \in [\mathsf{s}(t_1)]$, then $\alpha \in [\mathsf{s}(t_2)]$.
  - The case $\alpha \to \alpha' \in [\mathsf{d}(t_1)]$ goes analogous to the previous case.
- The symmetric cases $B_1^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$, $B_0^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ and $B_*^{-1}(\langle t_1, \alpha \rangle, \langle t_2, \alpha \rangle)$ are easy because the line of reasoning used for each condition in the previous cases can be reversed.

This finishes the proof that B is a bisimulation. By definition, we have $B(\langle [\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle, \langle V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}, \alpha \rangle)$ for all $\alpha \in \mathcal{V}_{\mathsf{st}}$. So, we immediately conclude that $[\![V' \nabla H]\!]^{\mathrm{pa}}_{\mathrm{cc}} \leftrightarrow V' \nabla [\![H]\!]^{\mathrm{pa}}_{\mathrm{cc}}$. $\quad\square$

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